St: The expression for k_g is involving the angle θ which the curve under consideration makes with the parametric curves v = constant. Regarding θ as a fn. of 's' along the curve, then Liouville's formula is

$$k_g = \theta' + Pu' + Qv'$$

where,

$$P = \frac{1}{2HE}(2EF_1 - FE_1 - EE_2)$$

$$Q = \frac{1}{2HE}(EG_1 - FE_2)$$

Proof: The directional coefficients of the curve v = const & and the given curve are $\left(\frac{1}{\sqrt{E}}, 0\right)$ & (u', v')

WKT,

$$\cos\theta = Ell' + F(lm' + l'm) + Gmm'$$

$$\sin\theta = H(lm' - l'm)$$

ie,

$$\cos\theta = E\frac{1}{\sqrt{E}}u' + F\left(\frac{1}{\sqrt{E}}v' + u'(0)\right) + G(0)(v')$$

$$= \frac{1}{\sqrt{E}}(Eu' + Fv')$$

$$\therefore \cos\theta = \frac{1}{\sqrt{E}}\left(\frac{\partial T}{\partial u'}\right) \quad \text{—(1)}$$

$$\sin\theta = H\left(\frac{1}{\sqrt{E}}v' - u'(0)\right)$$

$$\therefore \sin\theta = \frac{Hv'}{\sqrt{E}} \quad \text{—(2)}$$

Diff. (1) w.r.t 's'

$$-\sin\theta \cdot \frac{d\theta}{ds} = \left(\frac{d}{ds}\left(\frac{1}{\sqrt{E}}\right)\right)\frac{\partial T}{\partial u'} + \frac{1}{\sqrt{E}}\frac{d}{ds}\left(\frac{\partial T}{\partial u'}\right)$$

$$= -\frac{1}{2}E^{-3/2}\frac{dE}{ds}\cdot\frac{\partial T}{\partial u'} + \frac{1}{\sqrt{E}}\left[U + \frac{\partial T}{\partial u}\right]$$

$$\left[\because U = \frac{d}{ds}\left(\frac{\partial T}{\partial u'}\right) - \frac{\partial T}{\partial u}\right.$$

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(147)

$$\Rightarrow -\frac{Hv}{\sqrt{E}}\frac{d\theta}{ds} = -\frac{1}{2E^{3/2}}(E_1u'+E_2v')\frac{\partial T}{\partial u'} + \frac{1}{\sqrt{E}}\left(U+\frac{\partial T}{\partial u}\right)$$

$$\left[\because \frac{dE}{ds} = \frac{\partial E}{\partial u}\frac{du}{ds} + \frac{\partial E}{\partial v}\frac{dv}{ds} = E_1u' + E_2v'\right]$$

$$-\frac{Hv'}{\sqrt{E}}\frac{d\theta}{ds} = -\frac{1}{2E^{3/2}}(E_1u'+E_2v')(Eu'+Fv') + \frac{1}{\sqrt{E}}\left(U+\frac{\partial T}{\partial u}\right)$$

$$\Rightarrow Hv'\theta' = U - \frac{1}{2E}(E_1u'+E_2v')(Eu'+Fv') + \frac{\partial T}{\partial u}$$

$$= U - \frac{1}{2E}\left[EE_1u'^2 + E_1Fu'v' + EE_2u'v' + E_2Fv'^2\right] + \frac{\partial T}{\partial u}$$

$$\left[\because T = \tfrac{1}{2}(Eu'^2 + 2Fu'v' + Gv'^2)\right]$$

$$= U - \frac{1}{2E}\left[EE_1u'^2 + E_1Fu'v' + EE_2u'v' + E_2Fv'^2\right] + \frac{1}{2}\left[E_1u'^2 + 2F_1u'v' + G_1v'^2\right]$$

$$= U - \frac{1}{2E}\left[\cancel{EE_1u'^2} + E_1Fu'v' + EE_2u'v' + E_2Fv'^2\right] + \frac{1}{2E}\left[\cancel{EE_1u'^2} + 2EF_1u'v' + G_1Ev'^2\right]$$

$$-\theta' = \frac{U}{Hv'} + \frac{1}{2EHv'}\left[2EF_1u'v' - E_1Fu'v' - EE_2u'v' + EG_1v'^2 - E_2Fv'^2\right]$$

$$= -k_g + \frac{1}{2EH}\left[(2EF_1 - E_1F - EE_2)u' + (EG_1 - FE_2)v'\right]$$

$$k_g = \theta' + \frac{u'}{2EH}(2EF_1 - FE_1 - EE_2) + \frac{v'}{2EH}(EG_1 - FE_2)$$

$$k_g = \frac{-H\lambda}{Fu' + Gv'} \quad \therefore k_g = \theta' + Pu' + Qv'$$

$$k_g = \frac{1}{H}\left[V\frac{\partial T}{\partial u'} - U\frac{\partial T}{\partial v'}\right] = \frac{1}{H}\left[V(Eu'+Fv') - U(Fu'+Gv')\right]$$

$$= \frac{-H}{Fu'+Gv'}\left(\frac{1}{H}\frac{U}{v'}\frac{\partial T}{\partial v'}\right) \quad \text{where } P = \frac{1}{2HE}(2EF_1 - FE_1 - EE_2)$$

$$= -U/H\theta' \qquad Q = \frac{1}{2HE}(EG_1 - FE_2)$$

small $-k_g = \dfrac{U}{H\theta'}$

Example - 15.2: Prove that if the orthogonal trajectories of the curves v = constant are geodesics, then H^2/E is independent of u.

Proof: The orthogonal trajectories satisfy $\theta = \pi/2$ & are geodesics if $k_g = 0$. —(1)

By Liouville's formula, $k_g = \theta' + Pu' + Qv'$ —(2)

$2HE$

$Q = \frac{1}{2HE}(EG_1 - FE_2)$

Sub. (1) in (2), we get

$$0 = 0 + Pu' + Qv'$$

$$Pu' = -Qv'$$

$$\frac{u'}{v'} = -Q/P \quad \text{—(3)}$$

The direction cosines of the parametric curve $v = c$ & the given curve are $\left(\frac{1}{\sqrt{E}}, 0\right)$ (u', v')

WKT, $\cos\theta = Ell' + \cancel{2}F(lm' + l'm) + Gmm'$

$\sin\theta = H(lm' - l'm)$

$$\cos\theta = E\left(\frac{1}{\sqrt{E}}\right)(u') + \cancel{2}F\left(\frac{1}{\sqrt{E}}v' + 0\right) + 0 = \frac{1}{\sqrt{E}}(Eu' + Fv')$$

$$\therefore \cos \pi/2 = \frac{1}{\sqrt{E}}(Eu' + Fv')$$

$$0 = Eu' + Fv' \Rightarrow Eu' = -Fv'$$

$$\frac{u'}{v'} = -F/E \quad \text{—(4)}$$

From (3) & (4) $\Rightarrow$ $Q/P = F/E \Rightarrow QE = PF$

$$QE - PF = 0 \quad \text{—(5)}$$

Sub. the values of P & Q in (5)

$$\frac{1}{2HE}(EG_1 - FE_2)E - \frac{1}{2HE}(2EF_1 - FE_1 - EE_2)F = 0$$

$$E^2G_1 - \cancel{FE_2}F - 2EF_1F + F^2E_1 + \cancel{EE_2}F = 0$$

$$E^2G_1 + F^2E_1 - 2EF_1F = 0$$

$$G_1 + \frac{F^2}{E^2}E_1 - \frac{2EF_1}{E^2}F = 0$$

$$G_1 - \left(\frac{2EF_1F - F^2E_1}{E^2}\right) = 0$$

$$\frac{\partial G_1}{\partial u} - \frac{\partial}{\partial u}\left(\frac{F^2}{E}\right)$$

$$\frac{\partial}{\partial u}\left(G - F^2/E\right) = 0 \Rightarrow \frac{\partial}{\partial u}\left(\frac{EG - F^2}{E}\right) = 0$$

$$\frac{\partial}{\partial u}\left(\frac{H^2}{E}\right) = 0$$

$\therefore \frac{H^2}{E}$ is independent of u

16. Gauss - Bonnet theorem:

Prove that

What is a simply connected region? Give an example.

Simply connected region

2 mark

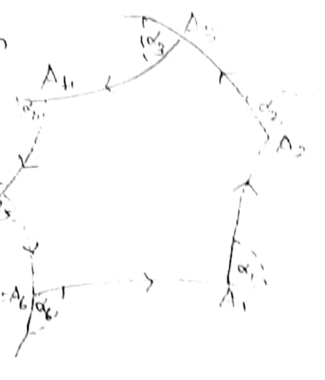
Every closed curve lying in R can be contracted continuously into a point without leaving R.

Now, consider a surface of class 2, with parameter system u, v and let a closed curve C be the Boundary of a simply connected region R of the surface.

Suppose that C consists of n arcs

$A_0 A_1, A_1 A_2, \ldots A_{n-1} A_n$ $(A_n = A_0)$, The arcs are in counter clockwise direction which is +ve

where n is finite, and that each arc is of class 2.



The vertices $A_0, A_1, \ldots$ are taken in order along C to agree with the positive sense of description of C.

This is usually described as the sense which "leaves the interior on the left".

ie., A positive rotation of $\pi/2$ from the tangent gives the normal which points to the interior region R.

At the vertex A_r $(r = 1, 2, \ldots n)$ let α_r be the angle b/w the tangents to the arcs $A_{r-1} A_r$ and $A_r A_{r+1}$, measured with the usual convention at A_r so that $-\pi < \alpha_r < \pi$.

At A_n, α_0 is the angle b/w the tangents to $A_{n-1} A_n$ and $A_n A_1$

Regarding C as a curvilinear polygon, $\alpha_1, \alpha_2, \dots \alpha_n$ are the exterior angles at the vertices $A_1, \dots A_n$.

The geodesic curvature exists at every point of C except possibly at the vertices and the line integral $\int_C k_g \, ds$ can be calculated.

The excess of C is defd. as

$$\text{ex } C = 2\pi - \sum_{r=1}^{n} \alpha_r - \int_C k_g \, ds$$

The ex C is called intrinsic.

This is an invariant, independent of the particular parameter system for the surface.

Note:

Excess of $C = 2\pi - \sum \alpha_r -$

i) For a rectilinear polygon, $k_g = 0$ at every point and $\sum \alpha_r$ is the sum of the exterior angles, ie, 2π

$$\therefore \text{ ex } C = 2\pi - \sum \alpha_r$$
$$= 2\pi - 2\pi$$
$$\text{ex } C = 0$$

ii) On any surface isometric with the plane, the excess of a simple closed curve is zero.

Repeated 10 mark

Gauss - Bonnet theorem:

2 mark

(St: For any curve C which encloses a simply connected region R, the excess of C is equal to the total curvature of R.)

Proof: WKT, the Liouville's formula for k_g is

$$k_g = \theta' + Pu' + Qv' \quad \text{——} (1)$$

where
$$P = \frac{1}{2EH}\left[2EF_1 - FE_1 - EE_2\right]$$

$$Q = \frac{1}{2EH}\left[EG_1 - FE_2\right]$$

$$K_g = \frac{d\theta}{ds} + P\frac{du}{ds} + Q\frac{dv}{ds}$$

$$K_g\, ds = d\theta + P\,du + Q\,dv$$

$$\int_C K_g\, ds = \int_C (d\theta + P\,du + Q\,dv) \quad \text{---} (2)$$

where θ is the angle which C makes with the parametric curve v = constant and P & Q are certain functions of u & v.

Since the curves v = constant form a family in the region R bounded by C, the tangent to C turns through 2π relative to these curves.

$$\text{ie, } \int_C d\theta + \sum_{r=1}^{n} \alpha_r = 2\pi$$

$$\int_C d\theta = 2\pi - \sum_{r=1}^{n} \alpha_r \quad \text{---} (3)$$

WKT, the ex C is defd. as $ex\ C = 2\pi - \Sigma\alpha_r - \int_C K_g\, ds$ —(4)

Sub. (2) & (3) in (4),

$$ex\ C = \int_C \cancel{d\theta} - \int_C (\cancel{d\theta} + P\,du + Q\,dv)$$

$$ex\ C = -\int (P\,du + Q\,dv) \quad \text{---} (5)$$

By Green's theorem, since R is simply connected & P & Q are differentiable functions of u & v in R.

$$\therefore \int_C (P\,du + Q\,dv) = \int_R \left(\frac{\partial Q}{\partial u} - \frac{\partial P}{\partial v}\right) du\,dv$$

since $ds = H\,du\,dv$ for the surface element

$$\Rightarrow \frac{ds}{H} = du\,dv$$

$$\therefore \int (P\,du + Q\,dv) = \int_R \left(\frac{\partial Q}{\partial u} - \frac{\partial P}{\partial v}\right) \frac{ds}{H}$$

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Sub. ⑥ in ⑤

$$ex\ C = -\int_R \frac{1}{H}\left(\frac{\partial Q}{\partial u} - \frac{\partial P}{\partial v}\right) ds$$

$$= \int_R k\, ds \qquad \left[\because k = -\frac{1}{H}\left(\frac{\partial Q}{\partial u} - \frac{\partial P}{\partial v}\right)\right]$$

$$\therefore ex\ C = \int_R k\, ds \quad \text{—— (7)}$$

Eqn. (7) shows that there is a certain function k of u & v which is determined by E, F and G and that the excess of any curve C which encloses a simply connected region R is equal to the surface integral of k over R.

<u>To show that k is uniquely determined:</u>

Suppose that $\bar{k}$ is a second function which also satisfies eqn. (7) & is independent of C.

$$\text{ie, } ex\ C = \int_R \bar{k}\, ds \quad \text{—— (8)}$$

<u>To prove:</u> Then for every region R,

$$\int_R (\bar{k} - k)\, ds = 0 \quad \text{—— (9)}$$

Suppose $\bar{k} \neq k$ there are 2 cases arrives at some point.

<u>case (i)</u>: $\bar{k} > k$

Since $\bar{k} - k$ is continuous, there is a region R which contains P & in which $\bar{k} - k > 0$ at every point

$$\text{ie, } \int_R (\bar{k} - k)\, ds > 0$$

which is contradiction to eqn. (9).

Case (ii) : $\bar{k} < k$

Since $\bar{k} - k$ is continuous, there is a region R which contains P in which $\bar{k} - k < 0$ at every point.

$$\therefore \int_R (\bar{k} - k)\, ds < 0$$

which is $\rightarrow\leftarrow$ to eqn. (9)

$\therefore$ From (9) $\bar{k} = k$ at every point.

ie, k is uniquely determined as function of u & v.

$\Rightarrow$ k is invariant, at every point the value of k is independent of the parameter system.

Also k is intrinsic, since it can be calculated when the metric is known.

Thus k is an intrinsic geometrical invariant, it is called the Gaussian curvature of the surface.

$\therefore$ For any region R, whether simply connected or not $\int_R k\,ds$ is called the total curvature of R.

ie, ex C = Total curvature of R.

Gauss - Bonnet theorem for a geodesic triangle:

For a geodesic triangle ABC, formed by geodesic arcs AB, BC, CA and enclosing a simply connected region, the excess is

$$\text{ex } C = 2\pi - \Sigma \alpha_r - \int k_g\, ds$$

$$= 2\pi - (\pi - A) - (\pi - B) - (\pi - C)$$

$$= A + B + C - \pi$$

where A, B, C are the interior angles of the triangle.

$\therefore$ The total curvature of a geodesic triangle ABC $= A + B + C - \pi$

17. Gaussian Curvature:

1) Defn - Gaussian curvature:

It follows from the Gauss-Bonnet theorem for a geodesic triangle.

If P is a given point and Δ the area of a geodesic triangle ABC which contains P, then at P

$$K = \lim \frac{A+B+C-\pi}{\Delta}$$

where the limit is taken ~~over~~ as all vertices tend to P.

Result 3:

Prove that for the sphere of radius 'a' along the Gaussian curvature is $1/a^2$.

Proof: The eqn. of the sphere is

$$r = (a \sin u \cos v,\ a \sin u \sin v,\ a \cos u)$$

Notes

Result:

Prove that the curvature is $K = -\frac{1}{2H}\left\{\frac{\partial}{\partial u}\left(\frac{G_1}{H}\right) + \frac{\partial}{\partial v}\left(\frac{E_2}{H}\right)\right\}$ when the parametric curves are orthogonal.

Proof: WKT, By Gaussian curvature,

$$K = -\frac{1}{H}\left(\frac{\partial Q}{\partial u} \mp \frac{\partial P}{\partial v}\right) \quad \text{—(1)}$$

where $P = \frac{1}{2HE}(2EF_1 - FE_1 - EE_2)$

$$Q = \pm\frac{1}{2HE}(EG_1 - FE_2)$$

Hence the parametric curves are orthogonal

$\Rightarrow F = 0$ i.e., $F_1 = 0$

$\therefore P = -\frac{EE_2}{2HE} = -\frac{E_2}{2H}$ & $Q = \frac{EG_1}{2HE} = \frac{+G_1}{2H}$

Sub. the values of P & Q in (1)

$$K = -\frac{1}{H}\left[\frac{\partial}{\partial u}\left(\frac{+G_1}{2H}\right) + \frac{\partial}{\partial v}\left(-\frac{E_2}{2H}\right)\right]$$

$$K = -\frac{1}{2H}\left[\frac{\partial}{\partial u}\left(\frac{G_1}{H}\right) + \frac{\partial}{\partial v}\left(\frac{E_2}{H}\right)\right]$$

Bookwork:

Prove that for the sphere of radius 'a' along the Gaussian curvature is $1/a^2$.

Proof: The eqn. of the sphere is

$$r = (a\sin u\cos v,\ a\sin u\sin v,\ a\cos u)$$

To find r_1, r_2, E, F, G

$$r_1 = (a\cos u\cos v,\ a\cos u\sin v,\ -a\sin u)$$

$$r_2 = (-a\sin u\sin v,\ a\sin u\cos v,\ 0)$$

$$E = r_1^2 = a^2\cos^2 u\cos^2 v + a^2\cos^2 u\sin^2 v + a^2\sin^2 u = a^2$$

$$F = r_1\cdot r_2 = -a^2\sin u\cos u\sin v\cos v + a^2\sin u\cos u\sin v\cos v = 0$$

$$G = r_2^2 = a^2\sin^2 u\sin^2 v + a^2\sin^2 u\cos^2 v = a^2\sin^2 u$$

WKT, $H^2 = EG - F^2 = a^4\sin^2 u - 0$

$$H = \sqrt{EG - F^2} = \sqrt{a^4\sin^2 u}$$

$$H = a^2\sin u$$

$$G_1 = \frac{\partial G}{\partial u} = \frac{\partial(a^2\sin^2 u)}{\partial u} = +2a^2\sin u\cos u$$

$E_1 = 0$ & $E_2 = 0$

WKT, $K = -\frac{1}{2H}\left[\frac{\partial}{\partial u}\left(\frac{G_1}{H}\right) + \frac{\partial}{\partial v}\left(\frac{E_2}{H}\right)\right]$

$$= \frac{-1}{2a^2\sin u}\left[\frac{\partial}{\partial u}\left(\frac{+2a^2\sin u\cos u}{a^2\sin u}\right) + 0\right]$$

$$= -\frac{1}{a^2 \sin u}\left[\frac{\partial}{\partial u}(\cos u)\right] = -\frac{1}{a^2 \cancel{\sin u}}(-\cancel{a} \cancel{\sin u})$$

$$\therefore K = 1/a^2.$$