UNIT – IV

Taylor's series

Taylor's theorem:-

Let $f(z)$ be analytic in a region D containing z_0. Then $f(z)$ can be represented as a power series in $(z-z_0)$ given by

$$f(z) = f(z_0) + \frac{f'(z_0)}{1!}(z-z_0) + \frac{f''(z_0)}{2!}(z-z_0)^2 + \cdots +$$

$$\frac{f^n(z_0)}{n!}(z-z_0)^n + \cdots$$ the expression

is valued in the largest open disc with centre z_0 contained in D.

Proof:

Let $r>0$ be such that the disc $|z-z_0|<r$ is contained in D.

Let, $0<r_1<r$. Let c_1 be the circle $|z-z_0| = r_1$ by cauchy's integral formula

$f(z)$

$$f(z) = \frac{1}{2\pi i}\int_{c_1} \frac{f(\xi)}{(\xi - z)}\, d\xi \rightarrow (1)$$

Also by theorem an higher derivatives we've

$$f^n(z) = \frac{n!}{2\pi i}\int_{c_1} \frac{f(\xi)}{(\xi - z)^{n+1}}\, d\xi \rightarrow (2)$$

Now $$\frac{1}{\xi - z} = \frac{1}{(\xi - z_0) - (z-z_0)}$$

$$= \frac{1}{(\xi - z_0)\left[1 - \frac{z-z_0}{\xi - z_0}\right]}$$

(2) $= \frac{1}{\zeta - z_0}\left[1 + \left(\frac{z-z_0}{\zeta-z_0}\right) + \left(\frac{z-z_0}{\zeta-z_0}\right)^2 + \cdots + \left(\frac{z-z_0}{\zeta-z_0}\right)^{n-1} + \frac{\left(\frac{z-z_0}{\zeta-z_0}\right)^n}{1-\left(\frac{z-z_0}{\zeta-z_0}\right)}\right]$

$$\left(\frac{1}{1-\alpha} = 1+\alpha+\alpha^2+\cdots+\alpha^{n-1}+\frac{\alpha^n}{1-\alpha}\right)$$

$$= \frac{1}{\zeta-z_0} + \frac{z-z_0}{(\zeta-z_0)^2} + \frac{(z-z_0)^2}{(\zeta-z_0)^3} + \cdots + \frac{(z-z_0)^{n-1}}{(\zeta-z_0)^n} + \frac{(z-z_0)^n}{(\zeta-z_0)^n(\zeta-z)}$$

Now, Multiplying $\frac{f(\zeta)}{2\pi i}$, Integrating over C_1 and using eqn ① & ② we get.

$$f(z) = f(z_0) + f'(z_0)(z-z_0) + \frac{f''(z_0)}{2!}(z-z_0)^2 + \cdots + \frac{f^{n-1}(z_0)}{(n-1)!}(z-z_0)^{n-1} + R_n \rightarrow ③$$

Where $R_n = \frac{(z-z_0)^n}{2\pi i}\int_{C_1}\frac{f(\zeta)\,d\zeta}{(\zeta-z)(\zeta-z_0)^n}$

Here ζ lies on C_1 and z lies in the interior of C_1 so that $|\zeta-z_0| = r_1$ and $|z-z_0| < r_1$

$$|\zeta-z| = |(\zeta-z_0)-(z-z_0)| \geq |\zeta-z_0| - |z-z_0|$$
$$= r_1 - |z-z_0|$$

$$\frac{1}{|\zeta-z|} \leq \frac{1}{r_1-|z-z_0|}$$

Let M denote the maximum value of $|f(z)|$ on C_1

Then $|R_n| \leq \frac{|z-z_0|^n}{2\pi}\frac{M\,2\pi r_1}{(r_1-|z-z_0|)r_1^n}$

$$= \frac{M|z-z_0|}{(r_1-|z-z_0|)}\left(\frac{|z-z_0|}{r_1}\right)^{n-1}$$

Also $\left|\frac{z-z_0}{r_1}\right| < 1$

Hence $\lim_{n\to\infty} R_n = 0$

Taking limit as $n \to \infty$ in eqn ③ we get (3)

$$f(z) = f(z_0) + \frac{f'(z_0)}{1!}(z - z_0) + \cdots + \frac{f^n(z_0)}{n!}(z - z_0)^n + \cdots$$

Hence the theorem.

Expand $f(z) = \frac{z-1}{z+1}$ as a taylor's series

i) about the point $z=0$

ii) about the point $z=1$. Determine the region of convergence in each case.

Soln:-

i) $f(z) = \frac{z-1}{z+1}$

$= (z-1)(z+1)^{-1}$

$= (z-1)(1 - z + z^2 - z^3 \dots)$ if $|z|<1$

$= (z - z^2 + z^3 - z^4 \dots) - (1 - z + z^2 - z^3 + \dots)$

$f(z) = -1 + 2z - 2z^2 + 2z^3 - \dots$

The region of convergence is $|z|<1$

ii) $f(z) = \frac{z-1}{z+1}$

$= \frac{z-1}{(2+z-1)} = \frac{z-1}{2\left(1+\frac{z-1}{2}\right)}$

$$= \frac{z-1}{2}\left[1+\frac{z-1}{2}\right]^{-1}$$

$$= \frac{z-1}{2}\left[1-\frac{z-1}{2}+\left(\frac{z-1}{2}\right)^2+\cdots\right] \text{ if } \left|\frac{z-1}{2}\right|<1$$

$$f(z) = \frac{z-1}{2} - \frac{(z-1)^2}{2^2} + \frac{(z-1)^3}{2^3} \cdots$$

The region of convergence is given by $\left|\frac{z-1}{2}\right|<1$ which is same as the circular disc $|z-1|<2$.

Show that,

i) $\frac{1}{z^2} = 1 + \sum_{n=1}^{\infty} (n+1)(z+1)^n$ when $|z+1|<1$

ii) $\frac{1}{z^2} = \frac{1}{4} + \frac{1}{4}\sum_{n=1}^{\infty} (-1)^n (n+1)\left(\frac{z-2}{2}\right)^n$ when $|z-2|<2$.

soln

i) $$\frac{1}{z^2} = \frac{1}{[1-(z+1)^2]}$$

$$= [1-(z+1)]^{-2}$$

$$= 1+2(z+1) + 3(z+1)^2 + \cdots \quad \text{if } |z+1|<1$$

$$\frac{1}{z^2} = 1+\sum_{n=1}^{\infty}(n+1)(z+1)^n \quad \text{when } |z+1|<1$$

ii) $$\frac{1}{z^2} = \frac{1}{(z-2+2)^2}$$

$$= \frac{1}{\left[2\left(1+\frac{z-2}{2}\right)\right]^2}$$

$$= \frac{1}{4}\left(1+\frac{z-2}{2}\right)^{-1}$$

$$= \frac{1}{4}\left[1-2\left(\frac{z-2}{2}\right)+3\left(\frac{z-2}{2}\right)^2-\cdots\right] \text{ if } \left|\frac{z-2}{2}\right|<1$$

$$= \frac{1}{4} - \frac{1}{4}\times 2\left(\frac{z-2}{2}\right) + \frac{1}{4}\times 3\left(\frac{z-2}{2}\right)^2 - \cdots$$

$$= \frac{1}{4} + \frac{1}{4}\sum_{n=1}^{\infty}(-1)^n(n+1)\left(\frac{z-2}{2}\right)^n$$

Here the region of convergence is $\left|\frac{z-2}{2}\right|<1$. which is the same as the circular disc $|z-2|<2$

Expand ze^{2z} in a Taylor's series about $z=-1$ and determine the region of convergence.

soln:-

$$\text{Let } f(z) = ze^{2z}$$

$$= ze^{2(z+1)-2}$$

$$= ze^{2(z+1)}e^{-2}$$

$$f(z) = \frac{1}{e^2}\left[(z+1)\,e^{2(z+1)} - e^{2(z+1)}\right]$$

$$= \frac{1}{e^2}\left[(z+1)\left\{1+\frac{2(z+1)}{1!}+\frac{4(z+1)^2}{2!}+\cdots\right\} - \left\{1+\frac{2(z+1)}{1!}+\frac{4(z+1)^2}{2!}+\cdots\right\}\right]$$

$$= \frac{1}{e^2}\left[\left\{(z+1)+\frac{2(z+1)^2}{1!}+\frac{2^2(z+1)^3}{2!}+\cdots\right\} - \left\{1+\frac{2(z+1)}{1!}+\frac{2^2(z+1)^2}{2!}+\cdots\right\}\right]$$

$$= \frac{1}{e^2}\left[-1+\left(1-\frac{2}{1!}\right)(z+1)+\left(\frac{2}{1!}-\frac{2^2}{2!}\right)(z+1)^2+\left(\frac{2^2}{2!}-\frac{2^3}{3!}\right)(z+1)^3+\cdots\right]$$

The expansion is valid throughout the complex plane.

Find the Taylor's series to represent $\frac{z^2-1}{(z+2)(z+3)}$ in $|z|<2$.

soln

By partial fraction

$$\frac{z^2-1}{(z+2)(z+3)} = 1+\frac{3}{z+2}-\frac{8}{z+3}$$

$$= 1+\frac{3}{2(1+\frac{z}{2})}-\frac{8}{3(1+\frac{z}{3})}$$

$$= 1+\frac{3}{2}(1+z/2)^{-1}-\frac{8}{3}(1+z/3)^{-1}$$

$$= 1+\frac{3}{2}\left(1-z/2+\frac{z^2}{2^2}-\cdots\right)-\frac{8}{3}\left[1-\frac{z}{3}+\frac{z^2}{3^2}-\cdots\right.$$

$$= \left(1 + \frac{3}{2} - \frac{8}{3}\right) + \left(\frac{-3}{2^2} + \frac{8}{3^2}\right) z + \left(\frac{3}{2 \cdot 2^2} - \frac{8}{3 \cdot 3^2}\right) z^2 + \cdots$$

$$= -\frac{1}{6} + \sum_{n=1}^{\infty} (-1)^{n+1} \left(\frac{8}{3^{n+1}} - \frac{3}{2^{n+1}}\right) z^n$$ and the expansion is valid in $|z| < 2$.

[illegible]

Laurent's theorem

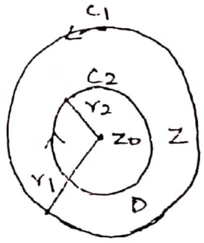
Let c_1 & c_2 denote respectively. The concentric circles $|z - z_0| = r_1$ and $|z - z_0| = r_2$ with $r_1 < r_2$. Let $f(z)$ be analytic in a region containning the circular annulus $r_1 < |z - z_0| < r_2$. Then $f(z)$ can be represented as a convergent series of +ve & −ve Powers of $z - z_0$ given by

$$f(z) = \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n} + \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

Where $b_n = \frac{1}{2\pi i} \int_{c_1} \frac{f(\xi)\, d\xi}{(\xi - z_0)^{-n+1}}$ and $a_n = \frac{1}{2\pi i} \int_{c_2} \frac{f(\xi)\, d\xi}{(\xi - z_0)^{n+1}}$

proof:

Let z be any point in the circular annulus $r_1 < |z - z_0| < r_2$



we know that

$$f(z) = \frac{1}{2\pi i} \int_{c_2} \frac{f(\xi)\, d\xi}{\xi - z} - \frac{1}{2\pi i} \int_{c_1} \frac{f(\xi)\, d\xi}{\xi - z}$$

$$f(z) = \frac{1}{2\pi i} \int_{c_2} \frac{f(\xi)\, d\xi}{\xi - z} + \frac{1}{2\pi i} \int_{c_1} \frac{f(\xi)}{z - \xi} d\xi \to (1)$$

As in the Proof of Taylor's theorem we have,

$$\frac{1}{2\pi i} \int_{c_2} \frac{f(\xi)\, d\xi}{\xi - z} = a_0 + a_1 (z - z_0) + a_2 (z - z_0)^2 + \cdots + a_{n-1} (z - z_0)^{n-1} + R_n(z) \to (2)$$

where $a_n = \frac{1}{2\pi i} \int_{c_2} \frac{f(\xi)\, d\xi}{(\xi - z_0)^{n+1}}$

and $R_n(z) = \frac{(z - z_0)^n}{2\pi i} \int_{c_2} \frac{f(\xi)\, d\xi}{(\xi - z_0)^n (\xi - z)}$

Now $\frac{1}{-\xi + z} = \frac{1}{z - z_0 + z_0 - \xi}$

(6)

$$= \frac{1}{(z-z_0)-(\xi-z_0)}$$

$$= \frac{1}{(z-z_0)\left[1-\frac{(\xi-z_0)}{z-z_0}\right]}$$

$$\frac{1}{z-\xi} = \frac{1}{z-z_0}\left[1+\left(\frac{\xi-z_0}{z-z_0}\right)+\cdots\left(\frac{\xi-z_0}{z-z_0}\right)^{n-1}+\frac{\left(\frac{\xi-z_0}{z-z_0}\right)^n}{1-\left(\frac{\xi-z_0}{z-z_0}\right)}\right]$$

Multiplying by $\frac{f(\xi)}{2\pi i}$ and integrating over c_1 we get

$$\int_{c_1}\frac{f(\xi)\,d\xi}{z-\xi} = \frac{b_1}{z-z_0}+\frac{b_2}{(z-z_0)^2}+\cdots+\frac{b_{n-1}}{(z-z_0)^{n-1}}+S_n(z) \rightarrow (3)$$

where $b_n = \frac{1}{2\pi i}\int_{c_1}\frac{f(\xi)\,d\xi}{(\xi-z_0)^{-n+1}}$ &

$$S_n = \frac{1}{2\pi i(z-z_0)^n}\int_{c_1}\frac{f(\xi)(\xi-z_0)^n}{z-\xi}\,d\xi$$

From (1), (2) & (3) we get,

$$f(z) = a_0+a_1(z-z_0)+\cdots+a_{n-1}(z-z_0)^{n-1}+\frac{b_1}{z-z_0}+\frac{b_2}{(z-z_0)^2}+\cdots+\frac{b_{n-1}}{(z-z_0)^{n-1}}+R_n(z)+S_n(z) \rightarrow (4)$$

the required result follows if we can prove that $R_n \to 0$ and $S_n \to 0$ as $n \to \infty$

Now, if $\xi \in C_1$ then $|\xi-z_0| = r_1$ &

$$|z-\xi| = |(z-z_0)-(\xi-z_0)| \geq |z-z_0|-r_1$$

if $\xi \in C_2$ then $|\xi-z_0| = r_2$ &

$$|\xi-z| = |(\xi-z_0)-(z-z_0)| \geq r_2-|z-z_0|$$

Now let M denote the maximum value of $|f(z)|$ in $C_1 \cup C_2$

Then

$$|R_n| \leq \frac{|z-z_0|^n}{2\pi}\,\frac{M(2\pi r_2)}{(r_2-|z-z_0|)\,r_2^n}$$

$$|R_n| \leq \frac{M|z-z_0|}{(r_2-|z-z_0|)}\left(\frac{|z-z_0|}{r_2}\right)^{n-1}$$

(7) since, $\left|\frac{z-z_0}{r_2}\right| < 1$ $R_n \to 0$ as $n \to \infty$

Hence by taking limit $n\to\infty$ in (4) we get,

$$f(z) = \sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n} + \sum_{n=0}^{\infty} a_n (z-z_0)^n$$

Hence the theorem.

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Find the Laurent's series expansion of $f(z) = z^2 e^{1/z}$, $z=0$

soln

$$f(z) = z^2 e^{1/z}$$

clearly $f(z)$ is analytic at all points $z \neq 0$

Now $$f(z) = z^2\left[1 + \frac{1}{z} + \frac{1}{2!z^2} + \frac{1}{3!z^3} + \cdots\right]$$

$$= z^2 + z + \frac{1}{2} + \frac{1}{6z}$$

This is the required Laurent's series Expansion for $f(z)$ at $z=0$

Find the Laurent's series for $\frac{z}{(z+1)(z+2)}$ about $z=-2$

soln

Let $$f(z) = \frac{z}{(z+1)(z+2)}$$

$$\frac{A}{z+1} + \frac{B}{z+2} = z \Rightarrow \frac{A(z+2)+B(z+1)}{(z+1)\cancel{(z+2)}} = \frac{z}{(z+1)\cancel{(z+2)}}$$

$$A(z+2) + B(z+1) = z$$

$z=-2$

$+B = +2$

$z=-1$

$A = -1$

$$\frac{-1}{z+1} + \frac{2}{z+2}$$

$$\frac{-1}{(z+2)-1} + \frac{2}{z+2}$$

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$$= (1-(z+2))^{-1} + \frac{2}{z+2}$$

$$= (1 + (z+2) + (z+2)^2 + \cdots) + \frac{2}{z+2}$$

$$= \frac{2}{z+2} + 1 + 1(z+2) + \cdots$$

Expand $\frac{-1}{(z-1)(z-2)}$ as a power series the region

i) $|z|<1$ ii) $1<|z|<2$ iii) $|z|>2$

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soln

Let $f(z) = \frac{-1}{(z-1)(z-2)}$

$$\frac{A}{z-1} + \frac{B}{z-2} = \frac{-1}{(z-1)(z-2)}$$

$$A(z-2) + B(z-1) = -1$$

$$B = -1$$

$$A = 1$$

$$f(z) = \frac{1}{z-1} - \frac{1}{z-2}$$

$\cancel{f(z)}$

The only points where $f(z)$ is not analytic are one and two

hence $f(z)$ is analytic in $|z|<1$

And hence can be represented as a Taylor's series in $|z|<1$

$$f(z) = \frac{1}{z-1} - \frac{1}{z-2}$$

$$f(z) = -\frac{1}{z-1} + \frac{1}{2-z}$$

$$= -(1-z)^{-1} + \frac{1}{2}\left(1 - z/2\right)^{-1}$$

$$= -[1 + z + \cdots + z^n] + \frac{1}{2}\left[1 + z/2 + \frac{z^2}{4} + \cdots + \left(\frac{z}{2}\right)^n\right]$$

$$= -\sum_{n=0}^{\infty} \left(\frac{1}{2^{n+1}} - 1\right) z^n$$

ii) $f(z)$ is analytic in annular region $1<|z|<2$

And hence can be expanded as a Laurent's series in this region

$$f(z) = \frac{1}{z-1} - \frac{1}{z-2}$$

$$= \frac{1}{z(1-1/z)} + \frac{1}{2(1-z/2)}$$

$$= \frac{1}{z}(1-1/z)^{-1} + \frac{1}{2}(1-z/2)^{-1}$$

$$= \frac{1}{z}\left[1+\left(\frac{1}{z}\right)+\left(\frac{1}{z}\right)^2+\cdots\right] + \frac{1}{2}\left[1+\left(\frac{z}{2}\right)+\left(\frac{z}{2}\right)^2+\cdots\right]$$

$$= \sum_{n=0}^{\infty} \frac{1}{z^{n+1}} + \sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}}$$

This given the laurents series expansion $1<|z|<2$

iii) $f(z)$ is analytic in the domain $|z|>2$ and in thi. domain we've $|2/z|<1$

$$f(z) = \frac{1}{z}\left[\frac{1}{(1-(1/z))}\right] - \frac{1}{z}\left[\frac{1}{1-(2/z)}\right]$$

$$= \frac{1}{z}\left[1-\frac{1}{z}\right]^{-1} - \frac{1}{z}\left[1-2/z\right]^{-1}$$

$$= \frac{1}{z}\left[\left(1+1/z+(1/z)^2+\cdots\right) - \left(1+2/z+(2/z)^2+\cdots\right)\right]$$

$$= \sum_{n=0}^{\infty} \frac{1-2^n}{z^{n+1}}$$

Expand $\frac{1}{z(z-1)}$ as Laurents series

i) about $z=0$ in powers of z and

ii) about $z=1$ in powers $z-1$

iii) about $z=1$ in powers $z-1$. Also state the region of validity.

soln:-

i) The only points where $f(z)$ is not analytic are 0 and 1

Hence $f(z)$ can be expanded as a laurent's series in the annuals $0<|z|<1$

$$f(z) = \frac{1}{z(z-1)}$$

$$= -\frac{1}{z}(1-z)^{-1}$$

$$= -\frac{1}{z}(1+z+z^2+\dots+z^n) \quad (\because |z|<1)$$

$$= -\left(\frac{1}{z}+1+z+z^2+\dots+z^n+\dots\right)$$

This is the Laurent's series expansion of $f(z)$ in $0<|z|<1$

ii) $f(z)$ is analytic in $0<|z-1|<1$ & hence can be expanded as a Laurant's series in powers of $z-1$ in the region.

$$\frac{1}{z(z-1)} = \frac{1}{z-1}\left[\frac{1}{1+(z-1)}\right]$$

$$= \frac{1}{z-1}[1+(z-1)]^{-1}$$

$$= \frac{1}{z-1}[1-(z-1)+(z-1)^2 - (z-1)^3+\dots]$$

$|z-1|<1$

$$= \frac{1}{z-1} - 1 + (z-1) - (z-1)^2 + \dots$$

This given the Laurent's series expansion in $0<|z-1|<1$.

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Cauchy's Residue theorem ie, $Res\{f(z);0\} = \frac{3/5}{2} = \frac{3}{10}$

Let $f(z)$ be a function which is analytic inside and on a simple closed curve c. except for a finite number of singular point $z_1, z_2, \dots z_n$ inside c, then

$$\int_c f(z)dz = 2\pi i \sum_{j=1}^{n} Res\{f(z); z_j\}$$

Proof:

Let $c_1, c_2, \dots c_n$ be circles with centered $z_1, z_2, \dots z_n$ respectively such that all circles are interior to c and are disjoint with each other. (7)

By cauchy's theorem for multiply connected Regions,

$$\int_C f(z)dz = \int_{C_1} f(z)\,dz + \int_{C_2} f(z)dz + \cdots + \int_{C_n} f(z)\,dz$$

$$\text{Res}\{f(z); a\} = \frac{1}{2\pi i}\int_C f(z)dz$$

$$= 2\pi i \ \text{Res}\{f(z); z_1\} + \cdots + 2\pi i \ \text{Res}\{f(z); z_n\}$$

$$\int_C f(z)dz = 2\pi i \sum_{j=1}^{n} \text{Res}\{f(z); z_j\}$$

Hence the theorem.

Evaluate $\int_C \frac{z^2\,dz}{(z-2)(z+3)}$ where c is the circle $|z|=4$

Soln:-

Let $f(z) = \frac{z^2}{(z-2)(z+3)}$

$z=2$, $z=-3$ are poles of $f(z)$ & both of them lie inside $|z|=4$

$$\text{Res}\{f(z); 2\} = \lim_{z\to 2} (z-2)\frac{z^2}{(z-2)(z+3)}$$

$$= \lim_{z\to 2} \frac{z^2}{(z+3)}$$

$$\text{Res}\{f(z); 2\} = 4/5$$

$$\text{Res}\{f(z); -3\} = \lim_{z\to 3} (z+3)\frac{z^2}{(z-2)(z+3)}$$

$$= \lim_{z\to 3} \frac{z^2}{(z-2)}$$

$$= \frac{9}{-5}$$

$$\int_C \frac{z^2\,dz}{(z-2)(z+3)} = 2\pi i\left[\text{Res}\{f(z); 2\} + \text{Res}(f(z); -3)\right]$$

$$= 2\pi i\left(4/5 - 9/5\right)$$

$$= 2\pi i(-5/5)$$

$$= -2\pi i$$

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Theorem

State and prove Argument theorem:-

Let f be a function which is analytic inside and on a simple closed curve c, except for a finite no of poles inside c. Also Let $f(z)$ have no zero's on c. Then $\frac{1}{2\pi i}\int_c \frac{f'(z)}{f(z)}dz = N-P$.

where N is the no of zero's $f(z)$ inside c and P is the no of poles of $f(z)$ inside c.

[4 poles are zero of order m is counted m times]

proof:-

The singularities of the function $\frac{f'(z)}{f(z)}$ inside c are poles and zero's of $f(z)$ lying inside c.

Let z_0 be $(z-z_0^n)$ a zero of order n for $f(z)$

Let c_1 be a circle with centre z_0 such that it's the only zero of $f(z)$ inside c_1, Then

$$f(z) = (z-z_0)^n g(z) \rightarrow ①$$

where $g(z)$ is analytic and non-zero inside c_1

Hence $f'(z) = n(z-z_0)^{n-1}g(z) + (z-z_0)^n g'(z)$

$$\frac{f'(z)}{f(z)} = \frac{n(z-z_0)^{n-1}\cancel{g(z)}}{\cancel{(z-z_0)}^n \cancel{g(z)}} + \frac{\cancel{(z-z_0)^n}g'(z)}{\cancel{(z-z_0)^n}g(z)}$$

$$= n(z-z_0)^{-1} + \frac{g'(z)}{g(z)}$$

$$\frac{f'(z)}{f(z)} = \frac{n}{(z-z_0)} + \frac{g'(z)}{g(z)} \rightarrow ①$$

Since $g(z)$ is analytic and non-zero Inside c, $\frac{g'(z)}{g(z)}$ is also analytic and hence can be expanded as a taylor's series about z_0.

$$\text{Res}\left\{\frac{f'(z)}{f(z)}; z_0\right\} = \text{co-efficient of } \frac{1}{z-z_0} \text{ in } ①$$

$\text{Res}\left\{\frac{f'(z)}{f(z)}; z_0\right\} = n.$

Similarly if z_1 is a pole of order P for $f(z)$

Then $\text{Res}\left\{\frac{f'(z)}{f(z)}; z_1\right\} = -P$

Hence by cauchy's Residue theorem

$$\frac{1}{2\pi i}\int_c \frac{f'(z)}{f(z)}\,dz = N - P$$

where N is the no of zero's and P is the no of Pole of $f(z)$ with in c.

$$\frac{1}{2\pi i}\int_c \frac{f'(z)}{f(z)}\,dz = N.$$

proof

W.K.T $\frac{1}{2\pi i}\int_c \frac{f'(z)}{f(z)}\,dz = N-P$

$P=0$, $\frac{1}{2\pi i}\int_c \frac{f'(z)}{f(z)}\,dz = N$

Since the number of Poles is zero we have $P=0$

$$\frac{1}{2\pi i}\int_c \frac{f'(z)}{f(z)}\,dz = N$$

Hence the proof.

Theorem

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State and Prove Roche's theorem :.

If $f(z)$ & $g(z)$ are analytic inside and on a simple closed curve c if $|g(z)| < |f(z)|$ on c then $f(z)+g(z)$ and $f(z)$ have the same no of zero's inside c.

proof:

$$f(z)+g(z) = f(z)\left(1 + \frac{g(z)}{f(z)}\right)$$

$f(z)+g(z) = f(z)\,\phi(z)$ where $\phi(z) = 1 + \frac{g(z)}{f(z)}$ $\to$ (1)

diff both sides

Hence $[f(z)+g(z)]' = f'(z)+g'(z)$

$$= f'(z)\phi(z) + f(z)\phi'(z) \rightarrow (2)$$

$$\frac{(2)}{(1)} = \frac{f'(z)+g'(z)}{f(z)+g(z)} = \frac{f'(z)\phi(z)+f(z)\phi'(z)}{f(z)\phi(z)}$$

$$= \frac{f'(z)\cancel{\phi(z)}}{f(z)\cancel{\phi(z)}} + \frac{\cancel{f(z)}\phi'(z)}{\cancel{f(z)}\phi(z)}$$

$$\frac{f'(z)+g'(z)}{f(z)+g(z)} = \frac{f'(z)}{f(z)} + \frac{\phi'(z)}{\phi(z)} \rightarrow (3)$$

Multiplying $\frac{1}{2\pi i}\int_c$ on both sides

$$\frac{1}{2\pi i}\int_c \frac{f'(z)+g'(z)}{f(z)+g(z)}\,dz = \frac{1}{2\pi i}\int_c \frac{f'(z)}{f(z)}\,dz + \frac{1}{2\pi i}\int_c \frac{\phi'(z)}{\phi(z)}\,dz \rightarrow (4)$$

by hypothesis

Since $|g(z)| < |f(z)|$

$\left|\frac{g(z)}{f(z)}\right| < 1$ on $c \Rightarrow |\phi(z)-1| < 1$ on c

Hence by maximum modulos theorem

$\phi(z) = 1 + \frac{g(z)}{f(z)}$

$\frac{g(z)}{f(z)} = \phi(z)-1$

$|\phi(z)-1| < 1$ for every point z inside c.

Since $\phi(z) \neq 0$ for every point inside c

Hence $\int_c \frac{\phi'(z)}{\phi(z)}\,dz =$ number of zero's of $\phi(z)$ with in c.

$f(z) = 1 + \frac{g(z)}{f(z)}$

$= 0$

$\frac{g(z)}{f(z)} < 1$

$f(z)-1 < 1$

From (4),

$$\frac{1}{2\pi i}\int_c \frac{f'(z)+g'(z)}{f(z)+g(z)}\,dz = \frac{1}{2\pi i}\int_c \frac{f'(z)}{f(z)}\,dz$$

$N_1 = N_2$. where N_1 & N_2 denote respectively the no of zero's of $f(z)+g(z)$ inside c.

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Theorem :-

State and prove Fundamental theorem of Algebra

A polynomial of degree n with complex co-efficient has n zero's in c.

proof:-

Let $a_0 + a_1 z + a_2 z^2 + \dots + a_n z^n$ (where $a_n \neq 0$)

be a polynomial of degree n

Let $f(z) = a_n z^n$ and $g(z) = a_0 + a_1 z + a_2 z^2 + \dots + a_n z^n$

$$\frac{g(z)}{f(z)} = \frac{a_0}{a_n z^n} + \frac{a_1 z}{a_n z^n} + \dots + \frac{a_{n-1} z^{n-1}}{a_n z^n}$$

$$= \frac{a_0}{a_n z^n} + \frac{a_1}{a_n z^{n-1}} + \dots + \frac{a_{n-1}}{a_n z^1}$$

$$\lim_{z \to \infty} \frac{g(z)}{f(z)} = 0$$

Hence there exist a +ve Real number.

Such that $\left| \frac{g(z)}{f(z)} \right| < 1 \; \forall z$, with $|z| > r$

Hence by Rouche's theorem $f(z)$ & $g(z) + f(z)$ have the same the no of zero's inside the circle $|z| = r+1$

But 0 is a zero of multiply n for $f(z)$.

Hence the given polynomial $f(z) + g(z)$ also has 'n' zeros.

Evaluate $\int_c \frac{1}{2z+3} dz$, where c is $|z| = 2$

soln

Let $f(z) = \frac{1}{2z+3}$

$h(z) = 1$, $k(z) = 2z+3$

$k'(z) = 2$

$2z+3 = 0$

$2z = -3$

$z = -3/2$

$z = -3/2$ is a pole for $f(z)$

which lies inside the circle $|z| = 2$

We know that

$$\text{Res}\{f(z); a\} = \lim_{z\to a} \frac{h(z)}{k'(z)} \Rightarrow \lim_{z\to -3/2} \frac{1}{2}$$

$$\text{Res}\{f(z); -3/2\} = \lim_{z\to -3/2} \frac{1}{2}$$

By Residue theorem

$$\int_c f(z)\,dz = 2\pi i \ \text{Res}\{f(z); -3/2\}$$

$$= 2\pi i \times \frac{1}{2}$$

$$\int_c f(z)\,dz = \pi i$$

Evaluate $\int_c \frac{dz}{z^2 e^z}$, where $c = \{z, |z| = 1\}$

Soln

Let $f(z) = \frac{1}{z^2 e^z}$

$z=0$ is a pole of order 2 at $z=0$

which lies inside the circle $|z|=1$

Let $g(z) = \frac{1}{e^z} = e^{-z}$

$$g'(z) = -e^{-z}$$

using Lemma 4,

$$\text{Res}\{f(z); a\} = \frac{g^{m-1}a}{(m-1)!}$$

$$\text{Res}\{f(z); 0\} = \frac{g'(z)}{1} = -e^{-0} = -1$$

By Residue theorem

$$\int_c f(z)\,dz = 2\pi i \ \text{Res}\{f(z); 0\}$$

$$= 2\pi i(-1)$$

$$= -2\pi i$$

Evaluate $\int_0^{2\pi} \frac{d\theta}{5+4\sin\theta}$ (13)

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Soln Let $I = \int_0^{2\pi} \frac{d\theta}{5+4\sin\theta}$

put $z = e^{i\theta}$

$dz = ie^{i\theta}\, d\theta$

$d\theta = \frac{dz}{ie^{i\theta}} = \frac{dz}{iz}$

Let $\sin\theta = \frac{z-z^{-1}}{2i}$

$I = \int \frac{dz}{iz\left[5+4\left(\frac{z-z^{-1}}{2i}\right)\right]}$ where c is the unit circle $|z|=1$

$= \int \frac{dz}{z(5i+2z-2z^{-1})}$

$= \int_c \frac{dz}{2z^2+5iz-2}$

$2z^2+5iz-2$

$a=2,\ b=5i,\ c=-2$

$\frac{-b\pm\sqrt{b^2-4ac}}{2a}$

$\frac{-5i+\sqrt{-25+16}}{4}$

$\frac{-5\pm\sqrt{-9}}{4}$

$\frac{-5\pm 3i}{4}$

$\frac{-5-3i}{4} = \frac{-8i}{4} = -2i$

$\frac{-5+3i}{4} = \frac{-2i}{4} = \frac{-i}{2}$

Let $f(z) = \frac{1}{2z^2+5iz-2}$

$= \frac{1}{2(z+2i)(z+i/2)}$

$-2i$ & $-i/2$ are simple poles of $f(z)$ and the pole $-i/2$ lies inside c

Also,

$\text{Res}\{f(z); -i/2\} = \lim_{z\to i/2} \frac{1}{2(z+2i)}$

$= \frac{1}{2(-i/2+2i)}$

$= \frac{1}{-i+4i}$

$= \frac{1}{3i}$

Cauchy's Residue theorem

$\int_c f(z)\,dz = 2\pi i\ \text{Res}\{f(z); -i/2\}$

$= 2\pi i\ (1/3i)$

$\int_c f(z)\,dz = \frac{2\pi}{3}$

Using contour integration evaluate $\int_0^{2\pi} \frac{d\theta}{13+5\sin\theta}$

Soln

$$\text{Let } I = \int_0^{2\pi} \frac{d\theta}{13+5\sin\theta}$$

Put $z = e^{i\theta}$

$$dz = ie^{i\theta}\, d\theta$$

$$dz = iz\, d\theta$$

$$d\theta = \frac{dz}{iz}$$

$$\sin\theta = \frac{z-z^{-1}}{2i}$$

$$I = \int_0^{2\pi} \frac{dz}{iz\left(13+5\left(\frac{z-z^{-1}}{2i}\right)\right)}$$

$$= \int_0^{2\pi} \frac{dz}{\cancel{i}z\,\frac{(26i+5z-z^{-1}5)}{2\cancel{i}}}$$

$$= \int_0^{2\pi} \frac{dz}{5z^2+26iz-5}$$

$$f(z) = \frac{2}{5z^2+26iz-5}$$

$$= \frac{2}{5z^2+25iz+iz-5}$$

$$= \frac{2}{5z(z+5i)+i(z+5i)}$$

$$= \frac{2}{(z+5i)(5z+i)}$$

$-i/5, -5i$ are Poles for $f(z)$

$-i/5$ is only the pole inside the circle

$$\text{Res}\{f(z); -i/5\} = \lim_{z\to -i/5} \frac{h(z)}{k'(z)}$$

$h(z) = 2$

$k(z) = 5z^2+26iz-5$

$k'(z) = 10z+26i$

$$\text{Res}\{f(z); -i/5\} = \lim_{z\to i/5} \frac{2}{10z+26i}$$

$$= \frac{2}{10(-i/5)+26i}$$

$$= \frac{1}{12i}$$

$\therefore$ By Cauchy's Residue thm,

$$\oint_C f(z)\,dz = 2\pi i \left(\text{Res}\{f(z); -i/5\} \right)$$

$$= 2\pi i \times 1/12i$$

$$\int_C f(z)\,dz = \pi/6$$

Use cantous integration technique to find the value of $\int_0^{2\pi} \frac{d\theta}{2+\cos\theta}$

soln

Let, $I = \int_0^{2\pi} \frac{d\theta}{2+\cos\theta}$

put $z = e^{i\theta}$ $\qquad \cos\theta = \frac{z+z^{-1}}{2}$

$dz = ie^{i\theta}d\theta$

$d\theta = \frac{dz}{iz}$

$$I = \int_0^{2\pi} \frac{dz}{iz\left[2+\left(\frac{z+z^{-1}}{2}\right)\right]}$$

$$= \int_0^{2\pi} \frac{dz}{iz\left[\frac{4+z+z^{-1}}{2}\right]}$$

$$= \int_0^{2\pi} \frac{-2i\,dz}{z^2+4z+1}$$

$$f(z) = \frac{-2i}{z^2+4z+1}$$

$$= \frac{-2i}{(z+2)^2-3}$$

$$= \frac{-2i}{(z+2)^2-(\sqrt{3})^2}$$

$$f(z) = \frac{-2i}{(z+2+\sqrt{3})(z+2-\sqrt{3})}$$

$z+2+\sqrt{3} = 0 \qquad z+2-\sqrt{3} = 0$

$z = -2-\sqrt{3} \qquad z = -2+\sqrt{3}$

$-2+\sqrt{3}$, $-2-\sqrt{3}$ are simple poles of $f(z)$

The poles $-2+\sqrt{3}$ lies inside c

$$\operatorname{Res}\{f(z); -2+\sqrt{3}\} = \lim_{z \to -2+\sqrt{3}} \left(\frac{h(z)}{k'(z)} \right)$$

$$h(z) = -2i$$

$$k(z) = z^2+4z+1$$

$$k'(z) = 2z+4$$

$$\operatorname{Res}\{f(z); -2+\sqrt{3}\} = \lim_{z \to -2+\sqrt{3}} \left(\frac{-2i}{2z+4} \right)$$

$$= \frac{-2i}{2(-2+\sqrt{3})+4}$$

$$= \frac{-2i}{-4+2\sqrt{3}+4}$$

$$= -i/\sqrt{3}$$

Cauchy's residue theorem

$$I = 2\pi i \left(\operatorname{Res}\{f(z); -2+\sqrt{3}\} \right)$$

$$= +2\pi i \left(-i/\sqrt{3}\right)$$

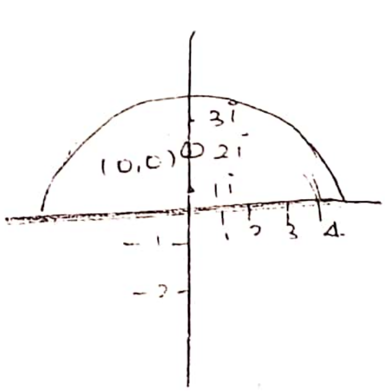
$$I = 2\pi/\sqrt{3}$$

Type II

Using the method of contour integration

evaluate $\int_{-\infty}^{\infty} \frac{x^2}{(x^2+1)(x^2+4)}$

soln



Let $f(z) = \frac{z^2}{(z^2+1)(z^2+4)}$

$z^2+1=0$ $\quad$ $z^2+4=0$

$z^2=-1$ $\quad$ $z^2=-4$

$z=\pm i$ $\quad$ $z=\pm 2i$

The poles of $f(z)$ are $i, -i, 2i, -2i$

The poles i and $2i$ lies with in c

By Residue theorem,

$$\int_C f(z)dz = 2\pi i \left[\text{Res}\{f(z); i\} + \text{Res}\{f(z); 2i\} \right] \quad \to (1)$$

We find the residues of $f(z)$

$$\operatorname{Res}\{f(z); i\} = \lim_{z \to i} \frac{h(z)}{k'(z)}$$

$$h(z) = z^2$$

$$\begin{aligned} k(z) &= (z^2+1)(z^2+4) \\ &= z^4 + z^2 + 4z^2 + 4 \\ &= z^4 + 5z^2 + 4 \end{aligned}$$

$$k'(z) = 4z^3 + 10z$$

$$\begin{aligned} \operatorname{Res}\{f(z); i\} &= \lim_{z \to i} \left(\frac{z^2}{4z^3 + 10z} \right) \\ &= \frac{(i)^2}{4i^3 + 10i} \\ &= \frac{-1}{-4i + 10i} \\ &= \frac{-1}{+6i} \\ &= \frac{i}{6}. \end{aligned}$$

$$\begin{aligned} \operatorname{Res}\{f(z); 2i\} &= \lim_{z \to 2i} \frac{h(z)}{k'(z)} \\ &= \lim_{z \to 2i} \left(\frac{z^2}{4z^3 + 10z} \right) \\ &= \frac{4i^2}{4(2i)^3 + 10(2i)} \\ &= \frac{-4}{-32i + 20i} \\ &= \frac{-4}{-12i} \\ &= \frac{1}{3i} \end{aligned}$$

$$\operatorname{Res}\{f(z); 2i\} = -i/3$$

By using cauchy's Residue theorem,

From ① $\int_C f(z)\,dz = 2\pi i \left\{ \operatorname{Res}\{f(z); i\} + \operatorname{Res}\{f(z); 2i\} \right\}$

$$\begin{aligned} &= 2\pi i \left[i/6 - i/3 \right] \\ &= 2\pi i \, (-i/6) \end{aligned}$$

$$\int_C f(z)\,dz = \frac{\pi}{3} \rightarrow ②$$

From (1) & (2)

$$\int_C f(z)\,dz = \int_{-r}^{r} \frac{x^2}{(x^2+1)(x^2+4)}\,dx + \int_{C_1} f(z)\,dz$$

$$\int_{-r}^{r} \frac{x^2}{(x^2+1)(x^2+4)}\,dx + \int_{C_1} f(z)\,dz = \pi/3 \rightarrow (3)$$

As $r \to \infty$, $\int_{C_1} f(z)\,dz \to 0$.

$$\int_{-\infty}^{\infty} \frac{x^2}{(x^2+1)(x^2+4)}\,dx = \pi/3$$

Evaluate $\displaystyle\int_{-\infty}^{\infty} \frac{x^2}{(x^2+a^2)(x^2+b^2)}\,dx = \pi/a+b$

soln

Let $f(z) = \dfrac{z^2}{(z^2+a^2)(z^2+b^2)}$

$z^2+a^2=0$ $z^2+b^2=0$

$z^2=-a^2$ $z^2=-b^2$

$z=\pm ai$. $z=\pm bi$

The poles of $f(z)$ are $ai, -ai, bi, -bi$

The poles ai and bi lie within c

By Residue theorem,

$$\int_C f(z)\,dz = 2\pi i\left[\operatorname{Res}\{f(z); ai\} + \operatorname{Res}\{f(z); bi\}\right] \rightarrow (1)$$

we find residues of $f(z)$

$$\operatorname{Res}\{f(z); ai\} = \lim_{z\to ai} \frac{h(z)}{k'(z)}$$

$h(z) = z^2$

$k(z) = (z^2+a^2)(z^2+b^2)$

$= z^4 + z^2b^2 + a^2z^2 + a^2b^2$

$k(z) = z^4 + (a^2+b^2)z^2 + a^2b^2$

$k'(z) = 4z^3 + 2z(a^2+b^2)$

$$\operatorname{Res}\{f(z); ai\} = \lim_{z\to ai} \frac{h(z)}{k'(z)}$$

$$= \lim_{z\to ai} \frac{z^2}{4z^3+2(a^2+b^2)z}$$

$$= \frac{(ai)^2}{4(ai)^3 + 2(a^2+b^2)ai}$$

$$= \frac{-a^2}{-4a^3i + 2a^3i + 2b^2ai}$$

$$= \frac{-a^2}{-ai[4a^2 - 2a^2 - 2b^2]}$$

$$= \frac{-ai}{2a^2 - 2b^2}$$

$$\text{Res}\{f(z); ai\} = \frac{-ai}{2(a^2-b^2)}$$

$$\text{Res}\{f(z); bi\} = \lim_{z \to bi} \frac{h(z)}{k'(z)}$$

$$= \lim_{z \to bi} \frac{z^2}{4z^3 + 2(a^2+b^2)z}$$

$$= \frac{(bi)^2}{4(bi)^3 + 2(a^2+b^2)bi}$$

$$= \frac{-b^2}{-4b^3i + 2a^2bi + 2b^3i}$$

$$= \frac{\cancel{-}b^{\cancel{2}}}{\cancel{-}\cancel{b}i[4b^2 - 2a^2 - 2b^2]}$$

$$\text{Res}\{f(z); bi\} = \frac{-bi}{2b^2 - 2a^2)}$$

By using cauchy's Residues theorem,

From ①

$$\int_C f(z)dz = 2\pi i\left[\frac{-ai}{2(a^2-b^2)} - \frac{bi}{2(b^2-a^2)}\right]$$

$$= \frac{\cancel{2}\pi i(-i)}{\cancel{2}}\left[\frac{a+b}{a^2-b^2 \quad -(a^2-b^2)}\right]$$

$$= \pi\left(\frac{a}{a^2-b^2}\right) - \left(\frac{b}{a^2-b^2}\right)$$

$$= \pi\frac{(a-\cancel{b})}{(a^2-\cancel{b^2})(a+b)}$$

$$\int_C f(z)\,dz = \frac{\pi}{a+b} \rightarrow ②$$

From ① & ②

$$\int_C f(z)\,dz = \int_{-r}^{r} \frac{x^2}{(x^2+a^2)(x^2+b^2)}\,dx + \int_{C1} f(z)\,dz$$

$$\int_{-r}^{r} \frac{x^2}{(x^2+a^2)(x^2+b^2)}\,dx + \int_{C1} f(z)\,dz = \frac{\pi}{a+b} \rightarrow ③$$

As, $r \to \infty$, $\int_{C1} f(z)\,dz \to 0$

$$\int_{-\infty}^{\infty} \frac{x^2}{(x^2+a^2)(x^2+b^2)}\,dx = \frac{\pi}{a+b}$$

Evaluate $\int_{-\infty}^{\infty} \frac{x^2-x+2}{x^4+10x^2+9}\,dx$

soln

Let $f(z) = \frac{z^2-z+2}{z^4+10z^2+9}$

$z^4+10z^2+9 = 0$

$z^4+9z^2+z^2+9=0$

$z^2(z^2+9)+1(z^2+9)=0$

$(z^2+9)(z^2+1)=0$

$z^2+9=0$ $\quad$ $z^2+1=0$

$z^2=-9$ $\quad$ $z^2=-1$

$z=\pm 3i$ $\quad$ $z=\pm i$

The poles of $f(z)$ are $3i, -3i, i, -i$

The poles $3i$ and i lie within c.

By Residue theorem,

$$\int_C f(z)\,dz = 2\pi i\left[\operatorname{Res}\{f(z); i\} + \operatorname{Res}\{f(z); 3i\}\right]$$

we find residues of $f(z)$

$$\operatorname{Res}\{f(z); i\} = \lim_{z\to i} \frac{h(z)}{k'(z)}$$

$h(z) = z^2 - z + 2$

$k(z) = z^4+10z^2+9$

$k'(z) = 4z^3+20z$

$$\operatorname{Res}\{f(z); i\} = \lim_{z \to i} \frac{h(z)}{k'(z)}$$

$$= \lim_{z \to i} \left(\frac{z^2 - z + 2}{4z^3 + 20z} \right)$$

$$= \frac{(i)^2 - i + 2}{4(i)^3 + 20(i)}$$

$$= \frac{-1 - i + 2}{-4i + 20i}$$

$$\operatorname{Res}\{f(z); i\} = \frac{1 - i}{16i}$$

$$\operatorname{Res}\{f(z); 3i\} = \lim_{z \to 3i} \left(\frac{z^2 - z + 2}{4z^3 + 20z} \right)$$

$$= \frac{(3i)^2 - (3i) + 2}{4(3i)^3 + 20(3i)}$$

$$= \frac{-9 - 3i + 2}{-108i + 60i}$$

$$\operatorname{Res}\{f(z); 3i\} = \frac{\cancel{+}(7 + 3i)}{\cancel{+}48i}$$

By using cauchy's residue theorem.

$$\text{From (1)} \Rightarrow \int_C f(z)\,dz = 2\pi i \left[\operatorname{Res}\{f(z); i\} + \operatorname{Res}\{f(z); 3i\} \right]$$

$$= 2\pi i \left[\frac{1 - i}{16i} + \frac{7 + 3i}{48i} \right]$$

$$= 2\pi i \left[\frac{3 - 3i + 7 + 3i}{48i} \right]$$

$$= \cancel{2}\pi \cancel{i} \left(\frac{\cancel{10}^{5}}{\cancel{48}\cancel{i}_{24}} \right)$$

$$\int_C f(z)\,dz = \frac{5\pi}{12} \rightarrow (2)$$

From (1) & (2)

$$\int_C f(z)\,dz = \int_{-r}^{r} \frac{x^2 - x + 2}{x^4 + 10x^2 + 9}\,dx + \int_{C_1} f(z)\,dz$$

$$\int_{-r}^{r} \frac{x^2 - x + 2}{x^4 + 10x^2 + 9}\,dx + \int_{C_1} f(z)\,dz = \frac{5\pi}{12} \rightarrow (3)$$

As $r \to \infty$, $\int_{C_1} f(z)\,dz \to 0$

$$\int_{-\infty}^{\infty} \frac{x^2 - x + 2}{x^4 + 10x^2 + 9}\,dx = \frac{5\pi}{12}$$