Mathematical statistics

Unit - I

Central limit Theorem:

If $X_i (i = 1, 2, \ldots n)$ be independent random variables such that $E(X_i) = \mu_i$ and $V(X_i) = \sigma_i^2$ then under certain very general conditions, the random variable $S_n = X_1 + X_2 + \cdots + X_n$ is asymptotically normal with μ and standard deviation σ, where

$$\mu = \sum_{i=1}^{n} \mu_i \text{ and } \sigma^2 = \sum_{i=1}^{n} \sigma_i^2$$

De-Moivre's Laplace Theorem:

$$\text{If } X_i = \begin{cases} 1, & \text{with probability } p. \\ 0, & \text{with probability } q. \end{cases}$$

Then the distribution of the random variable $S_n = X_1 + X_2 + \cdots + X_n$, where X_i's are independent is asymptotically normal as $n \to \infty$.

Characteristic function:

$$\phi_X(t) = E(e^{itx}) = \begin{cases} \int e^{itx} f(x)\,dx, & \text{for continuous} \\ \sum_x e^{itx} p(x), & \text{for discrete} \end{cases}$$

Properties of characteristic function.

Property 1: For all real t, we've

(i) $\phi(0) = \int_{-\infty}^{\infty} dF(x) = 1.$

(ii) $|\phi(t)| \leq 1 = \phi(0)$

property : 2:

$\phi(t)$ is continuous everywhere,

i,e,, $\phi(t)$ is continuous function of t in $(-\infty, \infty)$. Rather $\phi(t)$ is uniformly continuous in 't'.

Proof:

For $h \neq 0$,

$$|\phi_X(t+h) - \phi_X(t)| = \left|\int_{-\infty}^{\infty} \left[e^{i(t+h)x} - e^{itx}\right] dF(x)\right|$$

$$\leq \int_{-\infty}^{\infty} |e^{itx}(e^{ihx} - 1)|\, dF(x) = \int_{-\infty}^{\infty} |e^{ihx} - 1|\, dF(x) \quad \to ①$$

The last integral does not depend on t. If it tends to zero as $h \to 0$. then $\phi_X(t)$ is uniformly continuous in 't'.

Now

$$|e^{ihx} - 1| \leq |e^{ihx}| + 1 \leq 1 + 1 = 2.$$

$$\int_{-\infty}^{\infty} |e^{ihx} - 1|\, dF(x) \leq 2 \int_{-\infty}^{\infty} dF(x) = 2.$$

Hence by Dominated convergence theorem (D.C.T.), taking the limit inside the integral sign in ①,

we get.

$$\lim_{h \to 0} |\phi_X(t+h) - \phi_X(t)| \leq \int_{-\infty}^{\infty} \lim_{h \to 0} |e^{ihx} - 1|\, dF(x) = 0.$$

$$\Rightarrow \lim_{h \to 0} \phi_X(t+h) = \phi_X(t), \ \forall t.$$

Hence $\phi_X(t)$ is uniformly continuous in 't'.

Property: 3.

$\phi_X(-t)$ and $\phi_X(t)$ are conjugate functions, i.e., $\phi_X(-t) = \overline{\phi_X(t)}$, where $\bar{a}$ is the complex conjugate of a.

Proof:

$$\phi_X(t) = E(e^{itX}) = E[\cos tX + i\sin tX]$$

$$\therefore \overline{\phi_X(t)} = E(\cos tX - i\sin tX)$$

$$= E\{\cos(-t)X + i\sin(-t)X\}$$

$$= E(e^{-itX}) = \phi_X(-t)$$

$$\overline{\phi_X(t)} = \phi_X(-t)$$

Property: 4

If the distribution function of a r.v. X is symmetrical about zero, i.e., if $1 - F(x) = F(-x) \Rightarrow f(-x) = f(x)$, then $\phi_X(t)$ is real valued and even function of t.

Proof:

By def we have

$$\phi_X(t) = \int_{-\infty}^{\infty} e^{itx} f(x)\,dx$$

$$= \int_{-\infty}^{\infty} e^{-ity} f(-y)\,dy \qquad (x = -y)$$

$$= \int_{-\infty}^{\infty} e^{-ity} f(y)\,dy \qquad [\because f(-y) = f(y)]$$

$$= \phi_X(-t)$$

$\Rightarrow \phi_X(t)$ is an even function of t.

From W.K.T.

$$\phi_x(-t) = \overline{\phi_x(t)}$$

we get $\phi_x(t) = \phi_x(-t) = \overline{\phi_x(t)}$.

Hence $\phi_x(t)$ is a real valued and even function of t.

Property 5:

If X is some r.v. with characteristic function $\phi_x(t)$, and if $\mu_r' = E(X^r)$ exists, then $\mu_r' = (-i)^r \left| \frac{\partial^r}{\partial t^r} \phi(t) \right|_{t=0}$

Proof:

$$\phi(t) = \int_{-\infty}^{\infty} e^{itx} f(x)\,dx$$

Differentiating 'r' times w.r.to t, we get.

$$\frac{\partial^r}{\partial t^r} \phi(t) = \int_{-\infty}^{\infty} (ix)^r e^{itx} f(x)\,dx$$

$$= (i)^r \int_{-\infty}^{\infty} x^r e^{itx} f(x)\,dx.$$

$$\left| \frac{\partial^r}{\partial t^r} \phi(t) \right|_{t=0} = (i)^r \left| \int_{-\infty}^{\infty} x^r e^{itx} f(x)\,dx \right|_{t=0}.$$

$$= (i)^r \int_{-\infty}^{\infty} x^r f(x)\,dx = i^r E(x^r)$$

$$= i^r \mu_r'$$

$$\text{Hence } \mu_r' = \left(\frac{1}{i}\right)^r \left| \frac{\partial^r}{\partial t^r} \phi(t) \right|_{t=0}$$

$$\mu_r' = (-i)^r \left| \frac{\partial^r}{\partial t^r} \phi(t) \right|_{t=0}$$

Hence proved.

Property - 6

$\phi_{cX}(t) = \phi_X(ct)$, c being a Constant.

Property - 7.

If X_1 and X_2 are independent random variable (r.v), then

$$\phi_{X_1+X_2}(t) = \phi_{X_1}(t) \cdot \phi_{X_2}(t)$$

More generally, for independent r.v. X_i, $i = 1, 2, \ldots n$, we have

$$\phi_{X_1+X_2+\cdots+X_n}(t) = \phi_{X_1}(t) \cdot \phi_{X_2}(t) \cdots \phi_{X_n}(t)$$

Property - 8.

Effect of change of orgin and scale on characteristic Function. If $U = \frac{X-a}{h}$, a and h being constant,

Then $\phi_U(t) = e^{-iat/h} \phi_X(t/h)$.

In particular if we take $a = E(X) = \mu$ and $h = \sigma_X = \sigma$ Then the characteristic function of the standard variable

$Z = \frac{X - E(X)}{\sigma_X} = \frac{X-\mu}{\sigma}$ is given by

$$\phi_Z(t) = e^{-i\mu t/\sigma} \phi_X(t/\sigma).$$

Property - 9: If $|\phi_X(s)| = 1$. for some $s \neq 0$, then for some real a, $X-a$ is a lattice variable with mesh, $h = 2\pi/|s|$.

Proof:

Consider any fixed t, we can write

$\varphi_X(t) = |\varphi_X(t)| e^{iat}$, ($a$ dependent on t)

Since any complex number z can be written as $z = |z| e^{i\theta}$.

$|\varphi_X(t)| = e^{-iat}\varphi_X(t) = \varphi_{X-a}(t)$

$= E[\cos t(X-a) + i \sin t(X-a)]$

$= E[\cos t(X-a)]$

Since left-Hand side being real, we must have $E\{\sin t(X-a)\} = 0$.

$\therefore 1 - |\varphi_X(t)| = E\{1 - \cos t(X-a)\}$ —— (1)

If $|\varphi_X(s)| = 1$, for some $s \neq 0$, then for a some a dependent on s, we have from (1), $E\{1 - \cos s(X-a)\} = 0$ —— (2)

But since $1 - \cos s(X-a)$ is a non-negative random variable, (2) $\Rightarrow$

$P\{1 - \cos s(X-a) = 0\} = 1$.

$\Rightarrow P\{\cos s(X-a) = 1\} = 1$

$\Rightarrow P\{s(X-a) = 2n\pi\} = 1$

$\therefore P\left[(X-a) = \frac{2n\pi}{|s|}\right] = 1$, for some $n = 0, 1, 2 \ldots$

Thus $(X-a)$ is a Lattice variable with mesh $h = \frac{2\pi}{|s|}$.

Uniqueness Theorem:

Characteristic function uniquely determines the distribution, i.e., a necessary and sufficient condition for two distributions with p.d.f's $f_1(\cdot)$ and $f_2(\cdot)$ to be identical is that their characteristic functions $\phi_1(t)$ and $\phi_2(t)$ are identical.

Proof:

If $f_1(\cdot) = f_2(\cdot)$, then from the definition of characteristic function, we get

$$\phi_1(t) = \int_{-\infty}^{\infty} e^{itx} f_1(x)\,dx$$

$$= \int_{-\infty}^{\infty} e^{itx} f_2(x)\,dx = \phi_2(t),$$

Conversely, if $\phi_1(t) = \phi_2(t)$ then

$$f_1(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-itx} \phi_1(t)\,dt$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-itx} \phi_2(t)\,dt = f_2(x).$$

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Continuity Theorem:

For a Sequence of distribution functions $\{F_n(x)\}$ with the Corresponding Sequence of characteristic functions $\{\phi_n(t)\}$, a necessary and sufficient Condition that $F_n(x) \to F(x)$ at all points of Continuity of F is that for every real t, $\phi_n(t) \to \phi(t)$, which is continuous at $t=0$ and $\phi(t)$ is the characteristic function corresponding to F.

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