5) General preliminaries on Banach Algebra

Algebra:

An algebra is a linear space whose vector can be multiplied in such away that

(1) $x(yz) = (xy)z$

(2) $x(y+z) = xy + xz$ and $(x+y)z = xz + yz$

(3) $\alpha(xy) = (\alpha x)y$ (or) $x(\alpha y)$ $\forall$ scalar α.

Here x, y, z are points of a linear space. The scalars are real numbers then the algebra is called a real algebra and if the scalars are complex numbers then it is called Complex algebra.

Commutative algebra:

A Commutative algebra is an algebra whose multiplication satisfying the following condition $xy = yx$.

Algebra with identify:

An algebra with identify is

algebra which possesses the following property.

There exist a non zero element 1 is called the identity element such that $x.1 = 1.x = x \quad \forall x$.

Banach Algebra: 2m

A Banach algebra is a complex banach space which is also an algebra with identity 1 in which the multiplicative structure is related with a norm by the following requireme

(1) $\|xy\| \le \|x\| \|y\|$ (2) $\|1\| = 1$.

Banach sub algebra:

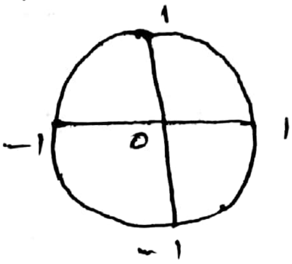
A banach sub algebra of a banach algebra A is a closed sub algebra of a which contains the identity 1.

Example for Banach algebra: 2m

(1) The set $C(x)$ of all bounded continuous complex functions defined by a topological space X is a Banach algebra.

(2) Consider the closed unit disc

$D = \{ z \,/\, \|z\| \leq 1 \}$ in the complex plane.



The subset of $C(D)$ which consists of all functions analytic in the interior of the sub algebra which contains the identity.

(x)

Theorem: [illegible]

Every element x, for which $\|x-1\| <$ is regular and its inverse is given by the formula

$$x^{-1} = 1 + \sum_{n=1}^{\infty} (1-x)^n.$$

Proof:

put $r = \|x-1\|$, $r < 1$

Now $\|1-x\| \leq \|1-x\|^n \leq r^n$.

$\therefore \|(1-x)^n\| \leq r^n$.

$\therefore$ The partial sum of the series $\sum_{n=1}^{\infty} (1-x)^n$ forms a Cauchy sequence in

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But A is complete (Being a Banach space)

$\therefore$ These partial sums converge to an element in A. let it be $\sum_{n=1}^{\infty}(1-x)^n$.

Now we define y by $y = 1+\sum_{n=1}^{\infty}(1-x)^n$

Then the joint continuity of multiplication yields

$$y - xy = (1-x)y$$

$$= (1-x)\left(1+\sum_{n=1}^{\infty}(1-x)^n\right)$$

$$= (1-x) + \sum_{n=2}^{\infty}(1-x)^n$$

$$= \sum_{n=1}^{\infty}(1-x)^n \quad (1-x)\left[(1-x)^1 + (1-x)^2 \ldots\right.$$

$$= y-1 \quad (1-x) \, (1-x)^2 + \ldots$$

$$y - xy - y = -1$$

$$\Rightarrow \quad xy = 1$$

IIIly we can prove $yx = 1$

This prove that y is the inverse of x.

ie, $y = x^{-1}$

$$= 1 + \sum^{\infty}(1-x)^n$$

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Theorem:

G is an open set and s is a closed set.

Proof:

G is the set of all regular elements in the Banach algebra A.

To prove G is open, we have to prove that for any point $x_0 \in G$, $\exists$ an open sphere centered at x_0 and contained in G.

Let $x_0 \in G$ and x be any other point in G, such that $\|x - x_0\| < \frac{1}{\|x_0^{-1}\|}$

Now Consider,

$$\|x_0^{-1} x^{-1}\| = \|x_0^{-1} x - x_0^{-1} x_0\|$$

$$= \|x_0^{-1}(x - x_0)\|$$

$$\leq \|x_0^{-1}\| \, \|x - x_0\|$$

$$\leq \|x_0^{-1}\| \frac{1}{\|x_0^{-1}\|} \leq 1$$

$$\|x_0^{-1} x^{-1}\| < 1.$$

$\therefore$ By previous theorem, $x_0^{-1}x$ is regular.

ie, $x_0^{-1} x \in G$

Since G is a group,

$x_0 \in G$, $x_0^{-1}x \in G$

$$x_0(x_0^{-1}x) \in G$$

$$\Rightarrow (x_0 x_0^{-1})x \in G$$

$$\Rightarrow 1 \cdot x \in G$$

$$\therefore x \in G.$$

Since x is any arbitrary element in A such that $\|x - x_0\| < \frac{1}{\|x_0^{-1}\|}$

ie, An open sphere centered at $x_0 \in G$ is contained in G.

$\therefore$ G is an open set.

$\therefore$ S lieing the complement of G is closed.

Hence the theorem.

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(X) Theorem:

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The mapping $x \to x^{-1}$ of G into G is continuous and therefore a homomorphism of G onto itself.

Proof:

Let $x_0 \in G$ be any element and x be any other element in A such that

$$\|x - x_0\| < \frac{1}{2\|x_0^{-1}\|}$$

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Now $\|x_0^{-1}x - 1\| = \|x_0^{-1}x - x_0^{-1}x_0\|$ (7)

$$= \|x_0^{-1}(x - x_0)\|$$

$$\leq \|x_0^{-1}\| \, \|x - x_0\|$$

$$\leq \|x_0^{-1}\| \frac{1}{2\|x_0^{-1}\|}$$

$$= \frac{1}{2} < 1.$$

$\therefore \|x_0^{-1}x - 1\| < 1.$

$\therefore$ By theorem $x_0^{-1}x \in G$.

Its inverse is given by

$$(x_0^{-1}x)^{-1} = x^{-1}x_0$$

$$(x_0^{-1}x)^{-1} = 1 + \sum_{n=1}^{\infty} (1 - x_0^{-1}x)^n \quad \text{------- (1)}$$

Now Consider,

$(\|x+y\| \leq \|x\| + \ldots$

$$\|x^{-1} - x_0^{-1}\| = \|(x^{-1}x_0 - 1)\, x_0^{-1}\|$$

$(\|xy\| \leq \|x\|\|y\|$ [illegible]

$$\leq \|x_0^{-1}\| \, \|x^{-1}x_0 - 1\|$$

$$\leq \|x_0^{-1}\| \, \left\|1 + \sum_{n=1}^{\infty} (1 - x_0^{-1}x)^n - 1\right\| \quad [\because \text{from (1)}$$

$$\leq \|x_0^{-1}\| \, \left\|\sum_{n=1}^{\infty} (1 - x_0^{-1}x)^n\right\|$$

$$= \|x_0^{-1}\| \, \|1 - x_0^{-1}x\| \, \left\|\sum_{n=0}^{\infty} (1 - x_0^{-1}x)^n\right\|$$

$$= \|x_0^{-1}\| \, \|1 - x_0^{-1}x\| \frac{1}{1 - \|1 - x_0^{-1}x\|}$$

$$\|x^{-1} - x_0^{-1}\| < \frac{\|x_0^{-1}\| \, \|1 - x_0^{-1}x\|}{1 - 1/2}$$

$$< 2\|x_0^{-1}\| \, \|1 - x_0^{-1}x\|$$

$$= 2\|x_0^{-1}\| \, \|x_0^{-1}(x_0 - x)\|$$

$$< 2\|x_0^{-1}\| \, \|x_0^{-1}\| \, \|(x_0 - x)\|$$

$$\|x^{-1} - x_0^{-1}\| < 2\,\|x_0^{-1}\|^2 \, \|x_0 - x\| \quad \text{-----} \; (2)$$

Whenever $x \to x_0$, $\|x - x_0\| \to 0$.

$\therefore$ In the above inequality the R.H.S $\to 0$.

$\therefore$ The L.H.S of equation (2)

$\|x^{-1} - x_0^{-1}\| \to 0$

$\therefore x^{-1} \to x_0^{-1}$

$\therefore$ whenever $x \to x_0 \Rightarrow x^{-1} \to x_0^{-1}$

$\therefore$ The mapping $x \to x^{-1}$ is continuous.

IIIly The mapping $x^{-1} \to (x^{-1})^{-1} = x$.

$x^{-1} \to x$ is also continuous

$\therefore$ The mapping $x \to x^{-1}$ is a homomorphism.

Hence the theorem.

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Topological Divisors of Zero

Let A be a Banach algebra. An element $z \in A$ is called a topological divisor of zero, if $\exists$ a sequence $\{z_n\} \in A \ni \|z_n\| = 1$ and either $z z_n \to 0$ (or) $z_n z \to 0$.

The set of all topological divisors of zero is denoted by Z.

Theorem :

Z is a subset of S.

Proof :

Let $z \in Z$.

Let $\{z_n\}$ be a sequence in Z such that $\|z_n\| = 1$ and $z z_n \to 0$. (say)

We have to prove that $z \in S$ so that $Z \subset S$.

On the contrary assume that $z \in G$.

$\therefore \exists z^{-1}$. Then by the joint continuity of multiplication,

$$z^{-1}(z z_n) = (z^{-1} z)(z_n)$$
$$= 1 . z_n = z_n \to 0$$

This contradicts $\|z_n\| = 1$.

$\therefore z \notin G$

$\therefore z \in S$

$\therefore Z \subset S$.

Hence the theorem.

Theorem : APR-05

The boundary of S is a subset of Z.

Proof:

Since S is closed, its boundary consists of all points in S, which are limit points of the convergent sequence in G.

We will show that if $z \in S$ and $\exists$ a seq $\{x_n\} \in G$ such that $x_n \to z$, then $z \in Z$.

Now, $$x_n^{-1}z - 1 = x_n^{-1}z - x_n^{-1}x_n$$

$$= x_n^{-1}(z - x_n)$$

$$\to 0 \quad [\because x_n \to z \Rightarrow (x_n - z) \to 0]$$

Also the sequence $\{x_n^{-1}\}$ is unbounded.

Otherwise, if it is bounded

$\|x_n^{-1}z - 1\| < n$, for some n.

$\therefore r_n^{-1} z$ is regular.

$\therefore y = r_n (r_n^{-1} z)$ is regular.

$\Rightarrow z$ is regular.

Which is a contradiction to $z \in S$.

$\therefore$ The sequence $\{r_n^{-1}\}$ is unbounded.

$\therefore$ Assume that $\| r_n^{-1} \| \to \infty$

Now define z_n by $z_n = \dfrac{r_n^{-1}}{\| r_n^{-1} \|}$

$$\| z_n \| = \left\| \frac{r_n^{-1}}{\| r_n^{-1} \|} \right\| = \frac{\| r_n^{-1} \|}{\| r_n^{-1} \|} = 1.$$

Now,

$$z z_n = z \cdot \frac{r_n^{-1}}{\| r_n^{-1} \|}$$

$$= \frac{1 + (z - r_n) r_n^{-1}}{\| r_n^{-1} \|}$$

$$= \frac{1}{\| r_n^{-1} \|} + \frac{(z - r_n) r_n^{-1}}{\| r_n^{-1} \|}.$$

But $\| r_n^{-1} \| \to \infty \quad \Rightarrow \dfrac{1}{\| r_n^{-1} \|} \to 0$

and $r_n \to z \Rightarrow (r_n - z) \to 0$

ie, $(z - r_n) \to 0$

RHS $\to 0$

$\therefore z z_n \to 0$

$\therefore$ We have $\|z_n\| = 1$ and $z z_n \to 0$

$\therefore$ $z \in Z$.

$\therefore$ The boundary of S is a subset of Z.

Hence the theorem.

Spectrum:

Define:

Let T be a operator on a Hilbert space H, then the spectrum of T is defined by

$$\sigma(T) = \{\lambda / T - \lambda I \text{ is singular}\}.$$

Spectrum of an element:

Let A be a Banach algebra and let $x \in A$. The spectrum of x is the subset of the complex plane

$$\sigma(x) = \{\lambda / (x - \lambda) \text{ is singular}\}$$

Properties:

(1) Since the set of singular elements in A is closed, $\sigma(x)$ is also closed.

(2) $\sigma(x)$ is the subset of the closed disc $\{z / |z| \leq \|x\|\}$.

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Theorem : X.

$\sigma(x)$ is non-empty.

Proof:

Let f be a functional defined on A.

ie, $f \in A^*$

Define $f(\lambda) = f(x(\lambda))$

Here $f(\lambda)$ is a complex function which is defined and continuous on the resolvent set ρ.

Now from the resolvent equation we have

$$x(\lambda) - x(\mu) = (\lambda - \mu) \cdot x(\lambda) \cdot x(\mu)$$

$$f(x(\lambda) - x(\mu)) = f((\lambda-\mu) \cdot x(\lambda) \cdot x(\mu))$$

$$f(x(\lambda)) - f(x(\mu)) = (\lambda-\mu)[f(x(\lambda) \cdot x(\mu))]$$

$$f(\lambda) - f(\mu) = (\lambda-\mu) f(x(\lambda) \cdot x(\mu))$$

$$\frac{f(\lambda) - f(\mu)}{\lambda - \mu} = f(x(\lambda) \cdot x(\mu)).$$

Take $\lim \lambda \to \mu$ on both sides

$$\lim_{\lambda \to \mu} \frac{f(\lambda) - f(\mu)}{\lambda - \mu} = \lim_{\lambda \to \mu} f\left(x(\lambda) \cdot x(\mu)\right)$$

$$= f\left(x(\mu) \cdot x(\mu)\right).$$

$$= f\left(x(\mu)\right)^2 \quad \text{------} \; (1)$$

$\therefore f(\lambda)$ has derivative at each point of $\rho(x)$.

Further, $\quad |f(\lambda)| = |f(x(\lambda))|$

$$\leq \|f\| \, \|x(\lambda)\|$$

Since $f \in A^*$, $\|f\|$ is bounded.

and $\|x(\lambda)\| \to 0$ as $\lambda \to \infty$

$\therefore f(\lambda) \to 0$ as $\lambda \to \infty$. $\quad$ ------ (2)

On the contrary, assume that $\sigma(x)$ is empty.

$\rho(x) = [\sigma(x)]^c$ is the entire complex plane.

$\therefore f(\lambda)$ is an entire function in the whole complex plane such that $f(\lambda) \to 0$ as $\lambda \to \infty$

[$\because$ from (1) & (2)

$\therefore f(\lambda)$ is bounded entire function.

$\therefore$ By Liouville's theorem

$f(\lambda) = 0 \quad \forall \lambda \in \rho(x)$.

ie, $f(x(\lambda)) = 0 \quad \forall \lambda$.

But $f \in A^*$ is an arbitrary function

$x(\lambda) = 0$

$(x - \lambda e)^{-1} = 0.$

This is impossible since inverse of an element is never zero.

$\therefore$ Are assumption that $\sigma(x)$ is empty is away.

$\therefore$ $\sigma(x)$ is non-empty.

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Spectral radius:

$\sigma(x)$ is non-empty and it is a Compac subspace of the complex plane, the number of $r(x)$ defined by

$$r(x) = \sup \{ |\lambda| = \lambda \in \sigma(x) \}$$ is called the spectral radius of x.

It is clear that $0 \leq r(x) \leq \|x\|$.

Division algebra:

The division algebra is an algebra with identity in which every non-zero element is regular.

Formula for the spectral radius:

Let A be a Banach algebra and $x \in$. Then its spectral radius $r(x)$ is defined by

$$r(x) = \sup \{ |\lambda| : \lambda \in \sigma_A(x) \}.$$

Theorem:

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The spectral radius $r(x) = \lim \|x^n\|^{1/n}$.

Proof:

To prove the theorem we need the following lemma.

Lemma:

$$\sigma(x^n) = [\sigma(x)]^n.$$

Proof:

Let λ be a non-zero complex number and $\lambda_1, \lambda_2, \ldots, \lambda_n$ be its distinct n roots so that $x^n - \lambda 1 = (x - \lambda_1 1)(x - \lambda_2 1) \ldots (x - \lambda_n 1)$

$x^n - \lambda 1$ is singular iff $(x - \lambda_i 1)$ is singular for atleast one i.

ie, $\{\lambda \mid (x^n - \lambda 1)\}$ is singular.

$\Leftrightarrow \{\lambda_i \mid (x - \lambda_i 1)$ is singular for $i\}$.

$\sigma(x) = \{\lambda \mid (x - \lambda 1)$ is singular$\}$

$\therefore \sigma(x^n) = [\sigma(x)]^n.$

Now let us prove the theorem :

By the lemma $\sigma(x^n) = [\sigma(x)]^n$.

ie, $r(x^n) = [r(x)]^n$.

But $r(x^n) \leq \|x^n\|$

ie, $r(x^n) = [r(x)]^n \leq \|x^n\|$.

Taking n^{th} root on both sides

$$\{[r(x)]^n\}^{1/n} = \|x^n\|^{1/n}$$

$$r(x) \leq \|x^n\|^{1/n} \quad \forall n.$$

To conclude the proof it suffice to show that if 'a' is any real number, such that $r(x) < a$ then $\|x^n\|^{1/n} \leq a$ for all but a finite number of n's.

Now if $\lambda > |x|$, then

$$x(\lambda) = (x - \lambda e)^{-1}$$

$$= \left[\lambda\left(\frac{x}{\lambda} - 1\right)\right]^{-1}$$

$$= \lambda^{-1}\left(\frac{x}{\lambda} - 1\right)^{-1}$$

$$= -\lambda^{-1}\left(1 - \frac{x}{\lambda}\right)^{-1}$$

ie, $x(\lambda) = -\lambda^{-1}\left[1 + \sum_{n=1}^{\infty} \frac{x^n}{\lambda^n}\right]$ ------ (1)

If f is any arbitrary functional in A.

$$f[x(\lambda)] = f\left[-\lambda^{-1}\left(1+\sum_{n=1}^{\infty}\frac{x^n}{\lambda^n}\right)\right]$$

$$= -\lambda^{-1}\left[f(1)+\sum_{n=1}^{\infty} f\left(\frac{x^n}{\lambda^n}\right)\right]$$

$$= -\lambda^{-1}\left[f(1)+\sum_{n=1}^{\infty} f(x^n)\lambda^{-n}\right] \quad \forall\, \lambda > \|x\| \quad \text{---} \text{ (2)}$$

Now $f[x(\lambda)]$ is an analytic function in the region $\|\lambda\| > r(x)$ and equation (2) its expansion for $|\lambda| > \|x\|$.

Now let α be any complex number such that $r(x) < \alpha < a$, then the series

$$\sum_{n=1}^{\infty} f\left(\frac{x^n}{\alpha^n}\right) \text{ converges.}$$

$\therefore$ Its term form a bounded sequence.

By a known theorem

$\left\{\frac{x^n}{\alpha^n}\right\}$ form a bounded sequence in A.

$$\therefore \left\|\frac{x^n}{\alpha^n}\right\| \leq k.$$

$$\therefore \|x^n\| \leq k\alpha^n.$$

$\|x^n\|^{1/n} \leq k^{1/n}\alpha$ for some positive constant k and for every n.

But $k^{1/n}\alpha \leq a$ for sufficiently large n.

$\because \|x^n\|^{1/n} \leq a$ for all but a finite number of n's.

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$\therefore r(x) = \lim \|x^n\|^{1/n}$.

Hence the theorem.