4. The General Form of Cauchy's Theorem

4.1. Chains and cycles

Defn:- Let $\gamma_1, \gamma_2, \cdots, \gamma_n$ form a subdivison of the arc γ. Then the formula sums $\gamma_1 + \gamma_2 + \cdots + \gamma_n$ of arcs are called a chain.

Note:
1) Two chains should be considered identically, if they yield the same line integrals for all functions f.
2) The following operations do not charge the identity of a chain:
   - i) Permutation of two arcs.
   - ii) Subdivision of an arc.
   - iii) fusion of sub arcs to a single arc.
   - iv) reparametrization of an arc.
   - v) Cancellation of opposite, arcs.

Defn:- A chain is a cycle if it can be represented as a sum of closed curves.

Note:- The integral of an exact differential over any ~~ar~~ cycle of zero.

Defn:- A region is simply connected if its complement with ~~represent~~ respect to the extended plane is connected.

eg:- A disc, a half plane and a parallel strip are simply connected.

Theorem:- Any region is simply connected iff $n(\gamma, a) = 0$ for all cycles γ in Ω and all points 'a' which do not belongs to Ω.

proof:- (Necessary Part)

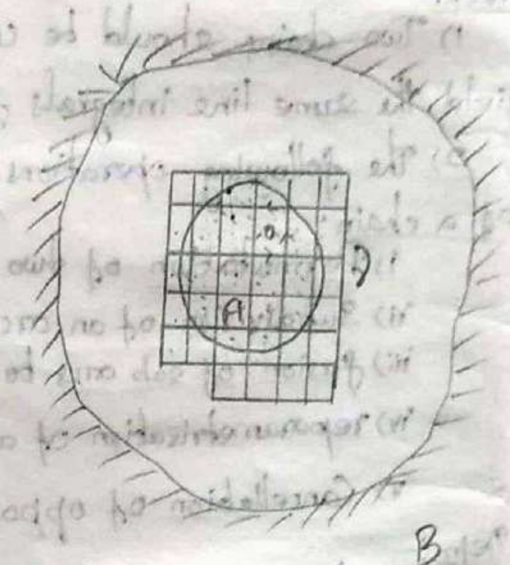
Given that any region is simply connected.

We prove that $n(\gamma, a) = 0$.

Let γ be any cycle in Ω. If the complement of Ω is connected, it must be contained in one of the regions determind by γ, and in as must as ∞ belongs to the complement this must be the unbounded region.

$\therefore n(\gamma, a) = 0$ for all finite points 'a' in the complement.

Sufficient Part:-



We assume that the complement of Ω can be represented as the union $A \cup B$ of two disjoint closed sets. One of these sets contains ∞, and other is consequently bounded.

Let A be the bounded set. The sets A & B have a shortest distance $\delta > 0$. Cover the whole plane with a set of squars Q of sid $< \delta/\sqrt{2}$. Let $a \in A$ be a point which is taken to be the centre of a square. The boundary of Q be denoted by ∂Q. We assume that the squares Q are closed and that the interior of Q lies to the left of the directed line segments which form ∂Q.

Consider the cycle $\gamma = \sum_j \partial Q_j$; where the sum ranges over all squares Q_j in the net which have a common with A. Since, 'a' lies in any one of these squares.

$\therefore n(\gamma, a) = 1$

A on γ does not meet B. But if the cancellations are carried out, it is clear that γ does not meet A.

Hence, γ is contained in Ω.

$\therefore \Omega$ is simply connected.

Remark:-

Cauchy's theorem is certainly not valied for regions which are not simply connected. If there is a cycle γ in Ω such that $n(\gamma, a) \neq 0$ for some a outside of Ω, then $\frac{1}{z-a}$ is analytic in Ω. Then,

$$\frac{1}{2\pi i}\int_\gamma \frac{dz}{z-a} = n(\gamma, a)$$

(or)

$$\int_\gamma \frac{dz}{z-a} = 2\pi i \; n(\gamma, a) \neq 0.$$

Homology:-

A cycle γ in an open set Ω is said to be homologous to zero with respect to Ω if $n(\gamma, a) = 0$ for all points 'a' in the complement of Ω. In symbols we write $\gamma \sim 0 \pmod{\Omega}$.

Note:-

1) The notation $\gamma_1 \sim \gamma_2$ is equivalent to $\gamma_1 - \gamma_2 \sim 0$.
2) Homologies can be added and subtracted and $\gamma \sim 0 \pmod{\Omega}$ implies $\gamma \sim 0 \pmod{\Omega'}$ for all $\Omega' \supset \Omega$.

The General Statement of Cauchy's Theorem:-

If $f(z)$ is analytic in Ω, then $\int_\gamma f(z)dz = 0$ for every cycle γ which is homologous to zero in Ω.

proof:- case:(i) Let Ω be bounded.

We cover the plane by a net of squares of side $\delta > 0$. Let Q_j, $j \in J$ be the closed squares in the net contained in Ω and since Ω is bounded, J is finite, for sufficiently small δ. J is nonempty.

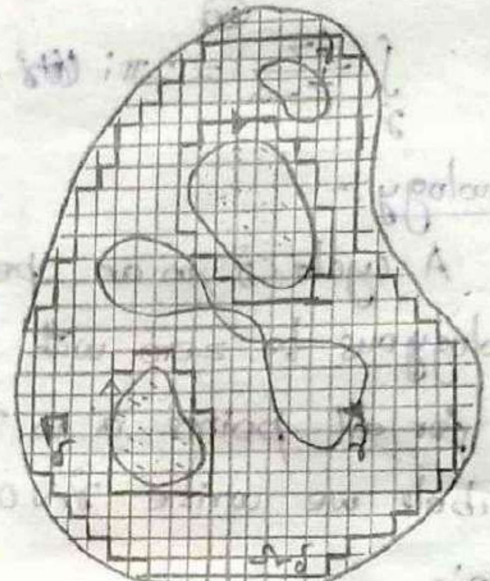
Let $\Gamma_\delta = \sum_{j \in J} \partial Q_j$

Where ∂Q_j is the boundary of Q_j.

$\therefore \Gamma_\delta$ is a sum of oriented boundaries which are sides of exactly one Q_j. Let Ω_δ be the interior of $\cup Q_j$.

Let $\gamma \equiv 0 \pmod{\delta}$. Also, δ is choosen such that $\gamma \subset \Omega_\delta$. Let $q \in \Omega - \Omega_\delta$.

$\therefore q$ belongs to atleast one Q and is not a Q_j. Let $q_0 \in Q$ which is not in Ω. Then it is possible to join q & q_0 by a line segment not meeting Ω_δ. Since $\gamma \subset \Omega_\delta$ (γ being the point set), it follows that



$$n(\gamma, q) = 0 = n(\gamma, q_0)$$

(ie) $n(\gamma, q) = 0 \ \forall q \in \Gamma_\delta$.

If a function $f(z)$ is analytic in Ω and if z_0 is an interior point of Q_{j_0}, then

$$\frac{1}{2\pi i} \int_{\partial Q_j} \frac{f(q)}{q - z}\, dq = \begin{cases} f(z) & , \text{if } j = j_0 \\ 0 & , \text{if } j \neq j_0 \end{cases}$$

$$\therefore f(z) = \frac{1}{2\pi i} \int_{\partial Q_j} \frac{f(q)\, dq}{q - z} = \frac{1}{2\pi i} \int_{\Gamma_\delta} \frac{f(q)\, dq}{q - z} \quad \forall z \in \Omega_\delta.$$

because of continuity of the function on both sides.

$$\therefore \int_\gamma f(z)\, dz = \int_\gamma \left(\frac{1}{2\pi i} \int_{\Gamma_\delta} \frac{f(q)\, dq}{q - z} \right) dz$$

$$= \int_{\Gamma_\delta} \left(\frac{1}{2\pi i} \int_\gamma \frac{dz}{q - z} \right) f(q)\, dq$$ (by changing the order of integration

$$\int_{\gamma} f(z)\,dz = \int_{\Gamma_\gamma} \left(-n(\gamma,q)\right) f(q)\,dq$$

$$= 0 \qquad (\because n(\gamma, q) = 0)$$

Case (ii) Let Ω be unbounded.

Also, let $\Omega' = \Omega \cap |z| < R$, R large enough to contain γ.

If $a \in$ the complement of Ω', then it lies either in the complement of Ω or lies outside the disk.

In either case

$n(\gamma, a) = 0$, so that $\gamma \sim 0 \pmod{\Omega'}$

Now, the proof done under case (i) is applicable to Ω' and the theorem is valid for unbounded Ω.

Hence by case (i) and case (ii) the theorem is valid for arbitrary Ω.

Locally Exact Differentials:

Defn:- A differential $p\,dx + q\,dy$ is said to be locally exact in Ω, if it is exact in same neighbourhood of each point in Ω.

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Theorem:- If $p\,dx+q\,dy$ is locally exact in Ω, then $\int_{\gamma} p\,dx + q\,dy = 0$ for every cycle $\gamma \sim 0$ in Ω.

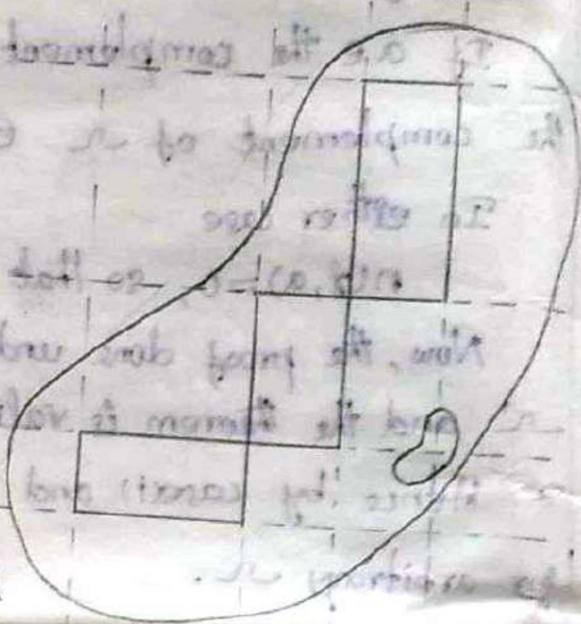
proof:- Let the distance from γ to the complement of Ω be ρ.

Let $\gamma : z = z(t)$, $z(t)$ being uniformly continuous on $[a,b]$ be divided into subintervals of length $< \delta$. This δ is determind by $|z(t) - z^{\circ}(t')| < \rho$ for $|t-t'| < \delta$.

The sub arcs γ_i of γ, associated with the subintervals of $[a,b]$ each will lie with in a disc of radius ρ which lies entirely in Ω. The end points of γ can be joined by a polygon σ_i consisting of a horizontal and a vertical segment and lies completely with in that disc. Since $pdx+qdy$ is exact in the disc, (6)

$$\int_{\sigma_i} pdx+qdy = \int_{\gamma_i} pdx+qdy$$

$$\therefore \int_{\sigma} pdx+q\,dy = \int_{\gamma} pdx+q\,dy \quad \to (1)$$



where, $\sigma = \sum_i \sigma_i$ and

$\gamma = \sum_i \gamma_i$.

Extending the segments of the polygon to infinite lines we get the plane divided into some finite rectangles R_i and some unbounded regions R_j', which may be regarded as infinite rectangles.

Let $a_i \in R_i$ and $\sigma_i = \sum_i n(\sigma, a_i)\,\partial R_i$ be a cycle. $\to (2)$

Here, the sum of ranges over all finite rectangles $n(\sigma, a_i)$ are well determined, for no a_i lies on σ.

Let $a_j' \in R_j'$.

Then, $n(\partial R_i, a_k) = \begin{cases} 1, & \text{if } k=i \\ 0, & \text{if } k \neq i \end{cases} \quad \to (3)$

and $n(\partial R_i, a_j') = 0 \;\forall j \quad \to (4)$

$\therefore$ From (2), $n(\sigma, a_i) = \sum_i n(\sigma, a_i)\, n(\partial R_i, a_i)$

$= n(\sigma, a_i)(1)$ (using (3))

$\to (5)$

and also $n(\sigma_0, a_j') = \sum_i n(\sigma, a_i)\, n(\partial R_i, a_j')$ ... ⑦

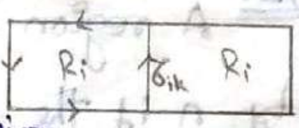
$= 0 \;\forall j \quad [\because n(\partial R_i, a_j') = 0 \;\forall j]$ → ⑥

and $n(\sigma, a_j') = 0$ → ⑦

for the interior of R_j' belongs to the unbounded region determind by σ. Thus, combining ⑤, ⑥ & ⑦, we have

$n(\sigma - \sigma_0, a) = (0, 0) \;\forall\; a = a_j' - a_i$ → ⑧

This shows that $\sigma_0 = \sigma$ upto the segments that cancel each other. Let σ_{ik} be the common side of two adjacent rectangles R_i, R_k. Let the orientation be choosen, such that R_i lies to left



of σ_{ik}. Suppose the reduced expression of $\sigma - \sigma_0$ contains the multiple of $c\sigma_{ik}$. Then the cycle $\sigma - \sigma_0 - c\partial R_i$ does not contain σ_{ik}.

$\therefore n(\sigma - \sigma_0 - c\partial R_i, a_i) = n(\sigma - \sigma_0 - c\partial R_i, a_k)$

(ie) $n(\sigma - \sigma_0, a_i) - c\, n(\partial R_i, a_i) = n(\sigma - \sigma_0, a_k) - c\, n(\partial R_i, a_k)$

(ie) $0 - c \cdot 1 = 0 - c \cdot 0$

(ie) $c = 0$.

The same reasoning applies if σ_{ij} is the common side of a finite rectangles R_i and an infinite rectangles R_j'.

Suppose that a point 'a' in the closed rectangle R_i, where not in Ω.

Then $n(\sigma, a) = 0 \quad [\because \sigma \sim 0 \,(\mathrm{mod}\ \Omega)]$

That is, the line segment between a and a_i does not intersect σ, and hence

$n(\sigma, a_i) = n(\sigma, a) = 0$

Therefore, by the local exactness of $pdx + qdy$.

$\int_\sigma pdx + qdy = 0$ with $\sigma = \sum_i n(\sigma, a_i)\, \partial R_i$

Hence the theorem

<u>Multiply Connected Regions</u>:-

<u>Defn</u>:- A region which is not simply connected is called multiply connected.

<u>Defn</u>:- A region Ω is said to have the finite connectivity n if the complement of Ω has exactly n components and infinite connectivity if the complement has infinitely many components.

<u>Note</u>:1 Every cycle is homologous to a ~~cycle~~ unique linear combination of the cycles $\gamma_1, \gamma_2, \cdots, \gamma_{n-1}$ called a homology basis for the region Ω. Also every homology basis has the same number of elements.

<u>Note</u>:2 Every region with a finite homology basis has finite connectivity and the no. of basis elements is one less than the connectivity.

<u>Defn</u>:- The numbers $P_i = \int_{\gamma_i} f\,dz$, that depend only on the function and not on γ, are called modulus of periodicity of the differential $f\,dz$.