(ODE, PDE, LAPLACE TRANSFORMS AND VECTOR ANALYSIS)

(16SACMM3) Unit – II.

Formation of Partial differential equations by eliminating constants and by elimination of arbitrary functions.

(i) Formation of Partial differential equation

1. Elimination of arbitrary Constant

2. Elimination of arbitrary functions.

(ii). Important symbal

$$p = \frac{\partial z}{\partial x}, \quad q = \frac{\partial z}{\partial y}, \quad r = \frac{\partial^2 z}{\partial x^2}$$

$$s = \frac{\partial^2 z}{\partial x \, \partial y}, \quad t = \frac{\partial^2 z}{\partial y^2}$$

Problem : 1

Solve $z = f(ax+y)$

Solution:-

$z = f(ax+y)$, 'a' is Constant

$z_x = f'(ax+y)\,[a] \Rightarrow p = f'a \rightarrow (1)$

Differential both side w.r. to 'y'

$z_y = f'(ax+y)\,[1] \Rightarrow q = f' \rightarrow (2)$

Put, f' from (2) in (1)

$$\boxed{p = aq}$$

//.

2). Solve: $z = f\left(\frac{x}{y}\right)$

Solution:- Given, $z = f\left(\frac{x}{y}\right)$

$$z_x = f'\left(\frac{x}{y}\right)\left[\frac{1}{y}\right] \Rightarrow p = f'\frac{1}{y} \longrightarrow (1)$$

Differential both side w.r.to 'y'

$$z_y = f'\left(\frac{x}{y}\right)\left[x\left(-\frac{1}{y^2}\right)\right]$$

$$\Rightarrow q = \frac{-x}{y^2} f' \longrightarrow (2)$$

from (1) $f' = py$, put in (2)

$$q = \frac{-x}{y^2} - py \Rightarrow yq = -xp$$

$$\therefore \boxed{xp + yq = 0}$$

3). Solve: $z = xy + f(x^2 - y^2)$

Solution:- Given, $z = xy + f(x^2 - y^2)$

$$z_x = y + f'(x^2 - y^2)[2x]$$

$$p = y + 2x f' \Rightarrow f' = \frac{p-y}{2x} \longrightarrow (1)$$

Differential both side w.r.to 'y'

$$z_y = x + f'(x^2 - y^2)[-2y]$$

$$q = x - 2y f' \longrightarrow (2)$$

$$q = x - \cancel{2}y\left[\frac{p-y}{\cancel{2}x}\right]$$

$$\Rightarrow q = \frac{x^2 - y[p-y]}{x}$$

$$qx = x^2 - y\,[p - y]$$

$$qx = x^2 - py + y^2$$

$$\Rightarrow \boxed{qx + py = x^2 + y^2}$$

//.

4). Solve: $z = f(x+ct) + g(x-ct)$

Solution:- Given, $z = f(x+ct) + g(x-ct)$

$$z_x = f'(x+ct) + g'(x-ct) \quad \because \text{C-Constant}$$

Differential w.r.to 'y'

$$z_t = cf'(x+ct) - cg'(x-ct)$$

$$z_{xx} = f''(x+ct) + g''(x-ct) \rightarrow (1)$$

$$z_{tt} = c^2 f'(x+ct) + c^2 g''(x-ct) = c^2[f''(x+ct) + g''(x-ct)]$$

$$\boxed{z_{tt} = c^2 z_{xx}}$$

//.

5). Form the partial differential equation the arbitrary function from $z = y^2 + 2f\left(\frac{1}{x} + \log y\right)$.

Solutions:- Given equation is

$$z = y^2 + 2f\left(\frac{1}{x} + \log y\right) \rightarrow (1)$$

Partially differentiating equation (1) w.r.to x and y respectively,

$$p = \frac{\partial z}{\partial x} = 2f'\left(\frac{1}{x} + \log y\right)\left(-\frac{1}{x^2}\right)$$

$$2f'\left(\frac{1}{x} + \log y\right) = -px^2 \rightarrow (2)$$

and,
$$q = \frac{\partial z}{\partial y} = 2y + 2f'\left(\frac{1}{x} + \log y\right)\frac{1}{y}$$

$$q = 2y - \frac{px^2}{y}$$

$$\Rightarrow qy = 2y^2 - px^2$$

$$\boxed{px^2 + qy - 2y^2 = 0}$$

—x—

II. <u>Definition of general, particular and Complete solutions</u>:

A solution $z = z(x,y)$ of a pde $F(x,y,z,p,q) = 0$, when interpreted as a surface in 3 dimensional space will be called as integral surface of the pde.

(i). Complete Integral (or) complete solution :-

Definition:-

An expression of the type solution of pde $f(x,y,z,a,b) = 0$, which is a solution of pde $F(x,y,z,p,q) = 0$ and a and b are arbitrary constants, is said to be a complete integral of the first order pde $F(x,y,z,p,q) = 0$.

Example:-

The expression $f(x,y,z,a,b) = z - (ax + by + a^2 + b^2) = 0$ has 2 arbitrary constants a and b. And it is also a solution of the pde $F(x,y,z,p,q) = z - px - py - p^2 - q^2 = 0$. So it is complete integral of the above mentioned pde.

(ii). General integral (or) General solution :-

Definition :-

An expression of the type $f(\phi, \psi) = 0$ where ϕ and ψ are two known functions and f is an unknown functions, and is also a solution of a first order pde $F(x, y, z, p, q) = 0$, is said to be a General integral of that First order pde $F(x, y, z, p, q) = 0$.

Example :-

Consider the pde $F(x, y, z, p, q) = z - xp - yq$. Let $\phi = \frac{y}{x}$ and $\psi = \frac{z}{x}$, then for any arbitrary function f, $f(\phi, \psi) = 0$ is a solution of given pde. Thus $f(\phi, \psi) = 0$ is a General integral of given pde.

(iii). Singular integral (or) Singular solution :-

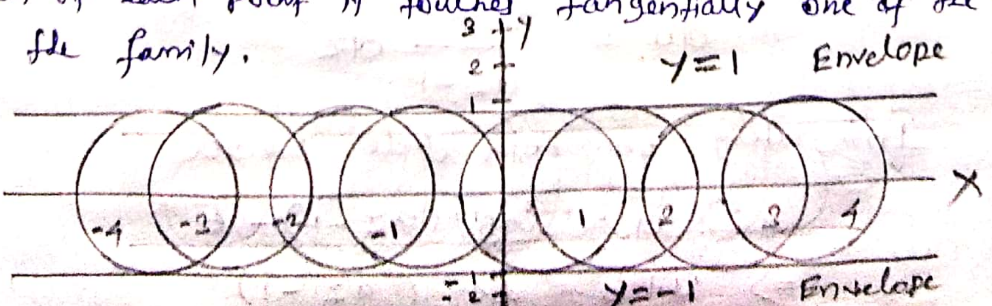
Definition :-

Let $f(x, y, z, a, b) = 0$ be complete integral of pde $F(x, y, z, p, q) = 0$. Now the Envelope of the complete integral (if exists) is also a solution of the pde $F(x, y, z, p, q) = 0$, and that Envelope is called the singular integral of that pde.

Example :-

(Envelope of a family of Curves)

The envelope of a family of curves is a curve such that at each point it touches, tangentially one of the curves of the family.



III. Solutions of first order equation in the standard forms :- $f(p,q)=0$, $f(x,p,q)=0$, $f(y,p,q)=0$, $f(z,p,q)=0$, $f_1(x,p)=f_2(y,q)$, $z=xp+yq+f(p,q)$ is presented in problems.

Problem :- (1)

Solve $p(1+q)=qz$

Solution :-

Given; $p(1+q)=qz \longrightarrow (1)$

This equation is of the form $f(z,p,q)=0$

Let $u = x+ay$

$$\frac{\partial u}{\partial x}=1, \quad \frac{\partial u}{\partial y}=a$$

$$p=\frac{dz}{du}, \quad q=a\frac{dz}{du}$$

$\therefore$ Equation (1) we get $\Rightarrow \frac{dz}{du}\left(1+a\frac{dz}{du}\right)=a\frac{dz}{du}z$

$$1+a\frac{dz}{du}=az$$

$$a\frac{dz}{du}=az-1$$

$$\frac{dz}{du}=\frac{az-1}{a}$$

$$\Rightarrow \frac{du}{dz}=\frac{a}{az-1}$$

$$du=\frac{a}{az-1}dz$$

Integrating on both side, we get

$$u=\log(az-1)-b$$

$$\Rightarrow u+b=\log(az-1)$$

Hence, the Complete solution is

$$x + ay + b = \log\left[(az-1)\right]$$

Since, the number of a.c = number of I.V

General integral can be found out in a usual way. //.

②. Solve :-

$$p(1+q^2) = q(z-a)$$

Solution :-

Given $p(1+q^2) = q(z-a) \longrightarrow ①$

∴ This equation is of the form $f(z,p,q)=0$

Let $u = x + by$

$$p = \frac{dz}{du}$$

$$q = b\frac{dz}{du}$$

Substituting these values of p & q in (1), we get

$$\frac{dz}{du}\left[1+b^2\left(\frac{dz}{du}\right)^2\right] = b\frac{dz}{du}(z-a)$$

$$1+b^2\left(\frac{dz}{du}\right)^2 = b(z-a)$$

$$b^2\left(\frac{dz}{du}\right)^2 = bz-ab-1$$

$$\frac{dz}{du} = \frac{1}{b}\sqrt{bz-ab-1}$$

$$\int \frac{b\,dz}{\sqrt{bz-ab-1}} = \int du.$$

$$2\sqrt{bz-ab-1} = u+c$$

$$4(bz-ab-1) = (u+c)^2$$

Hence, the Complete solution is

$$4(bz-ab-1) = (x+by+c)^2$$

Here, a is a given constant, b and c are arbitrary Constants.

Since, the number of a.c = number of i.v

General integral can be found out in a usual way //.

③. Solve :-

$$z^2 = 1 + p^2 + q^2$$

Solution:-

Given, $z^2 = 1 + p^2 + q^2 \longrightarrow (1)$

The given problem is of the type $f(z, p, q) = 0$.

Let $u = x + ay$

$$\frac{\partial u}{\partial x} = 1, \quad \frac{\partial u}{\partial y} = a$$

$$p = \frac{\partial z}{\partial x} = \frac{dz}{du}\frac{\partial u}{\partial x} = \frac{dz}{du}$$

$$q = \frac{\partial z}{\partial y} = \frac{dz}{du}\frac{\partial u}{\partial y} = a\frac{dz}{du}$$

Substitute in (1), we get

$$z^2 = 1 + \left[\frac{dz}{du}\right]^2 + a^2\left[\frac{dz}{du}\right]^2$$

ie, $z^2 - 1 = \left[\frac{dz}{du}\right]^2 [1 + a^2]$

$$\left[\frac{dz}{du}\right]^2 = \frac{z^2 - 1}{1 + a^2}$$

$$\frac{dz}{du} = \sqrt{\frac{z^2 - 1}{1 + a^2}}$$

$$\frac{dz}{\sqrt{z^2 - 1}} = \frac{du}{\sqrt{1 + a^2}}$$

Integrating on both sides, we get

$$\cosh^{-1} z = \frac{1}{\sqrt{1 + a^2}} u + b$$

Hence, the complete solution is

$$\cosh^{-1} z = \frac{1}{\sqrt{1 + a^2}} \cdot (x + ay) + b$$

Since, the number of a.c = number of i.v

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4. Solve :- $9(p^2 z + q^2) = 4$.

Solution:- Given, $9(p^2 z + q^2) = 4 \rightarrow (1)$

This equation is of the form $f(z, p, q) = 0$

Let $u = x + ay$

$$\frac{\partial u}{\partial x} = 1; \quad \frac{\partial u}{\partial y} = a$$

$$p = \frac{\partial z}{\partial x} = \frac{dz}{du}\frac{\partial u}{\partial x} = \frac{dz}{du}$$

$$q = \frac{\partial z}{\partial y} = \frac{dz}{du}\frac{\partial u}{\partial y} = a\frac{dz}{du}$$

Substituting in (1), we get ;

$$9\left[\left(\frac{dz}{du}\right)^2 z + \left(a\frac{dz}{du}\right)^2\right] = 4$$

$$\left[\frac{dz}{du}\right]^2 = \frac{4}{9(z+a^2)}$$

$$\frac{dz}{du} = \frac{2}{3}\frac{1}{\sqrt{z+a^2}}$$

$$3\sqrt{z+a^2}\, dz = 2du$$

Integrating on both sides, we get

$$3\frac{(z+a^2)^{3/2}}{(3/2)} = 2u + 2b$$

$$\Rightarrow (z+a^2)^{3/2} = x + ay + b$$

Hence, the complete solution is

$$\Rightarrow (z + a^2)^3 = (x + ay + b)^2$$

Since, the number of a.c = number of i.v

General integral can be found out in a usual way. //

5. Solve:- $z = p^2 + q^2$.

Solution:-

Given, $z = p^2 + q^2 \longrightarrow (1)$

This equation is of the form $f(z, p, q) = 0$.

Let $u = x + ay$

$$\frac{\partial u}{\partial x} = 1, \quad \frac{\partial u}{\partial y} = a$$

$$p = \frac{\partial z}{\partial x} = \frac{dz}{du}\frac{\partial u}{\partial x} = \frac{dz}{du}$$

$$q = \frac{\partial z}{\partial y} = \frac{dz}{du}\frac{\partial u}{\partial y} = a\frac{dz}{du}$$

Substituting in (1), we get

$$z = \left[\frac{dz}{du}\right]^2 + a^2\left[\frac{dz}{du}\right]^2$$

$$z = \left[\frac{dz}{du}\right]^2\left[1 + a^2\right]$$

$$\frac{z}{1+a^2} = \left[\frac{dz}{du}\right]^2$$

$$\frac{dz}{du} = \sqrt{\frac{z}{1+a^2}}$$

$$\frac{dz}{\sqrt{z}} = \frac{du}{\sqrt{1+a^2}}$$

Integrating on both sides, we get

$$2\sqrt{z} = \frac{u}{\sqrt{1+a^2}} + b$$

Hence, the complete solution is

$$2\sqrt{z} = \frac{x + ay}{\sqrt{1+a^2}} + b$$

Since, the number of a.c = number of i.v

General integral can be found out in a usual way. //

6. Solve:- $Ap + Bq + Cz = 0$.

Solution:- Given $Ap + Bq + Cz = 0 \longrightarrow (1)$

This equation is of the form $f(z, p, q) = 1$.

Let $u = x + ay$

$$\frac{\partial u}{\partial x} = 1, \quad \frac{\partial u}{\partial y} = a$$

$$p = \frac{\partial z}{\partial x} = \frac{dz}{du}\frac{\partial u}{\partial x} = \frac{dz}{du}$$

$$q = \frac{\partial z}{\partial y} = \frac{dz}{du}\frac{\partial u}{\partial y} = a\frac{dz}{du}$$

Substituting in (1), we get

$$A\frac{dz}{du} + Ba\frac{dz}{du} + Cz = 0$$

$$\frac{dz}{du} = \frac{-Cz}{A+Ba}$$

$$\frac{dz}{z} = -\frac{C}{A+Ba}\,du.$$

Integrating on both sides, we get

$$\log z = -\frac{C}{A+Ba}(x+ay) + b$$

Hence, the complete solution.

Here, A, B, C are given constants.

Since, the number of a.c = number of i.v

General integral can be found out in a usual way.

7. Solve :- $p(1-q^2) = q(1-z)$.

Solution :-

Given $p(1-q^2) = q(1-z) \rightarrow (1)$

This equation is of the form $f(z, p, q) = 0$

Let $u = x + ay$

$$\frac{\partial u}{\partial x} = 1, \quad \frac{\partial u}{\partial y} = a$$

$$p = \frac{dz}{du}, \quad q = a\frac{dz}{du}$$

Equation (1) we get

$$(1) \Rightarrow \frac{dz}{du}\left[1 - a^2\left(\frac{dz}{du}\right)^2\right] = a\frac{dz}{du}[1-z].$$

$$1 - a^2\left(\frac{dz}{du}\right)^2 = a(1-z)$$

$$1 - a(1-z) = a^2\left(\frac{dz}{du}\right)^2$$

$$1 - a + az = a^2\left(\frac{dz}{du}\right)^2$$

$$\left(\frac{dz}{du}\right)^2 = \frac{1}{a^2}[1 - a + az]$$

$$\frac{dz}{du} = \frac{1}{a}\sqrt{1-a+az}$$

$$\int \frac{a}{\sqrt{1-a+az}}\, dz = \int du$$

$$2\sqrt{1-a+az} = u + b$$

Squaring in both sides, we get

$$4(1-a+az) = (u+b)^2$$

$$4(1-a+az) = (x+ay+b)^2$$

which gives the complete integral of the given equation. Since, the number of a.c = number of i.v General integral can be found out in a usual way.

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IV. Problems based on Type in $f(x, p, q) = 0$

① Solve :- $p = 2qx$.

Solution:-

Given, $p = 2qx \rightarrow (1)$

This equation is of the form $f(x, p, q) = 0$.

Let $q = a$, Then $p = 2ax$

But, $dz = 2ax \cdot dx + a\,dy$

Integrating on both sides, we get

$$z = ax^2 + ay + b \rightarrow (2)$$

Equation (2) is the complete integral of the given equation,

Since, the number of a.c = number of i.v

Differentiating partially w.r.to 'b' we get $1 = 0$.

Hence, there is no singular integral. //

② Solve :- $q = px + p^2$.

Solution:-

Given, $q = px + p^2 \rightarrow (1)$

This equation is of the form $f(x, p, q) = 0$.

Assume $q = a$ (constant)

Then $p^2 + px - a = 0$

$$p = \frac{-x \pm \sqrt{x^2 + 4a}}{2}$$

We know that, $dz = p\,dx + q\,dy$

$$\left[\frac{-x \pm \sqrt{x^2+4a}}{2}\right] dx + a\,dy$$

Integrating on both sides, we get

$$z = \frac{-x^2}{4} \pm \frac{1}{2}\int \sqrt{x^2+4a}\,.dx + ay + b$$

$$\frac{-x^2}{4} \pm \frac{1}{2}\left[2a\sinh^{-1}\left(\frac{x}{2\sqrt{a}}\right) + \frac{x}{2}\sqrt{x^2+4a}\right] + ay + b$$

This equation is the Complete solution.

Since, the number of a.c = number of i.v

Then, singular integral does not exist. //.

(3). Solve:- $\sqrt{p} + \sqrt{q} = \sqrt{x}$.

Solution:-

Given, $\sqrt{p} + \sqrt{q} = \sqrt{x} \longrightarrow (1)$

This equation is of the form $f(x, p, q) = 0$.

Assume $q = a$ (constant)

$$(1) \Rightarrow \sqrt{p} + \sqrt{a} = \sqrt{x}$$

$$\sqrt{p} = \sqrt{x} - \sqrt{a}$$

$$p = (\sqrt{x} - \sqrt{a})^2$$

$$= x + a - 2\sqrt{ax}$$

We know that, $dz = p\,dx + q\,dy$

$$dz = [x + a - 2\sqrt{a}\sqrt{x}]\,dx + a\,dy$$

Integrating on both sides, we get

$$z = \frac{x^2}{2} + ax - 2\sqrt{a}\,\frac{x^{3/2}}{(3/2)} + ay + b$$

Hence, the complete solution is

$$z = \frac{x^2}{2} + ax - \frac{4\sqrt{a}}{3}x^{3/2} + ay + b \quad //.$$

Since, the number of a.c = number of i.v.

④ . Solve :- Find the complete integral of $q = 2px$.

Solution:-

Given, $q = 2px$

This equation is of the form $f(x, p, q) = 0$.

Let $q = a$ Then $p = \frac{a}{2x}$

$$d = p\,dx + q\,dy$$

But,

$$dz = \frac{a}{2x}\,dx + a\,dy$$

Integrating on both sides, we get

$$\int dz = \int \frac{a}{2x}\,dx + \int a\,dy$$

Hence, the complete solution is

$$z = \frac{a}{2}\log x + ay + b$$

Since, the number of a.c = number of i.v. //.

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V. problems based on Type in $f(y, p, q) = 0$.

① . Solve :- $pq = y$.

Solution :-

Given, $pq = y \longrightarrow ①$

This equation is of the form $f(y, p, q) = 0$.

Assume $p = a$ (constant),

Then $aq = y$

$$q = \frac{y}{a}$$

$$dz = p\,dx + q\,dy$$

$$= a\,dx + \frac{y}{a}\,dy$$

Integrating, $z = ax + \frac{y^2}{2a} + b$ is the complete solution

Since, the number of a.c = number of i.v

differentiating p w.r.to b, we get $0 = 1$ (absurd)

There is no singular integral

put, $b = \phi(a)$.,

$$z = ax + \frac{y^2}{2a} + \phi(a) \longrightarrow (2)$$

differentiating (2) p w.r.to a, we get

$$0 = x - \frac{y^2}{2a^2} + \phi'(a) \longrightarrow (3)$$

Eliminate 'a' between (2) and (3) to get General solution.//.

(2) Solve:- $q = py + p^2$

Solution:- Given $q = py + p^2 \longrightarrow (1)$

This equation is of the form $f(y, p, q) = 0$.

Assume $p = a$ (constant)

$\therefore (1) \Rightarrow q = ay + a^2$

Since, $dz = p\,dx + q\,dy$

$$dz = a\,dx + (ay + a^2)\,dy$$

Integrating on both sides, we get

$$z = ax + \frac{ay^2}{2} + a^2y + b \longrightarrow (2)$$

differentiating p w.r.to b, we get $0 = 1$ (absurd)

There is no singular integral

Let $b = \phi(a)$

$$z = ax + \frac{ay^2}{2} + a^2 y + \phi(a) \longrightarrow (3)$$

differentiating (3) p w.r.to a, we get

$$0 = x + \frac{y^2}{2} + 2ay + \phi'(a) \longrightarrow (4)$$

Eliminate 'a' between (3) and (4), we get General solution. //.

VI. Problem based on Type iv Separable equations.

First order partial differential equation are separable. It can be written as $f(x,p) = \phi(y,q)$

put $f(x,p) = \phi(y,q) = a$

solving for p and q, we get $p = f_1(x,a)$ and $q = \phi_1(y,a)$, But

$$dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy$$

Hence, $dz = p\,dx + q\,dy$

$$= f_1(x,a)\,dx + \phi_1(y,a)\,dy$$

Integrating on both sides, we get

$$z = \int f_1(x,a)\,dx + \int \phi_1(y,a)\,dy + b$$

This equation contains two arbitrary constants and hence it is the complete integral.

The singular and general integrals are found out as usual.

1. Solve: - $p^2 y(1+x^2) = qx^2$.

Solution: -

Given $p^2 y(1+x^2) = qx^2 \longrightarrow (1)$

The equation is separable

$$p^2 \frac{(1+x^2)}{x^2} = \frac{q}{y} = a,$$

where a is an arbitrary constants.

Thus,

$$p^2 \frac{1+x^2}{x^2} = a$$

$$p = \frac{x\sqrt{a}}{\sqrt{1+x^2}}$$

$$q = ay$$

We know that, $dz = pdx + qdy$

$$= \frac{x\sqrt{a}}{\sqrt{1+x^2}} dx + ay\,dy$$

Integrating on both sides, we get

$$z = \sqrt{a} \int \frac{x}{\sqrt{1+x^2}} dx + a\int y\,dy$$

$$= \sqrt{a}\sqrt{1+x^2} + \frac{1}{2} ay^2 + b$$

This is the complete integral, where a and b are arbitrary constants.

Differentiating partially w.r.to b, we find that

There is no singular integral.

②. Find the Complete solution of $p+q=\sin x+\sin y$.

Solution:-

Given, $p+q=\sin x+\sin y$

The given differentiating equation can be written as

$$p-\sin x=\sin y-q$$

It is of the form $f(x,p)=\phi(y,q)$.

Let $p-\sin x=\sin y-q=a$

$p-\sin x=a, \quad \sin y-q=a$

$p=a+\sin x, \quad q=\sin y-a$

$dz=p\,dx+q\,dy$

$dz=(a+\sin x)dx+(\sin y-a)dy$

Integrating on both sides,

$$z=ax-\cos x+(-\cos y-ay)+b$$

$$z=a(x-y)-\cos x-\cos y+b$$

This is the complete integral. //.

③. Find the complete integral of $pq=xy$.

Solution:-

Given, $pq=xy$

Hence, $\frac{p}{x}=\frac{y}{q}$

It is of the form $f(x,p)=\phi(y,q)$.

Let $\frac{p}{x}=\frac{y}{q}=a$ (a is an arbitrary constant)

$\therefore p=ax$ and $q=\frac{y}{a}$

Hence, $dz=p\,dx+q\,dy$

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$$dz = ax\,dx + \frac{y}{a}\,dy$$

Integrating on both sides, we get

$$z = a\frac{x^2}{2} + \frac{y^2}{2a} + b$$

$2az = a^2x^2 + y^2 + 2ab$ is the required complete integral. //.

4. Find the complete integral of $\sqrt{p} + \sqrt{q} = 2x$.

Solution:-

Given: $\sqrt{p} + \sqrt{q} = 2x$

The given equation can be written as

$$\sqrt{p} - 2x = -\sqrt{q}$$

This is of the form $f(x,p) = \phi(y,q)$

Let $\sqrt{p} - 2x = -\sqrt{q} = a$

$$\sqrt{p} = a + 2x; \quad \sqrt{q} = -a$$

$$p = (a+2x)^2, \quad q = a^2$$

Now, $dz = p\,dx + q\,dy$

$$= (a+2x)^2\,dx + a^2\,dy$$

$$z = \frac{(a+2x)^3}{6} + a^2y + b$$

This is the required complete integral. //.

5. Find the complete integral of $p^{-1}x + q^{-1}y = 1$.

Solution:-

Given: $p^{-1}x + q^{-1}y = 1$.

The given equation can be written as

$$p^{-1}x - 1 = -q^{-1}y$$

This is of the form $f(x,p) = \phi(y,q)$

Let $p^{-1}x - 1 = -q^{-1}y = a$

$\therefore p = \dfrac{x}{a+1}$ and $q = \dfrac{-y}{a}$

Now, $dz = p\,dx + q\,dy$

$$dz = \frac{x}{a+1}\,dx - \frac{y}{a}\,dy$$

Integrating on both sides, we get

$$z = \frac{x^2}{2(a+1)} - \frac{y^2}{2a} + b$$

This is the complete integral.//

(b). Solve :- $p^2 + q^2 = z^2(x^2+y^2)$

Solution :-

Given, $p^2 + q^2 = z^2(x^2+y^2) \longrightarrow (1)$

$$\left[\frac{p}{z}\right]^2 + \left[\frac{q}{z}\right]^2 = x^2 + y^2$$

This equation is of the form $f_1(x, z^m p) = f_2(y, z^n q)$.

Here, $m = -1$.

put $Z = \log z$, then $z^{-1}p = P$, $z^{-1}q = Q$

Substitute in equation (1), we get

$$P^2 + Q^2 = x^2 + y^2$$

$$P^2 - x^2 = y^2 - Q^2$$

This equation is of the form $f_1(x,P) = f_2(y,Q)$.

$\therefore P^2 - x^2 = y^2 - Q^2 = a^2$

$p^2 - x^2 = a^2$; $y^2 - Q^2 = a^2$

$P = \sqrt{a^2+x^2}$ $Q = \sqrt{y^2-a^2}$

$dz = p\,dx + Q\,dy$

$dz = \sqrt{a^2+x^2}\,dx + \sqrt{y^2-a^2}\,dy$

$\int dz = \int \sqrt{a^2+x^2}\,dx + \int \sqrt{y^2-a^2}\,dy$

$$z = \frac{a^2}{2}\sinh^{-1}\frac{x}{a} + \frac{x}{2}\sqrt{a^2+x^2} + \frac{y}{2}\sqrt{y^2-a^2} - \frac{a^2}{2}\cosh^{-1}\frac{y}{a} + b .$$

$$\log z = \frac{a^2}{2}\sinh^{-1}\frac{x}{a} + \frac{x}{2}\sqrt{a^2+x^2} + \frac{y}{2}\sqrt{y^2-a^2} - \frac{a^2}{2}\cosh^{-1}\frac{y}{a} + b .$$

=x=

VII. Lagrange's Linear Equation :-

The equation of the form $Pp + Qq = R$ is known as Lagrange's equation when P, Q and R are functions of x, y and z .

$$P = \frac{\partial u}{\partial y}\frac{\partial v}{\partial z} - \frac{\partial u}{\partial z}\frac{\partial v}{\partial y} ; \quad Q = \frac{\partial u}{\partial z}\frac{\partial v}{\partial x} - \frac{\partial u}{\partial x}\frac{\partial v}{\partial z} ;$$

$$R = \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x} .$$

To solve this equation it is enough to solve the subsidiary equation .

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

First step : Write down the subsidiary equations

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

Second step : Solve the above subsidiary equations.

Let the two solutions be $u = a$ and $v = b$.

Third step : Then $f(u, v) = 0$ (or) $u = \phi(v)$ is the required solution of $Pp + Qq = R$

Generally the subsidiary equation can be solved in two ways.

1. Method of Grouping , 2. Method of Multipliers

1. Method of Grouping :-

In the subsidiary equation $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$ if the variables can be separated in any pair of equations, then we get a solution of the form $u(x,y) = a$ and $v(x,y) = b$.

2. Method of Multipliers :-

Choose any three multipliers l, m, n which may be constants (or) function of x, y, z, we have.

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} = \frac{l\,dx + m\,dy + n\,dz}{lP + mQ + nR}$$

If it is possible to choose l, m, n such that $lP + mQ + nR = 0$, then $l\,dx + m\,dy + n\,dz = 0$.

If $l\,dx + m\,dy + n\,dz$ is an exact differential then on integration, we get a solution $u = a$.

The multipliers l, m, n are called lagrangian multipliers.

(1). Solve :- (i) **Problems based on Lagrange's method of Grouping**

$Px + qy = z$.

Solution:-

Given, $Px + qy = z$ (ie), $xp + yq = z$ → (1)

This equation is of the form $Pp + Qq = R$.

Where $P = x$, $Q = y$, $R = z$

The Lagrange's subsidary equations are $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

(ie), $\frac{dz}{x} = \frac{dy}{y} = \frac{dz}{z}$

Take, $\frac{dx}{x} = \frac{dy}{y}$

$\int \frac{dx}{x} = \int \frac{dy}{dy}$

$\log x = \log y + \log c_1$

$\log x = \log (y c_1)$

$x = y c_1$

$\frac{x}{y} = c_1$

(ie), $u = \frac{x}{y}$

Take, $\frac{dx}{x} = \frac{dz}{z}$

$\int \frac{dx}{x} = \int \frac{dz}{z}$

$\log x = \log z + \log c_2$

$\log x = \log (z c_2)$

$x = z c_2$

$\frac{x}{z} = c_2$

(ie), $v = \frac{x}{z}$

Hence, the general solution is $f(u, v) = 0$

(ie), $f\left(\frac{x}{y}, \frac{x}{z}\right) = 0$, where f is arbitrary //.

Note:- (1). $\int \frac{1}{x} dx = \log x + c$ (2) $\int \frac{1}{x^2} dx = -\frac{1}{x} + c$

(3). $\int \frac{1}{\sqrt{x}} dx = 2\sqrt{x} + c$ (4) $\log (ab) = \log a + \log b$.

(2). Write the solution of $px^2 + qy^2 = z^2$.

Solution:-

Given, $px^2 + qy^2 = z^2$

(ie), $x^2 p + y^2 q = z^2$

This equation is of the form $Pp + Qq = R$.

Where $P = x^2$, $Q = y^2$, $R = z^2$

The Lagrange's subsidiary equations are $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

(ie), $\frac{dx}{x^2} = \frac{dy}{y^2} = \frac{dz}{z^2} \longrightarrow (1)$

Take, $\frac{dx}{x^2} = \frac{dy}{y^2}$

$\int \frac{dx}{x^2} = \int \frac{dy}{y^2}$

$-\frac{1}{x} = \frac{-1}{y} - c_1$

$\frac{1}{y} - \frac{1}{x} = -c_1$

$c_1 = \frac{1}{x} - \frac{1}{y}$

(ie), $u = \frac{1}{x} - \frac{1}{y}$

Take, $\frac{dy}{y^2} = \frac{dz}{z^2}$

$\int \frac{dy}{y^2} = \int \frac{dz}{z^2}$

$-\frac{1}{y} = \frac{-1}{z} - c_2$

$c_2 = \frac{1}{y} - \frac{1}{z}$

(ie), $v = \frac{1}{y} - \frac{1}{z}$

Hence, the general solution is $f(u,v) = 0$

(ie), $f\left(\frac{1}{x} - \frac{1}{y}, \frac{1}{y} - \frac{1}{z}\right) = 0$, where f is arbitrary.

(3). Find the solution of $p\sqrt{x} + q\sqrt{y} = \sqrt{z}$.

Solution:- Given, $p\sqrt{x} + q\sqrt{y} = \sqrt{z}$

(ie), $\sqrt{x}\, p + \sqrt{y}\, p = \sqrt{z}$

This equation is of the form $Pp + Qq = R$

Where $P = \sqrt{x}$, $Q = \sqrt{y}$, $R = \sqrt{z}$

The Lagrange's subsidiary equations are $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

(ie) $\frac{dx}{\sqrt{x}} = \frac{dy}{\sqrt{y}} = \frac{dz}{\sqrt{z}}$

Take, $\frac{dx}{\sqrt{x}} = \frac{dy}{\sqrt{y}}$

$\int \frac{dx}{\sqrt{x}} = \int \frac{dy}{\sqrt{y}}$

Take, $\frac{dy}{\sqrt{y}} = \frac{dz}{\sqrt{z}}$

$\int \frac{dy}{\sqrt{y}} = \int \frac{dz}{\sqrt{z}}$

$2\sqrt{x} = 2\sqrt{y} + 2c_1$

$\sqrt{x} = \sqrt{y} + c_1$

$c_1 = \sqrt{x} - \sqrt{y}$

(ie)., $u = \sqrt{x} - \sqrt{y}$

$2\sqrt{y} = 2\sqrt{z} + 2c_2$

$\sqrt{y} = \sqrt{z} + c_2$

$c_2 = \sqrt{y} - \sqrt{z}$

(ie), $v = \sqrt{y} - \sqrt{z}$

Hence, the general solution is $f(u,v) = 0$

(ie)., $f(\sqrt{x} - \sqrt{y}, \sqrt{y} - \sqrt{z}) = 0$,

where f is arbitrary. //.

④. Solve:- $x^2 p + y^2 q = z$.

Solution:- Given, $x^2 p + y^2 q = z$.

This equation is of the form $Pp + Qq = R$.

where $P = x^2$, $Q = y^2$, $R = z$.

The Lagrange's subsidiary equation are

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

(ie), $$\frac{dx}{x^2} = \frac{dy}{y^2} = \frac{dz}{z}$$

Take, $$\frac{dx}{x^2} = \frac{dy}{y^2}$$

$$\int \frac{dx}{x^2} = \int \frac{dy}{y^2}$$

$$-\frac{1}{x} = -\frac{1}{y} - c_1$$

$$c_1 = \frac{1}{x} - \frac{1}{y}$$

(ie), $u = \frac{1}{x} - \frac{1}{y}$

Take, $$\frac{dx}{x^2} = \frac{dz}{z}$$

$$\int \frac{dx}{x^2} = \int \frac{dz}{z}$$

$$-\frac{1}{x} = \log z - c_2$$

$$c_2 = \frac{1}{x} + \log z$$

(ie), $v = \frac{1}{x} + \log z$

Hence, the general solution is $f(u,v)=0$.

(ie), $f\left(\frac{1}{x}-\frac{1}{y}, \frac{1}{x}+\log z\right)=0$,

where f is arbitrary. //.

$\longrightarrow \times \longleftarrow$

(ii). **Problems based on Lagrange's method of multipliers:**

① Solve:- $x(y-z)p+y(z-x)q=z(x-y)$.

(Or)

$$\left(\frac{y-z}{yz}\right)p+\left(\frac{z-x}{zx}\right)q=\frac{x-y}{xy}$$

Solution:-

Given, $x(y-z)p+y(z-x)q=z(x-y)$.

This equation is of the form $Pp+Qq=R$

where $P=x(y-z)$, $Q=y(z-x)$, $R=z(x-y)$

The Lagrange's subsidiary equations are

$$\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}$$

(ie), $$\frac{dx}{x(y-z)}=\frac{dy}{y(z-x)}=\frac{dz}{z(x-y)} \longrightarrow (1)$$

Taking the Lagrange's multipliers are 1,1,1, we get each ratio in (1).

$$=\frac{dx+dy+dz}{x(y-z)+y(z-x)+z(x-y)}=\frac{d(x+y+z)}{0}$$

Hence, $d(x+y+z)=0$ Integrating, we get

$x+y+z=a$

Taking the Lagrange's multipliers $\frac{1}{x}, \frac{1}{y}, \frac{1}{z}$, we get each ratio in (1).

$$= \frac{\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz}{(y-z)+(z-x)+(x-y)} = \frac{\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz}{0}$$

(ie), $\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz = 0$

Integrating, we get

$$\log x + \log y + \log z = \log b$$

$$\log(xyz) = \log b$$

$$xyz = b$$

Hence, the general solution is $f(a,b) = 0$

(ie), $f(x+y+z, xyz) = 0$,

Where f is arbitrary. //.

(2) Solve:- $(mz-ny)p + (nx-lz)q = ly-mx$.

Solution:-

Given, $(mz-ny)p + (nx-lz)q = ly-mx$

This equation is of the form $Pp + Qq = R$

where $P = mz-ny$, $Q = nx-lz$, $R = ly-mx$

The Lagrange's subsidiary equations are $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

(ie), $\frac{dx}{mz-ny} = \frac{dy}{nx-lz} = \frac{dz}{ly-mx} \longrightarrow (1)$

Taking the Lagrange's multipliers are x, y, z, we get each ratio in (1).

$$= \frac{xdx + ydy + zdz}{x(mz-ny)+y(nx-lz)+z(ly-mx)} = \frac{xdx+ydy+zdz}{0}$$

Hence, $xdx + ydy + zdy = 0$

Integrating, we get

$lx + my + nz = b$

$lx + my + nz = b$

Hence, the general solution is $f(a, b) = 0$.

(ie), $f(x^2+y^2+z^2, \ lx+my+nz) = 0$,

Where f is arbitrary. //.

(3) Solve :-

$$(3z-4y)p - (4x-2z)q = 2y-3x.$$

Solution:-

Given, $(3z-4y)p - (4x-2z)q = 2y-3x$.

This equation is of the form $Pp+Qq = R$.

Where $P = 3z-4y$, $Q = 4x-2z$, $R = 2y-3x$

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

(ie), $$\frac{dx}{3z-4y} = \frac{dy}{4x-2z} = \frac{dz}{2y-3x} \longrightarrow (1)$$

use Lagrangian multipliers x, y, z, we get each ratio in (1),

$$= \frac{xdx+ydy+zdz}{x(3z-4y)+y(4x-2z)+z(2y-3x)} = \frac{xdx+ydy+zdy}{0}$$

(ie), $xdx+ydy+zdz = 0$ (by method of multipliers formula)

Integrating, we get $\int xdx + \int ydy + \int zdz = 0$

$$\frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} = \frac{a}{2} \quad \text{(ie)}, x^2+y^2+z^2 = a$$

Again use Lagrange's multipliers $2, 3, 4$, we get each ratio in (1),

$$= \frac{2dx+3dy+4dy}{6z-8y-12x-6z+8y+12x} = \frac{2dx+3dy+4dz}{0}$$

(ie), $2dx+3dy+4dz = 0$

Integrating, we get, $\int 2dx + \int 3dy + \int 4dz = 0$

$$2x + 3y + 4z = b$$

Hence, the general solution is $f(a, b) = 0$

(ie), $f(x^2+y^2+z^2,\ 2x+3y+4z) = 0$,

Where f is arbitrary. //.

④. Find the general solution of

$$x(y^2-z^2)p + y(z^2-x^2)q = z(x^2-y^2).$$

Solution:-

Given, $x(y^2-z^2)p + y(z^2-x^2)q = z(x^2-y^2)$ → (1)

This equation is of the form $Pp + Qq = R$

Where $P = x(y^2-z^2)$, $Q = y(z^2-x^2)$, $R = z(x^2-y^2)$

The Lagrange's subsidiary equation are $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$.

(ie), $$\frac{dx}{x(y^2-z^2)} = \frac{dy}{y(z^2-x^2)} = \frac{dz}{z(x^2-y^2)} \rightarrow (2)$$

use Lagrange's multipliers x, y, z; we get each ratio in (2)

$$= \frac{x\,dx + y\,dy + z\,dz}{x^2(y^2-z^2) + y^2(z^2-x^2) + z^2(x^2-y^2)} = \frac{x\,dx + y\,dy + z\,dz}{0}$$

(ie), $x\,dx + y\,dy + z\,dz = 0$

Integrating, we get

$$\int x\,dx + \int y\,dy + \int z\,dz = 0$$

$$\frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} = \frac{a}{2}$$

(ie), $x^2 + y^2 + z^2 = a$

use Lagrange's multipliers $\frac{1}{x}, \frac{1}{y}, \frac{1}{z}$, we get each ratio in (2).

$$= \frac{\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz}{y^2 - z^2 + z^2 - x^2 + x^2 - y^2} = \frac{\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz}{0}$$

(ie), $\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz = 0$

Integrating, we get

$$\int \frac{1}{x}dx + \int \frac{1}{y}dy + \int \frac{1}{z}dz = 0$$

$$\log x + \log y + \log z = \log b$$

$$\log(xyz) = \log b$$

(ie), $xyz = b$

Hence, the general solution is $f(a,b) = 0$,

(ie), $f(x^2+y^2+z^2, xyz) = 0$,

where f is arbitrary. //.

⑤ Solve :- $(y^2+z^2)p - xyq + xz = 0$.

Solution:-

Given, $(y^2+z^2)p - xyq + xz = 0$

This equation is of the form $Pp + Qq = R$.

where $P = y^2+z^2$, $Q = -xy$, $R = -xz$

The Lagrange's subsidiary equations are $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

$$\frac{dx}{y^2+z^2} = \frac{dy}{-xy} = \frac{dz}{-xz} \longrightarrow (A)$$

....(1) ....(2) ....(3)

Take, $\frac{dy}{-xy} = \frac{dz}{-xz}$ (Method of grouping for (2) & (3))

$$\frac{dy}{y} = \frac{dz}{z}$$

$$\int \frac{dy}{y} = \int \frac{dz}{z}$$

$\log y = \log z + \log c_1$

$\log y = (\log z\, c_1)$

$y = z\, c_1$

$c_1 = \frac{y}{z}$ (ie), $u = \frac{y}{z}$

use Lagrangian multipliers x, y, z, we get each ratio in (1)

$$= \frac{x\,dx + y\,dy + z\,dz}{x(y^2+z^2) - xy^2 - xz^2}$$

$$= \frac{x\,dx + y\,dy + z\,dz}{xy^2 + xz^2 - xy^2 - xz^2} = \frac{x\,dx + y\,dy + z\,dz}{0}$$

(ie), $x\,dx + y\,dy + z\,dz = 0$ (by Lagrangian multipliers formula).

$\int x\,dx + \int y\,dy + \int z\,dz = 0$

$$\frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} = \frac{c_2}{2}$$

$x^2 + y^2 + z^2 = c_2$

(ie), $v = x^2 + y^2 + z^2$

Hence, the general solution is $f(u, v) = 0$

(ie), $f\ \frac{y}{z}\,(x^2+y^2+z^2) \Rightarrow f\left(\frac{y}{z}, x^2+y^2+z^2\right) = 0$.

Where f is arbitrary. //.

(b) Solve:- $(z^2 - 2yz - y^2)p + (xy + zx)q = xy - zx$.

Solution:-

Given, $(z^2 - 2yz - y^2)p + (xy + zx)q = xy - zx$.

This equation is of the form $Pp + Qq = R$, Where, $P = x^2 - 2yz - y^2$, $Q = xy + zx$, $R = xy - zx$.

The Lagrange's subsidiary equations are,

$$\frac{dx}{z^2 - 2yz - y^2} = \frac{dy}{x(y+z)} = \frac{dz}{x(y-z)}$$

Take last two ratios,

$$\frac{dy}{x(y+z)} = \frac{dz}{x(y-z)}$$

$$\frac{dy}{y+z} = \frac{dz}{y-z}$$

$$(y-z)\,dy = (y+z)\,dz$$

$$y\,dy - z\,dy = y\,dz + z\,dz$$

$$y\,dy - z\,dy - y\,dz - z\,dz = 0$$

$$y\,dy - d(yz) - z\,dz = 0$$

Integrating, we get

$$\int y\,dy - \int d(yz) - \int z\,dz = \frac{c_1}{2}$$

$$\frac{y^2}{2} - yz - \frac{z^2}{2} = \frac{c_1}{2}$$

$$y^2 - 2yz - z^2 = c_1$$

Choose the multipliers x, y, z, we get

$$= \frac{x\,dx + y\,dy + z\,dz}{xz^2 - 2xyz - xy^2 + xy^2 + xyz + xyz - z^2x}$$

$$= \frac{x\,dx + y\,dy + z\,dz}{0}$$

(i), $x\,dx + y\,dy + z\,dz = 0$

Integrating, we get

$$\int x\,dx + \int y\,dy + \int z\,dz = \frac{c_2}{2}$$

$$\frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} = \frac{C_2}{2}$$

(ie), $C_2 = x^2 + y^2 + z^2$

Hence, the general solution is

$f(y^2 - 2yz - z^2, x^2 + y^2 + z^2) = 0$,

where f is arbitrary. //.

(7) Solve:- $(y+z)p + (z+x)q = x+y$.

Solution:- Given, $(y+z)p + (z+x)q = x+y$.

This equation is of the form $Pp + Qq = R$

where, $P = y+z$, $Q = z+x$, $R = x+y$

The Lagrange's subsidiary equations are

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\frac{dx}{y+z} = \frac{dy}{z+x} = \frac{dz}{x+y}$$

$$= \frac{dx - dy}{y-x} = \frac{dy - dz}{z-y} = \frac{dx + dy + dz}{2(x+y+z)}$$

Take, $$\frac{d(x-y)}{-(x-y)} = \frac{d(y-z)}{-(y-z)}$$

$$\int \frac{d(x-y)}{x-y} = \int \frac{d(y-z)}{y-z}$$

$$\log(x-y) = \log(y-z) + \log c_1$$

$$\log\left(\frac{x-y}{y-z}\right) = \log c_1$$

$$c_1 = \frac{x-y}{y-z}, \quad \text{(ie)}, \quad u = \frac{x-y}{y-z}$$

Take, $\frac{d(y-z)}{-(y-z)} = \frac{d(x+y+z)}{2(x+y+z)}$

$$-2\frac{d(y-z)}{y-z} = \frac{d(x+y+z)}{x+y+z}$$

$$-2\int\frac{d(y-z)}{y-z} = \int\frac{d(x+y+z)}{x+y+z}$$

$$-2\log(y-z) = \log(x+y+z) - \log C_2$$

$$\log C_2 = \log(x+y+z) + 2\log(y-z)$$

$$= \log(x+y+z) + \log(y-z)^2$$

$$= \log\left[(x+y+z)(y-z)^2\right]$$

$$C_2 = (x+y+z)(y-z)^2$$

$$v = (x+y+z)(y-z)^2$$

Hence, the general solution is $f(u,v) = 0$

(ie), $f\left[\left(\frac{x-y}{y-z}\right), (y-z)^2(x+y+z)\right] = 0$,

where f is arbitrary. //.

* //.— End —//. *