VECTOR CALCULUS

Gauss Divergence theorem

If F is a continuously differentiable vector function and s is a closed surface enclosing a region v, then

$$\iint_S \bar{F}\cdot\hat{n}\,ds = \iiint_V \mathrm{div}\,\bar{F}\,dV = \iiint_V \nabla\cdot\bar{F}\,dV.$$

problem:

Use Gauss divergence theorem, show that $\iint_S \bar{r}\cdot\hat{n}\,ds = 3V$ where $\bar{r} = x\bar{i}+y\bar{j}+z\bar{k}$ and s is a closed surface enclosing a volume V.

soln:

G.T $\bar{r} = x\bar{i}+y\bar{j}+z\bar{k}$

Now $\mathrm{div}\,\bar{r} = \nabla\cdot\bar{r}$

$= \left(i\frac{\partial}{\partial x}+j\frac{\partial}{\partial y}+k\frac{\partial}{\partial z}\right)\cdot(x\bar{i}+y\bar{j}+z\bar{k})$

$= \frac{\partial}{\partial x}(x)+\frac{\partial}{\partial y}(y)+\frac{\partial}{\partial z}(z)$

$= 1+1+1$

$\mathrm{div}\,\bar{r} = 3.$

WKT Gauss divergence theorem,

$$\iint_S \bar{F}\cdot\hat{n}\,ds = \iiint_V \mathrm{div}\,\bar{F}\,dV$$

Now $\iint_S \bar{r}\cdot\hat{n}\,ds = \iiint_V \mathrm{div}\,\bar{r}\,dV$

$= \int_V 3\,dV$

$= 3\int_V dV$

$\iint_S \bar{r}\cdot\hat{n}\,ds = 3V.$

problem:

If V is the volume enclosed by a closed surface s and $\bar{r} = x\bar{i}+2y\bar{j}+3z\bar{k}$, show that $\int_S \bar{r}\cdot\hat{n}\,ds = 6V.$

soln:

G.T $\bar{r} = x\bar{i}+2y\bar{j}+3z\bar{k}$

Now $\mathrm{div}\,\bar{r} = \nabla\cdot\bar{r}$

$= \left(i\frac{\partial}{\partial x}+j\frac{\partial}{\partial y}+k\frac{\partial}{\partial z}\right)\cdot(x\bar{i}+2y\bar{j}+3z\bar{k})$

$= \frac{\partial}{\partial x}(x)+\frac{\partial}{\partial y}(2y)+\frac{\partial}{\partial z}(3z)$

$= 1+2+3$

$\mathrm{div}\,\bar{r} = 6$

WKT Gauss divergence theorem

$$\iint_S \bar{F}\cdot\hat{n}\,ds = \iiint_V \mathrm{div}\,\bar{F}\,dv.$$

Now $\iint_S \bar{r}\cdot\hat{n}\,ds = \iiint_V \mathrm{div}\,\bar{r}\,dv$

$= \int_V 6\,dv$

$$= 6\int_V dv$$

$$\therefore \int_S \bar{r}\cdot\hat{n}\, ds = 6V.$$

Problem:

Use Gauss divergence theorem, prove that $\iint_S \frac{\bar{r}}{r^3}\cdot\hat{n}\, ds = 0$.

soln:

WKT $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$

$$\Rightarrow r^2 = x^2 + y^2 + z^2.$$

WKT $\frac{\partial \bar{r}}{\partial x} = \bar{i}, \quad \frac{\partial \bar{r}}{\partial y} = \bar{j}, \quad \frac{\partial \bar{r}}{\partial z} = \bar{k}$

and $\frac{\partial r}{\partial x} = \frac{x}{r}, \quad \frac{\partial r}{\partial y} = \frac{y}{r}, \quad \frac{\partial r}{\partial z} = \frac{z}{r}$

Now $\operatorname{div}\left(\frac{\bar{r}}{r^3}\right) = \operatorname{div}(\bar{r}\, r^{-3})$

$$= \nabla\cdot(\bar{r}\, r^{-3})$$

$$= \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)\cdot(\bar{r}\, r^{-3})$$

$$\operatorname{div}\left(\frac{\bar{r}}{r^3}\right) = i\frac{\partial}{\partial x}(\bar{r} r^{-3}) + j\frac{\partial}{\partial y}(\bar{r} r^{-3}) + k\frac{\partial}{\partial z}(\bar{r} r^{-3}) \quad \text{—— ①}$$

Now $\frac{\partial}{\partial x}(\bar{r} r^{-3}) = \bar{r}(-3) r^{-4}\frac{\partial r}{\partial x} + r^{-3}\frac{\partial \bar{r}}{\partial x}$

$$= -3\bar{r} r^{-4}\cdot\frac{x}{r} + r^{-3}\cdot\bar{i}$$

$$\frac{\partial}{\partial x}(\bar{r} r^{-3}) = -\frac{3\bar{r}x}{r^5} + \frac{\bar{i}}{r^3} \quad \text{—— ②}$$

III^{ly} $\frac{\partial}{\partial y}(\bar{r} r^{-3}) = -\frac{3\bar{r}}{r^5}y + \frac{\bar{j}}{r^3} \quad \text{—— ③}$

$$\frac{\partial}{\partial z}(\bar{r} r^{-3}) = -\frac{3\bar{r}}{r^5}z + \frac{\bar{k}}{r^3} \quad \text{—— ④}$$

Sub ②, ③ and ④ in ① we get

$$\operatorname{div}\left(\frac{\bar{r}}{r^3}\right) = i\left(-\frac{3\bar{r}x}{r^5} + \frac{\bar{i}}{r^3}\right) + \bar{j}\left(-\frac{3\bar{r}y}{r^5} + \frac{\bar{j}}{r^3}\right) + \bar{k}\left(-\frac{3\bar{r}z}{r^5} + \frac{\bar{k}}{r^3}\right)$$

$$= -\frac{3\bar{r}}{r^5}x\bar{i} + \frac{1}{r^3} - \frac{3\bar{r}y\bar{j}}{r^5} + \frac{1}{r^3} - \frac{3\bar{r}z\bar{k}}{r^5} + \frac{1}{r^3}$$

$$= -\frac{3\bar{r}}{r^5}(x\bar{i} + y\bar{j} + z\bar{k}) + \frac{3}{r^3}$$

$$= -\frac{3\bar{r}\cdot\bar{r}}{r^5} + \frac{3}{r^3}$$

$$= -\frac{3\cancel{r^2}}{\cancel{r^5}\,r^3} + \frac{3}{r^3}$$

$$= -\frac{3}{r^3} + \frac{3}{r^3}$$

$$\operatorname{div}\left(\frac{\bar{r}}{r^3}\right) = 0.$$

WKT Gauss divergence theorem

$$\iint_S \bar{F}\cdot\hat{n}\,ds = \iiint_V \operatorname{div}\bar{F}\,dV$$

Now $$\iint_S \frac{\bar{r}}{r^3}\cdot\hat{n}\,ds = \iiint_V \operatorname{div}\left(\frac{\bar{r}}{r^3}\right)dV$$

$$= \iiint_V 0\,dV$$

$$\therefore \iint_S \frac{\bar{r}}{r^3}\cdot\hat{n}\,ds = 0.$$

Green's theorem:

If $p(x,y)$ is a continuous function with continuous partial derivatives exists in a region R of the plane and boundary C then

$$\int_C p\,dx + Q\,dy = \iint_{R_{xy}} \left(\frac{\partial Q}{\partial x} - \frac{\partial p}{\partial y}\right)dx\,dy$$

and C is described in the +ve direction (Anticlockwise)

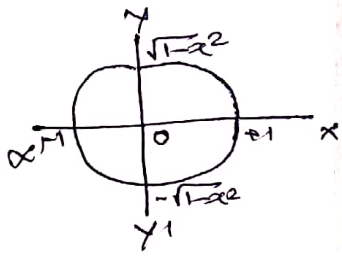
problem:

Use Green's theorem $\int_C y(2xy-1)\,dx + x(2xy+1)\,dy$, where C is the circle $x^2+y^2=1$.

soln:

G.T $\int_C y(2xy-1)\,dx + x(2xy+1)\,dy$

which is of the form $\int_C p\,dx + Q\,dy$



Here $p = y(2xy-1) = 2xy^2 - y$

$Q = x(2xy+1) = 2x^2y + x.$

Now $\frac{\partial p}{\partial y} = 2x(2y) - 1 = 4xy - 1$

$\frac{\partial Q}{\partial x} = 2y(2x) + 1 = 4xy + 1$

Now $\frac{\partial Q}{\partial x} - \frac{\partial p}{\partial y} = (4xy+1) - (4xy-1)$

$= \cancel{4xy} + 1 - \cancel{4xy} + 1$

$$\frac{\partial Q}{\partial x} - \frac{\partial p}{\partial y} = 2$$

G.T the equation of the circle $x^2+y^2=1$.

Treating x as a constant (ie $y=0$)

The limit of x is $x=-1$ to $x=+1$

The limit of y is $y=-\sqrt{1-x^2}$ to $y=+\sqrt{1-x^2}$

$\Rightarrow x^2+0^2=1$
$\Rightarrow x^2=1$
$\Rightarrow x=\pm 1$

WKT Green's theorem

$$\int_C p\,dx + Q\,dy = \iint_{R_{xy}} \left(\frac{\partial Q}{\partial x} - \frac{\partial p}{\partial y}\right)dx\,dy$$

Now $$\int_C y(2xy-1)\,dx + x(2xy+1)\,dy = \int_{x=-1}^{+1}\int_{y=-\sqrt{1-x^2}}^{\sqrt{1-x^2}} 2\,dx\,dy$$

$$= 2\int_{-1}^{+1}\left[\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dy\right] dx$$

$$= 2\int_{-1}^{1} (y)_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dx$$

$$= 2\int_{-1}^{1} [(\sqrt{1-x^2}) - (-\sqrt{1-x^2})]\, dx$$

$$= 2\int_{-1}^{1} 2\sqrt{1-x^2}\, dx$$

$$= 4\int_{-1}^{1} \sqrt{1-x^2}\, dx$$

$$= 4\left[\frac{x}{2}\sqrt{1-x^2} + \frac{1^2}{2}\sin^{-1}\left(\frac{x}{1}\right)\right]_{-1}^{1}$$

$$= 4\left[0 + \frac{1}{2}(\sin^{-1}(1) - \sin^{-1}(-1))\right]$$

$$= \frac{4}{2}\left[\sin^{-1}(\sin \pi/2) - \sin^{-1}(\sin(-\frac{\pi}{2}))\right]$$

$$= 2\left[\frac{\pi}{2} - (-\frac{\pi}{2})\right]$$

$$\therefore \oint_C y(2xy-1)dx + x(2xy+1)dy = 2\pi$$

Stoke's theorem

If S is an open surface bounded by a closed curve C and $\bar{F} = F_1\bar{i} + F_2\bar{j} + F_3\bar{k}$ be any continuously differentiable vector point function then

$$\iint_S \text{curl}\,\bar{F}\cdot\hat{n}\,ds = \int_C \bar{F}\cdot d\bar{r}$$

Problems

Evaluate by Stoke's theorem $\int_C e^x dx + 2y\,dy - dz$, where C is the curve $x^2+y^2=4$ and $z=2$.

Soln:

G.T the equation of the curve $x^2+y^2=4$.

Let $x = 2\cos\theta$ and $y = 2\sin\theta$

now $dx = -2\sin\theta\, d\theta$ and $dy = 2\cos\theta\, d\theta$

Now $e^x dx + 2y\,dy - dz = e^{2\cos\theta}(-2\sin\theta\, d\theta) + 2(2\sin\theta)(2\cos\theta\, d\theta) - 0.$

$$= -2e^{2\cos\theta}\sin\theta\, d\theta + 4(2\sin\theta\cos\theta)$$

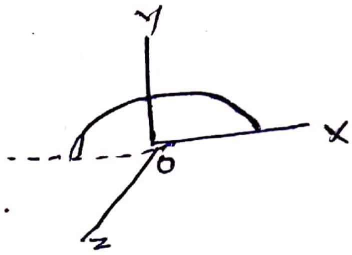
$z = 2$

$dz = 0$.

$$\therefore e^x dx + 2y\,dy - dz = -2e^{2\cos\theta}\sin\theta\, d\theta + 4\sin 2\theta\, d\theta$$

The limit of θ is $\theta = 0$ to $\theta = 2\pi$

Now $\int e^{x} dx + 2y\,dy - dz$



$$= \int_0^{2\pi} -2e^{2\cos\theta} \sin\theta \, d\theta + 4 \sin 2\theta \, d\theta$$

$$= -2\int_0^{2\pi} e^{2\cos\theta} \sin\theta \, d\theta + 4\int_0^{2\pi} \sin 2\theta \, d\theta$$

$$= \int_0^{2\pi} e^{x} dx + \cancel{4}^{2} \left(-\frac{\cos 2\theta}{\cancel{2}} \right)_0^{2\pi}$$

$$= (e^{x})_0^{2\pi} - 2(\cos 2\theta)_0^{2\pi}$$

$$= (e^{2\cos\theta})_0^{2\pi} - 2(\cos 2(2\pi) - \cos 2(0))$$

$$= e^{2\cos 2\pi} - e^{2\cos(0)} - 2(\cos 4\pi - \cos 0)$$

$$= e^{2(1)} - e^{2(1)} - 2(1-1)$$

$$= e^{2} - e^{2} - 0$$

$$= 0.$$

$$\therefore \int_C e^{x} dx + 2y\,dy - dz = 0.$$