4

Field Equations and Conservation Laws

INTRODUCTION :

[In chapter 1 and 3 we have summarised the mathematical description of static electric and magnetic fields. We now wish to consider the more general situation in which the field quantities may depend upon time. Under such conditions there is an interdependence of the field quantities and it is no longer possible to discuss separately the electric and magnetic fields and we are forced to consider the general concept of an electromagnetic field. The time dependent electromagnetic field equation are called Maxwell's equations. These equations are mathematical abstractions of experemental results.

In this chapter we seek to establish the formation of the field equations, to show that their solutions are unique, to discuss the scalar and vector potentials of the field and to consider the law of conservations of charge, energy and momentum].

§ 4·1. Equation of Continuity.

Under steady-state conditions the charge density in any given region will remain constant. We now relax the requirement of steady-state conditions and allow the charge density to become a function of time. It is experimentally verified that the net amount of electric charge in a closed system remains constant. Therefore if the net charge within a certain region decreases with time, this implies that a like amount of charge must appear in some other region. This transport of charge constitutes a current *i.e.*

$$I=-(dq/dt) \quad \text{...(1)}$$

$-$ive sign here indicates that charge contained in a specified volume decreases with time.

However by definition of current density

$$I=\oint \mathbf{J} \cdot d\mathbf{s} \quad \text{...(2)}$$

So from equation (1) and (2) we have—

$$\oint_S \mathbf{J} \cdot d\mathbf{s} = -\frac{dq}{dt}$$

i.e.
$$\oint_S \mathbf{J} \cdot d\mathbf{s} = -\frac{d}{dt}\int_\tau \rho \, d\tau \qquad \left[\text{as } q = \int_\tau \rho \, d\tau\right]. \qquad \ldots(2)$$

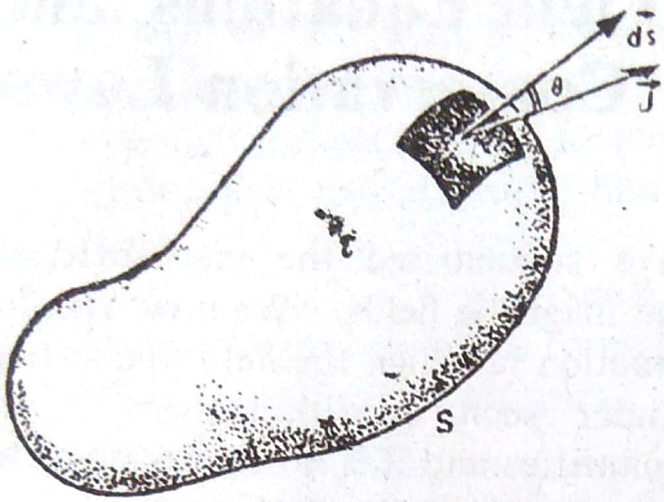

Fig. 4·1

If we hold the surface S fixed in space, the time variation of the volume integral must be solely due to the time variation of ρ. Thus

$$\oint_S \mathbf{J} \cdot d\mathbf{s} = \int \frac{\partial \rho}{\partial t} \, d\tau. \qquad \ldots(3)$$

But from Gauss's theorem

$$\oint_S \mathbf{J} \cdot d\mathbf{s} = -\int_\tau (\text{div } \mathbf{J}) \, d\tau \qquad \ldots(4)$$

So comparing expressions (3) and (4) we get

$$\int_\tau (\text{div } \mathbf{J}) \, d\tau = -\int_\tau \frac{\partial \rho}{\partial t} \, d\tau$$

i.e.
$$\int_\tau \left(\text{div } \mathbf{J} + \frac{\partial \rho}{\partial t} \right) d\tau = 0 \qquad \ldots(5)$$

Since equations (5) is true for any arbitrary finite volume, the integrand must vanish *i.e.*

$$\text{div } \mathbf{J} + \frac{\partial \rho}{\partial t} = 0. \qquad \ldots(A)$$

Equation (A) is called the *equation of continuity* and is an expression of the experimental fact that electric charge is conserved.

§ 4·2. Displacement Current.

We know that Ampere's circuital law in its most general form is given by

$$\oint_C \mathbf{H}.d\mathbf{l} = \int_S \mathbf{J} \cdot d\mathbf{s} \text{ [see equation } c \text{ of § 3·10 (a)]}$$

i.e.

$$\int_S \text{curl } \mathbf{H} \cdot d\mathbf{s} = \int_S \mathbf{J} \cdot d\mathbf{s}$$

or

$$\text{curl } \mathbf{H} = \mathbf{J} \quad ...(1)$$

Let us now examine the validity of this equation in the event that the fields are allowed to vary with time. If we take the divergence of both sides of equation (1) then

$$\text{div (curl } \mathbf{H}) = \text{div } \mathbf{J}. \quad ...(2)$$

Now as *div* of *curl* of any vector is zero, we get from equation (2) $\text{div } \mathbf{J} = 0$ ...(3)

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*In note (ii) of application (d) in § 2·3 we have shown that for electrostatic effects a conductor acts like a material of infinite dielectric constant. However in case of steady current as polarisation effects are completely overshadowed by dispersion of metals this result does not hold good. For purpose of estimation, in case of conduction through metal we usually take $\epsilon_r \to 1$ as discussed in § 7·9

Now the continuity equation in general states

$$\text{div } \mathbf{J} = -\frac{\partial \rho}{\partial t} \quad \text{...(4)}$$

and will therefore vanish only in the special case that the charge density is static. Consequently we must conclude that Ampere's law as stated in equation (1) is valid only for steady state conditions and is insufficient for the case of time-dependent fields. Because of this Maxwell assumed that equation (1) is not complete but should have something else to it. Let this 'something' be denoted be $\mathbf{J}_d$, then equation (1) can be rewritten as

$$\text{cur } \mathbf{H} = \mathbf{J} + \mathbf{J}_d. \quad \text{...(5)}$$

In order to identify $\mathbf{J}_d$, we calculate the divergence of equation (2) again and get

$$\text{div curl } \mathbf{H} = \text{div } (\mathbf{J} + \mathbf{J}_d)$$

i.e. $\quad \text{div } (\mathbf{J} + \mathbf{J}_d) = 0 \quad$ (as div curl $\mathbf{H}=0$)

or $\quad \text{div } \mathbf{J} + \text{div } \mathbf{J}_d = 0$

or $\quad \text{div } \mathbf{J}_d = -\text{div } \mathbf{J}$

i.e. $\quad \text{div } \mathbf{J}_d = \frac{\partial \rho}{\partial t} \quad$ [from equation (4)]

i.e. $\quad \text{div } \mathbf{J}_d = \frac{\partial}{\partial t} (\text{div } \mathbf{D}) \quad$ [as div $\mathbf{D} = \rho$]

i.e. $\quad \text{div}\left(\mathbf{J}_d - \frac{\partial \mathbf{D}}{\partial t} \right) = 0 \quad$ ...(6)

As equation (6) is true for any arbitrary volume

$$\mathbf{J}_d = \frac{\partial \mathbf{D}}{\partial t}. \quad \text{...(A)}$$

And so the modified form of Ampere's circuital law becomes

$$\text{curl } \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}. \quad \text{...(B)}$$

The term which Maxwell added to Ampere's law *viz.* $(\partial \mathbf{D}/\partial t)$ is called the *displacement* current to distinguish it from $\mathbf{J}$, the conduction current. By adding this term to Ampere's law, Maxwell assumed that the time rate of change of displacement produces a magnetic field just as a conduction current does.

Regarding displacement current it is worthy to note that :

(i) Displacement current is a current only in the sense that it produces a magnetic field. It has none of the other properties of current. For example displacement current can have a finite value in perfect vacuum where there are no charges of any type.

(ii) The magnitude of the displacement current is equal to the time rate of change of electric displacement vector **D**.

(iii) Displacement current serves to make the total current continuous across discontinuties in conduction current. (See example 2 and problem 4).

(iv) The displacement current in a good conductor is negligible as compared to the condition current at any frequency lower than the optical frequencies ($\sim 10^{15}$ Hertz). (See example 3).

(v) The addition of displacement current *i.e.* $(\partial \mathbf{D}/\partial t)$ to Ampere's law *i.e.* curl $\mathbf{H}=\mathbf{J}$ results in

$$\text{curl } \mathbf{H}=\mathbf{J}+\frac{\partial \mathbf{D}}{\partial t}$$

i.e. displacement current relates the electric field vector **E** (as $\mathbf{D}=\epsilon\mathbf{E}$) to the magnetic field vector H. This in turn implies that in case of time dependent fields it is not possible to deal with electric and magnetic fields separately, but the two fields are interlinked and give rise to what are know as electromagnetic fields *i.e. The addition of displacement current to Ampere's law result, in the unification of electric and magnetic phenomena.*

It must be emphasized here that the ultimate justification for Maxwell's assumption of displacement current is in the experimental verification. Indeed the effects of the displacement current are dificult to observe directly except at very high frequencies. However indirect verification is afforded by predictions of many effects particularly in electromagnetic theory of light which are confirmed by experiments. We may therefore consider that Maxwell's form of Ampere's law has been subjected to experimental tests and has been found to be generally valid.

§ 4·3. Maxwell's equations.

(A) The equation :

These are four fundamental equation of electromagnetism and corresponds to a generalisation of certain experimental observations-regarding electricity and magnetism. The following four laws of electricity and magnetism constitutes the so called '*differentital form*' of Maxwell's equations :

(i) *Gauss' law* for the electric field of charge yields

$$\text{div } \mathbf{D} = \nabla . \mathbf{D} = \rho$$

where **D** is electric displacement in coulombs/m^2 and ρ is the free charge density in coul/m^3.

(ii) *Gauss' law* for magnetic field yields

$$\text{div } \mathbf{B} = \nabla . \mathbf{B} = 0$$

where **B** is the magnetic induction in web/m^2.

(iii) *Ampere's law* in circuital form for the magnetic field accompanying a current when modifield by Maxwell yields

$$\text{curl } \mathbf{H} = \nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$$

where **H** is the magnetic field intensity in amperes/m and **J** is the current density in amp/m^2.

(iv) *Faraday's law* in circuital form for the induced electromotive force produced by the rate of change of magnetic flux linked with the path yields

$$\text{curl } \mathbf{E} = \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

where **E** is the electric field intensity in volts/m.

(B) Derivations :

(i) Let us consider a surface S bounding a volume τ within a dielectric. Originally the volume τcontains no net charge but we allow the dielectric to be polarised say by placing it in an electric field. We also deliberately place some charge on the dielectric body. Thus we have two type of charges :

(a) real charge of density ρ (b) bound charge density ρ'.

Gauss' law then can be written as

$$\oint_S \mathbf{E}\ ds = \frac{1}{\epsilon_0} \int (\rho + \rho')\ d\tau$$

i.e. $$\epsilon_0 \oint_S \mathbf{E}.ds = \int_\tau \rho\ d+\tau \int \rho'\ d\tau. \quad ...(1)$$

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But as the bound charge density ρ' is defined as $\rho' = -\text{div}\,\mathbf{P}$

and
$$\oint_S \mathbf{E} \cdot ds = \int_\tau \text{div}\,\mathbf{E}\, d\tau$$

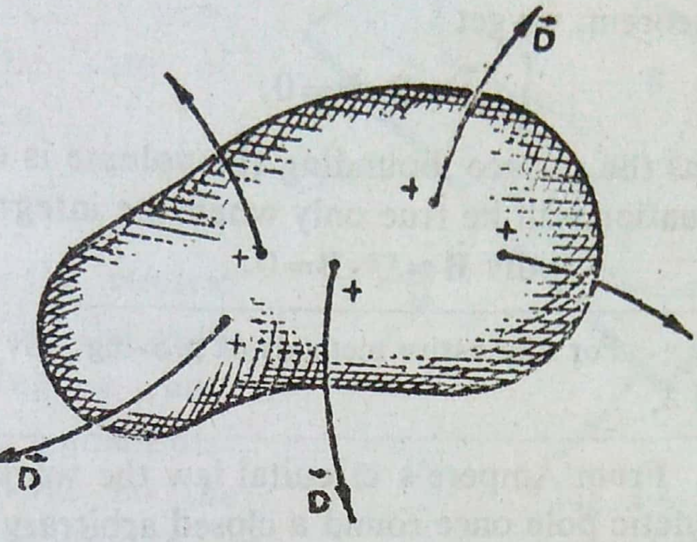

Fig. 4·3

So equation (1) becomes

$$\epsilon_0 \int_\tau \text{div}\,\mathbf{E}\, d\tau = \int_\tau \rho\, d\tau - \int_\tau \text{div}\,\mathbf{P}\, d\tau$$

i.e.
$$\int_\tau \text{div}\,(\epsilon_0 \mathbf{E} + \mathbf{P})\, d\tau = \int_\tau \rho\, d\tau$$

or
$$\int_\tau \text{div}\,\mathbf{D}\, d\tau = \int_\tau \rho\, d\tau \quad (\text{as } \mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P})$$

or
$$\int_\tau (\text{div}\,\mathbf{D} - \rho)\, d\tau = 0.$$

Since this equation is true for all volumes, the integrand must vanish. Thus we have

$$\text{div}\,\mathbf{D} = \nabla \cdot \mathbf{D} = \rho. \qquad \text{...(A)}$$

(ii) Experiments to-date have shown that magnetic monopoles do not exist. This in turn implies that the magnetic lines of force are either closed group or go off to infinity. Hence the number of magnetic lines of force entering any arbitrary closed surface is exactly the same leaving it. Therefore the flux of magnetic induction **B** across any closed surface is always zero *i.e.*

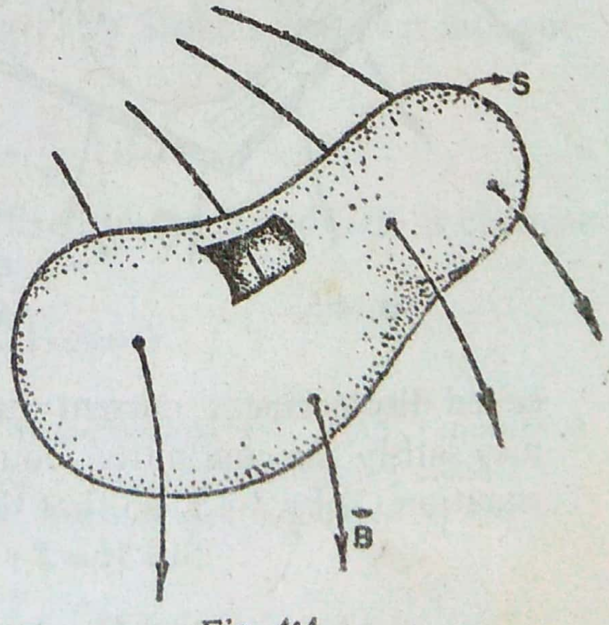

Fig. 4·4

$$\oint_S \mathbf{B}.d\mathbf{s}=0.$$

Transforming this surface integral into volume integral by Gauss' theorem, we get

$$\int_\tau \text{div } \mathbf{B}\, d\tau=0.$$

But as the surface bounding the volume is quite arbitrary the above equation will be true only when the integrand vanishes *i.e.*

$$\text{div } \mathbf{B}=\nabla\cdot\mathbf{B}=0. \tag{B}$$

Note : For alternative methods of proving div **B**=0 see example 1 in chapter 3.

(iii) From Ampere's circuital law the work done in carrying unit magnetic pole once round a closed arbitrary path linked with the current I is expressed by

$$\oint_C \mathbf{H}\cdot d\mathbf{l}=I$$

i.e.
$$\oint_C \mathbf{H}\cdot d\mathbf{l}=\int_S \mathbf{J}\cdot d\mathbf{s} \qquad \left(\text{as } I=\int \mathbf{J}\cdot d\mathbf{s}\right)$$

where S is the surface bounded by the closed path C.

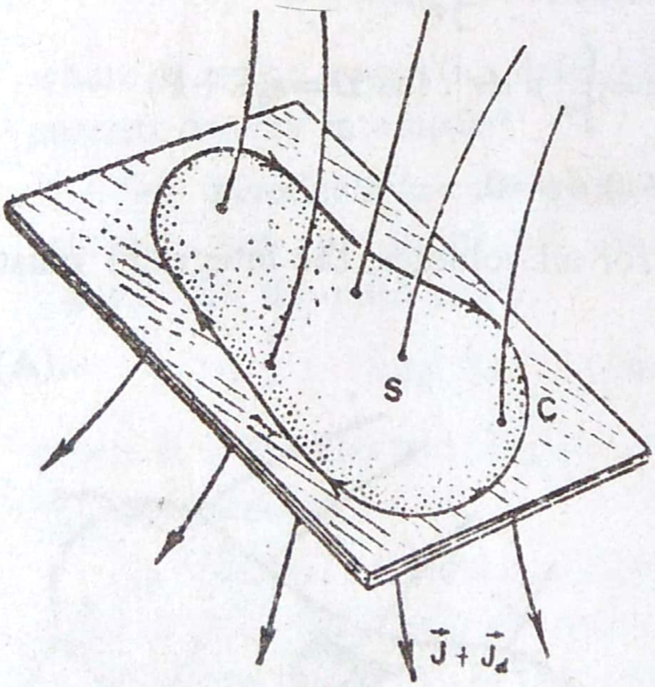

Fig. 4·5.

Now changing the line integral into surface integral by Stoke's theorem, we get

$$\int_S \text{curl } \mathbf{H}\cdot d\mathbf{s}=\int_S \mathbf{J}\cdot d\mathbf{s}$$

i.e. $\quad \text{curl } \mathbf{H}=\mathbf{J}. \quad$ ...(2)

But Maxwell found it to be incomplete for changing electric fields and assumed that a quantity

$$\mathbf{J}_d=\frac{\partial \mathbf{D}}{\partial t}$$

called displacement current must also be included in it so that it may satisfy the continuity equation *i.e.* **J** must be replaced in equation (2) by $\mathbf{J}+\mathbf{J}_d$ so that the law becomes

$$\text{curl } \mathbf{H}=\mathbf{J}+\mathbf{J}_d$$

i.e.
$$\text{curl } \mathbf{H}=\mathbf{J}+\frac{\partial \mathbf{D}}{\partial t} \qquad \text{...(C)}$$

(iv) According to Faraday's law of electromagnetic induction we know that the induced e.m.f. is proportional to the rate of change of flux *i.e.*

$$\in = -\frac{d\phi_B}{dt}. \qquad \ldots(3)$$

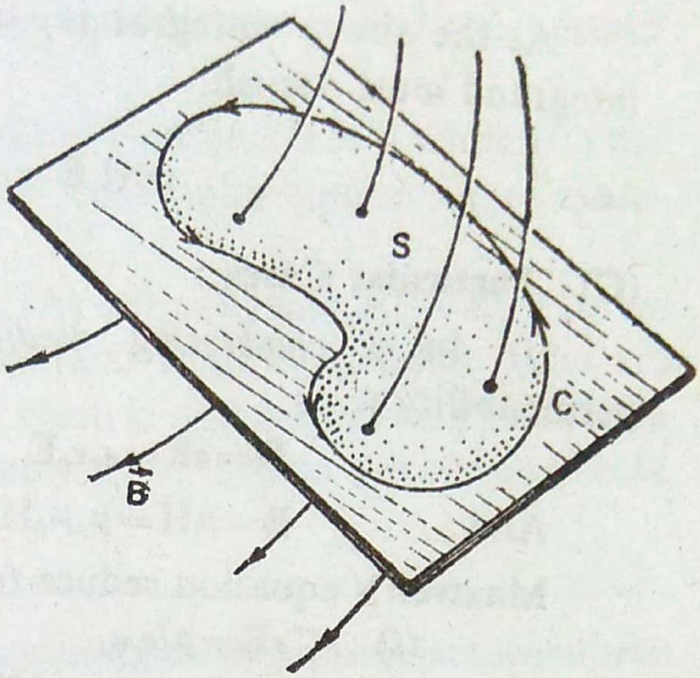

Fig. 4·6.

Now if **E** be the electric intensity at a point the work done in moving a unit charge through a small distance $d\mathbf{l}$ is $\mathbf{E}\cdot d\mathbf{l}$. So the work done in moving the unit charge once round the circuit is $\oint_C \mathbf{E}\cdot d\mathbf{l}$. Now as e.m.f. is defined as the amount of work done in moving a unit charge once round the electric circuit.

$$\in = \oint_C \mathbf{E}\cdot d\mathbf{l} \qquad \ldots(4)$$

So comparing equation (3) and (4), we get

$$\oint_C \mathbf{E}\cdot d\mathbf{l} = -\frac{d\phi_B}{dt}^*. \qquad \ldots(5)$$

But as

$$\phi_B = \int_S \mathbf{B}\cdot d\mathbf{s}$$

So $$\oint_C \mathbf{E}\cdot d\mathbf{l} = -\frac{d}{dt}\int \mathbf{B}\cdot d\mathbf{s}.$$

Transformating the line integral by Stoke's theorem into surface integral we get

$$\int_S \text{curl } \mathbf{E}\cdot d\mathbf{s} = -\frac{d}{dt}\int \mathbf{B}\cdot d\mathbf{s}.$$

Assuming that surface S is fixed in space and only **B** changes with time, above equation yields

$$\int_S \left(\text{curl } \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t}\right)\cdot d\mathbf{s} = 0$$

As the above integral is true for any arbitrary surface the integrand must vanish.

i.e. $$\text{curl } \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}. \qquad \text{...(D)}$$

(C) Particular Cases :

(i) In a *conducting medium* of relative permitivity ϵ_r and permeability μ_r as

$$\mathbf{D} = \epsilon\mathbf{E} = \epsilon_r\epsilon_0\mathbf{E}$$

And $$\mathbf{B} = \mu\mathbf{H} = \mu_r\mu_0\mathbf{H}.$$

Maxwell's equation reduce to

(*i*) $\nabla \cdot \mathbf{E} = \rho/\epsilon_r\epsilon_0$ (*ii*) $\nabla \cdot \mathbf{H} = 0$

(*iii*) $\nabla \times \mathbf{H} = \mathbf{J} + \epsilon_r\epsilon_0 \frac{\partial \mathbf{E}}{\partial t}$ (*iv*) $\nabla \times \mathbf{E} = -\mu_r\mu_0 \frac{\partial \mathbf{H}}{\partial t}$.

(ii) In a *non-conducting media* of relative permitive ϵ_r and permeability μ_r as

$$\rho = \sigma = 0$$

so $$\mathbf{J} = \sigma\mathbf{E} = 0$$

and hence Maxwell's equations become

(*i*) $\nabla \cdot \mathbf{E} = 0$ (*ii*) $\nabla \cdot \mathbf{H} = 0$

(*iii*) $\nabla \times \mathbf{H} = \epsilon_r\epsilon_0 \frac{\partial \mathbf{E}}{\partial t}$ (*iv*) $\nabla \times \mathbf{E} = -\mu_r\mu_0 \frac{\partial \mathbf{H}}{\partial t}$.

(iii) *In free space as*

$$\epsilon_r = \mu_r = 1$$
$$\rho = \sigma = 0.$$

Maxwell's equations become

(*i*) $\nabla \cdot \mathbf{E} = 0$ (*ii*) $\nabla \cdot \mathbf{H} = 0$

(*iii*) $\nabla \times \mathbf{H} = \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}$ (*iv*) $\nabla \times \mathbf{E} = -\mu_0 \frac{\partial \mathbf{H}}{\partial t}$.

(D) Discussion :

1. These equations are based on experimental observations. The equations : (A) and (C) correspond to electricity while (B) and (D) to magnetism.

2. These equations are general and apply to all electromagnetic phenomena in media which are at rest w.r.t. the coordinate system.

3. These equation are not independent of each other as from equation (D) we can derive (B) and from (C), (A) (see example-4). This is why equations (B) and (D) are called the first pair

of Maxwell's equations while (A) and (C) are called the second pair.

4. The equation (A) represents Coulomb's law while (C) the law of conservation of charge *i.e.* continuity equation (see example 5).

5. If we compare equation (A) with (B) and (C) with (D) we find that left hand sides are identical while right hand sides are not. This in turn implies that electric and magnetic phenomena are assymmetric and this asymmetry arises due to the non-existance of monopoles.

Note : This asymmetry of electro-magnetism suggests that monopoles (a particle having either north or south magnetic charge) should exist as the concept of magnetic monopoles would bring to electricity and magnetism a symmetry to which nature loves and is lacking in our present picture. Dirac has also proved on theoretical grounds that monopole should exist and predicted their properties. But so far the magnetic monopole has frustated all its investigators. The experiments have failed to find any sign of these. The theorists on the other hand have failed to find any good reason why monopoles should not exist.

Recently, American Institute of Physics and the University of California at Berkley jointly announced that monopole has been observed by a group of physists. If confirmed, the detection of monopoles will have a major impact on Physics and Technology.

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(6) The correspondance of **B** and **H** with **E** and **D** through Maxwell equations (D) and (C) respectively implies that in case of time dependent fields the electric and magnetic fields are inseparably linked with each other giving rise to what is known as electromagnetic field and it is not possible to deal separately with electric and magnetic fields in this situation.

(E) Physical Significance (or Integral Form)

By means of Gauss' and Stoke's Theorems we can write the Maxwell's field equations in integral form and hence obtain their physical significance.

(i) Integrating Maxwell's first equation div $\mathbf{D}=\rho$ over an arbitrary volume τ we get

$$\int_\tau \nabla \cdot \mathbf{D}\, d\tau = \int_\tau \rho\, d\tau$$

changing the vol. integral of L. H. S. into surface integral by Gauss' divergence theorem and keeping in mind that $\int \rho \, d\tau = q$ we get

$$\oint \mathbf{D} \cdot d\mathbf{s} = q \qquad \ldots(A_1)$$

So Maxwell's first equation signifies that *the total flux of electric displacement linked with a closed surface is equal to the total charge enclosed by the closed surface.*

(ii) Integrating Maxwell's second equation div $\mathbf{B}=0$ over an arbitrary vol. τ we get

$$\int_\tau \nabla \cdot \mathbf{B} \, d\tau = 0.$$

Converting the vol. integral into surface integral with the help of Gauss' theorem we get

$$\oint \mathbf{B} \cdot d\mathbf{s} = 0 \qquad \ldots(B_1)$$

So Maxwell's II equation signifies that *the total flux of magnetic induction linked with a closed surface is zero.*

(iii) Integrating Maxwell's III equation curl $\mathbf{H} = \mathbf{J} + (\partial \mathbf{D}/\partial t)$ over a surface S bounded by the loop C we get

$$\int \text{curl } \mathbf{H} \cdot d\mathbf{s} = \int \left(\mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \right) \cdot d\mathbf{s}$$

Converting the surface integral of L. H. S. into line integral with the help of stoke's theorem we get

$$\oint \mathbf{H} \cdot d\mathbf{l} = \int \left(\mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \right) \cdot d\mathbf{s} \qquad \ldots(C_1)$$

which signifies that *magnetomotive force around a closed path* $\left[\oint \mathbf{H} \cdot d\mathbf{l} \right]$ *is equal to the conduction current plus displacement current linked with that path.*

(iv) Integrating Maxwell's IV equation curl $\mathbf{E} = -(\partial \mathbf{B}/\partial t)$ over a surface S bounded by the loop C we get

$$\int \text{curl } \mathbf{E} \cdot d\mathbf{s} = -\int \frac{\partial \mathbf{B}}{\partial t} . d\mathbf{s}$$

Converting the surface integral of L. H. S. into line integral with the help of stoke's Theorem we get

$$\oint \mathbf{E}.d\mathbf{l} = -\frac{\partial}{\partial t} \int \mathbf{B}.d\mathbf{s} \qquad \ldots(D_1)$$

which signifies that *the electromotive force i.e. line integral of electric intensity around a closed path is equal to the negative rate of change of magnetic flux linked with the path.*

§ 4·4. Energy in Electromagnetic fields. (Poynting's theorem)

From Maxwell's equations it is possible to derive an important expression which we shall recognise at the energy principle in an electromagnetic field.

For this consider Maxwell's equations (C) and (D) *i.e.* Ampere's and Faraday's laws in differential forms

$$\text{curl } \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \qquad \ldots(1)$$

and

$$\text{curl } \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}. \qquad \ldots(2)$$

If we take the scalar product of equation (1) with **E** and of equation (2) with (−H) we get

$$\mathbf{E} \cdot \text{curl} \, \mathbf{H} = \mathbf{E} \cdot \mathbf{J} + \mathbf{E} \cdot \frac{\partial \mathbf{D}}{\partial t} \qquad \ldots(3)$$

and $$-\mathbf{H}\cdot\text{curl }\mathbf{E}=+\mathbf{H}\cdot\frac{\partial \mathbf{D}}{\partial t}.$$ ...(4)

adding equations (3) and (4) we get

$$-\mathbf{H}\cdot\text{curl }\mathbf{E}+\mathbf{E}\cdot\text{curl }\mathbf{H}=\mathbf{J}\cdot\mathbf{E}+\left[\mathbf{E}\cdot\frac{\partial \mathbf{D}}{\partial t}+\mathbf{H}\cdot\frac{\partial \mathbf{B}}{\partial t}\right]$$

But by the vector identity

$$\mathbf{H}\cdot\text{curl }\mathbf{E}-\mathbf{E}\cdot\text{curl }\mathbf{H}=\text{div }(\mathbf{E}\times\mathbf{H})$$

The above equation reduces to

$$-\text{div }(\mathbf{E}\times\mathbf{H})=\mathbf{J}\cdot\mathbf{E}+\left[\mathbf{E}\cdot\frac{\partial \mathbf{D}}{\partial t}+\mathbf{H}\cdot\frac{\partial \mathbf{B}}{\partial t}\right]. \quad ...(5)$$

Now as $\mathbf{E}\cdot\frac{\partial \mathbf{D}}{\partial t}=\epsilon_r\epsilon_0\,\mathbf{E}\,\frac{\partial \mathbf{E}}{\partial t}=\frac{1}{2}\,\epsilon_r\epsilon_0\,\frac{\partial}{\partial t}\,(\mathbf{E}\cdot\mathbf{E})=\frac{1}{2}\frac{\partial}{\partial t}\,(\mathbf{E}\cdot\mathbf{D})$

and $\mathbf{H}\cdot\frac{\partial \mathbf{B}}{\partial t}=\mu_r\mu_0\mathbf{H}\cdot\frac{\partial \mathbf{H}}{\partial t}=\frac{1}{2}\mu_r\mu_0\,\frac{\partial}{\partial t}\,(\mathbf{H}\cdot\mathbf{H})=\frac{1}{2}\,\frac{\partial}{\partial t}\,(\mathbf{H}\cdot\mathbf{B})$

So equation (5) reduces to

$$\mathbf{J}\cdot\mathbf{E}+\frac{1}{2}\frac{\partial}{\partial t}(\mathbf{E}\cdot\mathbf{D}+\mathbf{H}\cdot\mathbf{B})+\text{div }(\mathbf{E}\times\mathbf{H})=0. \quad ...(6)$$

Each term in the above equation can be given some physical meaning if it is multiplied by an element of volume $d\tau$ and integrated over a volume τ whose enclosing surface is S. Thus the result is

$$\int_\tau (\mathbf{J}\cdot\mathbf{E})\,d\tau+\int_\tau \frac{1}{2}\,\frac{\partial}{\partial t}(\mathbf{E}\cdot\mathbf{D}+\mathbf{H}\cdot\mathbf{B})\,d\tau+\int_\tau \text{div }(\mathbf{E}\times\mathbf{H})\,d\tau=0$$

But as $$\int_\tau \text{div }(\mathbf{E}\times\mathbf{H})\,d\tau=\oint_S (\mathbf{E}\times\mathbf{H})\cdot d\mathbf{s}$$

so $$\int_\tau (\mathbf{J}\cdot\mathbf{E})\,d\tau+\int_\tau \frac{1}{2}\,\frac{\partial}{\partial t}(\mathbf{E}\cdot\mathbf{D}+\mathbf{H}\cdot\mathbf{B})\,d\tau+\oint_S (\mathbf{E}\times\mathbf{H})\cdot d\mathbf{s}=0 \quad ...(A)$$

To understand what equation (A) means, let us now interpret various term in it—

(A) Interpretation of $\int_\tau \mathbf{J}.\mathbf{E}\,d\tau$:

The current distribution represented by the vector J can be considered as made up of various changes q_i moving with velocity $\mathbf{v}_i$ so that

$$\int \mathbf{J}\cdot\mathbf{E}\,d\tau=\int I\,d\mathbf{l}\cdot\mathbf{E} \qquad [\text{as } \mathbf{J}\,d\tau=I\,d\mathbf{l}]$$

$$= \int dq\, \mathbf{v} \cdot \mathbf{E} \qquad [\text{as } I\, d\mathbf{l} = (dq/dt)\, d\mathbf{l} = dq\, \mathbf{v}]$$

$$= \Sigma q_i\, (\mathbf{v}_i \cdot \mathbf{E}_i) \qquad \ldots(7)$$

where $\mathbf{E}_i$ denotes the electric field at the position of charge q_i.

Now electromagnetic force on the ith charged particle is given by the *Lorentz expression*

$$\mathbf{F}_i = q_i\, (\mathbf{E}_i + \mathbf{v}_i \times \mathbf{B}_i).$$

So the work done per unit time on the charge q_i by the field will be

$$\frac{\partial W_i}{\partial t} = \mathbf{F}_i \cdot \mathbf{v}_i \qquad \left[\frac{dW}{dt} = \frac{\mathbf{F}.d\mathbf{l}}{dt} = \mathbf{F} \cdot \mathbf{v}\right]$$

$$= q_i\, (\mathbf{E}_i + \mathbf{v}_i \times \mathbf{B}_i) \cdot \mathbf{v}_i \qquad (\text{as } \mathbf{F}_i = q_i\, (\mathbf{E}_i + \mathbf{v}_i \times \mathbf{B}_i)$$

i.e. $$\frac{\partial W_i}{\partial t} = q_i \mathbf{v}_i \cdot \mathbf{E}_i \qquad [\text{as } \mathbf{v}_i \cdot (\mathbf{v}_i \times \mathbf{B}_i) = (\mathbf{v}_i \times \mathbf{v}_i) \cdot \mathbf{B}_i = 0]$$

So the rate at which the work is done by the field on the charges is

$$\frac{\partial W}{\partial t} = \Sigma\, \frac{\partial W_i}{\partial t} = \Sigma\, q_i \mathbf{v}_i \cdot \mathbf{E}_i. \qquad \ldots(8)$$

Comparing equation (7) and (8) we find that

$$\int \mathbf{J} \cdot \mathbf{E}\, d\tau = \frac{dW}{dt} \qquad \ldots(9)$$

i.e. the first term $\int (\mathbf{J} \cdot \mathbf{E})\, d\tau$ *represents the rate at which work is done by the field on the charges*

(B) Interpretation of $\int_\tau \frac{1}{2} \frac{\partial}{\partial t} (\mathbf{E} \cdot \mathbf{D} + \mathbf{H} \cdot \mathbf{B}) \, d\tau$

If we allow the volume τ to be arbitrary large the surface integral in eqn. (A) can be made to vanish by placing the surface S sufficiently far away so that the field cannot propagate to this distance in any finite time *i.e.* $\oint_S (\mathbf{E} \times \mathbf{H}) \cdot d\mathbf{s} = 0$. So under these circumstances equation (A) reduces to

$$\frac{\partial}{\partial t} \int_{\text{all space}} \frac{1}{2} (\mathbf{E} \cdot \mathbf{D} + \mathbf{H} \cdot \mathbf{B}) \, d\tau + \frac{\partial W}{\partial t} = 0$$

i.e.

$$\frac{\partial}{\partial t} \left[\int_{\text{all space}} \frac{1}{2} (\mathbf{E} \cdot \mathbf{D} + \mathbf{H} \cdot \mathbf{B} \, d\tau + W \right] = 0$$

Thus the quantity in the square bracket is conserved. Now consider a closed system in which the total energy is assumed to be constant. The system consists of the electromagnetic field and of all the charged particles present in the field. The term W represents the total kinetic energy of the particles. We are therefore led to associate the remaining energy term

$$\int_{\text{all space}} \frac{1}{2} (\mathbf{E} \cdot \mathbf{D} + \mathbf{H} \cdot \mathbf{B}) \, d\tau$$

with the energy of the eletcro-magnetic field, *i.e.*

$$U = \int_{\text{all space}} \frac{1}{2} (\mathbf{E} \cdot \mathbf{D} + \mathbf{H} \cdot \mathbf{B}) d\tau. \quad \text{...(10)}$$

the quantity U may be considered to be a kind of potential energy, One need not ascribe this potential energy to the charged particles and must consider this term as a field energy. A concept such as energy stored in the field itself rather than residing with the particles is a basic concept of the theoryof etectromagnetism.

Note : If we write equation (10) as

$$U = \int_{\text{all space}} u \, d\tau$$

where $u = \frac{1}{2}(\mathbf{E} \cdot \mathbf{D} + \mathbf{H} \cdot \mathbf{B})$ may be thought of as the energy density of the electromagnetic field.

Further as

$$u = \tfrac{1}{2}\, \mathbf{E} \cdot \mathbf{D} + \tfrac{1}{2}\, \mathbf{H} \cdot \mathbf{B}$$

first term on R.H.S.contains only electrical quantities while the second, one magnetic, we can have

$$u = u_e + u_m$$

with $u_e = \frac{1}{2}\mathbf{E} \cdot \mathbf{D} = \frac{1}{2}\epsilon_r \epsilon_0 E^2 =$ energy density of electric field

ane $u_m = \frac{1}{2}\mathbf{H} \cdot \mathbf{B} = \frac{1}{2}\mu_r \mu_0 H^2 =$ energy density of magnetic field

---

(C) Interpretation of $\oint_S (\mathbf{E} \times \mathbf{H}) \cdot ds$.

Instead of taking the volume integral in equations (A) over all space, let us now consider a finite volume. In this case the surface integral of $(\mathbf{E} \times \mathbf{H})$ will not in general vanish and so this term must be retained. Let us construct the surface S in such a way that in the interval of time under consideration, none of the charged particles will cross this surface. Then for the conservation of energy

$$\frac{\partial U}{\partial t} + \frac{\partial W}{\partial t} = -\oint_S (\mathbf{E} \times \mathbf{H}) \cdot ds \qquad \text{...(11)}$$

The left hand side is the time rate of change of the energy of the field and of the particles contained within the volume τ. Thus the surface intergral $\oint_S (\mathbf{E} \times \mathbf{H}) \cdot ds$ must be considered as the energy flowing out of the volume bounded by the surface S per sec. But by hypothesis no pnrticles are crossing the surface, so the vector $(\mathbf{E} \times \mathbf{H})$ is to be interpreted as the amount of the field energy passing through unit area of the surface in unit time which is normal to the direction of energy flow. The vector $(\mathbf{E}\times\mathbf{H})$ is called poynting vector* and is represented by **S** *i.e.*

$$\mathbf{S}=(\mathbf{E}\times\mathbf{H}). \quad ...(12)$$

Interpretation of the Energy Equation.

In the light of above, equation (6) in differential form can be written as

$$\mathbf{J}\cdot\mathbf{E}+\frac{\partial u}{\partial t}+\nabla\cdot\mathbf{S}=0. \quad ...(13)$$

In the event that the medium has zero conductivity *i.e.* $\mathbf{J}=\sigma\mathbf{E}=0$, the above equation becomes exactly of the same form as the continuity equation which expresses the law of conservation of charge. We are led by this analogy that the physical meaning of equation 13, 11 or (A) is to represent the law of conservation of energy for electromagnetic phenomena. According to equation (11) *the time rate of change of electromagnetic energy within a certain volume plus the rate at which the work is done by the field on the charges is equal to the energy flowing into the system through its bounding surface per unit time.*

§ 4·5. Poynting Vector.

In § 4·4 we have seen that according to the law of conservation of energy in an electromagnetic field

$$\frac{\partial U}{\partial t}+\frac{\partial W}{\partial t}+\int (\text{div } \mathbf{S})\, d\tau=0 \qquad \text{...(1)}$$

with

$$\mathbf{S}=\mathbf{E}\times\mathbf{H}. \qquad \text{...(2)}$$

The vector **S** is known as Poynting vector. It is interpreted as *the amount of the field energy passing through unit area of the surface in a direction perpendicular to the plane containing* **E** *and* **H** *per unit time*. For example as in a plane electro-magnetic wave **E** and **H** are perpendicular to each other and also to the direction of wave propagation, S has a magnitude $EH \sin 90 = EH$ and points in the direction of wave propagation. The dimensions of Poynting vector are (energy/area × time) so the units will be *Joule*/$m^2 \times sec$ or $watt/m^2$.

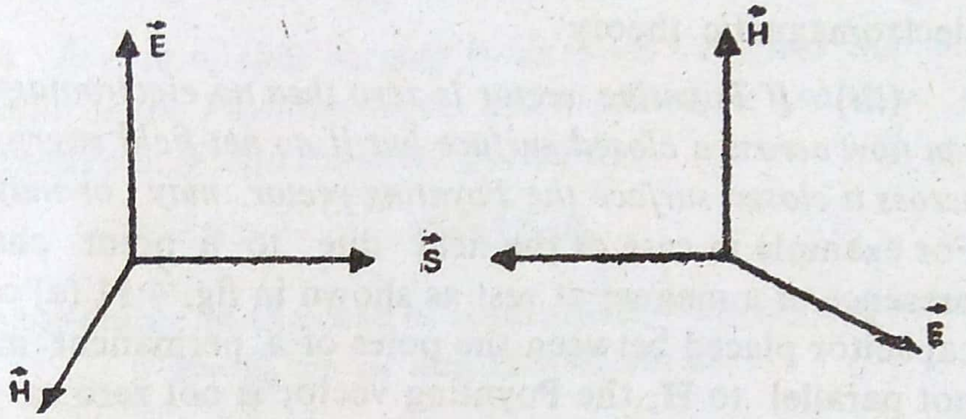


Fig 4·9

Regarding Poynting vector it is worthy to note that—

(*i*) *Poynting vector at any arbitrary point in the field varies inversely as the square of the distance from the point source of radiation.* To understand it consider a source L of electromagnetic radiations which is emitting radiations at the rate of P watts and imagine two concentric spherical surface A and B of radii r_1 and r_2 respectively with source being at their common centre. If S_1 and S_2 are the Poynting vector at any point on A and B respectively then

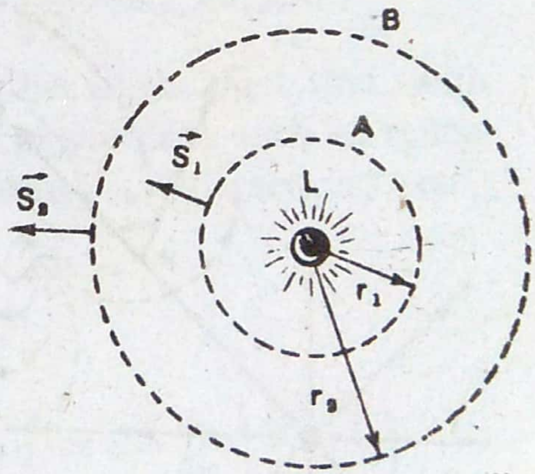


Fig. 4·10

$$S_1 \times 4\pi r_1^2 = S_2 \times 4\pi r_2^2 = P$$

i.e. $$S_1 = (P/4\pi r_1^2)$$

and $$S_2 = (P/4\pi r_2^2).$$

So in general $S \propto (1/r^2)$.

(*ii*) *The definition of Poynting vector is not a mandatory.* Since this vector has been introduced only by way of its divergence, the curl of any arbitrary vector can be added to it without altering the physical facts of the case *i.e.* it is arbitrary to the extent that curl or any vector field can be added to it *i.e.*

$$\mathbf{S} \equiv \mathbf{E} \times \mathbf{H} + \mathbf{G}$$

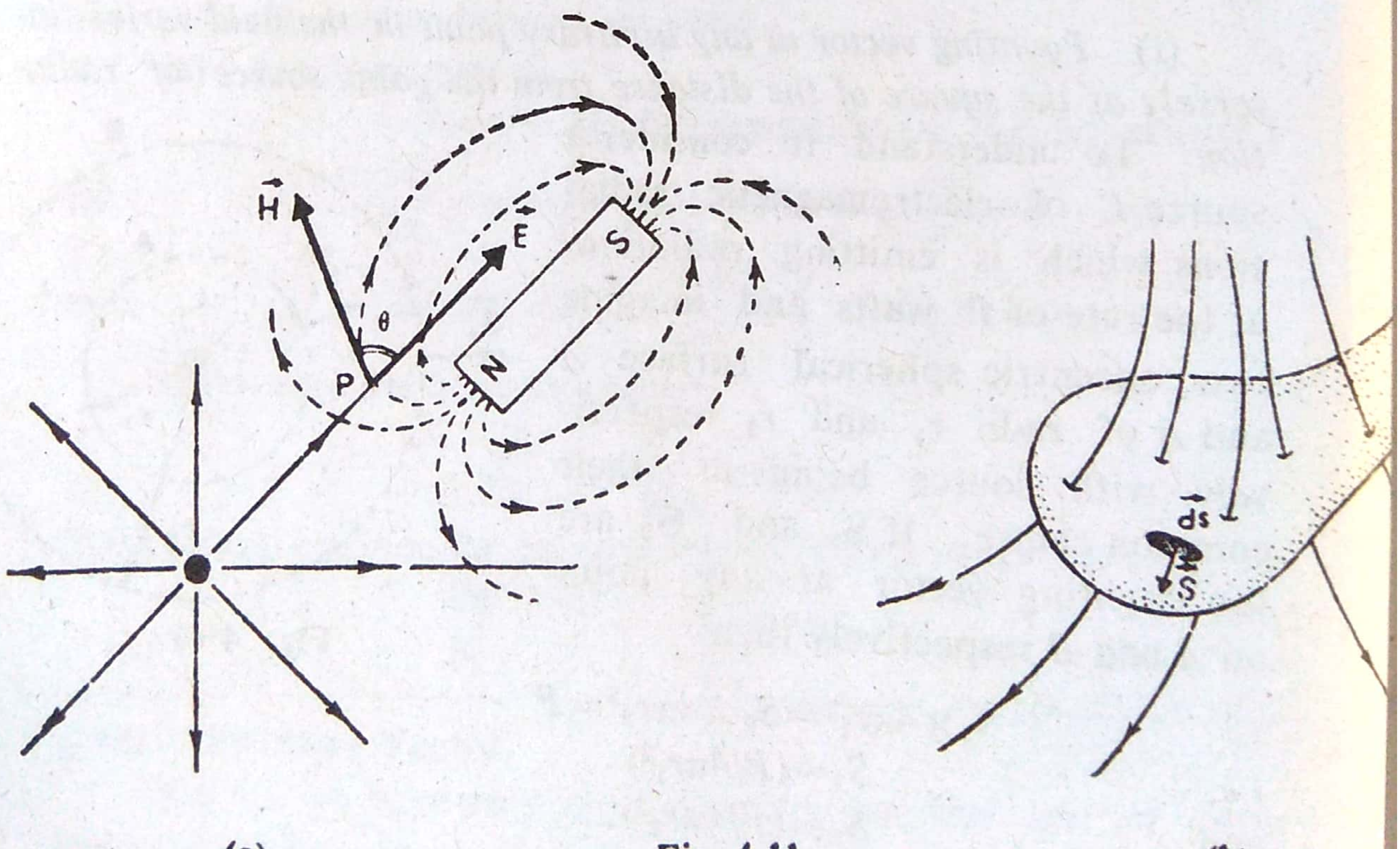
where $\mathbf{G} = \text{curl } \mathbf{M}$

$\mathbf{M}$ = any arbitrary field vector.

However on various grounds, such an additional term has no physical consequence and the definition of **S** is retained as such. This definition is also found to be convenient particularly in electromagnetic theory.

(*iii*) *If Poynting vector is zero then no electromagnetic energy can flow across a closed surface but if no net field energy is flowing across a closed surface the Poynting vector may or may not be zero.* For example in case of the field due to a point charge in the presence of a magnet at rest as shown in fig. 4·11 (a) or a charged capacitor placed between the poles of a permanent magnet if **E** is not parallel to **H**, the Poynting vector is not zero as

$$|\mathbf{S}| = |\mathbf{E} \times \mathbf{H}| = EH \sin \theta \neq 0.$$



(a) Fig. 4 11. (b)

For steady fields $\mathbf{S} \neq 0$ *i.e.* $\mathbf{S} \cdot d\mathbf{s} \neq 0$ but

there is no flow of energy *i.e.* $\oint \mathbf{S} \cdot d\mathbf{s} = 0$

While the flow of energy across any closed surface is zero as

$$\oint_S \mathbf{S}\cdot d\mathbf{s} = \int_\tau \nabla\cdot\mathbf{S}\, d\tau = \int_\tau \nabla\cdot(\mathbf{E}\times\mathbf{H})\, d\tau$$

i.e. $$\oint_S \mathbf{S}\cdot d\mathbf{s} = \int_\tau (\mathbf{H}\cdot\text{curl}\,\mathbf{E} - \mathbf{E}\cdot\text{curl}\,\mathbf{H})\, d\tau = 0$$

(as **E** and **H** are constant for steady fields).

This is because, **S** does not determine the rate of flow through small element ds at a point but implies that only the flux of **S** across a closed surface is significant.

(*iv*) *In case of time varying fields* $\mathbf{S}=\mathbf{E}\times\mathbf{H}$ *gives the instantaneous value of the Poynting vector and the average value is defined as the average over one complete period i.e.*

$$<\mathbf{S}> = \frac{1}{T}\int_0^{2\pi} (\mathbf{E}\times\mathbf{H})\, dt.$$

For example in the case of sinusoidally varying fields *i.e.*

$$\mathbf{E}=\mathbf{E}_0 \sin\omega t \text{ and } \mathbf{H}=\mathbf{H}_0 \sin\omega t$$

$$<\mathbf{S}> = \frac{1}{T}\int_0^T (\mathbf{E}_0 \sin\omega t)\times(\mathbf{H}_0 \sin\omega t)\, dt$$

i.e. $$<\mathbf{S}> = \frac{1}{T}(\mathbf{E}_0\times\mathbf{H}_0)\int_0^{2\pi/\omega} \sin^2\omega t\, dt^*$$

i.e. $$<\mathbf{S}> = \tfrac{1}{2}(\mathbf{E}_0\times\mathbf{H}_0) = \frac{\mathbf{E}_0}{\sqrt{2}}\times\frac{\mathbf{H}_0}{\sqrt{2}} = \mathbf{E}_{av}\times\mathbf{H}_{av}.$$

The importance of Poynting vector lies in the fact that with its help we can interpret various optical phenomena such as reflection, refraction, dispersion, scattering, differaction, etc., electromagnetically.

(v) We know that

$$\mathbf{B}=\mu_0\mathbf{H}$$

i.e. $$\frac{\mu_0}{4\pi}\int I\frac{d\mathbf{l}\times\mathbf{r}}{r^3} = \mu_0\mathbf{H}, \qquad \left(\text{as } \mathbf{B}=\frac{\mu_0}{4\pi}\int\frac{Id\mathbf{l}\times\mathbf{r}}{r^3}\right)$$

or $$\mathbf{H} = \frac{1}{4\pi}\int I\frac{d\mathbf{l}\times\mathbf{r}}{r^3}$$

$$^*\ \frac{1}{T}\int_0^{2\pi/\omega} \sin^2\omega t\, dt = \frac{\omega}{2\pi}\int_0^{2\pi/\omega}\frac{(1-\cos 2\omega t)}{2}\, dt$$

$$= \frac{\omega}{4\pi}\left[t - \frac{\sin 2\omega t}{2\omega}\right]_0^{2\pi/\omega} = \frac{\omega}{4\pi}\times\frac{2\pi}{\omega} = \frac{1}{2}$$

or $(\mathbf{H}) = \dfrac{[A]\,[L]\,[L]}{[L^3]} = [AL^{-1}]$

and $\mathbf{E} = (\mathbf{F}/q)$

i.e. $[E] = \dfrac{[MLT^{-2}]}{[AT]} = [MLT^{-3}\,A^{-1}]$

So $[S] = [E]\,[H] = [MT^{-3}]$

i.e. $[S] = \dfrac{[ML^2\,T^{-2}]}{L^2T} = \dfrac{\text{Energy}}{\text{area} \times \text{time}} = \dfrac{\text{Power}}{\text{area}}$

§ 4·7. Electromagnetic Potentials A and ϕ.

The analysis of an electromagnetic field is often facilitated by the use of auxilary functions known as electromagnetic potentials. At every point of space the field vectors satisfy the equations

$$\text{div } \mathbf{D} = \rho \qquad \text{...(A)}$$

$$\text{div } \mathbf{B} = 0 \qquad \text{...(B)}$$

$$\text{curl } \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \qquad \text{...(C)}$$

$$\text{curl } \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}. \qquad \text{...(D)}$$

According to equation (B) the field of vector **B** is always *solenoidal*, consequently **B** can be represented as the curl of another vector say **A** *i.e.*

$$\mathbf{B} = \text{curl } \mathbf{A} \quad \text{...(1)}$$

where **A** is a vector which is function of space (x, y, z) and time (t) both.

Now substituting the value of **B** in equation (D) from (1), we get

$$\text{curl } \mathbf{E} = -\frac{\partial}{\partial t} (\text{curl } \mathbf{A})$$

or

$$\text{curl} \left(\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t} \right) = 0$$

i.e. the field of the vector $\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t}$ is *irrotational* and must be equal to the gradient of some scalar function *i.e.*

$$\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t} = -\text{grad } \phi$$

or

$$\mathbf{E} = -\text{grad } \phi - \frac{\partial \mathbf{A}}{\partial t}. \quad \text{...(2)}$$

Thus we have introduced a vector **A** and a scalar ϕ both being functions of position and time. These are called electromagnetic potentials. The scalar ϕ is called the scalar potential and the vector **A**, vector potential. Regarding electromagnetic potentials it is worth noting that

(i) These are mathematical functions which are not physically measurable.

(ii) These are not independent of each other.

(iii) These define the field vector **E** and **B** uniquely but themselves are nonunique (See § 4·7).

(iv) These play an important role in relativistic electrodynamics (See chapter 10).

This is the desired ...

§ 4·8. Maxwell equations in terms of Electromagnetic Potentials.

Now consider the Maxwell's equation (C) *i.e.*

$$\mu \text{ curl } \mathbf{H}=\mu \mathbf{J}+\mu \frac{\partial \mathbf{D}}{\partial t}$$

or

$$\text{curl } \mathbf{B}=\mu \mathbf{J}+\epsilon\mu \frac{\partial \mathbf{E}}{\partial t}$$

substituting **B** and **E** from equations (1) and (2) of § 4·7 in above we get

$$\text{curl (curl } \mathbf{A}) = \mu \mathbf{J} + \mu\epsilon \frac{\partial}{\partial t}\left(-\text{grad } \phi - \frac{\partial \mathbf{A}}{\partial t}\right)$$

i.e. $$\text{grad div } \mathbf{A} - \nabla^2 \mathbf{A} = \mu \mathbf{J} - \mu\epsilon \frac{\partial}{\partial t}(\text{grad } \phi) - \mu\epsilon \frac{\partial^2 \mathbf{A}}{\partial t^2}$$

i.e. $$\nabla^2 \mathbf{A} - \mu\epsilon \frac{\partial^2 \mathbf{A}}{\partial t^2} - \text{grad}\left(\text{div } \mathbf{A} + \mu\epsilon \frac{\partial \phi}{\partial t}\right) = -\mu \mathbf{J}. \qquad \ldots(3)$$

Similarly if we consider equation (A) *i.e.*

$$\text{div } \mathbf{D} = \rho$$

i.e. $$\epsilon \text{ div } \mathbf{E} = \rho$$

i.e. $$\text{div}\left(-\text{grad } \phi - \frac{\partial \mathbf{A}}{\partial t}\right) = \frac{\rho}{\epsilon}$$

i.e. $$\nabla^2 \phi + \frac{\partial}{\partial t}(\text{div } \mathbf{A}) = -\frac{\rho}{\epsilon}.$$

Adding and substracting $\mu\epsilon \frac{\partial^2 \phi}{\partial t^2}$ it becomes

$$\nabla^2 \phi - \mu\epsilon \frac{\partial^2 \phi}{\partial t^2} + \frac{\partial}{\partial t}\left(\text{div } \mathbf{A} + \mu\epsilon \frac{\partial \phi}{\partial t}\right) = -\frac{\rho}{\epsilon} \qquad \ldots(4)$$

Equation (3) and (4) are field equations in terms of electromagnetic potentials, as equations (B) and (D) are satisfied in defining the scalar and vector potentials. So Maxwell equations are reduced from four to two by electromagnetic potentials, however these are coupled.

§ 4·9. Non-uniqueness of Electromagnetic Potentials and concept of Gauge :

In terms of electromagnetic potentials field vectors are given by

$$\mathbf{B} = \text{curl } \mathbf{A} \quad \text{...(1)}$$

and

$$\mathbf{E} = -\text{grad } \phi - \frac{\partial \mathbf{A}}{\partial t}. \quad \text{...(2)}$$

From equations (1) and (2) it is clear that for a given **A** and ϕ, each of the field vectors **B** and **E** has only one value *i.e.* **A** and ϕ determine **B** and **E** uniquely. However the converse is not true *i.e.* field vectors do not determine the potentials **A** and ϕ completely. This in turn implies that for a given **A** and ϕ there will be only one **E** and **B** while for a given **E** and **B** there can be an infinite number of **A**'s and ϕ's. This is because the curl of the gradient of any scalar vanishes identically and hence we may add to **A** the gradient of a scalar Λ without affecting **B**. That is **A** may be replaced by

$$\mathbf{A}' = \mathbf{A} + \text{grad } \Lambda. \quad \text{...(3)}$$

But if this is done, equation (2) becomes

$$\mathbf{E} = -\text{grad } \phi - \frac{\partial}{\partial t}(\mathbf{A}' - \text{grad } \Lambda)$$

i.e.

$$\mathbf{E} = -\text{grad}\left(\phi - \frac{\partial \Lambda'}{\partial t}\right) - \frac{\partial \mathbf{A}'}{\partial t}.$$

So if we make the transformation given by eqn. (3) we must also replace ϕ by

$$\phi' = \phi - \frac{\partial \Lambda}{\partial t}. \quad \text{...(4)}$$

The expressions for field vectors **E** and **B** remain unchanged under transformations (3) and (4) *i.e.*

$$\mathbf{B}=\text{curl }\mathbf{A}=\text{curl }(\mathbf{A}'-\text{grad }\Lambda)=\text{curl }\mathbf{A}'$$

and $$\mathbf{E}=-\text{grad }\phi-\frac{\partial \mathbf{A}}{\partial t}=-\text{grad}\left(\phi'+\frac{\partial \Lambda}{\partial t}\right)-\frac{\partial}{\partial t}(\mathbf{A}'-\text{grad }\Lambda)$$

$$=\text{grad }\phi'-\frac{\partial \mathbf{A}'}{\partial t}$$

i.e. we get the same field vectors whether we use the set ($\mathbf{A}$, ϕ) or ($\mathbf{A}'$, ϕ'). *So electromagnetic potentials define the field vectors uniquely though they themselves are non-unique.*

The transformations given by equations (3) and (4) are called gauge transformations and the arbitrary scalar Λ gauge function. From the above it is also clear that even though we add the gradient of a scalar function, the field vectors remain unchanged. Now it is the field quantities and not the potentials that possess physical meaningfulness. We therefore say that the *field vectors are invarient to gauge transformations i.e.* they are gauge invarient.

Because of the arbitrariness in the choice of gauge *i.e.* non-uniqueness of potentials, we are free to impose an anditional condition on A. We may state this in other terms : *a vector is not completely specified by giving only its curl but if both the curl and the divergence of a vector are specified the vector is uniquely determined.* Clearly it is to our advantage to make a choice for div **A** in any convenient manner that will provide a simplification for the particular problem under consideration. Generally div **A** is chosen in two ways (described in following articles) according as the field contains charge or not.

§ 4·10. Lorentz Gauge :

The Maxwell's field equations in terms of electromagnetic potentials are

$$\nabla^2\mathbf{A}-\mu\epsilon\frac{\partial^2\mathbf{A}}{\partial t^2}-\text{grad}\left(\text{div }\mathbf{A}+\mu\epsilon\frac{\partial\phi}{\partial t}\right)=-\mu\mathbf{J} \qquad ...(1)$$

$$\nabla^2\phi-\mu\epsilon\frac{\partial^2\phi}{\partial t^2}+\frac{\partial}{\partial t}\left(\text{div }\mathbf{A}+\mu\epsilon\frac{\partial\phi}{\partial t}\right)=-\frac{\rho}{\epsilon} \qquad ...(2)$$

A casual glance at equations (1) and (2) reveals that these equations will be much more simplified (*i.e.* will become identical and uncoupled) if we choose

$$\text{div }\mathbf{A}+\mu\epsilon\frac{\partial\phi}{\partial t}=0. \qquad ...(3)$$

This requirement is called the Lorentz condition and when the vector and scalar potential satisfy it, the gauge is known as Lorentz gauge.

So with Lorentz condition field equation reduce to

$$\nabla^2 \mathbf{A} - \mu\epsilon \frac{\partial^2 \phi}{\partial t^2} = -\mu \mathbf{J} \quad \text{...(4)}$$

and $$\nabla^2 \phi - \mu\epsilon \frac{\partial^2 \phi}{\partial t^2} = -\rho/\epsilon. \quad \text{...(5)}$$

But as $\mu\epsilon = 1/v^2$*.

So equations (4) and (5) can be written as

$$\Box^2 \mathbf{A} = -\mu \mathbf{J} \quad \text{...(6)}$$

$$\Box^2 \phi = -\rho/\epsilon \quad \text{...(7)}$$

with $$\Box^2 = \nabla^2 - \frac{1}{v^2}\frac{\partial^2}{\partial t^2}.$$

Equations (6) and (7) are inhomogeneous wave equations and are known as D' Alembertian equations and can be solved in general by a method similar to that we use to solve Poisson's equation. The potentials obtained by solving these equations are called retarted potentials and are discussed in § 8·1.

In order to determine the requirement that Lorentz condition places on Λ, we substitute A′ and ϕ' from equations (3) and (4) of § 4·8 in eqn. (3) *i.e.*

$$\text{div}\,(\mathbf{A}' - \text{grad}\,\Lambda) + \mu\epsilon \frac{\partial}{\partial t}\left(\phi' + \frac{\partial \Lambda}{\partial t}\right) = 0$$

i.e. $$\text{div}\,\mathbf{A}' + \mu\epsilon \frac{\partial \phi'}{\partial t} = \nabla^2 \Lambda - \epsilon\mu \frac{\partial^2 \Lambda}{\partial t^2}$$

So A′ and ϕ' will also satisfy equation (3) *i.e.* Lorentz condition provided that

$$\nabla^2 \Lambda - \epsilon\mu \frac{\partial^2 \Lambda}{\partial t^2} = 0$$

i.e. $$\Box^2 \Lambda = 0. \quad \text{...(8)}$$

i.e., Lorentz condition is invariant under those gauge tranformations for which the gauge functions are solutions of the homogeneous wave equations.

The advantages of this particular gauge are :

(i) It makes the equations for **A** and ϕ independent of each other.

(ii) It leads to the wave equations which treat ϕ and **A** on equivalent footings.

(iii) It is a concept which is independent of the co-ordinate system chosen and so fits naturally into the considerations of special theory of relativity.

§ 4·11. Coulomb Gauge :

An inspection of field equations in terms of electromagnetic potentials *i.e.*

$$\nabla^2 \mathbf{A} - \mu\epsilon \frac{\partial^2 \mathbf{A}}{\partial t^2} - \text{gard}\left(\text{div } \mathbf{A} + \mu\epsilon \frac{\partial \phi}{\partial t}\right) = -\mu \mathbf{J}. \quad ...(1)$$

And $$\nabla^2 \phi - \mu\epsilon \frac{\partial^2 \phi}{\partial t^2} + \frac{\partial}{\partial t}\left(\text{div } \mathbf{A} + \mu\epsilon \frac{\partial \phi}{\partial t}\right) = -\frac{\rho}{\epsilon}$$

i.e. $$\nabla^2 \phi + \frac{\partial}{\partial t}(\text{div } \mathbf{A}) = -\frac{\rho}{\epsilon} \quad ...(2)$$

shows that if we assume

$$\text{div } \mathbf{A} = 0. \quad ...(3)$$

equation (2) reduces to Poisson's equation

$$\nabla^2 \phi_{(r,\, t)} = -\frac{\rho_{(r',\, t)}}{\epsilon} \quad ...(4)$$

whose solution is

$$\phi_{(r, t)} = \frac{1}{4\pi\epsilon} \int \frac{\rho_{(r',\, t)}}{R}^* \, d\tau' \quad ...(5)$$

i.e. the scalar potential is just the instantaneous Coulombian potential due to charge $\rho_{(x',\, y',\, z',\, t)}$. This is the origin of the name Coulomb gauge.

Equation (1) in the light of (3) reduced to

$$\nabla^2 \mathbf{A} - \frac{1}{v^2} \frac{\partial^2 \mathbf{A}}{\partial t^2} = -\mu \mathbf{J} + \mu\epsilon \nabla\left(\frac{\partial \phi}{\partial t}\right) \quad ...(6)$$

Now to express equation (6) in more convenient way we use Poisson's equation (4) which with the help of (5) can be written as

$$\nabla^2 \left\{ \frac{1}{4\pi\epsilon} \int \frac{\rho_{(r'\, t)}}{R} \, d\tau' \right\} = -\frac{\rho_{(r',\, t)}}{\epsilon} \quad ...(7)$$

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*For the solution of Poisson's equation see Appendix—II.

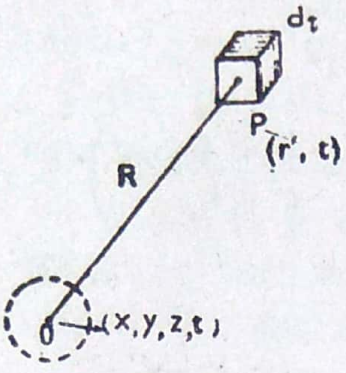

Fig. 4·14

Now as Poisson's equation holds good for scalar and vectors both, replacing $\rho(r', t)$ by $\mathbf{J}'$ we get

$$\nabla^2\left\{\frac{1}{4\pi\epsilon}\int\frac{\mathbf{J}'}{R}\,d\tau'\right\}=-\frac{\mathbf{J}'}{\epsilon} \qquad ...(8)$$

Now from the vector identity

$$\nabla\times\nabla\times\mathbf{G}=\nabla\,(\nabla\cdot\mathbf{G})-\nabla^2\mathbf{G}$$

$$\nabla^2\mathbf{G}=\nabla\,(\nabla\cdot\mathbf{G})-\nabla\times\nabla\times\mathbf{G}$$

Taking $\mathbf{G}=\int(\mathbf{J}/R)\,d\tau'$, we get

$$\nabla^2\int\frac{\mathbf{J}'}{R}\,d\tau'=\nabla\left(\nabla\cdot\int\frac{\mathbf{J}'}{R}\,d\tau'\right)-\nabla\times\nabla\times\int\frac{\mathbf{J}'}{R}\,d\tau'$$

which in the light of equation (8) reduces to

$$-4\pi\,\mathbf{J}'=\nabla\left(\nabla\cdot\int\frac{\mathbf{J}'}{R}\,d\tau'\right)-\nabla\times\nabla\times\int\frac{\mathbf{J}'}{R}\,d\tau'$$

i.e.

$$\mathbf{J}'=-\frac{1}{4\pi}\nabla\left[\nabla\cdot\int\frac{\mathbf{J}'}{R}\,d\tau'\right]+\frac{1}{4\pi}\nabla\times\nabla\times\int\frac{\mathbf{J}'}{R}\,d\tau' \qquad ...(9)$$

Now as $\nabla\cdot\int(\mathbf{J}'/R)\,d\tau'$

$$=\int\left[\frac{1}{R}\nabla\cdot\mathbf{J}'+\mathbf{J}'\cdot\nabla\left(\frac{1}{R}\right)\right]\left[\text{as } \nabla(S\mathbf{V})=S\nabla\cdot\mathbf{V}+\mathbf{V}\cdot\Delta S\right]$$

$$=\int\mathbf{J}'\cdot\nabla(1/R)\,d\tau' \qquad [\text{as } \mathbf{J}' \text{ is not a function } x, y \text{ and } z]$$

$$=-\int\mathbf{J}'\cdot\nabla'(1/R)\,d\tau' \qquad [\text{as } \nabla(1/R)=-\nabla'(1/R)]$$

$$=\int\left[\frac{\nabla'\cdot\mathbf{J}'}{R}-\nabla'\cdot\left(\frac{\mathbf{J}'}{R}\right)\right]d\tau'$$

$$\left[\text{as } \nabla'\cdot\left(\frac{\mathbf{J}'}{R}\right)=\left(\frac{1}{R}\right)\nabla'\cdot\mathbf{J}'+\mathbf{J}'\cdot\nabla'\left(\frac{1}{R}\right)\right]$$

$$=\int\frac{\nabla'\cdot\mathbf{J}'}{R}\,d\tau'-\oint_S\left(\frac{\mathbf{J}'}{R}\right).d\mathbf{s}$$

$$\left[\text{as } \int\nabla'\left(\frac{\mathbf{J}'}{R}\right)d\tau'=\oint_S\frac{\mathbf{J}'}{R}.d\mathbf{s}\right]$$

As $\mathbf{J}'$ is confined to the vol τ', the surface contribution will vanish so

$$\nabla.\int\left(\frac{\mathbf{J}'}{R}\right)d\tau'=\int\frac{\nabla'.\mathbf{J}'}{R}\,d\tau' \qquad ...(10)$$

And $\nabla\times\int(\mathbf{J}'/R)\,d\tau'$

$$=\int\left[\frac{\nabla\times\mathbf{J}'}{R}-\mathbf{J}'\times\nabla\left(\frac{1}{R}\right)\right]d\tau'$$

$$[\text{as curl } S\mathbf{V}=S \text{ curl } \mathbf{V}-V\times\text{grad } S]$$

$$= -\int [\mathbf{J}' \times \nabla(1/R)]\, d\tau'$$

[as $\mathbf{J}'$ is not a function of x, y and z]

$$= \int \mathbf{J}' \times \nabla'(1/R)\, d\tau' \qquad [\text{as } \nabla(1/R) = -\nabla'(1/R)]$$

$$= \int \left[\frac{\nabla' \times \mathbf{J}'}{R} - \nabla' \times \left(\frac{\mathbf{J}'}{R} \right) \right] d\tau'$$

$$\left[\text{as } \nabla \times \left(\frac{\mathbf{J}'}{R} \right) = \frac{1}{R} \nabla' \times \mathbf{J}' - \mathbf{J}' \times \nabla' \left(\frac{1}{R} \right) \right]$$

$$= \int \frac{\nabla' \times \mathbf{J}'}{R}\, d\tau' + \oint \frac{\mathbf{J}'}{R} \times d\mathbf{s}$$

$$\left[\text{As} \int \nabla \times \mathbf{V}\, d\tau' = -\oint_S \mathbf{V} \times d\mathbf{s}^* \right]$$

As $\mathbf{J}'$ is confined to vol τ', surface contribution will vanish so

$$\nabla \times \int \left(\frac{\mathbf{J}'}{R} \right) d\tau' = \int \frac{\nabla' \times \mathbf{J}'}{R}\, d\tau' \qquad \text{...(11)}$$

So eqn. (9) becomes

$$\mathbf{J}' = -\frac{1}{4\pi} \nabla \int \frac{\nabla' \cdot \mathbf{J}'}{R}\, d\tau' + \frac{1}{4\pi} \nabla \times \int \frac{\nabla' \times \mathbf{J}'}{R}\, d\tau'$$

i.e. $\mathbf{J}' = \mathbf{J}'_l + \mathbf{J}'_T$ ...(12)

with $\mathbf{J}'_l = -\frac{1}{4\pi} \nabla \int \frac{\nabla' \cdot \mathbf{J}'}{R}\, d\tau'$ and $\mathbf{J}'_T = \frac{1}{4\pi} \nabla \times \int \frac{\nabla' \times \mathbf{J}'}{R}\, d\tau'$ ...(13)

Now as

$$\nabla \times \mathbf{J}'_l = \nabla \times \left[-\frac{1}{4\pi} \nabla \int \frac{\nabla' \cdot \mathbf{J}'}{R}\, d\tau' \right]$$

i.e. $\nabla' \times \mathbf{J}'_l = 0$ (as curl grad $\phi = 0$) ...(14)

and $\nabla \cdot \mathbf{J}'_T = \nabla \cdot \left[\nabla \times \int \frac{\nabla' \times \mathbf{J}}{R}\, d\tau' \right]$

i.e. $\nabla \cdot \mathbf{J}'_T = 0$ (as div curl $\mathbf{V} = 0$) ...(15)

the first term on R.H.S. of equation (12) is irrotational and second is solenoidal. The first term is called longitudinal current aud the other transverse current.

So in the light of equation (12), (6) can be written as

$$\nabla^2\mathbf{A}-\frac{1}{v^2}\frac{\partial^2\mathbf{A}}{\partial t^2}=-\mu(\mathbf{J}_l+\mathbf{J}_T)+\mu\epsilon\nabla\left(\frac{\partial\phi}{\partial t}\right)$$

i.e. $$\nabla^2\mathbf{A}-\frac{1}{v^2}\frac{\partial^2\mathbf{A}}{\partial t^2}=-\mu\mathbf{J}_T-\mu\mathbf{J}_l+\mu\epsilon\nabla\frac{\partial}{\partial t}\left[\frac{1}{4\pi\epsilon}\int\frac{\rho(r',t)}{R}d\tau'\right]$$

[substituting ϕ from equation (5)]

i.e. $$\nabla^2\mathbf{A}-\frac{1}{v^2}\frac{\partial^2\mathbf{A}}{\partial t^2}=-\mu\mathbf{J}_T-\mu\mathbf{J}_l+\mu\frac{1}{4\pi}\nabla\int\frac{-\nabla'\cdot\mathbf{J}}{R}d\tau'$$

$\left[\text{as from continuity eqn. } \frac{\partial\rho(r',t)}{\partial t}=-\nabla'\cdot\mathbf{J}\right]$

or $$\nabla^2\mathbf{A}-\frac{1}{v^2}\frac{\partial^2\mathbf{A}}{\partial t^2}=-\mu\mathbf{J}_T-\mu\mathbf{J}_l+\mu\mathbf{J}_l \quad \text{[From equation (13)]}$$

i.e. $$\nabla^2\mathbf{A}-\frac{1}{v^2}\frac{\partial^2\mathbf{A}}{\partial t^2}=-\mu\mathbf{J}_T$$

i.e. $$\Box^2\mathbf{A}=-\mu\mathbf{J}_T. \qquad ...(16)$$

i.e. the equation for A can be expressed entirely in terms of the transverse current. So this gauge sometimes is also called transverse gauge.

The Coulomb gauge has a certain advantage. In it the scalar potential is exactly the electrostatic potential (equation 5) and electric field

$$\mathbf{E}=-\text{grad }\phi-\frac{\partial\mathbf{A}}{\partial t}$$

is separable into an electrostatic field $V=\phi$ and a wave field given by $-(\partial\mathbf{A}/\partial t)$.

This gauge is often used when no sources are present. Then according to equation (5) $\phi=0$ and A satisfies the homogeneous wave equation (16). The fields are given by

$$\mathbf{E}=-\frac{\partial\mathbf{A}}{\partial t} \text{ and } \mathbf{B}=\nabla\times\mathbf{A}.$$

Plane Electromagnetic Waves and their Propagation

INTRODUCTION :

[In this chapter we shall show that the Maxwell's field equations, predict the existence of electromagnetic waves and discuss the propagation of these waves in free space, non-conducting, conducting and ionized media. We shall also investigate the energy flow associated with their propagation.]

§ 5·1. Electromagnetic Waves in free space.*

We know that Maxwell's equations are

$$\left.\begin{aligned} \nabla\cdot\mathbf{D}&=\rho \\ \nabla\cdot\mathbf{B}&=0 \\ \nabla\times\mathbf{H}&=\mathbf{J}+\frac{\partial\mathbf{D}}{\partial t} \\ \nabla\times\mathbf{E}&=-\frac{\partial\mathbf{B}}{\partial t} \end{aligned}\right\} \text{ with} \begin{cases} \mathbf{J}=\sigma\mathbf{E} \\ \mathbf{B}=\mu\mathbf{H} \\ \mathbf{D}=\epsilon\mathbf{E} \end{cases} \quad \ldots(1)$$

and in free space *i.e.* vacuum

$$\rho=0 \qquad \epsilon_r=1$$
$$\sigma=0 \qquad \mu_r=1$$

So Maxwell's equations reduce to

$$\left.\begin{aligned} \nabla\cdot\mathbf{E}&=0 & \ldots(a) \\ \nabla\cdot\mathbf{H}&=0 & \ldots(b) \\ \nabla\times\mathbf{H}&=\epsilon_0\frac{\partial\mathbf{E}}{\partial t} & \ldots(c) \\ \Delta\times\mathbf{E}&=-\mu_0\frac{\partial\mathbf{H}}{\partial t}. & \ldots(d) \end{aligned}\right\} \quad \ldots(2)$$

Now if

(I) We take the curl of equation 2 (c) then

*Reader is advi...

$$\nabla\times(\nabla\times\mathbf{H})=\epsilon_0\ \nabla\times\left(\frac{\partial\mathbf{E}}{\partial t}\right)$$

i.e. $$[\nabla(\nabla\cdot\mathbf{H})-\nabla^2\mathbf{H}]=\epsilon_0\ \frac{\partial}{\partial t}\ (\nabla\times\mathbf{E}). \qquad ...(3)$$

But from equations 2 (b) and 2 (d)

$$\nabla\cdot\mathbf{H}=0 \quad \text{and} \quad \nabla\times\mathbf{E}=-\mu_0\ \frac{\partial\mathbf{H}}{\partial t}.$$

So eqn. (3) reduces to

$$\nabla^2\mathbf{H}-\frac{1}{c^2}\ \frac{\partial^2\mathbf{H}}{\partial t^2}=0 \text{ with } \mu_0\epsilon_0=\frac{1}{c^2}. \qquad ...(A)$$

(II) We take the curl of equation 2 (d), then

$$\nabla\times(\nabla\times\mathbf{E})=\nabla\times\left(-\mu_0\ \frac{\partial\mathbf{H}}{\partial t}\right)$$

i.e. $$[\nabla(\nabla\cdot\mathbf{E})-\nabla^2\mathbf{E}]=-\mu_0\ \frac{\partial}{\partial t}\ (\nabla\times\mathbf{H}). \qquad ...(4)$$

But from equation 2 (a) and 2 (c)

$$\nabla\cdot\mathbf{E}=0 \quad \text{and} \quad \nabla\times\mathbf{H}=\epsilon_0\ \frac{\partial\mathbf{E}}{\partial t}.$$

So equation (4) reduces to

i.e. $$\nabla^2\mathbf{E}-\frac{1}{c^2}\ \frac{\partial^2\mathbf{E}}{\partial t^2}=0 \text{ with } \mu_0\epsilon_0=\frac{1}{c^2}. \qquad ...(B)$$

A glance at differential equations (A) and (B) reveals that these are indential in form to the equation

$$\nabla^2\psi-\frac{1}{v^2}\ \frac{\partial^2\psi}{\partial t^2}=0. \qquad ...(5)$$

However equation (5) is a standard wave equation representing unattenuated wave travelling at a speed v*. So we conclude that *field vector* **E** *and* **H** *are propagated in free space as waves at a speed*

$$c=\frac{1}{\sqrt{(\epsilon_0\mu_0)}}=\sqrt{\left(\frac{4\pi}{4\pi\epsilon_0\mu_0}\right)}=\sqrt{(9\times10^9)}\times\sqrt{(10^7)}$$

$$=3\times10^8\ m/s$$

*i.e. the velocity of light.***

Further as equation (A) and (B) are vector wave equations their solution can be obtained in many forms, for instance either stationary or progressive waves or having wave fronts of particular types such as plane, cylindrical or spherical. Where no boundary conditions are imposed, as in this chapter, plane progressive solutions are most appropriate. So as the plane progressive solution of equation (5) is

$$\psi=\psi_0\, e^{-i(\omega t-\mathbf{k}\cdot\mathbf{r})}$$

the solutions of equations (A) and (B) will be of the form

$$\left.\begin{aligned}\mathbf{E}&=\mathbf{E}_0\, e^{-i(\omega t-\mathbf{k}\cdot\mathbf{r})}\\ \mathbf{H}&=\mathbf{H}_0\, e^{-i(\omega t-\mathbf{k}\cdot\mathbf{r})}\end{aligned}\right\} \quad \text{...(C)}$$

where **k** is the so called wave vector given by

$$\mathbf{k}=k\mathbf{n}=\frac{2\pi}{\lambda}\,\mathbf{n}=\frac{2\pi f}{c}\,\mathbf{n}=\frac{\omega}{c}\,\mathbf{n}$$

with **n** as a unit vector in the direction of wave propagation.

The form of field vectors **E** and **H** given by eqn. (C) suggets that in case of field vectors operator ∇ is equivalent to i **k** while $\partial/\partial t$ to $(-i\omega)$.* So Maxwell's equations in free space *i.e.* eqn. (2) in terms of operator (i **k**) and ($-i\omega$) can be written as

$$\left.\begin{aligned}\mathbf{k}\cdot\mathbf{E}&=0 &&\text{...(a)}\\ \mathbf{k}\cdot\mathbf{H}&=0 &&\text{...(b)}\\ -\mathbf{k}\times\mathbf{H}&=\omega\epsilon_0\mathbf{E} &&\text{...(c)}\\ \mathbf{k}\times\mathbf{E}&=\omega\mu_0\mathbf{H} &&\text{...(d)}\end{aligned}\right\} \quad \text{...(4)}$$

Regarding plane electromagnetic waves in free space it is worthy to note that :

(*i*) As according to eqn. 4 (a) the vector **E** is perpendicular to the direction of propagation while according to eqn. 4 (b) the vector **H** is perpendicular to the direction of propagation (*i.e.* in an electromagnetic wave both the vectors **E** and **H** are perpendicular to the direction of wave propagation), *electromagnetic waves are transverse in nature.*

Further as according to eqn. 4 (d) **H** is perpendicular to both **E** and **k** while according to eqn 4 (a) **E** is perpendicular to **k**. This all in turn implies that *in a plane electromagnetic waves vectors* **E**, **H** *and* **k** *are orthogonal* as shown in fig. 5·1.

(ii) As according to equation 4 (d)

$$\mathbf{k}\times\mathbf{E}=\omega\mu_0\mathbf{H}$$

i.e. $$\mathbf{H}=\frac{k}{\omega\mu_0}(\mathbf{n}\times\mathbf{E}) \quad (\text{as } \mathbf{k}=\mathbf{n}k)$$

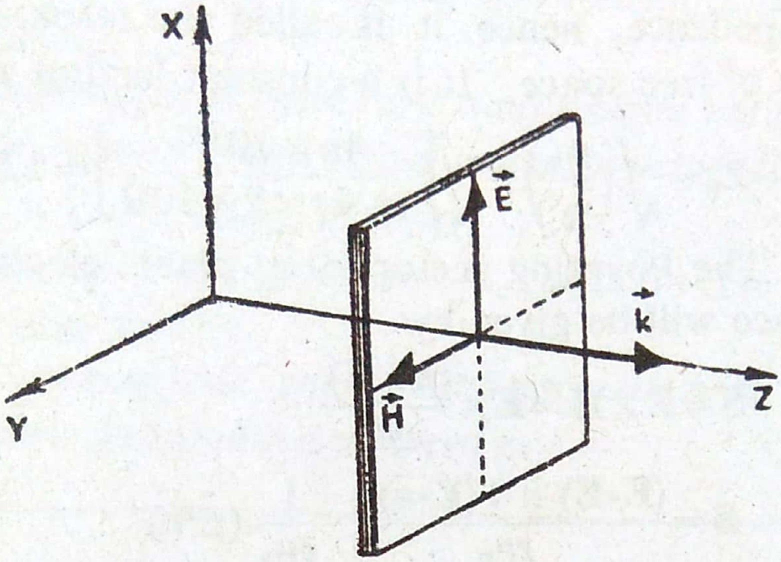

Fig. 5·1

i.e. $$\mathbf{H}=\frac{\mathbf{n}\times\mathbf{E}}{c\mu_0}=c\epsilon_0(\mathbf{n}\times\mathbf{E}) \quad \left(\text{as } k=\frac{\omega}{c} \text{ and } \epsilon_0\mu_0=\frac{1}{c^2}\right)$$

i.e. $$\mathbf{B}=\frac{\mathbf{n}\times\mathbf{E}}{c} \quad \text{...(D)}$$

and $$\left|\frac{\mathbf{E}}{\mathbf{H}}\right|=\frac{E_0}{H_0}=c\mu_0=\frac{1}{c\epsilon_0}=\sqrt{\left(\frac{\mu_0}{\epsilon_0}\right)}=Z_0 \quad \left(\text{as } \mu_0\epsilon_0=\frac{1}{c^2}\right)$$

As the ratio | **E**/**H** | is real and positive, *the vectors* **E** *and* **H** *are in phase.** *i.e.* when **E** has its maximum value **H** has also its maximum value. This is shown in fig. 5·2. From the above it is also clear that in an electromagnetic wave *the amplitude of electric vector* **E** *is* Z_0 *times that of the magnetic vector* **H**.

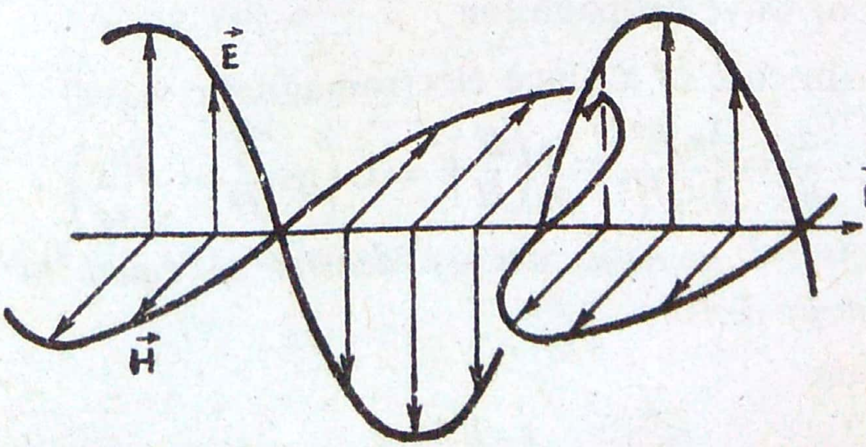

Fig. 5·2

The quantity Z_0 has the dimension

$$[Z_0]=\left[\sqrt{\left(\frac{\mu_0}{\epsilon_0}\right)}\right]=\sqrt{\left(\frac{H/m}{F/m}\right)}=\sqrt{\left(\frac{\text{ohm}\times\text{sec}}{\text{coul./volt}}\right)}$$

$$=\sqrt{\left(\frac{\text{ohm}\times\text{volt}}{\text{amp}}\right)}=\text{ohm}$$

i.e. of impedence, hence it is called the *intrinsic* or *characteristic impedence* of free space. It is a constant having value

$$Z_0=\sqrt{\left(\frac{\mu_0}{\epsilon_0}\right)}=\sqrt{\left[\frac{4\pi\times10^{-7}}{(1/4\pi\times9\times10^9)}\right]}=120\pi\approx377\,\Omega$$

(*iii*) The Poynting vector for a plane electromagnetic wave in free space will be given by

$$\mathbf{S}=\mathbf{E}\times\mathbf{H}=\mathbf{E}\times\frac{(\mathbf{n}\times\mathbf{E})}{c\mu_0}$$

i.e. $$\mathbf{S}=\frac{(\mathbf{E}\cdot\mathbf{E})\,\mathbf{n}-(\mathbf{E}\cdot\mathbf{n})}{c\mu_0}=\frac{1}{c\mu_0}\,(E^2\mathbf{n})$$

(as $\mathbf{E}\cdot\mathbf{n}=0$ because $\mathbf{E}$ is $\perp$ to $\mathbf{n}$)

or $$\mathbf{S}=\epsilon_0\,cE^2\mathbf{n}=\frac{1}{Z_0}\,E^2\,\mathbf{n}\left(\text{as }\frac{1}{c\mu_0}=c\epsilon_0=\frac{1}{Z_0}\right)$$

or $$<\mathbf{S}>=\epsilon_0 c<E^2>\mathbf{n}=\frac{1}{Z_0}<E^2>\mathbf{n}.$$

But as

$$<E^2>=<[E_0e^{-i(\omega t-\mathbf{k}\cdot\mathbf{r})}]^2>=E_0^2<\cos^2(\omega t-\mathbf{k}\cdot\mathbf{r})>$$

i.e. $$<E^2>=\frac{E_0^2}{2}=\left(\frac{E_0}{\sqrt{2}}\right)\left(\frac{E_0}{\sqrt{2}}\right)=E^2_{rms}\ [\text{as }<\cos^2\theta>)=\tfrac{1}{2}]$$

So $$<\mathbf{S}>=\epsilon_0 cE^2_{rms}\,\mathbf{n}=\frac{1}{Z_0}\,E^2_{rms}\,\mathbf{n} \quad \text{...(E)}$$

i.e. the flow of energy in a plane wave in free space is in the direction of wave propagation.

(iv) In case of a plane electromagnetic wave

$$\frac{u_e}{u_m}=\frac{\frac{1}{2}\epsilon_0E^2}{\frac{1}{2}\mu_0H^2}=\frac{\epsilon_0}{\mu_0}\left(\frac{E}{H}\right)^2=1\ \left(\text{as }\frac{E}{H}=\sqrt{\frac{\mu_0}{\epsilon_0}}\right)$$

i.e. the electromagnetic energy density is equal to the magnetostatic energy density.

Further

$$\frac{<\mathbf{S}>}{<u>}=\frac{\epsilon_0cE^2_{rms}\mathbf{n}}{\epsilon_0E^2_{rms}}=c\mathbf{n}$$

i.e. $$\mathbf{S}=cu\mathbf{n}$$

In case of propagation E. M. W. in free space.

(*i*) *The wave propagates with a speed equal to that of light in free space.*

(*ii*) *The electromagnetic waves are transverse in nature.*

(*iii*) *The wave vectors* **E** *and* **H** *are mutually perpendicular.*

(*iv*) *The vector* **E** *and* **H** *are in phase.*

(*v*) *The electrostatic energy density is equal to the magnetostatic energy density.*

(*vi*) *The electromagnetic energy is transmitted in the direction of wave propagation at speed c.*

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§ 5·2. Propagation of E.M.W. in Isotropic Dielectrics.*

We know that Maxwell's field equations are

$$\left.\begin{aligned} \nabla \cdot \mathbf{D} &= \rho \\ \nabla \cdot \mathbf{B} &= 0 \\ \nabla \times \mathbf{H} &= \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \\ \nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t} \end{aligned}\right\} \quad \text{with} \begin{cases} \mathbf{J} = \sigma \mathbf{E} \\ \mathbf{B} = \mu \mathbf{H} \\ \mathbf{D} = \epsilon \mathbf{E} \end{cases} \qquad \ldots(1)$$

and in isotropic dielectrics

$$\sigma = 0 \text{ and } \rho = 0.$$

So Maxwell's equations reduce to

$$\left.\begin{aligned} \nabla \cdot \mathbf{E} &= 0 & \ldots(a) \\ \nabla \cdot \mathbf{H} &= 0 & \ldots(b) \\ \nabla \times \mathbf{H} &= \epsilon \frac{\partial \mathbf{E}}{\partial t} & \ldots(c) \\ \nabla \triangle \mathbf{E} &= -\mu \frac{\partial \mathbf{H}}{\partial t} & \ldots(d) \end{aligned}\right\} \qquad \ldots(2)$$

Now if

(I) We take the curl of equation 2 (c) then

$$\nabla \times (\nabla \times \mathbf{H}) = \epsilon \; \nabla \times \left(\frac{\partial \mathbf{E}}{\partial t} \right)$$

or $$\nabla(\nabla \cdot \mathbf{H})-\nabla^2\mathbf{H}=\epsilon\frac{\partial}{\partial t}(\nabla \times \mathbf{E}).$$ ...(3)

But from equations 2 (b) and 2 (d)

$$\nabla \cdot \mathbf{H}=0 \quad \text{and} \quad \nabla \times \mathbf{E}=-\mu\frac{\partial \mathbf{H}}{\partial t}.$$

So equation (3) reduces to

$$\nabla^2\mathbf{H}-\mu\epsilon\frac{\partial^2\mathbf{H}}{\partial t^2}=0$$

i.e. $$\nabla^2\mathbf{H}-\frac{1}{v^2}\frac{\partial^2\mathbf{H}}{\partial t^2}=0 \text{ with } \mu\epsilon=1/v^2$$ ...(A)

(II) We take the curl of eqn. 2 (d) then

$$\nabla \times(\nabla \cdot \mathbf{E})=\nabla \times\left(-\mu\frac{\partial \mathbf{H}}{\partial t}\right)$$

i.e. $$\nabla(\nabla \cdot \mathbf{E})-\nabla^2\mathbf{E}=-\mu\frac{\partial}{\partial t}(\nabla \times \mathbf{H})$$ ...(4)

But from equations 2 (a) and 2 (c)

$$\nabla \cdot \mathbf{E}=0 \text{ and } \nabla \times \mathbf{H}=\epsilon\frac{\partial \mathbf{E}}{\partial t}$$

So equation (4) reduces to

$$\nabla^2\mathbf{E}-\mu\epsilon\frac{\partial^2\mathbf{E}}{\partial t^2}=0$$

i.e. $$\nabla^2\mathbf{E}-\frac{1}{v^2}\frac{\partial^2\mathbf{E}}{\partial t^2}=0 \text{ with } \mu\epsilon=1/v^2$$ ..(B)

A glance at equation (A) and (B) reveals that these are identical in form to the equation

$$\nabla^2\psi-\frac{1}{v^2}\frac{\partial^2\psi}{\partial t^2}=0.$$ ...(5)

However equation (5) is a standard wave equation representing an unattenuated wave travelling at a speed v. So we conclude that field vectors **E** and **H** propagate in isotropic dielectric as waves given by

$$\begin{Bmatrix}\mathbf{E}\\ \mathbf{H}\end{Bmatrix}=\begin{Bmatrix}\mathbf{E}_0\\ \mathbf{H}_0\end{Bmatrix}e^{-i(\omega t-\mathbf{k}\cdot\mathbf{r})}$$ ...(C)

at a speed

$$v=\frac{1}{\sqrt{(\epsilon\mu)}}=\frac{1}{\sqrt{(\epsilon_r\mu_r\epsilon_0\mu_0)}} \qquad (\text{as } \epsilon=\epsilon_r\epsilon_0 \text{ and } \mu=\mu_r\mu_0)$$

i.e. $$v=\frac{c}{\sqrt{(\epsilon_r\,\mu_r)}}<c\ [\text{as } \epsilon_0\mu_0=1/c^2\ ;\ \epsilon_r \text{ and } \mu_r \triangleright 1]$$ ...(6)

i.e. the speed of electromagnetic wave in isotropic dielectrics is less than the speed of electromagnetic waves in free space.

Further as index of refraction is defined as

$$n=(c/v)$$

So in this particular case

$$n=\sqrt{(\epsilon_r\mu_r)} \text{ [as } v=c/\sqrt{(\epsilon_r\mu_r)}]$$

and as in a non-magnetic medium $\mu_r=1$

$$n=\sqrt{(\epsilon_r)} \; i.e. \; n^2=\epsilon_r \qquad ...(7)$$

Equation (7) is called Maxwell's relation and has been actually confirmed by experiments for long waves *i.e.* radio frequency and slow infrared oscillations. In visible region of the spectrum this relation is also fairly well satisfied for some substances such as H_2, CO_2, N_2 and O_2. But for many other substances it fails, when as a rule the substance shows infrared selective absorption. With water the failure is especially marked. For water $\mu_r \approx 1$, $\epsilon_r \approx 81$ so that $n \approx 9$. But it is well known that the index of refraction of water for light is very closely given by 4/3 *i.e.* 1·33. The solution of this appearent contradiction lies in the fact that our macroscopic formulation of electromagnetic theory gives no indication of the values to be expected for ϵ_r and μ_r and we must rely on experiment to obtain them. It turns out that these quantitites are not really constant for a given material but usually have a strong dependence on frequency due to dispersion*.

It is also worthy to note here that $\epsilon_r > 1$ the velocity of light in an isotropic dielectric medium.

$$v=\frac{c}{n}=\frac{c}{\sqrt{(\epsilon_r)}}$$

is always less then c as $\epsilon_r > 1$. ...(8)

It is therefore possible for high energy particles to have velocities in excess of v. When such particles pass through a dielectric a bluish light known as *Cerenkov-radition* is emitted due to the interaction of uniformly moving charged particles with the medium.

Further as the form of field vector **E** and **H** given by equation (C) suggests that

$$\nabla \rightarrow i\mathbf{k} \text{ and } \frac{\partial}{\partial t} \rightarrow -i\omega$$

So in terms of these operators eqn. (2) reduces to

$$\left.\begin{aligned} \mathbf{k}\cdot\mathbf{E}&=0 & \ldots(a)\\ \mathbf{k}\cdot\mathbf{H}&=0 & \ldots(b)\\ -\mathbf{k}\times\mathbf{H}&=\omega\epsilon\mathbf{E} & \ldots(c)\\ \mathbf{k}\times\mathbf{E}&=\omega\mu\mathbf{H} & \ldots(d)\end{aligned}\right\} \quad \ldots(9)^*$$

From this form of Maxwell's equation it is self evident that in a plane electromagnetic wave propagating through isotopic dielectric –

(*i*) *The vectors* **E**, **H** *and* **k** *are orthogonal i.e. the electromagnetic wave is transverse in nature and in it the electric and magnetic vectors are also mutually orthogonal.* This is because

according to 9 (a) **E** is ⊥ to **k**
according to 9 (b) **H** is ⊥ to **k**
according to 9 (c) **E** is ⊥ to both **k** and **H**
and according to 9 (d) **H** is ⊥ to both **k** and **E**

(*ii*) *The vectors* **E** *and* **H** *are in phase and their magnitudes are related to each other by the relation.*

$$\left|\frac{\mathbf{E}}{\mathbf{H}}\right|=\frac{E_0}{H_0}=\sqrt{\left(\frac{\mu_r}{\epsilon_r}\right)}\,Z_0=Z$$

where Z is called the impendence of the medium.

This is because according to equation 9 (d).

$$\mathbf{H}=\frac{k}{\omega\mu}(\mathbf{n}\times\mathbf{E})=\frac{1}{\mu v}(\mathbf{n}\times\mathbf{E}) \qquad \left(\text{as } k=\frac{\omega}{v}\right)$$

i.e. $$\mathbf{H}=\sqrt{\left(\frac{\epsilon}{\mu}\right)}(\mathbf{n}\times\mathbf{E})=\frac{(\mathbf{n}\times\mathbf{E})}{\mathbf{Z}} \qquad \left(\text{as } v=\frac{1}{\sqrt{(\mu\epsilon)}}\right)$$

with $$\mathbf{Z}=\sqrt{\left(\frac{\mu}{\epsilon}\right)}=\sqrt{\left(\frac{\mu_r}{\epsilon_r}\frac{\mu_0}{\epsilon_0}\right)}=\frac{\mu_r\mathbf{Z}_0}{n} \qquad \left(n=\sqrt{(\mu_r\epsilon_r)}\right)$$

or $$\left|\frac{\mathbf{E}}{\mathbf{H}}\right|=\left|\frac{\mathbf{E}_0}{\mathbf{H}_0}\right|=Z=\text{real quantity.} \qquad \ldots(10)$$

(*iii*) *The direction of flow of energy is the direction in which the wave propagates and the Poynting vector is* (n/μ_r) *times of the poynting vector if the same wave propagates, through free space.* It is because

$$\mathbf{S}=\mathbf{E}\times\mathbf{H}=\mathbf{E}\times\frac{(\mathbf{n}\times\mathbf{E})}{\mathbf{Z}}$$

i.e. $$\mathbf{S}=\frac{1}{Z}\left[(\mathbf{E}\cdot\mathbf{E})\,\mathbf{n}-(\mathbf{E}\cdot\mathbf{n})\mathbf{E}\right]$$

---

* In this case

$$\mathbf{k}=k\mathbf{n}=\frac{2\pi}{\lambda}\mathbf{n}=\frac{2\pi f}{v}\mathbf{n}=\frac{\omega}{v}\mathbf{n}$$

i.e. $\mathbf{S}=\frac{1}{Z}E^2\mathbf{n}$ [as $\mathbf{E}\cdot\mathbf{n}=0$ because $\mathbf{E}$ is $\perp$ to $\mathbf{n}$]

i.e. $\mathbf{S}=\frac{1}{Z}E^2\mathbf{n}=\frac{n}{\mu_r}[\epsilon_0 cE^2]\,\mathbf{n}$ $\left(\text{as } \frac{1}{Z}=\frac{n}{\mu_r}\frac{1}{Z_0}=\frac{n}{\mu_r}\epsilon_0 c\right)$

i.e. $<\mathbf{S}>=\frac{1}{Z}E^2_{rms}\,\mathbf{n}=\frac{n}{\mu_r}[\epsilon_0 cE^2_{rms}]\,\mathbf{n}$ ...(11)

(*iv*) *The electromagnetic energy density is equal to the magnetostatic energy density and the total energy density is* ϵ_r *times of the energy density if the same wave propagates through free space.*

This is because

$$\frac{u_e}{u_m}=\frac{\frac{1}{2}\epsilon E^2}{\frac{1}{2}\mu H^2}=\frac{\epsilon}{\mu}\left(\frac{E^2}{H^2}\right)=\frac{\epsilon}{\mu}(Z^2)=\frac{\epsilon}{\mu}\times\frac{\mu}{\epsilon}=1 \left(\text{as } |\mathbf{H}|=\frac{|\mathbf{E}|}{Z}\right)$$

and $u=u_e+u_m=\epsilon E^2=\epsilon_r(\epsilon_0 E^2)$

Further $\frac{<\mathbf{S}>}{<u>}=\frac{\frac{n}{\mu_r}[\epsilon_0 cE^2_{rms}]}{[\epsilon_r\epsilon_0 E^2_{rms}]}=\frac{nc}{\mu_r\epsilon_r}\,\mathbf{n}$

i.e. $<\mathbf{S}>=\frac{nc}{n^2}<u>\,\mathbf{n}$ [(as $n=\sqrt{(\mu_r\epsilon_r)}$)]

i.e. $<\mathbf{S}>=v<u>\,\mathbf{n}$ (as $c/n=v$)

i.e. electromagnetic energy is transmitted with the same velocity with which the fields do.

§ 5·3. Propagation of E.M.W. in Anisotropic Dielectric*.

In anistropic medium the relative permitivity is no longer a scalar and to deal with wave propagation we refer all fields to the principal axes so that

$$D_x=\epsilon_x\epsilon_0 E_x\,;\ D_x=E_y\epsilon_0 E_y \text{ and } D_z=\epsilon_z\epsilon_0 E_z \quad ...(1)$$

Further since the medium is non conducting *i.e.*

$$\mathbf{J}=0\,;\ \rho=0 \text{ and } \mu_r=1$$

So Maxwell's equation in an anisotropic dielectric medium reduce to

$$\left.\begin{aligned} &\text{div } \mathbf{D}=0 && (a)\\ &\text{div } \mathbf{H}=0 && (b)\\ &\text{curl } \mathbf{H}=\frac{\partial \mathbf{D}}{\partial t} && (c)\\ &\text{curl } \mathbf{E}=-\mu_0\frac{\partial \mathbf{H}}{\partial t} && (d)\end{aligned}\right\} \quad ...(2)$$

It is important to note that in this case though

$$\text{div } \mathbf{D}=0, \text{ div } \mathbf{E}\neq 0$$

because **D** in general is not in the direction of **E**.

Now consider a plane wave advancing with phase velocity v along the direction of wave normal **n** (*i.e.* wave vector **k**). Let it be

$$\begin{Bmatrix} \mathbf{E} \\ \mathbf{H} \end{Bmatrix} = \begin{Bmatrix} \mathbf{E}_0 \\ \mathbf{H}_0 \end{Bmatrix} e^{-i(\omega t - \mathbf{k}\cdot\mathbf{r})} \quad \text{...(3)}$$

So the operator ∇ and $\frac{\partial}{\partial t}$ will be

$$\nabla \to i\mathbf{k} \text{ and } \frac{\partial}{\partial t} \to (-i\omega).$$

And in terms of these operations equations (2) can be written as

$$\left.\begin{aligned} \mathbf{k}\cdot\mathbf{D}&=0 && \text{(a)} \\ \mathbf{k}\cdot\mathbf{H}&=0 && \text{(b)} \\ -\mathbf{k}\times\mathbf{H}&=\omega\mathbf{D} && \text{(c)} \\ \mathbf{k}\times\mathbf{E}&=\mu_0\omega\mathbf{H} && \text{(d)} \end{aligned}\right\} \quad \text{...(4)}$$

From this form of Maxwell's eqns. it is clear that

(*i*) *The E.M.W. are transverse it nature w.r.t* **D** *and* **H** (and not w.r.t. **E** and **H** as in a isotropic media). It is because accor-

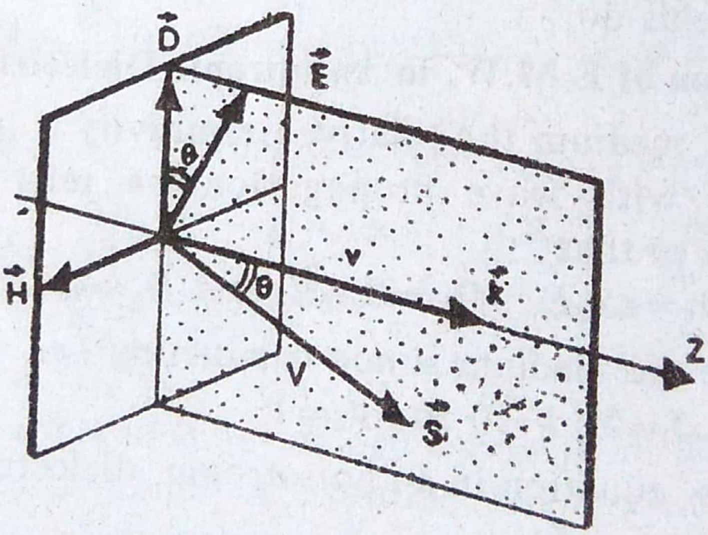


Fig. 5·3

ding to 4 (a) **k** is ⊥ to **D** while according to 4 (b) **k** is ⊥ to **H** *i.e.* **k** is ⊥ to both **H** and **D** as shown in fig. 5·5.

(*ii*) *The vectors* **D**, **H** *and* **k** *are orthogonal* because according o eqn. 4 (b) **k** is ⊥ to **H** while according to eqn. 4 (c) **D** is ⊥ to ›oth **k** and **H**.

(*iii*) *The vectors* **D**, **E** *and* **k** *are co-planer*. This is because ccording to equation 4 (c)

$$\mathbf{D}=-(\mathbf{k}\times\mathbf{H})/\omega \quad \text{...(5)}$$

while according to 4 (d)

$$\mathbf{H} = (\mathbf{k} \times \mathbf{E})/\mu_0\omega \qquad \text{...(6)}$$

So from equations (5) and (6)

$$\mathbf{D} = -[\mathbf{k} \times \mathbf{k} \times \mathbf{E})]/\mu_0\omega^2$$

i.e.

$$\mathbf{D} = -[\mathbf{k} \cdot \mathbf{E}) \mathbf{k} - k^2 \mathbf{E}]/\mu_0\omega^2 \qquad \text{...(7)}$$

(*iv*) *In an anisotropic medium energy is not propagated in general in the direction of wave propagation* (*i.e. the direction of* **k** *and* **S** *are not same*) *and the Poynting vector is coplaner with* **D, E** *and* **k**. This is because the Poynting vector is given by

$$\mathbf{S} = \mathbf{E} \times \mathbf{H}$$

i.e. **S** is normal to the plane of **E** and **H** and not to the plane of **D** and **H** (which is the direction of **k**).

§ 6·2. Reflection and refraction of E.M.W.

We now need to consider that what happens when plane electromagnetic waves which are travelling in one medium are incident upon an infinite plane surface separating this medium from another with different electromagnetic properties.

When an electric wave is travelling through space there is an exact balance between the electric and magnetic fields. Half of the energy of wave as a matter of fact is in electric field and half in the magnetic.* If the wave enters some different medium, there must be a new distribution of energy (due to the change in field vectors). Whether the new medium is a dielectric, a magnetic, a conducting or an ionised region, there will have to be a readjustment of energy relations as the wave reaches its surface. Since no energy can be added to the wave as it passes through the boundary surface, the only way that a new balance can be achieved is for some of the incident energy to be reflected. This is what actually happens. The transmitted energy constitutes the refracted wave and the reflected one the reflected wave.

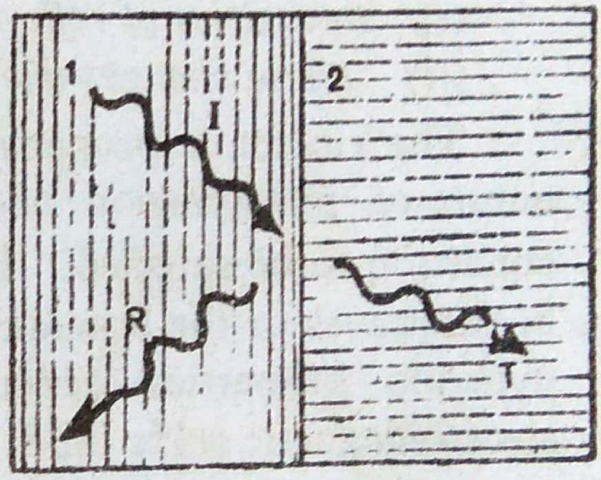


Fig. 6·3.

The reflection and refraction of light at a plane surface between two media of different dielectric properties is a familiar example of reflection and refraction of electromagnetic waves. The various aspects of the phenomenon divide themselves into two classes :

(A) Kinematic Properties :

Following are the kinematic properties of reflection and refraction :

(*i*) **Law of Frequency** : *The frequency of the wave remains unchanged by reflection or refraction.*

(*ii*) *The reflected and refracted waves are in the same plane as the incident wave and the normal to the boundary surface.*

(*iii*) **Law of Reflection** : *In case of reflection the angle of reflection is equal to the angle of incidence i e.*

$$\theta_i = \theta_R$$

(*iv*) **Snell's Law** ; *In case of refraction the ratio of the sin of the angle of refraction to the sin of angle of incidence is equal to the ratio of the refractive indices of the two media i.e.*

$$n_1 \sin \theta_i = n_2 \sin \theta_T$$

(B) Dynamic Properties :

These properties are concerned with the :—

(*i*) *Intensities of reflected and refracted waves.*

(*ii*) *Phase changes and polarisation of waves.*

The kinematic properties follow immediately from the wave nature of phenomenon and the fact that there are boundary condition to be satisfied. But they do not depend on the nature of the waves or the boundary conditions. On the other hand the dynamic properties depend entirely on the specific nature of electromagnetic fields and the boundary conditions. Kinematic properties are proved in example—1 while dynamic properties are discussed in details in forth-coming articles.

§ 6·7. Propagation of Electromagnetic Waves between parallel and perfectly conducting planes.*

We know that in a good conductor the field penetrates only to a small distance comparable with the skin depth, which reduces to zero in the limit of perfect conductivity. Thus, a perfect conductor behaves as a perfect reflector and hence allows no field to penetrate it. Consequently, the electric and magnetic fields are zero within the conducting plane and our boundary conditions becomes —

(i) *The tangential component of* **E** *at the surface of the conducting plane is zero*, since it must be zero within the plane and we know that the tangential component is continuous.

(ii) *The normal component of* **B** *at the surface of the conducting plane is zero*, since it must be zero within the plane and the normal component of **B** is continuous.

In the light of above boundary conditions consider the propagation of electromagnetic waves in the space between two parallel perfectly conducting planes separated by a distance d. Let the planes be in the x-z plane situated at $y=0$ at $y=d$ as shown in fig. 6·17. If an electromagnetic wave is incident on any plane it will be reflected from the wall whenever it strikes the wall and hence by a process of multiple reflections the wave will propagate in a direction parallel to planes. Since in fig. 6·17 the x and z directions are physically indistinguishable, no generality is lost if we consider only waves with wave vector $\mathbf{k}_0$ in y-z plane making an angle θ with y-axis. Such wave will strike on prefectly conducting surface at $y=d$ and will be reflected as waves whose propagation vector $\mathbf{k}'$ makes the angle θ with the minus y-axis (as $\theta_i=\theta_r$). When these waves are reflected again by the surface

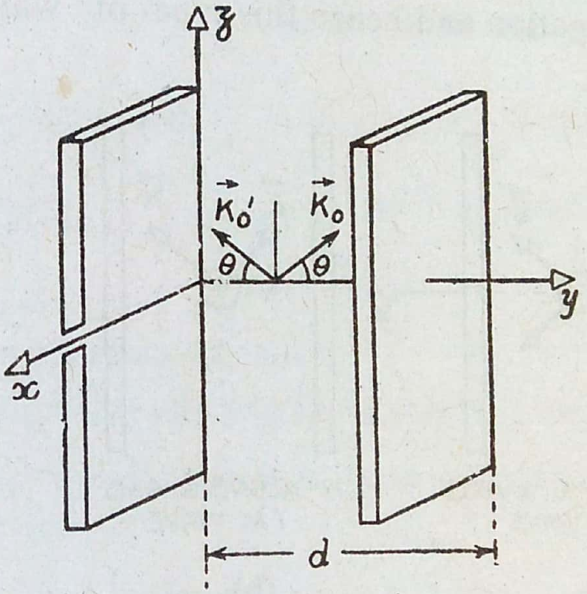


Fig. 6·17

at $y=0$. They become waves of initial type again and have propagation vector $\mathbf{k}_0$. Thus the propagation between two conducting planes can be described in terms of exponential factors.

$$e^{-i(\omega t-\mathbf{k}_0\cdot\mathbf{r})}=e^{-i\omega t}\,e^{ik_0(y\cos\theta+z\sin\theta)}$$

[for incident wave]

and

$$e^{-i(\omega t-\mathbf{k}'_0\cdot\mathbf{r})}=e^{-i\omega t}\,e^{ik_0(-y\cos\theta+z\sin\theta)}$$

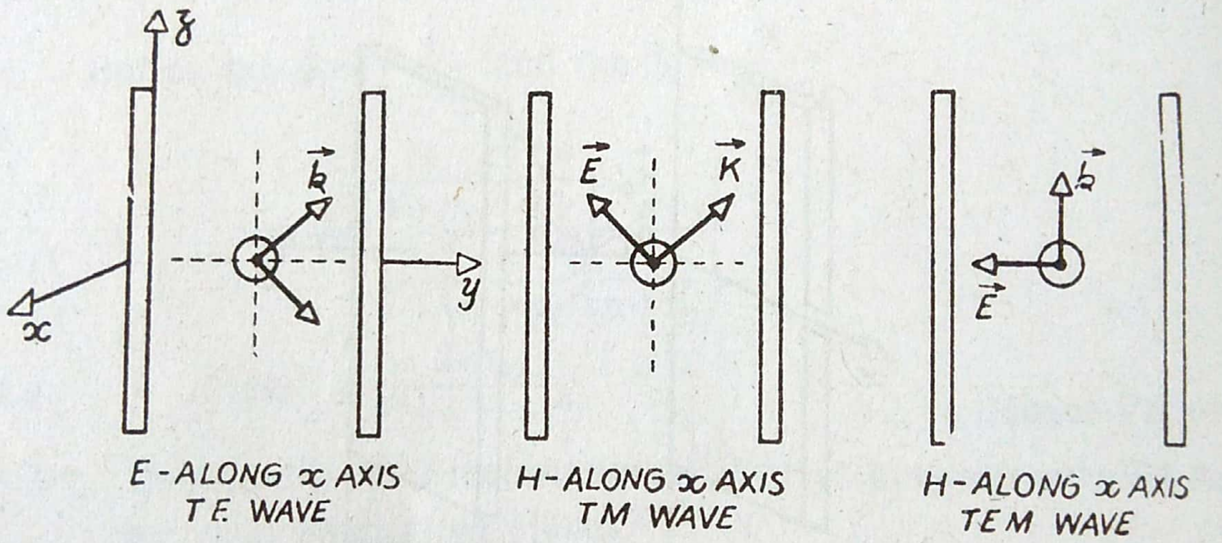
[for reflected wave]

where $|\mathbf{k}_0|=|\mathbf{k}_0'|=|2\pi/\lambda_0)=(\omega/c)$ and λ_0 is called the *free space wavelength*.

Now since in an E.M.W. **E**, **H** and **k** are orthogonal, in general there are three possible modes of propagation *viz.*

(A) T E Waves (or Mode) : This is characterised by an E.M.W. having an electric field **E** which is entirely in a plane transverse to the assumed axis of propagation (which is *z*-axis here). Only the magnetic field **H** has a component along the assumed axis of propagation and hence this type of wave is also known as *H-wave*. This is shown in fig. 6·18 (a) For *TE* wave it is possible to express all field components in terms of the axial magnetic field component H_z.

(B) TM wave (or Mode) : This is characterised by an E.M.W. having magnetic field **H** which is entirely in a plane transverse to the assumed axis of propagation (which is *z*-axis here). Only the electric field **E** has a component along the assumed axis of propagation and hence this type of wave is also known



(a) (b) (c)

Fig. 6·18

as *E*-wave. This is shown in fig. 6·18 (b). For TM wave it is possible to express all field components in terms of the axial electric field components E_z

(C) TEM wave (or Mode) : It is characterial by an E.M.W. having both the electric and magnetic fields entirely in a plane transverse to the assumed axis of propagation *i.e.* it is an electromagnetic wave in which the direction of wave motion is along the assumed axis of propagation. This is shown in fig. 6·18 (c) [In coaxial cables usually EMW are propagated in this mode].

As an example here *we shall discuss only TE wave.* The electric fiel ds for incident and reflected waves in *TE* case will be

$$\mathbf{E}_i = \mathbf{i}\, E_0 e^{-i\omega t}\, e^{ik_0 (y \cos\theta + z \sin\theta)}$$

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$$\mathbf{E}_R = \mathbf{i}\, E'_0\, e^{-i\omega t}\, e^{ik_0(-y\cos\theta + z\sin\theta)}$$

So by principle of super position the resultant electric field in the region between the planes in *TE* will be

$$\mathbf{E} = \mathbf{E}_i + \mathbf{E}_R = \mathbf{i}e^{-i\omega t}\,[E_0\, e^{ik_0(y\cos\theta + z\sin\theta)} + E_0'\, e^{ik_0(-y\cos\theta + z\sin\theta)}]$$

Now as by boundary condition that tangential component of *E* must vanishes at the surface of the conducting plane *i.e.*

$$\mathbf{E} = 0 \text{ at } y = 0$$

We get $\mathbf{i}\, e^{-i\omega t}\left[E_0\, e^{ik_0 z\sin\theta} + E_0' e^{ik_0 z\sin\theta}\right] = 0$

i.e. $E_0 + E'_0 = 0$ or $E'_0 = -E_0$

This condition simply indicates that *the reflection at the conducting plane involves a phase change of* π *and no change in amplitude*

And so $\mathbf{E} = \mathbf{i}\, E_0\, [e^{ik_0 y\cos\theta} - e^{-ik_0 y\cos\theta}]\, e^{-i(\omega t - k_0 z\sin\theta)}$

or $\mathbf{E} = \mathbf{i}\, E_0\, [2i\sin(k_0 y\cos\theta)]\, e^{-i(\omega t - k_0 z\sin\theta)}$

or $\mathbf{E} = \mathbf{i}\, 2iE_0\sin(k_c y)\, e^{-i(\omega t - k_g z)}$ ...(1)

with $k_c = k_0\cos\theta$ and $k_g = k_0\sin\theta$ ...(2)

This is the required result and from this it is clear that

(I) The resultant disturbance is propagating as a wave along *z*-axis with a wave length

$$\lambda_g = \frac{2\pi}{k_g} = \frac{2\pi}{k_0\sin\theta} \qquad (\text{as } k_g = k_0\sin\theta)$$

$$\lambda_g = \frac{\lambda_0}{\sin\theta} \qquad \left(\text{as } k_0 = \frac{2\pi}{\lambda_0}\right) \qquad ...(3)$$

λ_g is called the guide wavelength and is $> \lambda_0$ as $\sin\theta$ is < 1.

And so the velocity of the wave will be

$$v = \frac{\omega}{k_g} = \frac{\omega}{k_0\sin\theta} \qquad (\text{as } k_g = k_0\sin\theta)$$

$$v = \frac{c}{\sin\theta} \qquad \left(\text{as } k_0 = \frac{\omega}{c}\right) \qquad ...(4)$$

This velocity is called *phase velocity* and is greater than *c* as $\sin\theta < 1$. At first glance this appears to be in direct conflict with special theory of relatively according to which no signal can be i.e. λ_c is the largest wavelength or ω_c is the lowest freqeuncy which can be propagated. This is why λ_c is called cut off wave length and the given problem acts as high pass filter.

(IV) The velocity with which energy is propagated along the axis is called group velocity and is given by

$$v_z = \frac{\partial \omega}{\partial k_g}$$

But from equation (7)

$$k_0 = \sqrt{(k_c^2 + k_g^2)} \quad \text{or } \omega = c\sqrt{(k_c^2 + k_g^2)} \text{ [as } k_0 = (\omega/c)]$$

or $$v_z = \frac{\partial \omega}{\partial k_g} = c\tfrac{1}{2}(k_c^2 + k_g^2)^{-1/2} \times 2k_g$$

i.e. $$v_z = c\frac{k_g}{k_0} \text{ [as } k_0 = (k_c^2 + k_g)^{1/2}]$$

i.e. $$v_z = c \sin\theta \qquad \text{[as } k_g = k_0 \sin\theta] \qquad \ldots(9)$$

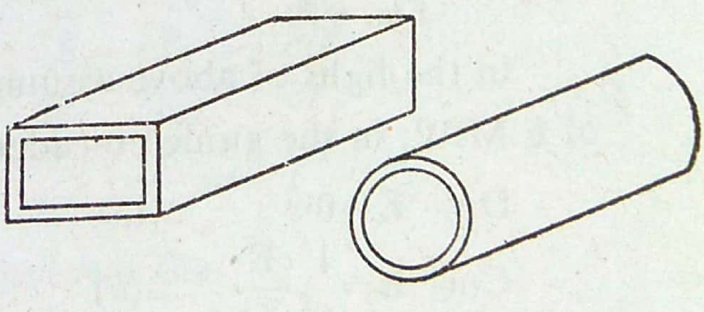
From expression (9) it is clear that the group velocity v_z with which energy is propagated along the axis is lesser than c as $\sin\theta < 1$. Further multiplying equation (4) and (9) we get

$$v\, v_z = c^2$$

a result which is expected but by no means apparent.

§ 6·8. Wave Guide (Rectangular)

A hollow conducting metallic tube of uniform cross section usually filled with air, for transmitting electromagnetic wave by sucessive reflections from inner walls of the tube is called a wave guide. If the cross section is rectangular it is called rectangular wave guide and if the cross section is circular it is called cylinderical wave guide.



Rectangular Wave Guide   Cylindrical Wave Guide

Fig. 6·19

It is used in U.H.F. and microwave region such as radar ($f > 3000$ MHz or $\lambda < 10$ cm) as an alternative to transmission lines as at these frequencies it can handle more power with lesser losses as compared to transmission lines.

Propagation of E.M.W. in wave guides can be considered as a phenomenon in which either *TE* or *TM* waves are reflected from wall to wall and hence pass down the wave guide in zig-zag fashion. [In transmission lines E.M.W. are usually propagated along the axis of cable as *TEM* waves.]

An essential feature of wave guide propagation is that it exhibits a cut off characteristic frequency similar to that of a high pass filter. At frequencies below the cut off value, the wave is simply reflected backwards and forwards across the wave guide and makes no forward progress. [Transmission line do not have any cut off frequency and are borad band devices.]

Theory :

For making the treatment simple we assume that

(i) *The walls of the guide are perfectly conducting* so that tangential component of **E** and normal component of **B** vanishes at its surface.

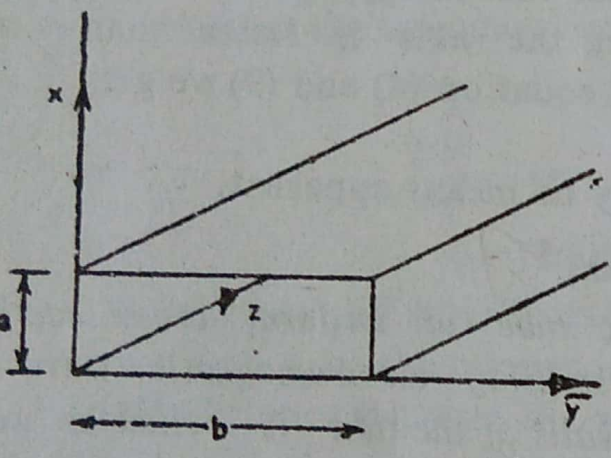


Fig. 6·20

(ii) *The interior of the wave guide is free space i.e.* vacuum so that

$\epsilon=\epsilon_0$, $\mu=\mu_0$,

$\sigma=0$ and $\rho=0$.

(iii) *The cross section of guide is uniform* and rectangular.

(iv) *The axis of wave guide is along z-direction* of right handed co-ordinate system.

In the light of above assumptions to discuss the propagation of E.M.W. in the guide consider Maxwell's eqns. in free space viz.

$$\left.\begin{array}{llll} \text{Div } \mathbf{E}=0 & \ldots(a) & \text{Div } \mathbf{B}=0 & \ldots(b) \\ \text{Curl } \mathbf{B}=\dfrac{1}{c^2}\dfrac{\partial \mathbf{E}}{\partial t} & \ldots(c) & \text{Curl } \mathbf{E}=-\dfrac{\partial \mathbf{B}}{\partial t} & \ldots(d) \end{array}\right] \ldots(1)$$

Taking the curl of eqn. 1 (d) we get

$$\nabla\times\nabla\times\mathbf{E}=-\frac{\partial}{\partial t}\text{Curl }(\mathbf{B})$$

or $$\nabla(\nabla\cdot\mathbf{E})-\nabla^2\mathbf{E}=-\frac{\partial}{\partial t}(\nabla\times\mathbf{B})$$

[as $\nabla\times\nabla\times\mathbf{V}=\nabla(\nabla\cdot\mathbf{V})-\nabla^2\mathbf{V}$]

which in the light of equations 1 (a) and (c) reduces to

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$$\nabla^2\mathbf{E}-\frac{1}{c^2}\frac{\partial^2\mathbf{E}}{\partial t^2}=0 \quad ...(2)$$

Similarly taking curl of eqn. 1 (c) and using 1 (b) and (d) we get

$$\nabla^2\mathbf{B}-\frac{1}{c^2}\frac{\partial^2\mathbf{B}}{\partial t^2}=0 \quad ...(3)$$

As equations (2) and (3) are of the form

$$\nabla^2\psi-\frac{1}{v^2}\frac{\partial^2\psi}{\partial t^2}=0$$

We come to the conclusion that *fields* **E** *and* **B** *are propagated as waves in the guide at a speed c.*

Now as the solution of above wave equation when it is propagating along z-axis is

$$\psi=\psi_0\, e^{-i\,(\omega t-kz)}$$

so if k_g is the *wave vector* or *propagation constant* along z-axis *i.e.* axis of guide the solution of equations (2) and (3) will be

$$\begin{Bmatrix}\mathbf{E}_{(r,\,t)}\\ \mathbf{B}_{(r,\,t)}\end{Bmatrix}=\begin{Bmatrix}\mathbf{E}_{(xy)}\\ \mathbf{B}_{(xy)}\end{Bmatrix}e^{-i\,(\omega t-k_g z)} \quad ...(4)$$

To determine how $\mathbf{E}_{(x,\,y)}$ and $\mathbf{B}_{(x,\,y)}$ vary with x and y we start with Maxwell's equations

$$\text{curl }\mathbf{B}=\frac{1}{c^2}\frac{\partial\mathbf{E}}{\partial t} \quad\text{and}\quad \text{curl }\mathbf{E}=-\frac{\partial\mathbf{B}}{\partial t}$$

which in terms of components can be written as

$$\begin{aligned}
&\frac{\partial B_z}{\partial y}-\frac{\partial B_y}{\partial z}=\frac{1}{c^2}\frac{\partial E_x}{\partial t} &&\frac{\partial E_z}{\partial y}-\frac{\partial E_y}{\partial z}=-\frac{\partial B_x}{\partial t}\\
&\frac{\partial B_x}{\partial z}-\frac{\partial B_z}{\partial x}=\frac{1}{c^2}\frac{\partial E_y}{\partial t} \quad\text{and} &&\frac{\partial E_x}{\partial z}-\frac{\partial E_z}{\partial x}=-\frac{\partial B_y}{\partial t} \quad ...(5)\\
&\frac{\partial B_y}{\partial x}-\frac{\partial B_x}{\partial y}=\frac{1}{c^2}\frac{\partial E_z}{\partial t} &&\frac{\partial E_y}{\partial x}-\frac{\partial E_x}{\partial y}=-\frac{\partial B_z}{\partial t}
\end{aligned}$$

But from equation (4) it is apparent that

$$\frac{\partial}{\partial z}\to ik_g \quad\text{and}\quad \frac{\partial}{\partial t}\to -i\omega\to -ik_0 c\left[\text{as } k_0=\frac{\omega}{c}\right]$$

So equation (5), reduces to

$$\left.\begin{aligned}
&\frac{\partial B_z}{\partial y}-ik_gB_y=-\frac{ik_0}{c}E_x \quad ...(i)\\
&ik_gB_x-\frac{\partial B_z}{\partial x}=-\frac{ik_0}{c}E_y \quad ...(ii)\\
&\frac{\partial B_y}{\partial x}-\frac{\partial B_x}{\partial y}=-\frac{ik_0}{c}E_z \quad ...(iii)
\end{aligned}\right\} \quad ...(6)$$

and
$$\left.\begin{aligned}\frac{\partial E_z}{\partial y}-ik_gE_y=ik_0cB_x \quad &\text{...(i)}\\ ik_gE_x-\frac{\partial E_z}{\partial x}=ik_0cB_y \quad &\text{...(ii)}\\ \frac{\partial E_y}{\partial x}-\frac{\partial E_x}{\partial y}=ik_0cB_z \quad &\text{...(iii)}\end{aligned}\right\} \quad \text{...(7)}$$

If we substitute the value of B_y from equation 7 (ii) in 6 (i), we get

$$\frac{\partial B_z}{\partial y}-ik_g\left(\frac{k_g}{k_0c}E_x-\frac{1}{ik_0c}\frac{\partial E_z}{\partial x}\right)=-\frac{ik_0}{c}E_x$$

i.e,
$$\frac{\partial B_z}{\partial y}+\frac{k_g}{k_0c}\frac{\partial E_z}{\partial x}=\left(\frac{ik_g^2}{ck_0}-\frac{ik_0}{c}\right)E_x$$

or
$$E_x=\frac{i}{[k_0^2-k_g^2]}\left[k_g\frac{\partial E_z}{\partial x}+k_0c\frac{\partial B_z}{\partial y}\right] \quad \text{...(A)}$$

And if we substitute the value of E_x from 6 (i) in 7 (ii), we get

$$B_y=\frac{i}{[k_0^2-k_g^2]}\left[\frac{k_0}{c}\frac{\partial E_z}{\partial x}+k_g\frac{\partial B_z}{\partial y}\right] \quad \text{...(B)}$$

Similarly eliminating B_x and E_y in turn, from 6 (ii) and 7 (i) we get

$$E_y=\frac{i}{[k_0^2-k_g^2]}\left[k_g\frac{\partial E_z}{\partial y}-k_0c\frac{\partial B_z}{\partial x}\right] \quad \text{...(C)}$$

and
$$B_x=\frac{i}{[k_0^2-k_g^2]}\left[-\frac{k_0}{c}\frac{\partial E_z}{\partial y}+k_g\frac{\partial B_z}{\partial x}\right] \quad \text{...(D)}$$

Examination of equations (A), (B), (C) and (D) shows that :

(*i*) If a electromagnetic wave is to be propagated along z axis then as $E_z=B_z=0$, the equations (A), (B), (C) and (D) vanish. Therefore there is no non-zero component of **E** or **B**. This in turn implies that *T E M waves cannot be propagated along the axis of a wave guide.*

(*ii*) If we set $k_0^2-k_g^2=k_c^2$ *i.e.*, $k_g^2=k_0^2-k_c^2$ we find that for $k_0<k_c$, k_g is imaginary which in turn results in the attenuation of **E** and **H** given by eqn. (4). This in turn means that we cannot propagate waves for which $k_0<k_c$ (or $f_0<f_c$) *i.e. a guide acts as a short of high pass filter in the sense that one can propagate waves along it whose frequencies are greater than cut off frequency.*

The equation

$$k_0^2 - k_g^2 = k_c^2 \quad i.e. \quad k_0^2 = k_g^2 + k_c^2$$

i.e. $$\frac{1}{\lambda_0^2} = \frac{1}{\lambda_g^2} + \frac{1}{\lambda_c^2} \qquad (\text{as} \quad k = 2\pi/\lambda)$$

is called guide equation. It relates the free space wave length λ_0 to cut off wave length λ_c and guide wave length λ_g. According to it

$$\lambda_g = \frac{\lambda_0}{\sqrt{\left\{1 - \left(\frac{\lambda_0}{\lambda_c}\right)^2\right\}}} \qquad \text{...(E)}$$

(iii) The phase velocity in the guide will be given by

$$v = \frac{\omega}{k_g} = c\frac{k_0}{k_g} \qquad \left[\text{as } k_0 = \frac{\omega}{c}\right]$$

or $$v = \frac{c\,k_0}{\sqrt{(k_0^2 - k_c^2}} = \frac{c}{\sqrt{[1 - (k_c/k_0)^2]}} \qquad [\text{as } k_g^2 = k_0^2 - k_c^2]$$

or $$v = \frac{c}{\sqrt{[1 - (\lambda_0/\lambda_c)^2]}} \qquad \left[\text{as} \quad k = \frac{2\pi}{\lambda}\right] \qquad \text{...(F)}$$

This result clearly shows that $v > c$ and for $\lambda_0 = \lambda_c$

$$v = \infty$$

i.e. phase velocity becomes infinite exactly at cut off.

(iv) As

$$k_0^2 = k_g^2 + k_c^2 \quad i.e. \quad \omega = c\,(k_g^2 + k_c^2)^{1/2} \qquad [\text{as} \quad k_0 = \omega/c]$$

The *group velocity* with which energy is propagated along the axis of the guide will be given by

$$v_z = \frac{\partial \omega}{\partial k_g} = \frac{\partial}{\partial k_g}\,[c\,(k_g^2 + k_c^2)^{1/2}]$$

i.e. $$v_z = c\,\tfrac{1}{2}\,(k_g^2 + k_c^2)^{-1/2}\,2\,k_g$$

i.e. $$v_z = c\,\frac{k_g}{k_0} = c\,\sqrt{[1 - (k_c/k_0)^2]} \qquad [\text{as } k_0^2 = k_g^2 + k_c^2]$$

or $$v_z = c\,\sqrt{[1 - (\lambda_0/\lambda_c)^2]} \qquad [\text{as } k = 2\pi/\lambda] \qquad \text{...(G)}$$

From this equation it is clear that $v_z < c$ and $vv_z = c^2$.

(v) Transverse components of the fields *i.e.* E_x, E_y, B_x and B_y of a guided wave are independent of one another and depend only on the values of the longitudinal components E_z or B_z of the guided wave, so it is possible to express them in terms of a linear superposition of two independent solutions, one for which $E_z = 0$ (*TE*) and one for which $B_z = 0$ (*TM*). Transverse electric waves are sometimes known as H wave and transverse magnetic waves as E-waves.

T E Waves :

For these as $E_z=0$ and $k_c^2=k_0^2-k_g^2$, equations (A), (B), (C) and (D) reduce to

$$\left.\begin{array}{ll} E_x=\dfrac{ik_0c}{k_c^2}\dfrac{\partial B_z}{\partial y} \quad \ldots(i) & B_x=\dfrac{ik_g}{k_c^2}\dfrac{\partial B_z}{\partial x} \quad \ldots(iii) \\ E_y=-\dfrac{ik_0c}{k_c^2}\dfrac{\partial B_z}{\partial x} \quad \ldots(ii) & B_y=\dfrac{ik_g}{k_c^2}\dfrac{\partial B_z}{\partial y} \quad \ldots(iv) \end{array}\right\} \quad \ldots(8)$$

Thus *in TE mode all the transverse components of* **E** *and* **B** *can be expressed in terms of longitudinal component of magnetic vector* B_z. In order to compute B_z we use equation (3) *i.e.*

$$\nabla^2\mathbf{B}-\frac{1}{c^2}\frac{\partial^2\mathbf{B}}{\partial t^2}=0$$

which in the light of eqn. (4) *i.e.*

$$B_{(r,\, t)}=B_{(x,\, y)}\, e^{-i(\omega t-k_g z)}$$

i.e. with $\dfrac{\partial}{\partial z}\to(i\,k_g)$ and $\dfrac{\partial}{\partial t}\to(-i\omega)$ becomes

$$\frac{\partial^2\mathbf{B}}{\partial x^2}+\frac{\partial^2\mathbf{B}}{\partial y^2}+(i\,k_g)^2\,\mathbf{B}-\frac{1}{c^2}\,(-i\omega)^2\,\mathbf{B}=0$$

i.e.

$$\frac{\partial^2\mathbf{B}}{\partial y^2}+\frac{\partial^2\mathbf{B}}{\partial y^2}+\left(\frac{\omega^2}{c^2}-k_g^2\right)\mathbf{B}=0$$

i.e.

$$\frac{\partial^2\mathbf{B}}{\partial x^2}+\frac{\partial^2\mathbf{B}}{\partial y^2}+k_c^2\mathbf{B}=0 \quad [\text{as } k_0=\omega/c \text{ and } k_0^2=k_g^2+k_c^2]$$

As above equation is a vector equation so must be satisfied for each component of **B**. For z-component of **B** it reduces to

$$\frac{\partial^2 B_z}{\partial x^2}+\frac{\partial^2 B_z}{\partial y^2}+k_c^2\, B_z=0 \quad \ldots(a)$$

with boundary condition $|\,\partial B_z/\partial n|_S=0$ *i.e.*

$$\frac{\partial B_z}{\partial x}=0 \quad \text{at} \quad x=0 \quad \text{and} \quad x=a.$$

and

$$\frac{\partial B_z}{\partial y}=0 \quad \text{at} \quad y=0 \quad \text{and} \quad y=b.$$

Such a solution is

$$B_z=B_0\cos\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right) \quad \ldots(H)$$

with

$$k_c^2=\pi^2\left[\frac{m^2}{a^2}+\frac{n^2}{b^2}\right] \quad \ldots(I)$$

where the indices m and n specify the mode. The cut of wavelength is given by

$$\left(\frac{1}{\lambda_c}\right)_{mn}=\frac{1}{2}\left[\frac{m^2}{a^2}+\frac{n^2}{b^2}\right]^{1/2} \qquad \left(\text{as } k=\frac{2\pi}{\lambda}\right)$$

i.e. $$(\lambda_c)_{mn}=\frac{2}{\sqrt{\left[\frac{m}{a}\right]^2+\left[\frac{n}{a}\right]^2}} \qquad \ldots(J)$$

while cut off frequency will be

$$\omega_{mn}=\pi c\left[\frac{m^2}{a^2}+\frac{n^2}{b^2}\right]^{1/2} \qquad \left[\text{as } \omega=\frac{2\pi c}{\lambda}\right] \qquad \ldots(K)$$

The modes corresponding to m and n are designated as TE_{mn} mode. The case $m=n=0$ gives a static field which do not represent a wave propagtion. So TE_{00} mode does not exist. If $a < b$ the lowest cut off frequency result for $m=0$ end $n=1$ i.e.

$$(\omega)_{01}=\frac{\pi c}{b} \text{ or } k_c=\frac{\pi}{b}$$

The TE_{01} mode is called the principal or dominant mode.

The fields in the guide for TE mode will be obtained from eqn. (8) by substituting the solution for B_z, which is

$$B_{z(r,\ t)}=B_{z(x,\ y)}\ e^{-i(\omega t-k_g z)}$$

i.e. $$B_{z(r,\ t)}=B_0 \cos\left[\frac{m\pi x}{a}\right]\cos\left[\frac{n\pi y}{b}\right] e^{-i(\omega t-k_g z)}$$

Thus we have

$$E_x=-\frac{in\pi ck_0}{k_c^2 b}B_0\cos\left[\frac{m\pi x}{a}\right]\sin\frac{n\pi y}{b}\ e^{-i(\omega t-k_g z)}$$

$$E_y=\frac{im\pi ck_0}{k_c^2 a}B_0\sin\frac{m\pi x}{a}\cos\frac{n\pi y}{b}\ e^{-i(\omega t-k_g z)}$$

$$B_x=-\frac{im\pi k_g}{k_c^2 a}B_0\sin\frac{m\pi x}{a}\cos\frac{n\pi y}{b}\ e^{-i(\omega t-k_g z)}$$

$$B_y=-\frac{in\pi k_g}{k_c^2 b}B_0\cos\left[\frac{m\pi x}{a}\right]\sin\left[\frac{n\pi y}{b}\right] e^{-i(\omega t-k_g z)}$$

TM Waves :

For there as $B_z=0$ and as $k_0^2-k_g^2=k_c^2$ equations A, B, C and D reduce to

$$\left.\begin{aligned} E_x&=\frac{ik_g}{k_c^2}\frac{\partial E_z}{\partial x} \quad \ldots(i) \qquad & B_x&=-\frac{ik_0}{ck_c^2}\frac{\partial E_z}{\partial y} \quad \ldots(iii)\\ E_y&=\frac{ik_g}{k_c^2}\frac{\partial E_z}{\partial y} \quad \ldots(ii) \qquad & B_y&=\frac{ik_0}{ck_c^2}\frac{\partial E_z}{\partial x} \quad \ldots(iv)\end{aligned}\right\} \ldots(10)$$

Thus in *TM mode, all the transverse components of* **E** and **B** *can be expressed in terms of longitudinal component of the electric field* E_z. E_z may be computed by using the eqn.(4) for z-component

i.e. $E_{z(r,t)}=E_{z(x,y)}\, e^{-i(\omega t-k_g z)}$

so that it satisfies eqn. (2) (for z component) *i.e.*

$$\nabla^2 E_z-\frac{1}{c^2}\frac{\partial^2 E_z}{\partial t^2}=0$$

i.e. $$\frac{\partial^2 E_z}{\partial x^2}+\frac{\partial^2 E_z}{\partial y^2}+(ik_g)^2\, E_z-\frac{(-i\omega)^2}{c^2}E_z=0$$

or $$\frac{\partial^2 E_z}{\partial x^2}+\frac{\partial^2 E_z}{\partial y^2}+k_c^2\, E_z=0 \qquad \left(\text{as } \frac{\omega^2}{c^2}-k^2_g=k^2_0-k^2_g=k^2_c\right)$$

with boundary conditions $E_z|s=0$ *i.e.*

$E_z=0$ at $x=0$ and $x=a$

and $E_z=0$ at $y=0$ and $y=b$

Such a solution is

$$E_z=E_0 \sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right) \qquad \text{...(L)}$$

with $$k_c^2=\pi^2\left[\frac{m^2}{a^2}+\frac{n^2}{b^2}\right] \qquad \text{...(M)}$$

which corresponds to a cut off wavelength

$$\left(\frac{1}{\lambda_c}\right)_{mn}=\frac{1}{2}\left[\frac{m^2}{a^2}+\frac{n^2}{b^2}\right]^{1/2} \qquad [\text{as } k=2\pi/\lambda]$$

and a cut off frequency

$$\omega_{mn}=\pi c\left[\frac{m^2}{a^2}+\frac{n^2}{b^2}\right]^{1/2} \qquad [\text{as } k=\omega/c]$$

Comparing eqn. (M) with (I) we find that in a rectangular waveguide *TE* and *TM* modes have the same set of cut off frequencies. However the cases $m=0$ and $n=1$ or $m=1$ and $n=0$ which were dominant in *TE* mode do not exist for *TM* wave because the field vanishes or m or $n=0$.

The value of the fields for *TM* mode will be obtained from eqn. (10) by substituting the solution for E_z, which is

$$E_{z(r,t)}=E_{z(x,y)}\, e^{-i(\omega t-k_g z)}$$

i.e. $$E_{z(r,t)}=E_0 \sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)e^{-i(\omega t-k_g z)}$$

Thus we have

$$E_x = \frac{im\pi k_g}{k_c^2 a} E_0 \cos\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) e^{-i(\omega t - k_g z)}$$

$$E_y = \frac{in\pi k_g}{k_c^2 b} E_0 \sin\left(\frac{m\pi x}{a}\right) \cos\left(\frac{n\pi y}{b}\right) e^{-i(\omega t - k_g z)}$$

$$B. = -\frac{in\pi k_0}{bck_c^2} E_0 \sin\left(\frac{m\pi x}{a}\right) \cos\left(\frac{n\pi y}{b}\right) e^{-i(\omega t - k_g z)}$$

and $$B_y = \frac{im\pi k_0}{ack_c^2} \cos\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) e^{-i(\omega t - k_g z)}$$

§ 6·9. Cavity Resonator :

A cavity resonator is an energy storing device, similar to a resonant circuit at low frequencies. Virtually any metallic enclosure, when properly excited will function as a cavity resonator or electromagnetic cavity. For certain specific frequencies electromagnetic field oscillations can be sustained within the enclosure with a very small expenditure of power loss in the cavity walls. Cavity resonators have the advantages of reasonable dimensions, simplicity, remarkable high Q and very high impedence.

A cavity resonator is usually superior to conventional L-C circuit by a factor of about 20. *i.e.* the fraction of the stored energy dissipated per cycle in a cavity resonator is about (1/20) the fraction dissipated per cycle in an L-C circuit. An additional advantage is that cavity resonators of practical size have resonant frequencies which range upward from a few hundred maga cycles just the region where it is almost impossible to construct a L-C circuit.

Cavity resonators are used as resonant circuit in high frequency tubes such as Klystron, for band pass filters and for wave meters to measure frequency.

Theory : Consider a rectangular cavity as shown in fig 6·21, with the assumptions.

(i) The walls are perfectly conducting.

(ii) The interior of cavity is free-space.

(iii) The cavity is rectangular.

(iv) The wave is advancing along z-axis.

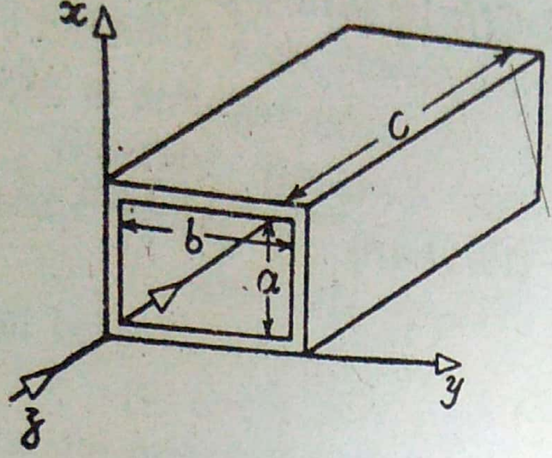

Fig 6·21

As there are two possible modes of propagation *TE* or *TM* in the cavity, we shall deal them separately.

Case I. TE Mode. In this mode $E_z=0$ so that the electric field propagating along +ive z-direction may be expressed as

$$\mathbf{E}_{i(r,t)}=\mathbf{E}_{(x,y)}\, e^{-i\,(\omega t-k_g z)}$$

The electric field of reflected wave propagating along z-axis will therefore be

$$\mathbf{E}_{r(r,t)}=\mathbf{E}_{(x,y)}\, e^{-i\,(\omega t+k_g z)}$$

So the resultant electric field

$$\mathbf{E}_{(r,t)}=\mathbf{E}'_{x,y)}e^{-i\,(\omega t-k_g z)}+\mathbf{E}'_{(x,y)}\, e^{-i\,(\omega t+k_g z)}$$

The boundary condition that tangential component of E is zero at the boundary $z=0$ (for all values of x, y and t) requires

$$\mathbf{E}+\mathbf{E}'=0 \quad i{\cdot}e. \quad \mathbf{E}'=-\mathbf{E}$$

so that

$$\mathbf{E}_{(r,t)}=E_{x,y)}\, e^{-i\omega t}\,[e^{ik_g z}-e^{-ik_g z}]$$

i.e. $$\mathbf{E}_{(r,t)}=2i\,\mathbf{E}_{(x,y)}\sin k_g z\, e^{-i\omega t}$$

the boundary condition $E_{(r,t)}=0$ at $z=d$ implies that

$$\sin k_g d=0 \qquad \text{or} \qquad k_g d=p\pi$$

i.e. $$k_g=(p\pi/d) \qquad \text{...(1)}$$

so that

$$E_{(r,t)}=2i\;\mathbf{E}_{(x,y)}\;\sin\left(\frac{p\pi z}{d}\right)\; e^{-i\omega t}$$

which in terms of components will be

$$\left.\begin{aligned} E_{x(r,t)}&=2i\;E_{x(x,y)}\;\sin\left(\frac{p\pi z}{d}\right)e^{-i\omega t} \\ \text{and}\quad E_{y(r,t)}&=2i\;E_{y(x,y)}\;\sin\left(\frac{p\pi z}{d}\right)e^{-i\omega t} \end{aligned}\right\} \qquad \text{...(2)}$$

In order to calculate E_x and E_y we write Maxwell's equations curl $\mathbf{B}=(1/c^2)\,(\partial\mathbf{E}/\partial t)$ and curl $\mathbf{E}=-(\partial\mathbf{B}/\partial t)$ in terms of components and solve to get

$$E_x=\frac{ik_0 c}{k_c^2}\frac{\partial B_z}{\partial y} \quad \text{and} \quad E_y=-\frac{ik_0 c}{k_c^2}\frac{\partial B_z{}^*}{\partial x} \qquad \text{...(3)}$$

Now B_z will be obtained by solving the z-component of wave equation for **B** *i.e.*

$$\left(\nabla^2-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)B_z=0$$

i.e. $$\frac{\partial^2 B_z}{\partial x^2}+\frac{\partial^2 B_z}{\partial y^2}+\frac{\partial^2 B_z}{\partial z^2}-\frac{1}{c^2}\frac{\partial^2 B_z}{\partial t^2}=0$$

But as for a wave propagating along z-axis

$$(\partial/\partial z)\to ik_g \qquad \text{and} \qquad (\partial/\partial t)\to(-i\omega)$$

so $$\frac{\partial^2 B_z}{\partial x^2}+\frac{\partial^2 B_z}{\partial y^2}+\left[\frac{\omega^2}{c^2}-k_g^2\right]B_z=0$$

or $$\frac{\partial^2 B_z}{\partial x^2}+\frac{\partial^2 B_z}{\partial y^2}+k_c^2 B_z=0$$

[with $\omega/c=k_0$ and $k_0^2-k_g^2=k_c^2$] ...(4)

The boundary conditions $|\partial B/\partial n|_S=0$ *i.e.*

$$\frac{\partial B_z}{\partial x}=0 \qquad \text{at} \quad x=0 \quad \text{and} \quad x=a$$

and $$\frac{\partial B_z}{\partial y}=0 \qquad \text{at} \quad y=0 \quad \text{and} \quad y=b$$

when applied to equation (4) yields

$$B_z=B_0\cos\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right) \qquad ...(5)$$

with $$k_c^2=\pi^2\left[\frac{m^2}{a^2}+\frac{n^2}{b^2}\right] \qquad ...(6)$$

So substituting the value of B_z from (5) in (3) we get

$$E_{x(x,y)}=-\frac{ik_0c}{k_c^2}\left(\frac{n\pi}{b}\right)B_0\cos\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)$$

$$E_{y(x,y)}=\frac{ik_0c}{k_c^2}\left(\frac{m\pi}{a}\right)B_0\sin\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)$$

The above equation when substituted in eqns. (2) results

$$E_{x(r,t}=\frac{2k_0c}{k_c^2}\left(\frac{n\pi}{b}\right)B_0\cos\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)\text{in}\left(\frac{p\pi z}{d}\right)e^{-i\omega t} \qquad ...(A)$$

$$E_{y(r,t)}=-\frac{2k_0c}{k_c^2}\left(\frac{m\pi}{a}\right)B_0\sin\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)\sin\left(\frac{p\pi z}{d}\right)e^{-i\omega t} \qquad (B)$$

with $E_{z(r,t)}=0$ as wave is *TE* ..(C)

The components of magnetic field in this case will be obtained by using the Maxwell's curl $\mathbf{E}=(-\partial\mathbf{B}/\partial t)$ in terms of components i.e.,

$$\left.\begin{array}{lll} \frac{\partial E_z}{\partial y}-\frac{\partial E_y}{\partial z}=-\frac{\partial B_x}{\partial t} & & \frac{-\partial E_y}{\partial z}=i\omega\ B_x \\ \frac{\partial E_x}{\partial z}-\frac{\partial E_z}{\partial x}=-\frac{\partial B_y}{\partial t} & \text{or} & \frac{\partial E_x}{\partial z}=i\omega\ B_y \\ \text{and}\quad \frac{\partial E_y}{\partial x}-\frac{\partial E_x}{\partial y}=-\frac{\partial B_z}{\partial t} & & \frac{\partial E_y}{\partial x}-\frac{\partial E_x}{\partial y}=i\omega\ B_z \end{array}\right\} \quad ...(7)$$

[as $E_z=0$ and $(\partial/\partial t)\rightarrow -i\omega$]

So eqn (7) in the light of (A) and (B) gives

$$B_x=-\frac{1}{i\omega}\frac{\partial E_y}{\partial z}=-\frac{2i}{k_c^2}\left(\frac{m\pi}{a}\right)\left(\frac{p\pi}{d}\right) B_0 \sin\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)\cos\left(\frac{p\pi z}{d}\right)e^{-i\omega t} \quad ...(D)$$

$$B_y=\frac{1}{i\omega}\frac{\partial E_x}{\partial z}=-\frac{2i}{k_c^2}\left(\frac{n\pi}{b}\right)\left(\frac{p\pi}{d}\right) B_0 \cos\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)\cos\left(\frac{p\pi z}{d}\right)e^{-i\omega t} \quad ...(E)$$

and $$B_z=\frac{1}{i\omega}\left[\frac{\partial E_y}{\partial x}-\frac{\partial E_x}{\partial y}\right]=\frac{2i}{k_c^2}B_0\left[\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2\right]\cos\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)\sin\left(\frac{p\pi z}{d}\right)e^{-i\omega t}$$

which in the light of condition given by eqn. (6) becomes

$$B_z=2iB_0\cos\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)\sin\left(\frac{p\pi z}{d}\right)e^{-i\omega t} \quad ...(F)$$

Discussion :

(1) Equation (A) to (F) express components of fields in the resonant cavity for TE mode. From these it is evident that TE_{000}, TE_{001}, TE_{010} or TE_{100} modes do not exist in the cavity. The physically possible lowest modes are TE_{101}, TE_{011}, or TE_{110}.

(2) To calcuate the resonant frequency of the cavity, we use the fact that in equation (4) k_c is defined as

$$k_0^2=k_g^2+k_c^2.$$

Above equation in the light of eqns. (1) and (6) reduce to

$$k_c^2=\left(\frac{\pi p}{d}\right)^2+\left[\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2\right]$$

or $$\omega=\pi c\left[\frac{m^2}{a^2}+\frac{n^2}{b^2}+\frac{p^2}{d^2}\right]^{1/2} \quad [\text{as } k_0=\omega/c] \quad ...(G)$$

Case II. TM Mode : In this mode $B_z=0$ and E_z can be computed by solving the z component of wave equation for **E** *i.e.*

$$\left(\nabla^2-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)E=0.$$

Proceeding as, in Case I we get

$$\frac{\partial^2 E_z}{\partial x^2}+\frac{\partial^2 E_z}{\partial y^2}+k_c^2\, E_z=0$$

subjected to the boundary conditions $|\, E_t|_S=0$ *i.e.*

$$E_z=0 \quad \text{at} \quad x=0 \quad \text{and} \quad x=a$$
$$E_z=0 \quad \text{at} \quad y=0 \quad \text{and} \quad y=b$$

Such a solution will be

$$E_{z(x,\,y)}=E_0 \sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right) \qquad \ldots(8)$$

with
$$k_c^2=\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2 \qquad \ldots(9)$$

In order to calculate $E_{x\,(x,\,y)}$ and $E_{y\,(x,\,y)}$ we write Maxwell equations curl $\mathbf{B}=(1/c^2)\,(\partial\mathbf{E}/\partial t)$ and curl $\mathbf{E}=(-\partial\mathbf{B}/\partial t)$ in terms of components and solve to get

$$E_x=\frac{i\,k_g}{k_c^2}\frac{\partial E_x}{\partial x} \quad \text{and} \quad E_y=\frac{i\,k_g}{k_c^2}\frac{\partial E_z'}{\partial y} \qquad \ldots(10)$$

Substituting the value of E_z from eqn. (9) in (10) we get

$$E_{x\,(x,\,y)}=\frac{i\,k_g}{k_c^2}\left(\frac{m\pi}{a}\right)E_0\cos\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)$$

$$E_{y\,(x,\,y)}=\frac{i\,k_g}{k_c^2}\left(\frac{n\pi}{a}\right)E_0\sin\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)$$

which in the light of equation (2) gives

$$E_{x\,(r,\,t)}=-\frac{2\,k_g}{k_c^2}\left(\frac{m\pi}{a}\right)E_0\cos\left(\frac{m\pi x}{a}\right)\sin\left(\frac{\pi\pi y}{b}\right)\sin\left(\frac{p\pi z}{d}\right)e^{-i\omega t} \qquad \ldots(H)$$

$$E_{y\,(r,\,t)}=-\frac{2k_g}{k_c^2}\left(\frac{n\pi}{b}\right)E_0\sin\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)\sin\left(\frac{p\pi z}{d}\right)e^{-i\omega t} \qquad \ldots(I)$$

The components $B_{x(r,\,t)}$, $B_{y(r,\,t)}$ and $E_{z}(r,\,t)$ will be obtained by using Maxwell equation (curl $\mathbf{B}=(1/c^2)\,(\partial\mathbf{E}/\partial t)$ in terms of components *i.e.*

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$$\left.\begin{aligned} \frac{\partial B_z}{\partial y}-\frac{\partial B_y}{\partial z}&=\frac{1}{c^2}\frac{\partial E_x}{\partial t} & \qquad \frac{\partial B_y}{\partial z}&=\frac{i\omega}{c^2}E_x \\ \frac{\partial B_x}{\partial z}-\frac{\partial B_z}{\partial x}&=\frac{1}{c^2}\frac{\partial E_y}{\partial t} \quad \text{or} & \qquad \frac{\partial B_x}{\partial z}&=-\frac{i\omega}{c^2}E_y \\ \frac{\partial B_y}{\partial x}-\frac{\partial B_x}{\partial y}&=\frac{1}{c^2}\frac{\partial E_z}{\partial t} & \qquad \frac{\partial B_y}{\partial x}-\frac{\partial B_x}{\partial x}&=-\frac{i\omega}{c^2}E_z \end{aligned}\right\} \quad \ldots(11)$$

[as $B_z=0$ and $(\partial/\partial t)\to -i\omega$]

So equation (11) in the light of (H) and (I) and with $\pi p/d=k_g$ yields

$$B_x=\frac{2i\omega}{k_c^2}\frac{E_0}{c^2}\left(\frac{n\pi}{b}\right)\sin\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)\cos\left(\frac{p\pi z}{d}\right)e^{-i\omega t} \quad \ldots(J)$$

$$B_y=-\frac{2i\omega}{k_c^2}\frac{E_0}{c^2}\left(\frac{m\pi}{a}\right)\cos\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)\cos\left(\frac{p\pi z}{d}\right)e^{-i\omega t} \quad \ldots(K)$$

and $$E_z=2E_0\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)\cos\left(\frac{p\pi z}{d}\right)e^{-i\omega t} \quad \ldots(L)$$

Equations (H) to (L) represents the components of field vectors and from these it is evident that modes TM_{000}, TM_{001}, TM_{100}, TM_{010}, TM_{011}, TM_{101} do not exist. The physically possible lowest mode is TM_{110}.

The resonant frequency will be given by the condition

$$k_0^2=k_g^2+k_c^2$$

i.e. $$\omega=\pi c\left[\left(\frac{m}{a}\right)^2+\left(\frac{n}{b}\right)^2+\left(\frac{p}{d}\right)^2\right]^{\frac{1}{2}}$$