(1)

Unit - I

Logarithmic Series.

Formulas:

When $-1 < x \leq 1$, we have

(i) $\log_e (1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$

(ii) $\log (1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \cdots$

Note:

$\log 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$

$-\log(1-x) = x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \cdots$

Problems:

(1) What is the sum of the series.

$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots ?$

Solution:

W.K.T.

$\log (1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$

put $x = 1$.

$\therefore \log (1+1) = 1 - \frac{1^2}{2} + \frac{1^3}{3} - \frac{1^4}{4} + \cdots$

$\log 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$

(2) If $|x| > 1$, then find $\frac{1}{x} + \frac{1}{3x^3} + \frac{1}{5x^5} + \cdots$

Soln

W.K.T.

$\frac{1}{2} \log \frac{1+x}{1-x} = x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots \quad |x| < 1$

Replacing x by $\frac{1}{x}$,

we get,

$$\frac{1}{x} + \frac{1}{3x^3} + \frac{1}{5x^5} + \cdots = \frac{1}{2} \log \frac{1+\frac{1}{x}}{1-\frac{1}{x}}$$

$$\frac{1}{x} + \frac{1}{3x^3} + \frac{1}{5x^5} + \cdots = \frac{1}{2} \log \frac{x+1}{x-1}$$

(3) Show that

$$\log 10 = 3\log 2 + \frac{1}{4} - \frac{1}{2}\cdot\frac{1}{4^2} + \frac{1}{3}\cdot\frac{1}{4^3} - \cdots \infty$$

Soln:

Let $\frac{1}{4} = x$, we've

$$\text{RHS} = 3\log 2 + x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots \infty$$

$$= 3\log 2 + \log(1+x)$$

$$= 3\log 2 + \log\left(1+\frac{1}{4}\right)$$

$$= \log 2^3 + \log \frac{5}{4}$$

$$= \log\left(2^3 \times \frac{5}{4}\right) = \log\left(\overset{2}{\cancel{8}} \times \frac{5}{\underset{1}{\cancel{4}}}\right)$$

$$= \log 10 = \text{LHS}.$$

$\therefore$ RHS = LHS

Hence proved.

(4) If $y = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$,

Show that $x = y + \frac{y^2}{2!} + \frac{y^3}{3!} + \frac{y^4}{4!} + \cdots$

Soln:

use $e^{\log f(x)} = f(x)$

Now, $y = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$

$\therefore y = \log(1+x)$

$\therefore e^y = e^{\log(1+x)} = 1+x$

(or) $1+x = e^y$

$$\therefore\ 1+x = 1 + \frac{y}{1!} + \frac{y^2}{2!} + \frac{y^3}{3!} + \frac{y^4}{4!} + \cdots$$

$$\therefore\ x = y + \frac{y^2}{2!} + \frac{y^3}{3!} + \frac{y^4}{4!} + \cdots$$

Hence the result.

(5) Sum the series $\log_{10} e - \log_{10^2} e + \log_{10^3} e - \cdots$

Soln:

changing the bases to 10,

$$\log_{10^2} e = \frac{\log_{10} e}{\log_{10}(10^2)} = \frac{\log_{10} e}{2}$$

$$\log_{10^3} e = \frac{\log_{10} e}{\log_{10}(10^3)} = \frac{\log_{10} e}{3}$$

$$\text{sum} = \log_{10} e - \frac{\log_{10} e}{2} + \frac{\log_{10} e}{3} - \cdots$$

$$= \log_{10} e \left[1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots\right]$$

$$= \log_{10} e \cdot \log_e 2 = \log_{10} 2.$$

(6) Show that

$$\frac{a-x}{a} + \frac{1}{2}\left(\frac{a-x}{a}\right)^2 + \frac{1}{3}\left(\frac{a-x}{a}\right)^3 + \cdots = \log a - \log x$$

Soln:

Let $\frac{a-x}{a} = y$, we've

$$\text{LHS} = y + \frac{y^2}{2} + \frac{y^3}{3} + \cdots = -\log(1-y)$$

$$= -\log\left(1 - \frac{a-x}{a}\right) = -\log\left(\frac{a-a+x}{a}\right)$$

$$= -\log\left(\frac{x}{a}\right) = \log\left(\frac{a}{x}\right)$$

$$= \log a - \log x = \text{RHS}.$$

LHS = RHS

Hence proved.

(7) Show that

$$\frac{1}{n+1}+\frac{1}{2(n+1)^2}+\frac{1}{3(n+1)^3}+\cdots=\frac{1}{n}-\frac{1}{2n^2}+\frac{1}{3n^3}-\cdots$$

Soln:

$$LHS=-\log\left(1-\frac{1}{n+1}\right)=-\log\left(\frac{n+1-1}{n+1}\right)$$

$$=-\log\left(\frac{n}{n+1}\right)=\log\left(\frac{n+1}{n}\right)$$

$$=\log\left(\frac{n}{n}+\frac{1}{n}\right)=\log\left(1+\frac{1}{n}\right)$$

$$=\log\left(1+\frac{1}{n}\right)=\frac{1}{n}-\frac{1}{2n^2}+\frac{1}{3n^3}-\cdots=RHS.$$

$\therefore$ LHS = RHS

Hence proved.

(8) If $x>0$, show that

$$\log x=\frac{x-1}{x+1}+\frac{1}{2}\cdot\frac{x^2-1}{(x+1)^2}+\frac{1}{3}\cdot\frac{x^3-1}{(x+1)^3}+\cdots\text{to }\infty$$

Soln:

Rearranging the terms of RHS, we've

$$RHS=\left[\left(\frac{x}{x+1}\right)+\frac{1}{2}\left(\frac{x}{x+1}\right)^2+\frac{1}{3}\left(\frac{x}{x+1}\right)^3+\cdots\right]$$

$$-\left[\left(\frac{1}{x+1}\right)+\frac{1}{2}\left(\frac{1}{x+1}\right)^2+\frac{1}{3}\left(\frac{1}{x+1}\right)^3+\cdots\right]$$

$$=\left[-\log\left(1-\frac{x}{x+1}\right)\right]-\left[-\log\left(1-\frac{1}{x+1}\right)\right]$$

$$=-\log\left(\frac{x+1-x}{x+1}\right)+\log\left(\frac{x+1-1}{x+1}\right)$$

$$=-\log\left(\frac{1}{x+1}\right)+\log\left(\frac{x}{x+1}\right)$$

$$=\log\left(\frac{x}{x+1}\times\frac{x+1}{1}\right)=\log x=LHS.$$

$\therefore$ RHS = LHS.

Hence proved.

(9) If α and β are the roots of the equation then $ax^2+bx+c=0$, show that.

$$\log(ax^2+bx+c) = \log a + 2\log x - \frac{\alpha+\beta}{x} - \frac{\alpha^2+\beta^2}{2x^2} \cdots$$

proveded $|x|>\alpha$, $|x|>\beta$.

<u>Soln:</u>

Sum of the roots $\alpha+\beta = -b/a$,

product of the roots $\alpha\beta = c/a$.

$$\text{RHS} = \log a + 2\log x - \left[\left(\frac{\alpha}{x}\right) + \frac{1}{2}\left(\frac{\alpha}{x}\right)^2 + \cdots\right] - \left[\left(\frac{\beta}{x}\right) + \frac{1}{2}\left(\frac{\beta}{x}\right)^2 + \cdots\right]$$

$$= \log a + \log x^2 - \left[-\log\left(1-\frac{\alpha}{x}\right)\right] - \left[-\log\left(1-\frac{\beta}{x}\right)\right]$$

$$\left(\because -\log(1-x) = x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots\right)$$

$$= \log\left[ax^2\right] + \log\left(1-\frac{\alpha}{x}\right) + \log\left(1-\frac{\beta}{x}\right)$$

$$= \log\left[ax^2\left(1-\frac{\alpha}{x}\right)\left(1-\frac{\beta}{x}\right)\right]$$

$$= \log\left[a\cancel{x^2}\left(\frac{x-\alpha}{\cancel{x}}\right)\left(\frac{x-\beta}{\cancel{x}}\right)\right]$$

$$= \log\left[a(x-\alpha)(x-\beta)\right]$$

$$= \log\left[a\{x^2 - (\alpha+\beta)x + \alpha\beta\}\right]$$

$$= \log\left[a\{x^2 - \left(-\frac{b}{a}\right)x + \frac{c}{a}\}\right]$$

$$= \log\left[ax^2 + \cancel{a}\left(\frac{b}{\cancel{a}}\right)x + \cancel{a}\left(\frac{c}{\cancel{a}}\right)\right]$$

$$= \log(ax^2 + bx + c) = \text{LHS}.$$

$\therefore$ RHS = LHS.

Hence proved.

Note. The expansions for $\log\left(1-\frac{\alpha}{x}\right)$ and $\log\left(1-\frac{\beta}{x}\right)$ are valid when $\left|\frac{\alpha}{x}\right| < 1$. and

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$|\beta/x| < 1$. Assuming that α and β are positive, we get $|x| > \alpha$ and $|x| > \beta$.

(10) Prove that

$$\log\sqrt{12} = 1 + \left(\frac{1}{2}+\frac{1}{3}\right)\frac{1}{4} + \left(\frac{1}{4}+\frac{1}{5}\right)\frac{1}{4^2} + \left(\frac{1}{6}+\frac{1}{7}\right)\frac{1}{4^3} \cdots$$

Proof:

Rearranging the terms of RHS, we've

$$RHS = \left(1 + \frac{1}{3}\cdot\frac{1}{4} + \frac{1}{5}\cdot\frac{1}{4^2} + \frac{1}{7}\cdot\frac{1}{4^3} + \cdots\right)$$
$$+ \left(\frac{1}{2}\cdot\frac{1}{4} + \frac{1}{4}\cdot\frac{1}{4^2} + \frac{1}{6}\cdot\frac{1}{4^3} + \cdots\right)$$

$$= 2\left[\left(\frac{1}{2}\right) + \frac{1}{3}\cdot\left(\frac{1}{2}\right)^3 + \frac{1}{5}\cdot\left(\frac{1}{2}\right)^5 + \frac{1}{7}\left(\frac{1}{2}\right)^7 + \cdots\right]$$
$$+ \frac{1}{2}\left[\left(\frac{1}{4}\right) + \frac{1}{2}\left(\frac{1}{4}\right)^2 + \frac{1}{3}\left(\frac{1}{4}\right)^3 + \cdots\right]$$

$$= \left[\log\frac{1+\frac{1}{2}}{1-\frac{1}{2}}\right] + \frac{1}{2}\left[-\log\left(1-\frac{1}{4}\right)\right]$$

$$= \left(\log\frac{\frac{3}{2}}{\frac{1}{2}}\right) + \frac{1}{2}\left[-\log\left(\frac{3}{4}\right)\right]$$

$$= \log\left(\frac{3}{\cancel{2}}\times\frac{\cancel{2}}{1}\right) + \frac{1}{2}\left[-\log\left(\frac{3}{4}\right)\right]$$

$$= \log 3 - \frac{1}{2}\log\frac{3}{4} = \frac{1}{2}\left[2\log 3 - \log\frac{3}{4}\right]$$

$$= \frac{1}{2}\left[\log 3^{\cancel{2}}\times\frac{4}{\cancel{3}}\right] = \frac{1}{2}\log 12$$

$$= \log\sqrt{12} = LHS.$$

$\therefore$ RHS = LHS.

Hence proved.

10 mark

(11) If a, b, c denote 3 consecutive integers

(X) Prove that

$$\log b = \frac{1}{2}\log a + \frac{1}{2}\log c + \frac{1}{2ac+1} + \frac{1}{3}\cdot\frac{1}{(2ac+1)^3} + \cdots$$

proof:

let $\frac{1}{2ac+1} = x$, we've

$$RHS = \frac{1}{2}\log a + \frac{1}{2}\log c + x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots$$

$$= \frac{1}{2}[\log a + \log c] + \frac{1}{2}\log\left(\frac{1+x}{1-x}\right)$$

$$= \frac{1}{2}\log(ac) + \frac{1}{2}\log\left(\frac{1+\frac{1}{2ac+1}}{1-\frac{1}{2ac+1}}\right)$$

$$= \frac{1}{2}\log ac + \frac{1}{2}\log\left(\frac{2ac+1+1}{2ac+1-1}\right)$$

$$= \frac{1}{2}\log ac + \frac{1}{2}\log\left(\frac{2ac+2}{2ac}\right)$$

$$= \frac{1}{2}\log ac + \frac{1}{2}\log\frac{\cancel{2}(ac+1)}{\cancel{2}ac}$$

$$= \frac{1}{2}\log ac + \frac{1}{2}\log\frac{ac+1}{ac}$$

$$= \frac{1}{2}\log\left[\cancel{ac} \times \frac{ac+1}{\cancel{ac}}\right]$$

$$= \frac{1}{2}\log(ac+1) \rightarrow (1)$$

But a, b, c are consecutive integers

so, we've

$a = b-1, \quad c = b+1$

$\therefore ac = (b-1)(b+1) = b^2 - 1.$

(or) $ac + 1 = b^2.$

Thus, substituting in (1),

we get.

$$RHS = \frac{1}{2}\log b^2 = \frac{1}{\cancel{2}} \times \cancel{2}\log b$$

$$= \log b = LHS.$$

$\therefore$ RHS = LHS.

Hence proved.

(12) prove that.

(8)

$$\log\left(\frac{1+2e^x}{3}\right) \simeq \frac{2x}{3} + \frac{x^2}{9},$$

neglecting x^3 and higher powers of x.

Proof:

Neglecting x^3 and higher powers of x,

we have.

$$\frac{1+2e^x}{3} = \frac{1}{3}(1+2e^x)$$

$$\simeq \frac{1}{3}\left[1+2\left(1+\frac{x}{1!}+\frac{x^2}{2!}\right)\right]$$

$$= \frac{1}{3}\left[1+2+\frac{2x}{1!}+\frac{2x^2}{2!}\right]$$

$$= \frac{1}{3}\left(1+2+2x+\frac{\cancel{2}x^2}{\cancel{2}}\right)$$

$$= \frac{1}{3}\left(3+(2x+x^2)\right)$$

$$= \frac{1}{\cancel{3}}(\cancel{3})+\frac{1}{3}(2x+x^2)$$

$$= 1+\frac{1}{3}(2x+x^2)$$

$$\therefore \log\left(\frac{1+2e^x}{3}\right) \simeq \log\left[1+\frac{1}{3}(2x+x^2)\right]$$

$$= \frac{1}{3}(2x+x^2) - \frac{1}{2}\left[\frac{1}{3}(2x+x^2)\right]^2 + \cdots$$

$$= \frac{1}{3}(2x+x^2) - \frac{1}{2}\left[\frac{1}{3}(2x)\right]^2 + \cdots$$

$$= \frac{1}{3}(2x+x^2) - \frac{1}{\cancel{2}}\left(\frac{\cancel{4}^{2}x^2}{9}\right)$$

$$= \frac{1}{3}(2x+x^2) - \frac{2x^2}{9}$$

$$= \frac{2x}{3} + \frac{x^2}{3} - \frac{2x^2}{9} = \frac{2x}{3} + \frac{3x^2-2x^2}{9}$$

$$= \frac{2x}{3} + \frac{x^2}{9}$$

$$\therefore \log\left(\frac{1+2e^x}{3}\right) \simeq \frac{2x}{3} + \frac{x^2}{9}$$

Hence proved

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(13) If n is large, prove that

(i) $\left(n-\frac{1}{3n}\right)\log\frac{n+1}{n-1} = 2+\frac{8}{45n^4}$ nearly.

(ii) $\left(\frac{n+1}{n-1}\right)^{n-(1/3n)} = e^2\left(1+\frac{8}{45n^4}\right)$ nearly.

Proof:

(i) $\left(n-\frac{1}{3n}\right)\log\frac{n+1}{n-1} = \left(n-\frac{1}{3n}\right)\log\frac{\left(1+\frac{1}{n}\right)}{\left(1-\frac{1}{n}\right)}$

$= \left(n-\frac{1}{3n}\right)\left[\log\left(1+\frac{1}{n}\right)-\log\left(1-\frac{1}{n}\right)\right]$

$= \left(n-\frac{1}{3n}\right)\cdot 2\left[\frac{1}{n}+\frac{1}{3n^3}+\frac{1}{5n^5}+\cdots\right]$

$= 2\left[\left(1+\frac{1}{3n^2}+\frac{1}{5n^4}+\cdots\right)-\left(\frac{1}{3n^2}+\frac{1}{9n^4}+\cdots\right)\right]$

$= 2\left[1+\cancel{\frac{1}{3n^2}}-\cancel{\frac{1}{3n^2}}+\frac{1}{5n^4}-\frac{1}{9n^4}+\cdots\right]$

$= 2\left[1+\frac{1}{n^4}\left(\frac{1}{5}-\frac{1}{9}\right)\right]$ nearly

$= 2\left[1+\frac{1}{n^4}\left(\frac{9-5}{45}\right)\right] = 2\left[1+\frac{1}{n^4}\cdot\frac{4}{45}\right]$ nearly

$= 2+\frac{8}{45n^4}$ nearly. $\rightarrow$ (1)

(ii) Let $x = \left(\frac{n+1}{n-1}\right)^{n-(1/3n)}$

Taking logarithm,

$\log x = \left(n-\frac{1}{3n}\right)\left[\log\frac{n+1}{n-1}\right]$

$= 2+\frac{8}{45n^4}$ by (1)

$\therefore\ x = e^{2+[8/(45n^4)]} = e^2 \cdot e^{8/(45n^4)}$

$\therefore\ x = e^2\left[1+\frac{8}{45n^4}+\frac{1}{2!}\left(\frac{8}{45n^4}\right)^2+\cdots\right]$

$\therefore x = e^2\left[1+\frac{8}{45n^4}\right]$ nearly.

(14) When n is large, show that $\left(1+\frac{1}{n}\right)^{n+\frac{1}{2}}$ is approximately equal to $e^{1+\frac{1}{12n^2}}$

Soln:

$$\left(1+\frac{1}{n}\right)^{n+\left(\frac{1}{2}\right)} = e^{\log\left(1+\frac{1}{n}\right)^{n+\left(\frac{1}{2}\right)}}$$

$$= e^{\left(n+\frac{1}{2}\right)\log\left(1+\frac{1}{n}\right)}$$

$$= e^{\left(n+\frac{1}{2}\right)\left[\frac{1}{n}-\frac{1}{2n^2}+\frac{1}{3n^3}-\frac{1}{4n^4}+\cdots\right]}$$

$$= e^{1+\left(\frac{1}{2n}-\frac{1}{2n}\right)-\frac{1}{4n^2}+\frac{1}{3n^2}-\frac{1}{6n^3}\cdots}$$

$$\simeq e^{1+\frac{1}{12n^2}}, \text{ neglecting } \frac{1}{n^3}, \frac{1}{n^4} \text{ etc.}$$

$$\therefore \left(1+\frac{1}{n}\right)^{n+\frac{1}{2}} = e^{1+\frac{1}{12n^2}} \text{ approximately.}$$

(15) When n is large, show that

$$\left(\frac{n+1}{n-1}\right)^{n/2} = e\left[1+\frac{1}{3n^2}\right] \text{ nearly}$$

Soln:

$$\left(\frac{n+1}{n-1}\right)^{n/2} = \left(\frac{1+\frac{1}{n}}{1-\frac{1}{n}}\right)^{n/2}$$

$$= e^{\log\left(\frac{1+1/n}{1-1/n}\right)^{n/2}} = e^{(n/2)\log\left(\frac{1+1/n}{1-1/n}\right)}$$

$$= e^{(n/2)\left[\log(1+1/n)-\log\left(1-\frac{1}{n}\right)\right]}$$

$$= e^{(n/2)\times 2\left(\frac{1}{n}+\frac{1}{3n^3}+\frac{1}{5n^5}+\frac{1}{7n^7}+\cdots\right)}$$

$$= e^{1+\frac{1}{3n^2}+\frac{1}{5n^4}+\frac{1}{7n^6}+\cdots}$$

$$= e^1 \cdot e^{\left(\frac{1}{3n^2}+\frac{1}{5n^4}+\cdots\right)}$$

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$$= e\left[1+\left(\frac{1}{3n^2}+\frac{1}{5n^4}+\cdots\right)+\frac{1}{2!}\left(\frac{1}{3n^2}+\frac{1}{5n^4}+\cdots\right)^2 + \cdots\right]$$

$$\simeq e\left[1+\frac{1}{3n^2}\right], \text{ neglecting } \frac{1}{n^4}, \frac{1}{n^6}, \text{ etc}$$

$$\left(\frac{n+1}{n-1}\right)^{n/2} = e\left[1+\frac{1}{3n^2}\right] \text{ nearly.}$$

(15) When n is large, show that.

$$\left(1+\frac{1}{n}\right)^n = e\left[1-\frac{1}{2n}+\frac{11}{24n^2}-\frac{7}{16n^3}+\cdots\right]$$

Soln :

$$\left(1+\frac{1}{n}\right)^n = e^{\log\left(1+\frac{1}{n}\right)^n} = e^{n\log\left(1+\frac{1}{n}\right)}$$

$$= e^{n\left(\frac{1}{n}-\frac{1}{2n^2}+\frac{1}{3n^3}-\frac{1}{4n^4}+\cdots\right)}$$

$$= e^{1-\frac{1}{2n}+\frac{1}{3n^2}-\frac{1}{4n^3}+\cdots}$$

$$= e^1 \cdot e^{-\left(\frac{1}{2n}-\frac{1}{3n^2}+\frac{1}{4n^3}-\cdots\right)}$$

$$= e\left[1-\frac{1}{1!}\left(\frac{1}{2n}-\frac{1}{3n^2}+\frac{1}{4n^3}-\cdots\right) + \frac{1}{2!}\left(\frac{1}{2n}-\frac{1}{3n^2}+\frac{1}{4n^3}-\cdots\right)^2 - \frac{1}{3!}\left(\frac{1}{2n}-\frac{1}{3n^2}+\frac{1}{4n^3}-\cdots\right)^3+\cdots\right]$$

$$= e\left[1-\frac{1}{2n}+\left(\frac{1}{3n^2}+\frac{1}{8n^2}\right)+\left(-\frac{1}{4n^3}-\frac{1}{6n^3}-\frac{1}{48n^3}\right)+\cdots\right]$$

$$= e\left[1-\frac{1}{2n}+\left(\frac{8+3}{24n^2}\right)+\left(\frac{-12-8-1}{48n^3}\right)+\cdots\right]$$

$$\left(1+\frac{1}{n}\right)^n = e\left[1-\frac{1}{2n}+\frac{11}{24n^2}-\frac{7}{16n^3}+\cdots\right]$$

Hence proved

(17) If x is small and $(1+x)^{1/x} = e(1+ax+bx^2)$ nearly, determine the constants a and b

Soln:

From problem (16) n is large and

$$\left(1+\frac{1}{n}\right)^n = e\left[1-\frac{1}{2}\frac{1}{n}+\frac{11}{24}\frac{1}{n^2}\right] \text{nearly}$$

If $x=\frac{1}{n}$, then x is small and

$$(1+x)^{1/x} \simeq e\left[1-\frac{1}{2}x+\frac{11}{24}x^2\right]$$

$\therefore$ $a = -\frac{1}{2}$ and $b = \frac{11}{24}$.

Binomial series

When n is a rational number and $-1<x<1$,

$$(1+x)^n = 1+\frac{n}{1!}+\frac{n(n-1)}{2!}x^2+\frac{n(n-1)(n-2)}{3!}x^3 + \dots$$

$$(1-x)^n = 1-\frac{n}{1!}+\frac{n(n-1)}{2!}x^2 - \frac{n(n-1)(n-2)}{3!}x^3 + \dots$$

Put $n=-n$, we have,

$$(1+x)^{-n} = 1+\frac{(-n)}{1!}x+\frac{(-n)(-n-1)}{2!}x^2 + \frac{(-n)(-n-1)(-n-2)}{3!}x^3+\dots$$

$$(1-x)^{-n} = 1-\frac{(-n)}{1!}(x)+\frac{(-n)(-n-1)}{2!}x^2 - \frac{(-n)(-n-1)(-n-2)}{3!}x^3+\dots$$

$$\therefore (1+x)^{-n} = 1 - \frac{n}{1!}x + \frac{n(n+1)}{2!}x^2 - \frac{n(n+1)(n+2)}{3!}x^3 + \cdots$$

$$(1-x)^{-n} = 1 + \frac{n}{1!}x + \frac{n(n+1)}{2!}x^2 + \frac{n(n+1)(n+2)}{3!}x^3 + \cdots$$

Exponential Series

Maclaurin's Series for e^x,

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$e^{-x} = 1 - \frac{x}{1!} + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots$$

put $x = 1$,

$$e^1 = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots$$

$$e^{-1} = 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots$$

Examples

1. Sum the series $\frac{1+3x}{1!} + \frac{(1+3x)^2}{2!} + \frac{(1+3x)^3}{3!} + \cdots$

Soln Let $1+3x = \alpha$

$$\therefore S = \frac{\alpha}{1!} + \frac{\alpha^2}{2!} + \frac{\alpha^3}{3!} + \cdots$$

$$= \left(1 + \frac{\alpha}{1!} + \frac{\alpha^2}{2!} + \frac{\alpha^3}{3!} + \cdots\right) - 1$$

$$= e^{\alpha} - 1 = e^{1+3x} - 1 = e \cdot e^{3x} - 1 \text{ "}$$

2. Sum the series $1 - \log_e 2 + \frac{(\log_e 2)^2}{2!} - \frac{(\log_e 2)^3}{3!} + \cdots$

Soln:

W.k.t. $e^{\log f(x)} = f(x)$

Now denoting $\log_e 2$ by x,

$$S = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots$$

$$= e^{-x}$$

$$= e^{-\log 2} = e^{\log(2^{-1})} = 2^{-1} = \frac{1}{2}$$ //

3. Show that $\frac{1}{2}\left(e - \frac{1}{e}\right) = 1 + \frac{1}{3!} + \frac{1}{5!} + \cdots$

Proof:

$$e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots \quad \text{—(1)}$$

$$e^{-1} = 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots \quad \text{—(2)}$$

$$\frac{(1)-(2)}{2} \Rightarrow \frac{1}{2}\left(e - \frac{1}{e}\right) = 1 + \frac{1}{3!} + \frac{1}{5!} + \cdots$$ //

4. Sum the series $1 + \frac{3^2}{2!} + \frac{3^4}{4!} + \frac{3^6}{6!} + \cdots$

Soln

W.k.T. $$\frac{e^x + e^{-x}}{2} = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots \quad \text{—(1)}$$

& $$\frac{e^x - e^{-x}}{2} = 1 + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots \quad \text{—(2)}$$

Put $x = 3$ in (1), we get.

$$1 + \frac{3^2}{2!} + \frac{3^4}{4!} + \frac{3^6}{6!} + \cdots = \frac{e^3 + e^{-3}}{2}$$ //

5. Show that $\dfrac{1+\frac{1}{2!}+\frac{1}{4!}+\cdots}{1+\frac{1}{3!}+\frac{1}{5!}+\cdots} = \dfrac{e^2+1}{e^2-1}$

Proof: W.K.T. $\dfrac{e+e^{-1}}{2} = 1+\dfrac{1}{2!}+\dfrac{1}{4!}+\dfrac{1}{6!}+\cdots$

$$\frac{e-e^{-1}}{2} = \frac{1}{1!}+\frac{1}{3!}+\frac{1}{5!}+\frac{1}{7!}+\cdots$$

$$L.H.S = \frac{\frac{1}{2}(e+e^{-1})}{\frac{1}{2}(e-e^{-1})} = \frac{e+1/e}{e-1/e}$$

$$= \frac{\frac{e^2+1}{e}}{\frac{e^2-1}{e}}$$

$$= \frac{e^2+1}{e^2-1}$$

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