Unit - I

Definiton :

A differential equation in an equation in which differential co-efficient occur.

Differential equation are of two types :

* Ordinary differential equation
* Partial differential equation

ODE :

An ordinary differential equation is one in which a single independent variable exters, either explicity or implicity.

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Example :

$$\frac{dy}{dx} = 2 \sin x \quad ; \quad \frac{d^2y}{dt^2} + m^2y = 0$$

$$y = 2^{x} ; \frac{dy}{dz} = 0 \qquad x = x(t) ; y = y(t)$$

PDE :

A partial differential equation one in which atleast two independent variable enter and the partial differential co-efficient occcuring in them have reference to any one of these variables.

Example :

$$\frac{x \partial z}{\partial x} + y \frac{\partial z}{\partial y} = z$$

Order :

The order of differential equation is the order of the highest derivative occuring in it.

Example : $\left(\frac{dy}{dx}\right)^2 + y^2 = A^2$

Here, **Order = 1**

Degree = 2

are cleared of radical and fractions.

Example :

Find the order and degree of

$$\frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{d^2y/dx^2} = a \rightarrow (1)$$

$$\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2} = a\left(\frac{d^2y}{dx^2}\right)$$

squaring on both sides, we get

$$\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3} = a^2\left(\frac{d^2y}{dx^2}\right)^2$$

Here, **Order = 2**

Degree = 2

Type - I :

Equations solvable for $\frac{dy}{dx} = p$

let the equation of the first order and of the n^{th} degree be

$$p^n + P_1 p^{n-1} + P_2 p^{n-2} + \dots + P_n = 0$$

(or)

$$\left(\frac{dy}{dx}\right)^n + P_1\left(\frac{dy}{dx}\right)^{n-1} + P_2\left(\frac{dy}{dx}\right)^{n-2} + \dots + P_n = 0$$

where $P_1, P_2, \dots, P_n$ are functions of x and y.

$$(P-R_1)(P-R_2)\dots(P-R_n) = 0$$

$$P-R_1 = 0, \quad P-R_2 = 0 \dots \quad P-R_n = 0$$

$$\therefore \phi_1(x, y, c_1) = 0 \; ; \; \phi_2(x, y, c_2) = 0 \dots \phi_n(x, y, c_n) = 0$$

where $c_1, c_2, \dots c_n$ are orbitary constants

Let $c_1, c_2, \dots c_n = c$

where c is orbitary constant

$\therefore$ The solution of the given differential equation is

$\phi(x,y,c)\ ;\ \phi_2(x,y,c) \dots \phi_n(x,y,c) = 0$

Example Sums :

1) Solve : $p^2 - 3p + 2 = 0$

Soln :

$$p^2 - 3p + 2 = 0$$

$\frac{-1}{p} \Big| \frac{-2}{p}$

$$(p-1)(p-2) = 0$$

$$p = 2 \quad ; \quad p = 1$$

$$\frac{dy}{dx} = 2 \quad ; \quad \frac{dy}{dx} = 1$$

$$dy = 2dx \quad ; \quad dy = 2dx$$

Integrating on both sides, we get

$$\int dy = 2\int dx \quad ; \quad \int dy = \int dx$$

$$y = 2x + c \quad ; \quad y = x + c$$

$$y - 2x - c = 0 \quad ; \quad y - x - c = 0$$

The solution of the given differential equation is $(y-2x-c)(y-x-c)$

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2) Solve : $x^2p^2 + 3xyp + 2y^2 = 0$

Soln :

$$p = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Given equation : $x^2p^2 + 3xyp + 2y^2 = 0$

Here $a = x^2$; $b = 3xy$; $c = 2y^2$

$$p = \frac{-3xy \pm \sqrt{9x^2y^2 - 4(x^2)(2y^2)}}{2x^2}$$

$$= \frac{-3xy \pm \sqrt{9x^2y^2 - 8x^2y^2}}{2x^2}$$

$$p = \frac{-3xy \pm \sqrt{x^2y^2}}{2x^2}$$

$$p = \frac{-3xy \pm \sqrt{x^2y^2}}{2x^2} \Rightarrow p = \frac{-3xy \pm xy}{2x^2}$$

$$p = \frac{-3xy + xy}{2x^2} \quad ; \quad p = \frac{-3xy - xy}{2x^2}$$

$$= \frac{-\cancel{2}xy}{\cancel{2}x^{\cancel{2}}} \quad ; \quad = \frac{-\cancel{4}^{2}x\cancel{y}y}{\cancel{2}x^2}$$

$$\frac{dy}{dx} = \frac{-y}{x} \quad ; \quad \frac{dy}{dx} = \frac{-2y}{x}$$

$$\frac{dy}{y} = \frac{-dx}{x} \quad ; \quad \frac{dy}{y} = -\frac{2dx}{x}$$

By integrating

$$\int \frac{1}{y}dy = -\int \frac{1}{x}dx \quad ; \quad \int \frac{1}{y}dy = -2\int \frac{1}{x}dx$$

$$\log y = -\log x + \log c \; ; \; \log y = -2\log x + \log c$$

$$\log y + \log x = \log c \; ; \; \log y + 2\log x = \log c$$

$$\log(xy) = \log c \; ; \; \log y + \log x^2 = \log c$$

$$xy = c \; ; \; \log(yx^2) = \log c$$

$$xy - c = 0 \; ; \; x^2y = c \Rightarrow x^2y - c = 0$$

$\therefore$ The solution of given equation is

$$(xy-c)(x^2y-c) = 0.$$

3) Solve : $p^2 - 5p + 6 = 0$

Soln :

$$(p-2)(p-3) = 0 \qquad \begin{array}{c|c} -2 & -3 \\ p & p \end{array}$$

$$p = 2 \; ; \; p = 3$$

$$\frac{dy}{dx} = 2 \; ; \; \frac{dy}{dx} = 3$$

$$dy = 2dx \; ; \; dy = 3dx$$

By integrating

$$\int dy = 2\int dx \; ; \; \int dy = 3\int dx$$

$$y = 2x + c \; ; \; y = 3x + c$$

$$y - 2x - c = 0 \; ; \; y - 3x - c = 0$$

The solution for the given equation is

$$(y - 2x - c)(y - 3x - c)$$

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4) Solve : $p^2 y + p(x-y) - x = 0$

Soln :

$$p = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

By equation :

$$a = y \; ; \; b = (x-y) \; ; \; c = -x$$

$$p = \frac{-(x-y) \pm \sqrt{(x-y)^2 - 4(y)(-x)}}{2y}$$

$$= \frac{(y-x) \pm \sqrt{x^2 + y^2 - 2xy + 4xy}}{2y}$$

$$= \frac{(y-x) \pm \sqrt{x^2 + y^2 + 2xy}}{2y}$$

$$= \frac{(y-x) \pm \sqrt{(x+y)^2}}{2y}$$

$$p = \frac{(y-x) \pm (x+y)}{2y}$$

$p = \frac{y - \cancel{x} + \cancel{x} + y}{2y}$; $p = \frac{y - x - x - y}{2y}$

$p = \frac{2y}{2y}$; $p = -\frac{\cancel{2}x}{\cancel{2}y}$

$p = 1$; $p = \frac{-x}{y}$

$\frac{dy}{dx} = 1$; $\frac{dy}{dx} = \frac{-x}{y}$

$dy = dx$; $dy = -\frac{x\,dx}{y}$

By integrating

$\int dy = \int dx$; $\int dy = -\frac{1}{y}\int x\,dx$

$y = x + c$; $\int dy\, y = -\int x\,dx$

$y - x - c = 0$; $\frac{y^2}{2} = \frac{-x^2}{2} + c$

; $y^2 = -\frac{2x^2}{2} + 2c$

; $y^2 = -x^2 + 2c$

; $y^2 + x^2 - 2c = 0$

The solution for the differential equation is

$(y - x - c)(y^2 + x^2 - 2c)$

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5) Solve : $p^2 + \underbrace{\left(x + y - \frac{2y}{x}\right)}_{b} p + \underbrace{xy + \frac{y^2}{x^2} - y - \frac{y^2}{x}}_{c} = 0$

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Soln:

$$p = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$a = 1$; $b = \left(x + y - \frac{2y}{x}\right)$; $c = xy + \frac{y^2}{x^2} - y - \frac{y}{x}$

$$p = \frac{-\left(x+y-\frac{2y}{x}\right) \pm \sqrt{\left(x+y-\frac{2y}{x}\right)^2 - 4\left(xy+\frac{y^2}{x^2}-y-\frac{y^2}{x}\right)}}{2(1)}$$

$$p = \frac{-\left(x+y-\frac{2y}{x}\right) \pm \sqrt{\left(x^2+y^2+\frac{4y^2}{x^2}+2xy+2y\left(-\frac{2y}{x}\right)+2\left(\frac{-2y}{x}\right)x-4xy-\frac{4y^2}{x^2}+4y+\frac{4y^2}{x}\right.}}{2}$$

$$= \frac{-\left(x+y-\frac{2y}{x}\right) \pm \sqrt{x^2+y^2+\cancel{\frac{4y^2}{x^2}}+2xy-\cancel{\frac{4y^2}{x}}-\cancel{4y}-4xy-\cancel{\frac{4y^2}{x^2}}+\cancel{4y}+\cancel{\frac{4y^2}{x}}}}{2}$$

$$= -\left(x+y-\frac{2y}{x}\right) \pm \sqrt{x^2+y^2-2xy}\,/2 \qquad \sqrt{(x-y)^2}$$

$$= -\left(x+y-\frac{2y}{x}\right) \pm (x-y)/2$$

$$p = \frac{-\left(x+y-\frac{2y}{x}\right)+(x-y)}{2} \quad ; \quad p = \frac{-\left(x+y-\frac{2y}{x}\right)-(x-y)}{2}$$

$$-\cancel{x}-y+\frac{2y}{x}+\cancel{x}-y$$

$$= \frac{-2y+2y/x}{2} \quad ; \quad = \frac{-2x+2y/x}{2}$$

$$= \frac{\cancel{2}(-y+y/x)}{\cancel{2}} \quad ; \quad = \frac{2(-x+y/x)}{2}$$

$$p = \frac{y}{x} - y \quad ; \quad p = \frac{y}{x} - x$$

$$\frac{dy}{dx} = \frac{y}{x} - y \quad ; \quad \frac{dy}{dx} = \frac{y}{x} - x$$

$$\frac{dy}{dx} = y\left(\frac{1}{x}-1\right) \quad ; \quad \frac{dy}{dx} = \frac{1}{x}(y-x^2)$$

$$\frac{dy}{y} = \left(\frac{1}{x}-1\right)dx \quad ; \quad \frac{dy}{dx} - \frac{y}{x} = -x$$

$\int$ ing on both sides

$$\int \frac{1}{y}\,dy = \int \left(\frac{1}{x}-1\right) dx$$

$$\log y = \log x - x + c$$

$$\log y - \log x = -x + c$$

$$\log (y/x) = -x + c$$

$$e^{\log(y/x)} = e^{-x+c}$$

$$y/x = e^{-x+c} \quad ; \quad \frac{dy}{dx} = \frac{y}{x} - x$$

$p = -1/x$

$$\frac{dy}{dx} - \frac{y}{x} = -x \quad ; \quad e^{\int p\,dx} = e^{\int -\frac{1}{x}dx}$$

$$= e^{-\log x}$$

$$= e^{\log x^{-1}}$$

$$= x^{-1}$$

$$e^{\int p\,dx} = \frac{1}{x}$$

$$y \cdot \frac{1}{x} = \int (-x) \cdot \frac{1}{x}\,dx + c$$

$$= -\int dx + c$$

$$\frac{y}{x} = -x + c$$

$\therefore$ The solution is $\left(\frac{y}{x} - e^{-x+c}\right)\left(\frac{y}{x} + x - c\right) = 0$

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6) **solve : $p^2 + 2y \cot x\, p = y^2$**

soln :

$$p^2 + 2y \cot x\, p - y^2 = 0$$

$$a = 1 \;;\; b = 2y\cot x \;;\; c = -y^2$$

$$p = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$p = \frac{-2y\cot x \pm \sqrt{4y^2\cot^2 x + 4y^2}}{2}$$

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$$p = \frac{-2y\cot x \pm 2y\sqrt{\cot^2 x+1}}{2}$$

$\cot^2 x + 1 = \operatorname{cosec}^2 x$

$$= \frac{-2y\cot x \pm 2y \operatorname{cosec} x}{2}$$

$$= 2(-y\cot x \pm y\operatorname{cosec} x)/2$$

$$p = -y\cot x \pm \operatorname{cosec} x \cdot y$$

$$p = -y\cot x + y\operatorname{cosec} x \rightarrow (1) \; ; \; p = -y\cot x - \operatorname{cosec} x \cdot y \rightarrow (2)$$

By (1);

$$\frac{dy}{dx} = y(-\cot x + \operatorname{cosec} x)$$

$$\frac{dy}{y} = (-\cot x + \operatorname{cosec} x)\,dx$$

By Integration

$$\int \frac{1}{y}\,dy = \int(-\cot x + \operatorname{cosec} x)\,dx$$

$$\log y = -\log \sin x - \log(\operatorname{cosec} x + \cot x) + \log c$$

$$= (-[\log \sin x + \log(\operatorname{cosec} x + \cot x)]) + \log c$$

$$= [-(\log \sin x + \log(\frac{1}{\sin x} + \frac{\cos x}{\sin x}))] + \log c$$

$$= -[\log \sin x + \log(\frac{1+\cos x}{\sin x})] + \log c$$

$$= -\log(\sin x \times \frac{1+\cos x}{\sin x}) + \log c$$

$$\log y = -\log(1+\cos x) + \log c$$

$$\log y + \log(1+\cos x) = \log c$$

i.e., $y(1+\cos x) = c$

$$y(1+\cos x) - c = 0$$

By (2);

$$p = -y\cot x - \operatorname{cosec} x \cdot y$$

$$\frac{dy}{dx} = y(-\cot x - \operatorname{cosec} x)$$

$$\frac{dy}{y} = (-\cot x - \operatorname{cosec} x)\,dx$$

By $\int$;

$$\int \frac{1}{y}\,dy = \int (-\cot x - \operatorname{cosec} x)\,dx$$

$$\log y = -\log \sin x + \log(\operatorname{cosec} x + \cot x) + \log c$$

$$= -\log \sin x + \log\left(\frac{1}{\sin x} + \frac{\cos x}{\sin x}\right) + \log c$$

$$= -\log \sin x + \log\left(\frac{1+\cos x}{\sin x}\right) + \log c$$

$$= \log\left(\frac{\left(\frac{1+\cos x}{\sin x}\right)}{\sin x}\right) + \log c$$

$$\therefore = \log\left(\frac{1+\cos x}{\sin^2 x}\right) + \log c$$

$$\log y = \log\left(\frac{1+\cos x}{1-\cos^2 x}\right) + \log c$$

$$= \log\left(\frac{\cancel{1+\cos x}}{\cancel{(1+\cos x)}(1-\cos x)}\right) + \log c$$

$$\log y = \log\left(\frac{1}{1-\cos x}\right) + \log c$$

$$\cancel{\log}\, y = \cancel{\log}\left[\left(\frac{1}{1-\cos x}\right)c\right]$$

$$y = \frac{c}{1-\cos x}$$

$$y(1-\cos x) = c$$

$$y(1-\cos x) - c = 0$$

$\therefore$ The solution of given equation is $[y(1+\cos x) - c][y(1-\cos x) - c] = 0$.

6) Solve :

$x^2p^2 + xyp - 6y^2 = 0$

soln :

$a = x^2, \ b = xy, \ c = -6y^2$

$p = -b \pm \sqrt{b^2 - 4ac} \ / 2a$

$= \dfrac{-xy \pm \sqrt{x^2y^2 + 24x^2y^2}}{2x^2}$

$= \dfrac{-xy \pm \sqrt{25x^2y^2}}{2x^2}$

$p = \dfrac{-xy \pm 5\sqrt{x^2y^2}}{2x^2}$

$p = \dfrac{-xy \pm 5xy}{2x^2}$

$p = \dfrac{-xy + 5xy}{2x^2}$; $p = \dfrac{-xy - 5xy}{2x^2}$

$\dfrac{dy}{dx} = \dfrac{\cancel{4}^{2}xy}{\cancel{2}x^{\cancel{2}}}$; $p = -\dfrac{\cancel{6}^{3}xy}{\cancel{2}x^{\cancel{2}}}$

$\dfrac{dy}{dx} = \dfrac{2y}{x}$; $\dfrac{dy}{dx} = -\dfrac{3y}{x}$

$\dfrac{dy}{y} = \dfrac{2dx}{x}$; $\dfrac{dy}{y} = -\dfrac{3dx}{x}$

By $\int$,

$\int \frac{1}{y} dy = 2\int \frac{1}{x} dx$; $\int \frac{1}{y} dy = -3\int \frac{1}{x} dx \rightarrow ①$

$\log y = 2\log x + \log c$

$\log y = \log x^2 + \log c$

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7) Solve :

$$xyp^2 + p(3x^2 - 2y^2) - 6xy = 0$$

soln :

$a = xy$; $b = 3x^2 - 2y^2$; $c = -6xy$

$$p = \frac{-(3x^2 - 2y^2) \pm \sqrt{9x^4 + 4y^4 - 12x^2y^2 + 24x^2y^2}}{2xy}$$

$$= -(+3x^2 - 2y^2) \pm \sqrt{(3x^2 + 2y^2)^2} \;/2xy$$

$$= -(+3x^2 - 2y^2) \pm (3x^2 + 2y^2)/2xy$$

$$p = \frac{-\cancel{3x^2} + 2y^2 + \cancel{3x^2} + 2y}{2xy}$$

$$= \overset{2}{\cancel{4}}y^2/\cancel{2}xy \quad ; \quad p = 2y/x$$

$dy/dx = 2y/x \quad ; \quad dy/y = 2dx/x$

By $\int$; $\int dy/y = 2\int \frac{1}{x} dx$

$\log y = 2\log x + \log c = \log x^2 + \log c$

$\log y = \log (x^2 c)$

$y = x^2 c$

$y - x^2 c = 0$

$p = -3x^2 + \cancel{2y^2} - 3x^2 - \cancel{2y^2} / 2xy$

$= -\cancel{6}^{3} \cancel{x^2}^{x} / \cancel{2}\cancel{x}y$

$p = -3x/y$

$\frac{dy}{dx} = -3x/y$

$ydy = -3xdx$

$\int$ By,

$\int ydy = -3\int xdx$

$y^2/2 = -\left[\frac{3x^2}{2} + c\right]$

$y^2/2 = -3x^2/2 + c/2$

$y^2 + 3x^2 - c = 0$

The solution for the given equation is

$(y - x^2 c)(y^2 + 3x^2 - c) = 0$.

Equations solvable for y :

$f(x, y, p) = 0$ can be put in the form. $y = F(x, p) \rightarrow (1)$

Diff (1). w. r. to 'x'

1) Solve : $xp^2 - 2yp + x = 0 \rightarrow (1)$

Soln: Given equ is

$$xp^2 - 2yp + x = 0$$

$$-2yp = -xp^2 - x$$

$$2yp = xp^2 + x$$

$$y = xp^2 + x / 2p \rightarrow (1) \qquad \frac{u}{v} = \frac{vdu - udv}{v^2}$$

Diff (1) . w. r. to 'x'

$$dy/dx = \frac{2p\,[x\,2p.dp/dx + p^2(1) + 1] - (xp^2 + x)2\frac{dp}{dx}}{4p^2(1)}$$

$$= \frac{4p^2x\,dp/dx + 2p^3 - 2p^2x\frac{dp}{dx} - 2x\frac{dp}{dx}}{4p^2}$$

$$\frac{dy}{dx} = \frac{2p^2x\,dp/dx + 2p^3 - 2x\,dp/dx}{4p^2}$$

$$p = \frac{2p^2x \cdot \frac{dp}{dx} + 2p^3 - 2x \cdot \frac{dp}{dx}}{4p^2}$$

$$4p^3 = 2p^2x \cdot \frac{dp}{dx} + 2p^3 + 2p - 2x \cdot \frac{dp}{dx}$$

$$4p^3 - 2p^3 = 2p^2 x \frac{dp}{dx} + 2p - 2x\frac{dp}{dx}$$

$$\cancel{2}p^3 = \cancel{2}\left[p^2 x \frac{dp}{dx} + p - x\frac{dp}{dx}\right]$$

$$p^3 - p = p^2 x \frac{dp}{dx} - x\frac{dp}{dx}$$

$$p\cancel{(p^2-1)} = x\frac{dp}{dx}\cancel{[p^2-1]}$$

$$p = x\frac{dp}{dx}$$

$$\frac{dx}{x} = \frac{dp}{p}$$

By $\int$;

$$\int \frac{1}{x}dx = \int \frac{1}{p}dp$$

$$\log x = \log p + \log c$$

$$\log x = \log(pc)$$

$$x = pc$$

$$p = \frac{x}{c} \quad ; \text{ let } \frac{1}{c} = c$$

ie, $p = cx$

sub $p = cx$ in ①, we get

$$x(cx)^2 - 2y(cx) + x = 0$$

$$c^2x^3 - 2cxy + x = 0$$

$$-2cxy = -x - c^2x^3$$

$$\cancel{+}2c\cancel{x}y = \cancel{+}\cancel{x}(c^2x^2+1)$$

ie, $2cy = c^2x^2 + 1$

The solution of given equ is

$$2cy = c^2x^2 + 1$$

let $f(x,y,p)=0$ can be put in the

Form $x = F(y,p) \rightarrow (1)$

Diff. w.r. to 'x'

$$\frac{dx}{dy} = \phi\left(y, p, \frac{dp}{dy}\right)$$

ie, $\frac{1}{p} = \phi\left(y, p, \frac{dp}{dy}\right) \rightarrow (2)$

solve equation (2), we get

$$\phi(y,p,c) = 0 \rightarrow (3)$$

eliminating p between (1) and (3) we get the solution.

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1) **Solve :**

$$x = y^2 + \log p$$

Soln :

Given equ is $x = y^2 + \log p \rightarrow (1)$

Diff (1). w. r. to 'y'

$$\frac{1}{p} = \frac{dx}{dy} = 2y + \frac{1}{p}\cdot\frac{dp}{dy}$$

$\times$ by $p \Rightarrow$ $1 = 2py + \frac{dp}{dy}$

$\frac{dy}{dx} + Py = Q$

$y \cdot e^{\int p\,dx} = e^{\int p\,dx} \cdot Q +$

ie, $\frac{dp}{dy} + \underbrace{2py}_{P = 2y} = 1$

Integrating factor $\Rightarrow$ I.F $= e^{\int P\,dy}$

$$= e^{\int 2y\,dy} = e^{\frac{2y^2}{2}}$$

2) Solve:

$$y = xp + x(1+p^2)^{1/2}$$

soln:

Given equ is $y = xp + x(1+p^2)^{1/2} \rightarrow ①$

Diff ①. w.r.to 'x'

$$\frac{dy}{dx} = x\frac{dp}{dx} + p\cancel{\frac{dx}{}}(1) + x \cdot \frac{1}{\cancel{2}}(1+p^2)^{1/2-1} \cdot \cancel{2}p \cdot \frac{dp}{dx} + (1+p^2)^{1/2}\cancel{dx}(1)$$

$$= x\frac{dp}{dx} + \cancel{p\,dx} + xp(1+p^2)^{-1/2}\frac{dp}{dx} + (1+p^2)^{1/2}\cancel{dx}$$

$$\cancel{p} = x\frac{dp}{dx} + p + xp\frac{dp}{dx}\frac{1}{(1+p^2)^{1/2}} + (1+p^2)^{1/2}\cancel{dx}$$

$$0 = x\frac{dp}{dx}\left(1 + \frac{p}{(1+p^2)^{1/2}}\right) + (1+p^2)^{1/2}$$

$$x\frac{dp}{dx}\left(1 + \frac{p}{\sqrt{1+p^2}}\right) = -\sqrt{1+p^2}$$

$$dp\left(1 + \frac{p}{\sqrt{1+p^2}}\right) \cdot \frac{1}{\sqrt{1+p^2}} = -\frac{dx}{x}$$

$$dp\left(\frac{1}{\sqrt{1+p^2}} + \frac{p}{(1+p^2)}\right) = -\int \frac{1}{x}dx$$

By $\int$,

$$\int \frac{1}{\sqrt{1+p^2}}\,dp + \int \frac{p}{\sqrt{1+p^2}}\,dp = -\int \frac{1}{x}\,dx$$

$$\log\left(p+\sqrt{1+p^2}\right) + \frac{1}{2}\log(1+p^2) = -\log x + \log c$$

$$\log\left(p+\sqrt{1+p^2}\right) + \log(1+p^2)^{1/2} + \log x = \log c$$

$$\log\left[\left(p+\sqrt{1+p^2}\right)\left(\sqrt{1+p^2}\right)x\right] = \log c$$

$$\left(p+\sqrt{1+p^2}\right)\left(\sqrt{1+p^2}\right)x = c \rightarrow (2)$$

i.e, $\left(p+\sqrt{1+p^2}\right)\left(\sqrt{1+p^2}\right)x = c$

Eliminating p b/w (1) and (2), we get the solution.

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3) Solve : $x^2 = (1+p^2)$

Soln :

$$x^2 = (1+p^2) \rightarrow (1)$$

$$x = \pm\sqrt{1+p^2}$$

Diff. w. r. to 'y'

$$\frac{dx}{dy} = \frac{1}{2\sqrt{1+p^2}} \cdot 2p \cdot \frac{dp}{dy}$$

$$\frac{1}{p} = \frac{p}{\sqrt{1+p^2}}\,\frac{dp}{dy}$$

$$dy = \frac{p^2}{\sqrt{1+p^2}}\,dp$$

$$\int \frac{1}{\sqrt{a^2+x^2}} = \log\left(x+\sqrt{a^2+x^2}\right)$$

$$\int \sqrt{a^2+x^2} = \frac{x}{2}\sqrt{a^2+x^2} + \frac{1}{2}\log\left(x+\sqrt{a^2+x^2}\right)$$

Integrating,

$$\int dy = \int\left(\frac{p^2}{\sqrt{1+p^2}}\right)dp$$

$$y + c = \int\left(\frac{p^2+1-1}{\sqrt{1+p^2}}\right)dp$$

$$= \int \frac{p^2+1}{\sqrt{1+p^2}} dp - \int \frac{1}{\sqrt{1+p^2}} dp$$

$$= \int \sqrt{p^2+1^2}\, dp - \int \frac{1}{\sqrt{1+p^2}} dp$$

$$= \frac{p}{2}\sqrt{1+p^2} + \frac{1}{2}\log\left[p+\sqrt{1+p^2}\right] - \log\left[p+\sqrt{1+p^2}\right]$$

$$y+c = \frac{p}{2}\sqrt{1+p^2} - \frac{1}{2}\log\left[p+\sqrt{1+p^2}\right]$$

$$y+c = \frac{1}{2}\left[p\sqrt{1+p^2} - \log\left[p+\sqrt{1+p^2}\right]\right] \rightarrow (2)$$

By eliminating p b/w (1) & (2), we get the solution.

---

4) solve : $y^2 = 1+p^2$

soln : $y^2 = 1+p^2 \rightarrow (1)$; $y = \pm\sqrt{1+p^2}$

Diff .w. r. to 'x'

$$\frac{dy}{dx} = \frac{1}{2}(1+p^2)^{-1/2} \cdot 2p \cdot dp/dx$$

$$\frac{dy}{dx} = \frac{1}{\cancel{2}} \cdot \frac{1}{\sqrt{1+p^2}} \cdot \cancel{2}p \cdot \frac{dp}{dx}$$

$$\cancel{p} = \frac{\cancel{p}}{\sqrt{1+p^2}} \cdot \frac{dp}{dx}$$

$$dx = \frac{1}{\sqrt{1+p^2}} dp$$

By $\int$

$$\int dx = \int \frac{1}{\sqrt{1+p^2}} dp$$

$$x+c = \log\left(p+\sqrt{1+p^2}\right) \rightarrow (2)$$

Eliminating p b/w (1) & (2), we get the solution.

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2M

Clairaut's form:

The equation known as clairauts is of the form.

$$y = px + f(p) \rightarrow (1)$$

Diff. w. r. to 'x'

$$\frac{dy}{dx} = p(1) + x\frac{dp}{dx} + f'(p)\frac{dp}{dx}$$

$$p = p + \frac{dp}{dx}(x + f'(p))$$

$$p - p = \frac{dp}{dx}(x + f'(p))$$

i.e, $\frac{dp}{dx}[x + f'(p)] = 0$

$\frac{dp}{dx} = 0$ (or) $x + f'(p) = 0$

$\Rightarrow dp = 0$

By $\int$ing

$p = c$

sub $p = c$ in (1), we get the solution

$$y = cx + f(c).$$

---

1) Solve : $y = (x-a)p - p^2$

soln : Given equation is

$$y = (x-a)p - p^2$$

$$y = xp - ap - p^2 \rightarrow (1)$$

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Clairaut's form $y = P(x) + f(P)$

Here ① is clairaut's form

$\therefore$ sub $p = c$ in ①, we get the solution

$$y = cx - ac - c^2$$

2) Solve : $p = \tan(px - y)$

Soln :

Given equation is

$$p = \tan(px - y)$$

$$\tan^{-1} p = px - y$$

$$y = px - \tan^{-1} p \rightarrow ①$$

Clairaut's form $y = px + f(P)$

Here is clairaut's form

$\therefore$ sub $p = c$ in ①, we get the solution

$$y = cx - \tan^{-1} c.$$

3) Solve : $y = 2px + y^2 p^3$

Soln :

Given equation is

$$y = 2px + y^2 p^3 \rightarrow ①$$

let $X = 2x$; $Y = y^2$

$dX = 2dx$; $dY = 2y\,dy$

$$P = \frac{dy}{dx}$$

$$= \frac{2ydy}{2dx}$$

$$P = \frac{ydy}{dx}$$

$$P = yp$$

$$p = \frac{P}{y}$$

$$① \Rightarrow y = px + yp^3$$

$$= \frac{P}{y}x + y\frac{P^3}{y^3}$$

$$= \frac{P}{y}x + y\frac{P^3}{y^2 \cdot y}$$

$$y = \frac{P}{y}x + \frac{yP^3}{y \cdot y}$$

$$y = \frac{P}{y}x + \frac{P^3}{y}$$

$$y^2 = Px + P^3$$

$$y = Px + P^3 \longrightarrow ②$$

Here ② is clairaut's for $y = P(x) + f(P)$

$\therefore$ Sub $p = c$ in ②, we get

$$y = cx + c^3$$

$\therefore$ The soln of given equation is

$$y^2 = 2cx + c^3$$

4) solve : $(px - y)(py + x) = 2p$

soln :

$$p^2xy + px^2 - py^2 - xy = 2p \rightarrow ①$$

let $X = x^2$; $Y = y^2$

$$dX = 2x\,dx \;;\; dY = 2y\,dy$$

$$P = \frac{dY}{dX} = \frac{2y\,dy}{2x\,dx}$$

$$= \frac{y\,dy}{x\,dx}$$

$$P = \frac{y}{x}p$$

$$p = \frac{xP}{y}$$

from ①; $X = x^2$; $Y = y^2$

$$p^2xy + pX - pY - xy = 2p$$

where $p = \frac{xP}{y}$

$$\frac{x^2P^2}{y^2}xy + \frac{xPX}{y} - \frac{xPY}{y} - xy = 2\frac{xP}{y}$$

$$\frac{x^3P^2 + xPX - xPY - xy^2}{y} = 2\frac{xP}{y}$$

$$x^3P^2 + xPX - xPY - xy^2 = 2xP$$

$$x(x^2P^2 + PX - PY - y^2) = 2xP$$

$$\underbrace{XP^2 + PX}_{PX} - \underbrace{PY - Y}_{-Y} = 2P$$

$$PX(P+1) - Y(P+1) = 2P$$

$$(P+1)(PX - Y) = 2P$$

$$PX - Y = \frac{2P}{P+1}$$

$$Y = PX - \frac{2P}{P+1} \rightarrow (2)$$

Here (2) is clairaut's form

$$y = p(x) + f(p)$$

Sub $p = c$ in (2), we get

$$Y = cX - \frac{2C}{C+1}$$

P = c 2m

clairet's = 5m

The solution of the given equ is

$$y^2 = cx^2 - \frac{2c}{c+1}.$$

Exact differential Equation

Let $Mdx + Ndy = 0$ be the differential equation

$Mdx + Ndy = 0$ is said to be exact if $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

Rule for solving an exact differential equation.

Integrate Mdx as if y is constant and integrate Ndy as if x is free from x.

The sum of these integrates equated to a constant gives the solution.

1) solve : $(a^2-2xy-y^2)dx-(x+y)^2dy=0$

soln : Given equation is

$$(a^2-2xy-y^2)dx-(x+y)^2dy=0 \rightarrow (1)$$

Here $M=a^2-2xy-y^2$; $N=-(x+y)^2$

x constant

y constant

$$\frac{\partial M}{\partial y}=-2x(1)-2y \quad ; \quad \frac{\partial N}{\partial x}=-2(x+y)(1+0)$$

$$\frac{\partial M}{\partial y}=-2x-2y \rightarrow (2) \qquad =-2x-2y \rightarrow (3)$$

from (2) and (3)

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

$$-2x-2y=-2x-2y$$

$\therefore$ (1) is exact.

x terms omitted

$\int$ing (1) , we get

x^2+y^2+2xy

$$\int(a^2-2xy-y^2)dx-\int(x+y)^2dy=c$$

free from x

$$a^2x-\frac{2x^2}{2}y-xy^2-\int y^2dy=c$$

$$a^2x-x^2y-xy^2-y^3/3=c$$

---

2) solve : $(x-y)\frac{dy}{dx}=2x-y$

soln :

Given equation is

$$(x-y)\frac{dy}{dx}=2x-y$$

$$(x-y)dy-(2x-y)dx=0 \rightarrow (1)$$

$\div$by$(-)$ $(2x-y)dx-(x-y)dy=0 \rightarrow (1)$

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Here $M = 2x - y$; $N = -(x-y)$

$\frac{\partial M}{\partial y} = 0 - 1$; $\frac{\partial N}{\partial x} = -1 - 0$

$\frac{\partial M}{\partial y} = -1 \rightarrow (2)$; $\frac{\partial N}{\partial x} = -1 \rightarrow (3)$

from (2) & (3)

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$$-1 = -1$$

$\therefore$ (1) is exact

$\int$ ing (1), we get

$$\int (2x-y)\,dx - \int (x-y)\,dy = c$$

$$\frac{\cancel{2}x^2}{\cancel{2}} - xy - \int (-y)\,dy = c$$

$$x^2 - xy + y^2/2 = c$$

---

1) solve: $y = px + \frac{ap}{(1+p^2)^{1/2}}$

soln:

Given equation

$$y = px + \frac{ap}{(1+p^2)^{1/2}} \rightarrow (1).$$

clairauts form $y = px + f(p)$

$p = c$ in (1)

$$y = cx + \frac{ax}{(1+c^2)^{1/2}}.$$

2) solve : $y = px + \frac{a}{p}$

Soln :

Given equation

$$y = px + \frac{a}{p} \rightarrow (1)$$

Clairauts form $y = px + f(p)$

$p = c$ in (1); $\therefore$ The solution of given equ is

$$y = cx + \frac{a}{c}$$

3) solve : $p = \log(px - y)$

Soln :

Given equ is $p = \log(px - y)$

$$\log(px - y) = p$$
$$px - y = \log^{-1} p$$
$$-y = \log^{-1} p - px$$
$$-y = e^{p} - px$$
$$y = px - e^{p} \rightarrow (1)$$

This is in clairauts form $p = c$,

sub in (1) ; $y = cx - e^{c}$

$\therefore$ This is the solution of given equation.

4) solve : $\sin px \cos y = \cos px \sin y + p$

Soln :

$$\sin px \cos y = \cos px \sin y + p$$
$$\sin px \cos y - \cos px \sin y = p$$
$$\sin(px - y) = p \quad [\sin(A-B) = \sin A \cos B - \cos A \sin B]$$
$$px - y = \sin^{-1} p$$
$$-y = \sin^{-1} p - px$$
$$y = px - \sin^{-1} x \rightarrow (1)$$

5) Solve : $(px-y)(py+x) = pa^2$.

Soln :

$$X = x^2 \; ; \; Y = y^2$$

$$dX = 2x\,dx \; ; \; dY = 2y\,dy$$

$$P = \frac{dY}{dX} = \frac{2y\,dy}{2x\,dx} = \frac{y}{x}p \; ; \; p = \frac{x}{y}P$$

$$(px-y)(py+x) = pa^2$$

$$p^2xy + px^2 - py^2 - xy = pa^2$$

$$xy(p^2-1) + p(\underbrace{x^2}_{X} - \underbrace{y^2}_{Y}) = pa^2$$

$$xy(p^2-1) + p(X-Y) = pa^2$$

$$xy\left[\frac{x^2}{y^2}P^2 - 1\right] + \frac{x}{y}P(X-Y) = \frac{x}{y}Pa^2$$

$$\frac{x}{y}\left[P^2X - Y\right] + \frac{x}{y}P(X-Y) = \frac{x}{y}Pa^2$$

$$P^2X - Y + P(X-Y) = Pa^2$$

$$P^2X - Y + PX - PY = Pa^2$$

$$-Y(1+P) + P(P+1) = Pa^2$$

$$(1+P)(PX-Y) = Pa^2$$

$$PX - Y = \frac{Pa^2}{1+P}$$

$$-Y = -PX + \frac{Pa^2}{1+P}$$

$$Y = PX - \frac{Pa^2}{1+P} \rightarrow (1)$$

This is in clairauts form,

Then sub, $p = c$ in ①

$$y = cx - \frac{ca^2}{1+c}$$

(i.e) $y^2 = cx^2 - \frac{ca^2}{1+c}$

The soln of the given equ,

$$y^2 = c\left(x^2 - \frac{a^2}{1+c}\right).$$

6) Solve : $y\,dx + (x+y)\,dy = 0$

Soln :

Given equ is $y\,dx + (x+y)\,dy = 0 \rightarrow ①$

Here

$M = y$, $N = x+y$

$\frac{\partial M}{\partial y} = 1 \rightarrow ②$; $\frac{\partial N}{\partial x} = 1 \rightarrow ③$

From ② & ③

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$$1 = 1$$

This is exact.

$\int ① \Rightarrow \int y\,dx + \int (x+y)\,dy = c$

$xy + y^2/2 = c$

$2xy + y^2 = 2c$

$\therefore$ This is the soln of given equ.

7) Solve : $(y\cos x + 1)\,dx + \sin x\,dy = 0$

Soln: Given equ is $(y\cos x + 1)\,dx + \sin x\,dy = 0 \rightarrow ①$

Here, $M = y\cos x + 1$; $N = \sin x$

$\frac{\partial M}{\partial y} = \cos x \rightarrow ②$; $\frac{\partial N}{\partial x} = \cos x \rightarrow ③$

From ② & ③

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$$\cos x = \cos x$$

This is exact.

$$\int ① \Rightarrow \int (y\cos x + 1)dx + \int \underbrace{\sin x}_{\text{free from } x} dy = 0$$

$$y\sin x + x = c$$

The soln of the given equ is,

$$y\sin x + x = c.$$

---

8) Solve : $x^2y^3dx + (x^3y^2 - 2)dy = 0$

Soln :

Given equ is $x^2y^3dx + (x^3y^2 - 2)dy = 0 \rightarrow ①$

$M = x^2y^3$; $N = x^3y^2 - 2$

$\frac{\partial M}{\partial y} = 3x^2y^2 \rightarrow ②$; $\frac{\partial N}{\partial x} = 3x^2y^2 \rightarrow ③$

from ② & ③

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$$3x^2y^2 = 3x^2y^2$$

This ① is exact.

$\int ①$, we get

$$\int x^2y^3\,dx + \int (\underbrace{\cancel{x^3y^2}}_{\text{free from } x} - 2)dy = 0$$

$$\left[\frac{x^3}{3}y^3\right] + (-2y) = c$$

The soln of given equ is,

$$x^3y^3 - 2y = 3c.$$

Rules for finding integrating factors.

1) When $Mx + Ny \neq 0$ and the equation is homogenous, $\frac{1}{Mx+Ny}$ is an integration factor of $Mdx + Ndy = 0$

2) When $Mx + Ny \neq 0$ and the equation is of the form $f_1(xy)y\,dx + f_2(xy)x\,dy = 0$.

$\frac{1}{Mx-Ny}$ is an integrating factor.

3) If $\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)$ is a function of x alone say. $f(x)$, then the integration factor is $e^{\int f(x)dx}$.

4) If $\frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)$ is a function of y alone say $f(y)$, then the integrating factor is $e^{\int f(y)dy}$.

1) Solve : $(x^2y - 2xy^2)dx - (x^3 - 3x^2y)dy = 0$.

$\frac{2+1}{3}$ $\frac{2+1}{3}$ $\frac{3}{3}$ $\frac{2+1}{3}$ so homogenous

Soln :

Given equ $(x^2y - 2xy^2)dx - (x^3 - 3x^2y)dy = 0 \rightarrow (1)$

$M = x^2y - 2xy^2$; $N = -(x^3 - 3x^2y)$

$\frac{\partial M}{\partial y} = x^2 - 4xy \rightarrow (2)$; $\frac{\partial N}{\partial x} = -(3x^2 - 6xy) \rightarrow (3)$

from (2) & (3)

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

$$x^2 - 4xy \neq -3x^2 + 6xy$$

(1) is not exact.

$$Mx + Ny = (x^2y - 2xy^2)x - (x$$

$$= x^3y - 2x^2y^2 - x^3y + 3x^2y^2$$

$$Mx+Ny = x^2y^2$$

$$\therefore Mx+Ny \neq 0$$

since ① is homogeneous

$\therefore \frac{1}{Mx+Ny}$ is an integrating factor

$\frac{1}{x^2y^2}$ is an integrating factor.

$\times$ by $\frac{1}{x^2y^2}$ in ①; we get,

$$\Rightarrow \left(\frac{x^2y - 2xy^2}{x^2y^2}\right)dx - \left(\frac{x^3 - 3x^2y}{x^2y^2}\right)dy = 0$$

$$\Rightarrow \left(\frac{xy(x-2y)}{x^2y^2}dx\right) - \left(\frac{x^2(x-3y)}{x^2y^2}dy\right) = 0$$

$$\Rightarrow \left(\frac{x-2y}{xy}\right)dx - \left(\frac{x-3y}{y^2}\right)dy = 0$$

$$\Rightarrow \left(\frac{x}{xy} - \frac{2y}{xy}\right)dx - \left(\frac{x}{y^2} - \frac{3y}{y^2}\right)dy = 0$$

$$\Rightarrow \left(\frac{1}{y} - \frac{2}{x}\right)dx - \left(\frac{x}{y^2} - \frac{3}{y}\right)dy = 0$$

By $\int$, we get

$$\int\left(\frac{1}{y} - \frac{2}{x}\right)dx - \int\left(\frac{x}{y^2} - \frac{3}{y}\right)dy = c$$

$$\frac{x}{y} - 2\log x + 3\log y = c$$

$\therefore$ The soln of given equ is

$$\frac{x}{y} - 2\log x + 3\log y = c.$$

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2) Solve: $y(xy+2x^2y^2)dx + x(xy-x^2y^2)dy = 0$

3 5 3 5

Soln: so not homogeneous

Given equation is

$$y(xy+2x^2y^2)dx + x(xy-x^2y^2)dy = 0 \rightarrow (1)$$

$$M = xy^2 + 2x^2y^3 \quad ; \quad N = x^2y - x^3y^2$$

$$\frac{\partial M}{\partial y} = 2xy + 6x^2y^2 \rightarrow (2) \quad ; \quad \frac{\partial N}{\partial x} = 2xy - 3x^2y^2 \rightarrow (3)$$

From (2) & (3)

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

$$2xy + 6x^2y^2 \neq 2xy - 3x^2y^2$$

(1) is not exact

$$Mx - Ny = xy(xy+2x^2y^2) - xy(xy-x^2y^2)$$

$$= \cancel{x^2y^2} + 2x^3y^3 - \cancel{x^2y^2} + x^3y^3$$

$$Mx - Ny = 3x^3y^3$$

$$\therefore Mx - Ny \neq 0$$

Since given equ is of the form

$$f_1(x,y)\, y\,dx + f_2(x,y)\, x\,dy = 0$$

$$\therefore \text{Integrating factor} = \frac{1}{Mx - Ny}$$

$$= \frac{1}{3x^3y^3}$$

$\Rightarrow (1) \times \frac{1}{3x^3y^3}$; we get

$$\frac{y(xy+2x^2y^2)}{3x^3y^3}dx + \frac{x(xy-x^2y^2)dy}{3x^3y^3} = 0$$

$$\left(\frac{xy^{\cancel{2}}}{3x^3y^{\cancel{3}2}} + \frac{2x^2y^3}{3x^3y^3}\right)dx + \left(\frac{x^2y}{3x^3y^3_2} - \frac{x^3y^2}{3x^3y^3}\right)dy = 0$$

$$\left(\frac{1}{3x^2y} + \frac{2}{3x}\right)dx + \left(\frac{1}{3xy^2} - \frac{1}{3y}\right)dy = 0 \rightarrow (4)$$

Here (4) is exact.

(4) By $\int$, we get;

$$\int \left(\frac{1}{3x^2y} + \frac{2}{3x}\right)dx + \int \left(\frac{1}{3xy^2} - \frac{1}{3y}\right)dy = \frac{c}{3}$$

$$\frac{-1}{3xy} + \frac{2}{3}\log x - \frac{1}{3}\log y = \frac{c}{3}$$

$$\frac{-1}{xy} + 2\log x - \log y = c$$

$$\frac{-1}{xy} + \log x^2 - \log y = c$$

$$\frac{-1}{xy} + \log\left(\frac{x^2}{y}\right) = c$$

---

3) solve: $(x^4e^x - 2mxy^2)dx + 2mx^2ydy = 0$

soln:

Given equ is

$$(x^4e^x - 2mxy^2)dx + 2mx^2ydy = 0 \rightarrow (1)$$

Here,

$$M = x^4e^x - 2mxy^2 \quad ; \quad N = 2mx^2y$$

$$\frac{\partial M}{\partial y} = 0 - 4mxy \quad ; \quad \frac{\partial N}{\partial x} = 4mxy$$

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

$$-4mxy \neq 4mxy$$

(1) is not exact

$$\frac{1}{N}\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right)=\frac{1}{2mx^2y}\left[-4mxy-4mxy\right]$$

$$=\frac{1}{2mx^2y}\left[-\cancel{8}^{4}mxy\right]$$

$$=\frac{-4}{x}\ [\text{say } f(x)]$$

Here

$\frac{1}{N}\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right)$ is a function of x alone.

$\therefore$ Integrating factor $= e^{\int f(x)dx}$

$$= e^{\int -\frac{4}{x}dx}$$

$$= e^{-4\log x}$$

$$= \cancel{e}^{\cancel{\log} x^{-4}}$$

$$= x^{-4}$$

$$= \frac{1}{x^4}$$

① $\times \frac{1}{x^4}$

$$\left(\frac{x^4e^x}{x^4}-2\frac{mxy^2}{x^4}\right)dx+\frac{2mx^2y}{x^4}\,dy=0$$

$$\left(e^x-\frac{2my^2}{x^3}\right)dx+\frac{2my}{x^2}\,dy=0 \rightarrow ②$$

Here ② is exact

$$\int\left(e^x-\frac{2my^2}{x^3}\right)dx+\int\frac{2my}{x^2}\,dy=c$$

$$e^x-\cancel{2}my^2\times\frac{x^{-2}}{-\cancel{2}}=c$$

$$e^x+\frac{my^2}{x^{-2}}=c$$

4) Solve :

$$\frac{dy}{dx} = \frac{x^2 - ay}{ax - y^2}$$

Soln :

$$\frac{dy}{dx} = \frac{x^2 - ay}{ax - y^2}$$

$$dy(ax - y^2) = (x^2 - ay)dx$$

$$(x^2 - ay)dx - (ax - y^2)dy = 0 \rightarrow ①$$

$$M = x^2 - ay \quad ; \; N = -ax + y^2$$

$$\frac{\partial M}{\partial y} = -a \quad ; \; \frac{\partial N}{\partial x} = -a$$

$$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$$-a = -a$$

① is exact.

① $\int$ ing $\Rightarrow$

$$\int (x^2 - ay)dx - \int (ax - y^2)dy = c$$

$$\frac{x^3}{3} - axy + \frac{y^3}{3} = c$$

$$x^3 - 3axy + y^3 = 3c$$

$\therefore$ The soln of given equ is

$$x^3 - 3axy + y^3 = 3c.$$

5) Solve:

$(2y\sin x - \cos y)dx + (x\sin y - 2\cos x)dy = 0$

Soln: Given equ is

$$(2y\sin x - \cos y)dx + (x\sin y - 2\cos x)dy = 0 \rightarrow (1)$$

$M = 2y\sin x - \cos y \quad ; N = x\sin y - 2\cos x$

$\frac{\partial M}{\partial y} = 2\sin x + \sin y \, ; \; \frac{\partial N}{\partial x} = \sin y + 2\sin x$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

(1) is exact

(1) $\int$ ing, we get [In M - y is constant
N - free from x]

$$\int (2y\sin x - \cos y)dx + \int (x\sin y - 2\cos x)dy = c$$

$$-2y\cos x - x\cos y = c$$

$$-(2y\cos x + x\cos y) = c$$

$$2y\cos x + x\cos y = -c$$

$$2y\cos x + x\cos y = c$$

$\therefore$ The solution of the given equation is,

$$2y\cos x + x\cos y = c$$

---

6) Solve: $(1+xy)y\,dx + (1-xy)x\,dy = 0$

Soln: Given equ is

$$(1+xy)y\,dx + (1-xy)x\,dy = 0 \rightarrow (1)$$

$M = (1+xy)y \quad ; N = (1-xy)x$

$M = y + xy^2 \quad ; \; N = x - x^2 y$

$\frac{\partial M}{\partial y} = 1 + 2xy$; $\frac{\partial N}{\partial x} = 1 - 2xy$

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

$\therefore$ The given equation is not exact

$$Mx - Ny = (y + xy^2)x - (x - x^2y)y$$
$$= \cancel{xy} + x^2y^2 - \cancel{xy} + x^2y^2$$
$$= 2x^2y^2$$

$Mx - Ny \neq 0$

Integrating factor $= \frac{1}{Mx - Ny}$

$$= \frac{1}{2x^2y^2}$$

① $\times \frac{1}{2x^2y^2} \Rightarrow$

$$\frac{(1+xy)y}{2x^2y^2}\,dx + \frac{(1-xy)x}{2x^2y^2}\,dx = 0$$

$$\left(\frac{y}{2x^2y^2} + \frac{xy^2}{2x^2y^2}\right)dx + \left(\frac{x}{2x^2y^2} - \frac{x^2y}{2x^2y^2}\right)dy =$$

$$\left(\frac{1}{2x^2y} + \frac{1}{2x}\right)dx + \left(\frac{1}{2xy^2} - \frac{1}{2y}\right)dy = 0 \longrightarrow ②$$

② Sing ; ② is exact

$$\int \left(\frac{1}{2x^2y} + \frac{1}{2x}\right)dx + \int \left(\frac{1}{2xy^2} - \frac{1}{2y}\right)dy = 0$$

$$-\frac{1}{2xy} + \frac{1}{2}\log x - \frac{1}{2}\log y = c$$

$$\frac{1}{2}\left(\frac{-1}{xy} + \log x - \log y\right) = c$$

$$-\frac{1}{xy} + \log x - \log y = 2c$$

$$\frac{-1}{xy} + \log(x/y) = 2c$$

$\therefore$ The solution of given equ is

$$\frac{-1}{xy} + \log\left(\frac{x}{y}\right) = 2c.$$

Integrable combinations

1) $d(xy) = y\,dx + x\,dy$

2) $d\left(\frac{x}{y}\right) = \frac{y\,dx - x\,dy}{y^2}$

3) $d\left(\frac{y}{x}\right) = \frac{x\,dy - y\,dx}{x^2}$

4) $d\left(\tan^{-1}\frac{y}{x}\right) = \frac{x\,dy - y\,dx}{x^2+y^2}$

5) $d\left(\tan^{-1}\frac{x}{y}\right) = \frac{y\,dx - x\,dy}{x^2+y^2}$

6) $d\left(\log\frac{x+y}{x-y}\right) = 2\left(\frac{x\,dy - y\,dx}{x^2-y^2}\right)$

1) Solve : $a(x\,dy + 2y\,dx) = xy\,dy$

Soln :

$\div(xy) \Rightarrow \quad a\left(\frac{x\,dy}{xy} + \frac{2y\,dx}{xy}\right) = \frac{xy\,dy}{xy}$

$$a\left(\frac{dy}{y} + 2\frac{dx}{x}\right) = dy$$

By $\int$ing

$$a\int \frac{1}{y}\,dy + 2a\int \frac{1}{x}\,dx = \int dy$$

2) Solve : $(y^2 e^x + 2xy)dx - x^2 dy = 0$

Soln :

Given equ is

$$(y^2 e^x + 2xy)dx - x^2 dy = 0$$

$$y^2 e^x dx + 2xy dx - x^2 dy = 0 \rightarrow (1)$$

Here $\overset{\checkmark dx\ term}{M} = y^2 e^x + 2xy$; $\overset{\checkmark dy\ term}{N} = -x^2$

$$\frac{\partial M}{\partial y} = 2y + 2c \quad ; \quad \frac{\partial N}{\partial x} = -2x$$

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

$\therefore$ (1) is not exact

(1) $\div y^2 \Rightarrow$

$$\frac{y^2 e^x}{y^2} dx + \frac{2xy\,dx}{y^2} - \frac{x^2}{y^2} dy = 0$$

$$e^x dx + \frac{2xy dx}{y^2} - \frac{x^2}{y^2} dy = 0$$

$$e^x dx + \frac{2xy\,dx - x^2 dy}{y^2} = 0$$

$$e^x dx + d\left(\frac{x^2}{y}\right) = 0 \qquad \frac{v\,du - u\,dv}{v^2}$$

$$e^x dx + d\left(\frac{x^2}{y}\right) = 0$$

$\int$ ing

$$\int e^x dx + \int d\left(\frac{x^2}{y}\right) = c$$

$$e^x + \frac{x^2}{y} = c$$

3) solve : $(y-3x^2)dx - x(1-xy^2)dy = 0$

soln :

Given equ is

$$(y-3x^2)dx - x(1-xy^2)dy = 0 \rightarrow (1)$$

$$y\,dx - 3x^2 dx - x\,dy + x^2y^2 dy = 0 \rightarrow (2)$$

(2) $\div x^2 \Rightarrow$

$$\frac{y}{x^2}dx - \frac{3\cancel{x^2}}{\cancel{x^2}}dx - \frac{x\,dy}{x^2} + \frac{\cancel{x^2}y^2}{\cancel{x^2}}dy = 0$$

$$\frac{y\,dx - x\,dy}{x^2} - 3dx + y^2 dy = 0$$

$$-\left(\frac{x\,dy - y\,dx}{x^2}\right) - 3dx + y^2 dy = 0$$

$$-d\left(\frac{y}{x}\right) - 3dx + y^2 dy = 0$$

$\int$ ing

$$\left(\frac{-y}{x}\right) - 3x + y^3/3 = c$$

$\therefore$ The solution of the given equ is

$$y^3/3 - y/x - 3x = c$$

4) Solve :

$$\frac{dy}{dx} = \frac{2x}{x^2+y^2-2y}$$

Soln :

$$\frac{dy}{dx} = \frac{2x}{x^2+y^2-2y} \rightarrow (1)$$

$$(x^2+y^2-2y)dy = 2x\,dx$$

$$(x^2+y^2-2y)dy - 2x\,dx = 0 \rightarrow (1)$$

Here $M = -2x$; $N = x^2+y^2-2y$

$$\frac{\partial M}{\partial y} = 0 \quad ; \quad \frac{\partial N}{\partial x} = 2x$$

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

$\therefore$ (1) is not exact

$$(1) \Rightarrow (x^2+y^2)dy - 2y\,dy - 2x\,dx = 0 \rightarrow (2)$$

$(2) \div (x^2+y^2) \Rightarrow$

$$\frac{(x^2+y^2)dy}{x^2+y^2} - \frac{2y\,dy}{x^2+y^2} - \frac{2x\,dx}{x^2+y^2} = 0$$

$$\Rightarrow dy - \left(\frac{2y\,dy - 2x\,dx}{x^2+y^2}\right) = 0$$

$$dy - \frac{d(x^2+y^2)}{x^2+y^2} = 0$$

$$\int dy - \int \frac{1}{x^2+y^2}\, d(x^2+y^2) = c$$

$$y - \log(x^2+y^2) = c$$

$\therefore$ The solution of the given soln equ is $y - \log(x^2+y^2) = c$.

5) solve :

$$y\,dx - x\,dy - 3x^2y^2e^{x^3}dx = 0$$

soln :

Given equ is

$$y\,dx - x\,dy - 3x^2y^2e^{x^3}dx = 0 \rightarrow (1)$$

$(1) \div y^2$

$$\frac{y\,dx - x\,dy}{y^2} - \frac{3x^2y^2e^{x^3}dx}{y^2} = 0$$

$$d\left(\frac{x}{y}\right) - \frac{3x^2e^{x^3}dx}{1} = 0$$

$e^{x^3}dx = 3x^2e^{x^3}$

$$d\left(\frac{x}{y}\right) - d\left(e^{x^3}\right) = 0$$

By $\int$ing

$$\int d\left(\frac{x}{y}\right) - \int d\left(e^{x^3}\right) = c$$

$$\frac{x}{y} - e^{x^3} = c$$

$\therefore$ The soln of the given equ is

$$\frac{x}{y} - e^{x^3} = c$$

---

6) solve :

$$(2xy + y - \tan y)\,dx + (x^2 - x\tan^2 y + \sec^2 y)\,dy = 0$$

soln :

Given equ is

$$(2xy + y - \tan y)\,dx + (x^2 - x\tan^2 y + \sec^2 y)\,dy = 0$$

$$2xy\,dx + y\,dx - \tan y\,dx + x^2dy - x\tan^2 y\,dy + \sec^2 y\,dy = 0$$

$$d(x^2y) + y\,dx - \tan y\,dx - x\tan^2 y\,dy + \sec^2 y\,dy + x\,dy - x\,dy = 0$$

$d(x^2y) + d(xy) - \tan y\, dx - x\,(\tan^2 y + 1)\, dy + \sec^2 y\, dy = 0$

$d(x^2y) + d(xy) - \tan y\, dx - x\sec^2 y\, dy + \sec^2 y\, dy$

$d(x^2y) + d(xy) - \tan y\, dx + \sec^2 y(1-x)\, dy = 0$

$d(x^2y) + d(xy) + d[\underbrace{\tan y}_{u}\, \underbrace{(1-x)}_{v}] = 0$

$\int d(x^2y) + \int d(xy) + \int d[\tan y\,(1-x)] = c$

$x^2y + xy + \tan y\,(1-x) = c$

$\therefore$ The solution of given equ is

$x^2y + xy + \tan y(1-x) = c$

---

7) Solve: $(y^2 e^{xy^3} + 4x^3)\,dx + (2xy\,e^{xy^2} - 3y^2)\,dy = 0$

Soln :

$(y^2 e^{xy^3} + 4x^3)\,dx + (2xy\,e^{xy^2} - 3y^2)\,dy = 0$

$y^2 e^{xy^3} dx + 4x^3 dx + 2xy\,e^{xy^2} dy - 3y^2 dy =$

$4x^3 dx - 3y^2 dy + e^{xy^2}(y^2 dx + 2xy\,dy) = 0$

$4x^3 dx - 3y^2 dy + d(e^{xy^2}) = 0$

$\int 4x^3 dx - \int 3y^2 dy + \int d(e^{xy^2}) = c$

$\frac{4x^4}{4} - \frac{3y^3}{3} + e^{xy^2} = c$

$x^4 - y^3 + e^{xy^2} = c$

$\therefore$ The solution of the given equ is $x^4 - y^3 + e^{xy^2} = c.$

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8) Solve :

$$ydx + (x+y)dy = 0$$

Soln :

Given equ is

$$ydx + (x+y)dy = 0 \rightarrow (1)$$

$$ydx + xdy + ydx = 0$$

$$d(xy) + ydy = 0$$

$$\int d(xy) + \int ydy = 0$$

$$xy + y^2/2 = c/2$$

$$2xy + y^2 = c$$

---

9) Solve : $x^2y^3dx + (x^3y^2 - 2)dy = 0$

Soln :

$$x^2y^3dx + (x^3y^2 - 2)dy = 0 \rightarrow (1)$$

Here, $M = x^2y^3$; $N = x^3y^2 - 2$

$$\frac{\partial M}{\partial y} = 3x^2y^2 \; ; \; \frac{\partial N}{\partial x} = 3x^2y^2$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$\therefore$ (1) is exact

By $\int$ing

$$\int x^2y^3dx + \int (x^3y^2 - 2)dy = c$$

$$\frac{y^3x^3}{3} - 2y = c$$

$\therefore$ The solution of given equ is

$$\frac{y^3x^3}{3} - 2y = c$$

3) $(1+xy^2)dx + (1+x^2y)dy = 0$

Soln:

Given equ is

$$(1+xy^2)dx + (1+x^2y)dy = 0$$

$$dx + xy^2dx + dy + x^2y\,dy = 0$$

$$dx + dy + xy^2dx + x^2y\,dy = 0$$

$$dx + dy + xy(y\,dx + x\,dy) = 0$$

$$dx + dy + xy\,d(xy) = 0$$

By $\int$ing

$$\int dx + \int dy + \int xy\,d(xy) = c$$

$$x + y + \frac{(xy)^2}{2} = c$$

$\therefore$ The solution of given equ

$$x + y + \frac{(xy)^2}{2} = c$$

---

4) Solve: $(x^2 - yx^2)\frac{dy}{dx} + (y^2 + x^2y^2) = 0$

Soln:

$$(x^2 - yx^2)\frac{dy}{dx} + (y^2 + x^2y^2) = 0$$

$$(x^2 - yx^2)dy + (y^2 + x^2y^2)dx = 0$$

$$x^2dy - yx^2dy + y^2dx + x^2y^2dx = 0 \rightarrow (1)$$

$(1) \times \frac{1}{x^2y^2} \Rightarrow$

$$\frac{x^2}{x^2y^2}dy - \frac{yx^2}{x^2y^2}dy + \frac{y^2}{x^2y^2}dx + \frac{x^2y^2}{x^2y^2}dx = 0$$

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$$\frac{1}{y^2}dy - \frac{1}{y}dy + \frac{1}{x^2}dx + dx = 0$$

By $\int$ing

$$\int\frac{1}{y^2}dy - \int\frac{1}{y}dy + \int\frac{1}{x^2}dx + \int dx = c$$

$$-\frac{1}{y} - \log y - \frac{1}{x} + x = c$$

$\therefore$ The solution of the given equ is

$$-\frac{1}{y} + \log y - \frac{1}{x} + x = c$$

---

5) solve : $(x^2 - x + y^2)dx - (ye^y - 2xy)dy = 0$

soln :

The given equ is

$$(x^2 - x + y^2)dx - (ye^y - 2xy)dy = 0$$

$$(x^2 - x + y^2)dx - (ye^y - 2xy)dy = 0$$

$$x^2dx - xdx + y^2dx - ye^ydy + 2xydy = 0$$

$$x^2dx - xdx + d(xy^2) - ye^ydy = 0$$

$\int u\,dv = uv - \int v\,du$

$u = y \quad dv = e^y dy$

$du = dy \quad v = e^y$

By $\int$ing

$$\int x^2dx - \int xdx + \int d(xy^2) - \int ye^ydy = c$$

$$\frac{x^3}{3} - \frac{x^2}{2} + xy^2 - [ye^y - \int e^ydy] = c$$

$$\frac{x^3}{3} - \frac{x^2}{2} + xy^2 - ye^y + e^y = c$$

$\therefore$ The solution of given equ is

$$\frac{x^3}{3} - \frac{x^2}{2} + xy^2 - ye^y + e^y = c$$

6) solve : $(x^2+y^2)(xdx+ydy) = a^2(xdy-ydx)$

Soln :

The given equ is

$(x^2+y^2)(xdx+ydy) + a^2(xdy-ydx) \rightarrow (1)$

$(1) \times \frac{1}{x^2+y^2} \Rightarrow \frac{(x^2+y^2)(xdx+ydy)}{x^2+y^2} = a^2(xdy-y$

$$xdx+ydy = a^2\frac{(xdy-ydx)}{x^2+y^2}$$

$$xdx+ydy = a^2 d[\tan^{-1} y/x]$$

By $\int$ing

$$\int xdx + \int ydy = a^2 \int d(\tan^{-1} y/x)$$

$$\frac{x^2}{2} + \frac{y^2}{2} = a^2 \tan^{-1}(y/x) + c$$

$\therefore$ The solution of the given equ is

$$\frac{x^2}{2} + \frac{y^2}{2} = a^2 \tan^{-1}(y/x) + c$$

---

7) Solve :

$[xy(\sin xy) + \cos xy]ydx + [xy\sin xy - \cos xy]xdy = 0$

Soln :

Given equ is

$[xy(\sin xy) + \cos xy]ydx + [xy\sin xy - \cos xy]xdy = 0 \rightarrow (1)$

Here,

$M = xy^2\sin xy + y\cos xy$; $N = x^2y\sin xy - x\cos xy$

$M_x - N_y = x^2y^2\sin xy + xy\cos xy - x^2y^2\sin xy + xy\cos xy$

$$M_x - N_y = 2xy\cos xy$$

$$I.F = \frac{1}{Mx - Ny}$$

$$= \frac{1}{2xy\cos xy}$$

$$(1) \times \frac{1}{2xy\cos xy}$$

$$\Rightarrow \left[\frac{xy^2\sin xy}{2xy\cos xy} + \frac{y\cos xy}{2xy\cos xy}\right]dx + \left[\frac{x^2y\sin xy}{2xy\cos xy} - \frac{x\cos xy}{2xy\cos xy}\right]dy = 0$$

$$\left(\frac{y}{2}\tan xy + \frac{1}{2x}\right)dx + \left(\frac{x}{2}\tan xy - \frac{1}{2y}\right)dy = 0$$

By $\int$ing

$$\frac{y}{2}\log(\sec xy) + \frac{1}{2}\log x - \frac{1}{2}\log y = c$$

$$y\log(\sec xy) + \log x - \log y = 2c$$

$$y\log(\sec xy) + \log(x/y) = c$$

$\therefore$ The solution of given equ is

$$y\log(\sec xy) + \log(x/y) = c$$

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Unit - II

Second order differential equations with constant co. efficients :

Form :

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = f(x) \text{ (or)}$$

$$(aD^2 + bD + c)y = f(x)$$

where $D = \frac{d}{dx}$; a, b, c are constant

Working rule :

$$(aD^2 + bD + c)y = 0$$

Auxillary equ is

$$am^2 + bm + c = 0$$

$$m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$m = m_1, m_2$$

case (i) :

m_1, m_2 are real and distinct

(i.e) $m_1 \neq m_2$

complementary function (C.F) $= Ae^{m_1 x} + Be^{m_2 x}$

case (ii) :

m_1, m_2 are real and equal

(i.e) $m_1 = m_2$

$$C.F = e^{m_1 x}(Ax + B)$$

case (iii) :

m_1, m_2 are complex roots

(i.e) $m_1 = \alpha + i\beta$, $m_2 = \alpha - i\beta$

$\therefore C.F = e^{\alpha x}(A\cos\beta x + B\sin\beta x)$

find particular of $(aD^2+bD+c)y = e^{ax}$

working rule :

case (i) : If $\alpha \neq m_1$ & $\alpha \neq m_2$

(i.e) $a\alpha^2 + b\alpha + c \neq 0$

then particular integral $(P.I) = \dfrac{e^{\alpha x}}{a\alpha^2+b\alpha+c}$

case (ii) :

If $\alpha = m_1$ (or) $\alpha = m_2$

(i.e) $a\alpha^2 + b\alpha + c = 0$

then $P.I = \dfrac{1}{(D-m_1)(D-m_2)} e^{\alpha x}$

$P.I = \dfrac{x}{\alpha - m_2} e^{\alpha x}$ (or) $\dfrac{x}{\alpha - m_1} e^{\alpha x}$

case (iii) :

If $\alpha = m_1$ & $\alpha = m_2$

(i.e) $a\alpha^2 + b\alpha + c = 0$

then $P.I = \dfrac{x^2}{2!} e^{\alpha x}$.

$\therefore$ The solution of $(aD^2+bD+c)y = f(x)$

is $y = C.F + P.I$

1) Solve : $(D^2 - 5D + 4)y = 0$

Soln :

Given equ is

$$(D^2 - 5D + 4)y = 0$$

Auxillary equ is

$$\frac{-1}{m} \Big| \frac{-4}{m}$$

$$m^2 - 5m + 4 = 0$$

$$(m-1)(m-4) = 0$$

$$m = 1, m = 4$$

$$m_1 = 1, m_2 = 4$$

Here m_1, m_2 are real and distin

$$\therefore C.F = Ae^{m_1 x} + Be^{m_2 x}$$

$$C.F = Ae^{x} + Be^{4x}$$

$\therefore$ The solution of given equ is

$$y = Ae^{x} + Be^{4x}$$

---

2) Solve : $(D^2 + 5D + 6)y = e^{x}$

x co.eff is 1

$1 \times x = x$

in P.I

Soln :

Given equ is

$$(D^2 + 5D + 6)y = e^{x}$$

Auxillary equ is

$$\frac{3}{m} \Big| \frac{2}{m}$$

$$m^2 + 5m + 6 = 0$$

$$(m+3)(m+2) = 0$$

$$m_1 = -3, m_2 = -2$$

Here m_1, m_2 are real and distinct

$\therefore C.F = Ae^{m_1 x} + Be^{m_2 x}$

$C.F = Ae^{-3x} + Be^{-2x}$

$P.I = \frac{1}{D^2+5D+6} e^{1(x)}$

$= \frac{1}{D^2+5D+6} e^{x}$

$\alpha = D$
$\alpha = 1$ so
$D = 1$

$= \frac{e^x}{(D+3)(D+2)}$

$= \frac{e^x}{(1+3)(1+2)}$

$= \frac{e^x}{(+3)(+4)}$

$P.I = \frac{e^x}{12}$

$\therefore$ The solution of given equ is

$y = C.F + P.I$

$y = Ae^{-3x} + Be^{-2x} + \frac{e^x}{12}$

---

3) Solve : $(D^2 - 4D + 4)y = 0$

Soln : Given equ is

$(D^2 - 4D + 4)y = 0$

Auxillary equ is

$m^2 - 4m + 4 = 0$

$\frac{-2}{m} \Big| \frac{-2}{m}$

$(m-2)(m-2) = 0$

$m_1 = 2, m_2 = 2$

$\therefore$ Here m_1, m_2 are real and equal

$$\therefore C.F = e^{m_1 x}(Ax+B)$$
where $m_1 = m_2$

$$C.F = e^{2x}(Ax+B)$$

$\therefore$ The solution of the given equ is
$$y = e^{2x}(Ax+B).$$

4) solve : $(2D^2 - 3D + 4)y = 0$

soln :

Given equ is
$$(2D^2 - 3D + 4)y = 0$$
Auxillary is equ is
$$2m^2 - 3m + 4 = 0$$
$$m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$a = 2 \; ; b = -3 \; ; \; c = 4$$
$$m = \frac{+3 \pm \sqrt{9 - 32}}{4}$$
$$= \frac{3 \pm \sqrt{-23}}{4}$$
$$m = \frac{3 \pm i\sqrt{23}}{4}$$
$$m_1 = \frac{3 + i\sqrt{23}}{4} \; ; \; m_2 = \frac{3 - i\sqrt{23}}{4}$$
$$\alpha = \frac{3}{4} \; ; \; \beta = \frac{\sqrt{23}}{4}$$

$\therefore$ Here m_1, m_2 are complex roots

$$\therefore C.F = e^{\alpha x}(A \cos\beta x + B \sin\beta x)$$

$$C.F = e^{3/4x}\left(A\cos\frac{\sqrt{23}}{4}x + B\sin\frac{\sqrt{23}}{4}x\right)$$

$\therefore$ The solution of the given equ

$$y = e^{\frac{3}{4}x}\left(A\cos\frac{\sqrt{23}}{4}x + B\sin\frac{\sqrt{23}}{4}x\right)$$

Solve : $(D^2 - 5D + 6)y = e^{4x}$

soln :

Given equ is

$$(D^2 - 5D + 6)y = e^{4x}$$

Auxillary equ is

$$m^2 - 5m + 6 = 0$$

$$\frac{-2}{m} \Big| \frac{-3}{m}$$

$$(m-2)(m-3) = 0$$

$$m_1 = 2 \; ; m_2 = 3$$

Here m_1, m_2 are real and distinct

$$\therefore C.F = Ae^{m_1 x} + Be^{m_2 x}$$

$$C.F = Ae^{2x} + Be^{3x}$$

$$P.I = \frac{1}{D^2 - 5D + 6} e^{4x}$$

$$= \frac{e^{4x}}{(D-2)(D-3)}$$

$$= \frac{e^{4x}}{(4-2)(4-3)}$$

$$= \frac{e^{4x}}{(2)(1)}$$

$$P.I = \frac{e^{4x}}{2}$$

$\therefore$ The solution of the given is

$$y = C.F + P.I$$

$$y = Ae^{3x} + Be^{2x} + \frac{e^{4x}}{2}$$

6) Solve : $(D^2 + 2D + 1)y = 2e^{3x}$

Soln :

Given equ is

$$(D^2 + 2D + 1)y = 2e^{3x}$$

Auxillary equ is

$$D = m = \alpha$$

$$\frac{1}{m} \Big| \frac{1}{m}$$

$$m^2 + 2m + 1 = 0$$

$$(m+1)(m+1) = 0$$

$$m_1 = -1 \ , \ m_2 = -1$$

Here m_1, m_2 are equal and real

$$C.F = e^{m_1 x}(Ax + B)$$

$$= e^{-x}(Ax + B)$$

$$P.I = \frac{1}{D^2 + 2D + 1} 2e^{3x}$$

$$= \frac{1}{(D+1)(D+1)} 2e^{3x}$$

$$= \frac{1}{(3+1)(3+1)} 2e^{3x}$$

$$= \frac{2e^{3x}}{(4)(4)}$$

$$= \frac{\cancel{2}e^{3x}}{\cancel{16}\,8}$$

$$P.I = \frac{e^{3x}}{8}$$

$\therefore$ The solution of the given equ is

$$y = C.F + P.I$$

$$y = e^{-x}(Ax+B) + \frac{e^{3x}}{8}$$

---

7) Solve : $(3D^2 + D - 14)y = 13e^{2x}$

Soln :

Given equ is

$$(3D^2 + D - 14)y = 13e^{2x}$$

Auxillary equ is.

$$D = m$$

$$3m^2 + m - 14 = 0$$

$$m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

where ; $a = 3, b = 1, c = -14$

$$m = \frac{-1 \pm \sqrt{1 - 4(3)(-14)}}{2(3)}$$

$$m = \frac{-1 \pm \sqrt{1+168}}{2(3)}$$

$$= \frac{-1 \pm \sqrt{169}}{2(3)}$$

$$= \frac{-1 \pm 13}{6}$$

$$m_1 = \frac{-1-13}{6} \quad ; \quad m_2 = \frac{-1+13}{6}$$

$$m_1 = \frac{-14}{6} = \frac{-7}{3} \quad ; \quad m_2 = \frac{12}{6} = 2$$

Here m_1 & m_2 are real and district

$$C.F = Ae^{m_1 x} + Be^{m_2 x}$$

$$C.F = Ae^{-7/3\,x} + Be^{2x}$$

$$P.I = \frac{1}{3D^2+D-14}\, 13e^{2x}$$

$$= \frac{13e^{2x}}{(D-2)(3D+7)}$$

$$= \frac{13e^{2x}}{(2-2)(6+7)} \qquad [\alpha = 2] \quad (\therefore D = 2)$$

$$= \frac{x\,\cancel{13}e^{2x}}{\cancel{13}}$$

$$P.I = xe^{2x}$$

The solution of the given equ is

$$y = C.F + P.I$$

$$y = Ae^{-7/3\,x} + Be^{2x} + xe^{2x}.$$

To find the P.I of

$$(AD^2 + BD + c)y = \cos ax \text{ (or) } \sin ax$$

case (i) :

If $\phi(-a)^2 \neq 0$

$$\frac{1}{\phi(D^2)} \sin ax = \frac{1}{\phi(-a^2)} \sin ax$$

Here $D^2 = a^2$

$$||^{ly} \quad \frac{1}{\phi(D^2)} \cos ax = \frac{1}{\phi(D^2)} \cos ax$$

case (ii) :

Let $\phi(-a^2) = 0$

$\therefore D^2 + a^2$ is a factor of $\phi(D^2)$

Hence $\phi(D^2) = (D^2 + a^2)\, \psi(D^2)$, where $\psi(-a^2) \neq 0$

$$\frac{1}{\phi(D^2)} \sin ax = \frac{1}{(D^2+a^2)\,\psi(D^2)} \sin ax$$

$$= \frac{1}{\psi(-a^2)(D^2+a^2)} \sin ax$$

Now $\frac{1}{D^2+a^2} \sin ax = \frac{1}{D^2+a^2}$ imaginary part of e^{iax}

[Euler's formula: $e^{iax} = \cos ax + i \sin ax$]

$\frac{1}{D^2+a^2} \sin ax$ = imaginary part of $\frac{1}{D^2+a^2} e^{iax}$

= imaginary part of $\frac{1}{(D+ia)(D-ia)} e^{iax}$

= imaginary part of $\frac{1}{(D-ai)2ai} e^{iax}$

$\times$ & $\div$ by (i) $\Rightarrow$ = imaginary part of $\frac{x}{2ai} e^{iax}$

= imaginary part of $-\frac{xi}{2a}(\cos ax + i \sin ax)$

$$\frac{1}{D^2+a^2} \sin ax = \frac{-x \cos ax}{2a}$$

$||^{ly}$ $$\frac{1}{D^2+a^2} \cos ax = \frac{x \sin ax}{2a}$$

---

1) Solve : $(D^2 - 8D + 9)y = 8 \sin 5x$

Soln :

Given equ is

$(D^2 - 8D + 9)y = 8 \sin 5x$

Auxillary equ is,

$D = m$

$m^2 - 8m + 9 = 0$

$$m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

where $a=1, b=-8, c=9$

$$m = \frac{8 \pm \sqrt{64-4(9)}}{2(1)}$$

$$= \frac{8 \pm \sqrt{64-36}}{2}$$

$$= \frac{8 \pm \sqrt{28}}{2} = \frac{8 \pm 2\sqrt{7}}{2}$$

$$m = 4 \pm \sqrt{7}$$

$m_1 = 4+\sqrt{7}$; $m_2 = 4-\sqrt{7}$

Here m_1 & m_2 are real and distinct

$$C.F = Ae^{m_1 x} + Be^{m_2 x}$$

$$C.F = Ae^{4+\sqrt{7}} + Be^{4-\sqrt{7}}$$

$$P.I = \frac{1}{D^2-8D+9} \, 8 \sin 5x$$

$$= \frac{1}{-25-8(5)+9} \, 8 \sin 5x \qquad \begin{aligned} D^2 &= -\alpha^2 \\ &= -5^2 \\ &= -25 \end{aligned}$$

$$= \frac{1}{-25-8D+9} \, 8 \sin 5x$$

$$= \frac{8 \sin 5x}{-16-8D}$$

$$= \frac{\cancel{8} \sin 5x}{-\cancel{8}(D+2)} = -\frac{\sin 5x}{D+2}$$

$\times$ & $\div$ by $(D-2)$ $$= \frac{-1\,(D-2)\sin 5x}{(D+2)(D-2)}$$

$$= \frac{-(D-2)\sin 5x}{D^2-4}$$

$$= \frac{-(D-2)\sin 5x}{-25-4}$$

$$= \frac{\cancel{-}(D-2)\sin 5x}{\cancel{-}29}$$

$$= \frac{D(\sin 5x) - 2\sin 5x}{29}$$

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$$= \frac{\cos 5x\,(5) - 2\sin 5x}{29}$$

$$P.I = \frac{5\cos 5x - 2\sin 5x}{29}$$

$\therefore$ The solution of the given equ is

$$y = C.F + P.I$$

$$y = Ae^{4+\sqrt{7}x} + Be^{4-\sqrt{7}x} + \frac{5\cos 5x - 2\sin 5x}{29}$$

---

2) Solve : $(D^2+D+1)y = \sin 2x$

Soln :

Given equ is

$$(D^2+D+1)y = \sin 2x$$

Auxillary equ is,

$$D = m$$

$$m^2+m+1=0$$

$$m = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

where; $a=1$; $b=1$; $c=1$

$$m = \frac{-1 \pm \sqrt{1-4(1)(1)}}{2(1)}$$

$$= \frac{-1 \pm \sqrt{1-4}}{2}$$

$$= \frac{-1 \pm \sqrt{-3}}{2}$$

$$m = \frac{-1 \pm i\sqrt{3}}{2}$$

Here m is complex,

$$\alpha = -1/2 \;;\; \beta = \sqrt{3}/2$$

$$C.F = e^{\alpha x}(A\cos\beta x + B\sin\beta x)$$

$$C.F = e^{-1/2\,x}\left(A\cos\frac{\sqrt{3}}{2}x + B\sin\frac{\sqrt{3}}{2}x\right)$$

$$P.I = \frac{1}{(D^2+D+1)}\sin 2x$$

$$D^2 = -4$$

$$= \frac{1}{(-4+D+1)}\sin 2x$$

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$$= \frac{1}{(D-3)} \sin 2x \quad \Rightarrow = \times \text{ \& } \div \text{ by } (D+3)$$

$$= \frac{(D+3)\sin 2x}{(D+3)(D-3)}$$

$$= \frac{D(\sin 2x) + 3\sin 2x}{D^2-9}$$

$$= \frac{\cos 2x(2) + 3\sin 2x}{(-4-9)}$$

$$P.I = \frac{2\cos 2x + 3\sin 2x}{-13}$$

∴ The solution of the given equ is

$$y = C.F + P.I$$

$$y = e^{-1/2x}\left(A\cos\frac{\sqrt{3}}{2}x + B\sin\frac{\sqrt{3}}{2}x\right) - \left(\frac{2\cos 2x + 3\sin 2x}{13}\right)$$

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3) solve : $(D^2+4)y = \sin 2x$

soln :

Given equ is

$(D^2+4)y = \sin 2x$

Auxillary equ is

$D = m$

$m^2+4=0$

$m^2 = -4$

$m = \pm\sqrt{-4}$ $\alpha:0$ real

$m = \pm i2 = \beta$ imaginary

Here m is complex

$C.F = e^{\alpha x}(A\cos\beta x + B\sin\beta x)$

$\alpha = 0$, $\beta = 2$

$= e^{0x}(A\cos 2x + B\sin 2x)$

$C.F = A\cos 2x + B\sin 2x$

$P.I = \frac{1}{D^2+4}\sin 2x$

$=$ imaginary part of $\frac{1}{D^2+4}e^{i2x}$

$D = 2i$ $=$ imaginary part of $\frac{e^{i2x}}{(D+2i)(D-2i)}$

$=$ imaginary part of $\frac{e^{i2x}}{(2i+2i)\underbrace{(2i-2i)}_{0=x}}$

$=$ imaginary part of $\frac{x\, e^{i2x}}{2i+2i}$

$\times$ & $\div$ by (i) $=$ imaginary part of $\frac{xi e^{i2x}}{4i \times i}$

$=$ imaginary part of $\frac{ix\, e^{i2x}}{4(i)^2}$

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$= \text{imaginary part of } \frac{-ix}{4} e^{i2x}$

$= \text{imaginary part of } \frac{-ix}{4} (\cos 2x + i \sin 2x)$

$= \text{imaginary part of } \left\{ -\frac{ix\cos 2x}{4} - \frac{i^2 x}{4} \sin 2x \right\}$

$= \text{imaginary part of } \left\{ \frac{-ix\cos 2x}{4} + \frac{x}{4} \sin 2x \right\}$

$$P.I = \frac{-x\cos 2x}{4}$$

$$y = C.F + P.I$$

$$y = A\cos 2x + B\sin 2x - \frac{x\cos 2x}{4}$$

---

4) Solve : $(D^2 - 8D + 9)y = 8\cos 5x$

Soln: Given equ is

$$(D^2 - 8D + 9)y = 8\cos 5x$$

Auxillary equ is,

$$D = m$$

$$m^2 - 8m + 9 = 0$$

$$m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

where $a = 1$, $b = -8$, $c = 9$

$$m = \frac{8 \pm \sqrt{64 - 4(1)(9)}}{2(1)}$$

$$= \frac{8 \pm \sqrt{64 - 36}}{2} = \frac{8 \pm \sqrt{28}}{2}$$

$$m = \frac{8 \pm 2\sqrt{7}}{2} = 4 \pm \sqrt{7}$$

$$m_1 = 4 + \sqrt{7}, \quad m_2 = 4 - \sqrt{7}$$

Here m_1 & m_2 are real and distinct

$$C.F = Ae^{m_1 x} + Be^{m_2 x}$$

$$C.F = Ae^{4+\sqrt{7}x} + Be^{4-\sqrt{7}x}$$

$$P.I = \frac{1}{(D^2-8D+9)} 8\cos 5x$$

$$= \frac{1}{-25-8D+9} 8\cos 5x \qquad D^2 = -25$$

$$= \frac{8\cos 5x}{-8D-16} = \frac{8\cos 5x}{-8(D+2)}$$

$\times$ & $\div$ by $(D-2)$ $$= -\frac{\cos 5x\,(D-2)}{(D-2)(D+2)}$$

$$= \frac{-D(\cos 5x) + 2\cos 5x}{(D^2-4)}$$

$$= \frac{\sin 5x(5) + 2\cos 5x}{-25-4}$$

$$P.I = \frac{5\sin 5x + 2\cos 5x}{-29}$$

$\therefore$ The solution of the given equ is,

$$y = C.F + P.I$$

$$y = Ae^{4+\sqrt{7}x} + Be^{4-\sqrt{7}x} - \left(\frac{5\sin 5x - 2\cos 5x}{29}\right)$$

To find the particular integral of x^n

The given differential equ is

$$(aD^2 + bD + c)y = x^n$$

$$P.I = \frac{x^n}{aD^2 + bD + c}$$

$$= \frac{x^n}{c\left(1 + \frac{b}{c}D + \frac{a}{c}D^2\right)}$$

$$= c\left(1+\frac{b}{c}D+\frac{a}{c}D^2\right)^{-1}(x^n)$$

---

1) solve : $(D^2+D+1)y = x$

Soln :

Given equ is

$$(D^2+D+1)y = x$$

Auxiliary equ is

$$D=m$$

$$m^2+m+1=0$$

$$m = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

where $a=1$; $b=1$; $c=1$

$$m = \frac{-1 \pm \sqrt{1-4}}{2}$$

$$m = \frac{-1 \pm i\sqrt{3}}{2}$$

$$\alpha = -1/2 \quad ; \beta = \sqrt{3}/2$$

Here m is complex

$$C.F = e^{\alpha x}(A\cos\beta x + B\sin\beta x)$$

$$= e^{-1/2\,x}\left(A\cos\frac{\sqrt{3}x}{2} + B\sin\frac{\sqrt{3}}{2}x\right)$$

$$P.I = \frac{1}{D^2+D+1}(x)$$

$(1+x)^{-1} = 1-x+x^2-\cdots$

$(1-x)^{-1} = 1+x+x^2+\cdots$

$$= (1+\underbrace{D+D^2}_{x})^{-1}x$$

$$= [1-(D+D^2)+(D+D)^2\ldots]x$$

$= x - D(x) - D^2(x) + \cdots$

$= x - 1 - 0$

$P.I = x - 1$

$\therefore$ The solution of the given equ is

$y = C.F + P.I$

$y = [e^{-1/2 x}]\left(A\cos\frac{\sqrt{3}}{2}x + B\sin\frac{\sqrt{3}}{2}x\right) + (x-1)$

---

2) solve : $(D^2-1)y = 2+5x$

soln :

Given equ is

$(D^2-1)y = 2+5x$

Auxiliary equ is

$D = m$

$m^2 - 1 = 0$

$m^2 = 1$ $\quad m_1 = 1$

$m = \pm\sqrt{1} \Rightarrow m = \pm 1$ $\quad m_2 = -1$

Here m_1 & m_2 are equal and distinct

$C.F = Ae^{m_1 x} + Be^{m_2 x}$

$C.F = Ae^{x} + Be^{-x}$

$P.I = \frac{1}{(D^2-1)}(2+5x)$

$= \frac{1}{(D^2-1)} 2e^{0x} + \frac{1}{(D^2-1)}(5x)$

$D = 0$

$= \frac{2e^{0x}}{(0-1)} + \frac{5x}{-(1-D^2)}$

$= -2\underbrace{e^{0x}}_{1} - (1-D^2)^{-1}(5x)$

$= -2 - [1 + D^2 + (D^2)^2 + \cdots](5x)$

3) solve : $(D^3 - D^2 - D + 1)y = 1 + x^2$

soln :

Given equ is

$(D^3 - D^2 - D + 1)y = 1 + x^2$

Auxillary equ is

$$\begin{array}{c|cccc} 1 & 1 & -1 & -1 & 1 \\ & 0 & 1 & 0 & -1 \\ \hline & 1 & 0 & -1 & \boxed{0} \end{array}$$

$m^3 - m^2 - m + 1 = 0$

$(m-1)(m^2-1) = 0$

$(m-1)(m-1)(m+1) = 0$

$m = 1, 1, -1$

$m_1 = 1 \; ; m_2 = 1 \; ; m_3 = -1$

Here m_1, m_2 are real and m_2 is distinct

$C.F = e^{x}(Ax + B) + Ce^{-x}$

$P.I = \frac{1}{[1 + (D^3 - D^2 - D)]}(1 + x^2)$

$= [1 + (D^3 - D^2 - D)]^{-1}(1 + x^2)$

$= [1 - (D^3 - D^2 - D) + (D^3 - D^2 - D)^2 \ldots]$
$[1 + x^2]$

$= [1 + x^2 - D^3(1 + x^2) + D^2(1 + x^2) +$
$D(1 + x^2) + D^2(1+ \ldots$

$= 1 + x^2 - 0 + 2 + 2x + 2$

$P.I = 5 + 2x + x^2$

$\therefore$ The solution of the given equ is

$y = C.F + P.I$

$y = e^x(Ax+B) + Ce^{-x} + (5+2x+x^2).$

---

4) solve : $(D^2+16)y = e^{-3x} + \cos 4x$

soln : Given equ is,

$(D^2+16)y = e^{-3x} + \cos 4x$

Auxillary equ is,

$D = m$

$m^2 + 16 = 0$

$m^2 = -16$

$m = \pm 4i$

Here m is complex

$C.F = e^{\alpha x}(A\cos\beta x + B\sin\beta x)$

$C.F = e^{0x}(A\cos 4x + B\sin 4x)$

$C.F = A\cos 4x + B\sin 4x$

$P.I = \frac{1}{D^2+16} e^{-3x} + \cos 4x$

$= \frac{e^{-3x}}{D^2+16} + \frac{\cos 4x}{D^2+16}$ $D^2 = -\alpha^2$

$= \frac{e^{-3x}}{9+16} + \frac{x}{2(4)}\sin 4x$ $D^2 = -9$

$P.I = \frac{e^{-3x}}{25} + \frac{x}{8}\sin 4x$

$\therefore$ The solution of the given equ is

$$y = C.F + P.I$$

$$y = A\cos 4x + B\sin 4x + \frac{e^{-3x}}{25} + \frac{x}{8}\sin 4x$$

5) solve : $(D^2 - 4D - 5)y = e^{2x} + 3\cos 4x$

Soln :

Given equ is

$$(D^2 - 4D - 5)y = e^{2x} + 3\cos 4x$$

Auxillary equ is

$$D = m$$

$$m^2 - 4m - 5 = 0$$

$$(m+1)(m-5) = 0 \qquad \frac{1}{m}\Big|\frac{-5}{m}$$

Here m_1 & m_2 are real and distinct

$$C.F = Ae^{m_1 x} + Be^{m_2 x}$$

$$C.F = Ae^{-x} + Be^{5x}$$

$$D^2 = -a^2$$
$$D^2 = -4^2$$
$$= -16$$

$$P.I = \frac{1}{D^2 - 4D - 5} + \frac{3\cos 4x}{D^2 - 4D - 5}$$

$D = 2$

$$= \frac{e^{2x}}{D^2 - 4D - 5} + \frac{3\cos 4x}{\underset{D^2}{\underbrace{-16}} - 4D - 5}$$

$$= \frac{e^{2x}}{4 - 4(2) - 5} + \frac{3\cos 4x}{-16 - 4D - 5}$$

$$= \frac{e^{2x}}{4 - 8 - 5} + \frac{3\cos 4x}{-21 - 4D}$$

$$= \frac{-e^{2x}}{9} + \frac{3\cos 4x}{-(4D + 21)}$$

$$= \frac{-e^{2x}}{9} - \frac{3\cos 4x}{4D + 21} \times \frac{4D - 21}{4D - 21}$$

$$= \frac{-e^{2x}}{9} - \frac{12D(\cos 4x) - 63\cos 4x}{16D^2 - 441}$$

$$= \frac{-e^{2x}}{9} - \left(\frac{-12\sin 4x(4) - 63\cos 4x}{16(-16) - 441}\right)$$

$$= \frac{-e^{2x}}{9} + \frac{48\sin 4x + 63\cos 4x}{-697}$$

$$P.I = \frac{-e^{2x}}{9} - \frac{48\sin 4x + 63\cos 4x}{697}$$

$\therefore$ The solution of the given equ is

$$y = C.F + P.I$$

$$y = Ae^{-x} + Be^{5x} - \frac{e^{2x}}{9} + \frac{1}{697}(48\sin 4x + 63\cos 4x)$$

---

6) Solve: $(D^3 - 2D + 4)y = e^x \cos x$.

Soln:

Given equ is

$$(D^3 - 2D + 4)y = e^x \cos x$$

Auxillary equ is,

$$\begin{array}{r|rrrr} -2 & 1 & 0 & -2 & 4 \\ & 0 & -2 & 4 & -4 \\ \hline & 1 & -2 & 2 & \boxed{0} \end{array}$$

$$D = m$$

$$m^3 - 2m + 4 = 0$$

$$(m+2)(m^2 - 2m + 2) = 0$$

$$m + 2 = 0 \quad ; m^2 - 2m + 2 = 0$$

$$m = -2$$

$$m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{2 \pm \sqrt{4 - 4(1)(2)}}{2(1)}$$

$$= \frac{2 \pm \sqrt{4-8}}{2} = \frac{2 \pm i\sqrt{4}}{2}$$

$= \frac{2 \pm i2}{2}$

$m = 1 \pm i$

Here m is complex

$C.F = Ae^{m_1 x} + e^{\alpha x}(B\cos\beta x + C\sin$

$= e^{\alpha x}(A\cos\beta x + B\sin\beta x + Ce$

$P.I = \frac{1}{D^3 - 2D + 4} e^x \cos x \qquad e^{ix}$

$= e^x \frac{1}{(D+1)^3 - 2(D+1) + 4} \cos x$

$= e^x \frac{1}{D^3 + 3D^2 + 3D + 1 - 2D - 2 + 4}$

$= e^x \frac{1}{D^3 + 3D^2 + D + 3} \cos x$

$P.I = e^x \frac{1}{D^2(D+3)(D+3)} \cos x$

$= e^x \frac{1}{(D+3)(D^2+1)} \cos x$

$\times$ by $(D-3)$ $= e^x \frac{(D-3)}{(D^2-9)(D^2+1)} \cos x$

$= e^x \frac{(D-3)}{(-1-9)(D^2+1)} \cos x$

$= \frac{e^x}{-10} \frac{(D-3)}{D^2+1} \cos x$

$P.I = \frac{e^x}{-10} \left\{ \text{Real part } \frac{D-3}{(D+i)(D-i)} e^{i} \right.$

$= \frac{-e^x}{10} \left\{ \text{Real part } \frac{x(D-3)}{(i+i)} e^{ix} \right.$

$$= \frac{-e^{x}}{10}\left\{\text{Real part } \frac{x(D-3)}{2i}(\cos x + i\sin x)\right\}$$

$$= \frac{-xe^{x}}{10}\left\{\text{Real part } \frac{(-\sin x + i\cos x - 3\cos x - 3i\sin x)}{2i}\right\}$$

$$= \frac{-xe^{x}}{10}\left\{\text{Real part } \frac{(-i\sin x - 3i\cos x - \cos x + 3\sin x)}{-2}\right\}$$

$$= \frac{-xe^{x}}{10}\left(\frac{-\cos x + 3\sin x}{-2}\right)$$

$$= \frac{xe^{x}}{20}(3\sin x - \cos x)$$

$\therefore$ The solution for given equ is

$$y = C.F + P.I$$

$$y = e^{\alpha x}(A\cos\beta x + B\sin\beta x + ce^{-2x}) + \frac{xe^{x}}{20}(3\sin x - \cos x).$$

---

7) Solve : $(D^2+1)y = x^2 e^{2x} + x\cos x$.

Soln :

The given equ is

$$(D^2+1)y = x^2e^{2x} + x\cos x$$

Auxillary equ is

$$m^2+1=0$$

$$m^2=-1 \Rightarrow m = \pm\sqrt{-1} \Rightarrow m = \pm i$$

$$C.F = e^{\alpha x}(A\cos\beta x + B\sin\beta x)$$

$$= A\cos x + B\sin x$$

$$P.I = \frac{1}{D^2+1}\, x^2e^{2x} + x\cos x$$

$$P.I_1 = \frac{1}{D^2+1} x^2 e^{2x} = e^{2x} \frac{1}{D^2+1} x^2 = e^{2x} \frac{1}{(D+2)^2+1} x^2$$

$$= e^{2x} \frac{1}{D^2+4D+4+1} x^2 = e^{2x} \frac{1}{D^2+4D+5} x^2$$

$$= e^{2x} \frac{1}{5\left(1+\frac{D^2+4D}{5}\right)} x^2 = \frac{e^{2x}}{5}\left(1+\frac{D^2+4D}{5}\right)^{-1} x^2$$

$$= \frac{e^{2x}}{5}\left[1-\left(\frac{D^2+4D}{5}\right)+\left(\frac{D^2+4D}{5}\right)^2\right] x^2$$

$$= \frac{e^{2x}}{5}\left[1-\left(\frac{D^2+4D}{5}\right)+\frac{D^4+16D^2+8D^3}{25}\right] x^2$$

$$= \frac{e^{2x}}{5}\left[x^2-\frac{1}{5}(2+8x)+\frac{16}{25}(2)\right]$$

$$= \frac{e^{2x}}{5}\left[x^2-\frac{2+8x}{5}+\frac{32}{25}\right]$$

$$= \frac{e^{2x}}{5}\left(\frac{25x^2-10-40x+32}{25}\right)$$

$$= \frac{e^{2x}}{125}\left(25x^2-40x-22\right)$$

$$P.I_2 = \frac{1}{D^2+1} x\cos x$$

$$= \text{Real part } \frac{1}{D^2+1} x \cdot e^{ix}$$

$$= \text{Real part } e^{ix} \frac{1}{(D+i)^2+1} x$$

$$= \text{Real part } e^{ix} \frac{1}{D^2+2i-1+1} x$$

$$= \text{real part } e^{ix} \frac{1}{D^2+2iD} x$$

$$= \text{Real part } e^{ix} \frac{1}{2iD\left(1+\frac{D^2}{2iD}\right)} x$$

$$= \text{Real part } e^{ix} \frac{1}{2iD\left(1+\frac{D}{2i}\right)} x$$

$$= \text{Real part } e^{ix} \frac{\left(1+\frac{D}{2i}\right)^{-1} x}{2iD}$$

$$= \text{Real part } e^{ix} \frac{\left(1-\frac{D}{2i}+\frac{D^2}{4i^2}-\frac{D^3}{(2i)^3}+\dots\right)}{2iD}$$

$$P.I_2 = \text{Real part } e^{ix}\left(\frac{i}{2iD} - \frac{D}{2i\times 2iD} + \frac{D^2}{4i^2\times 2iD} - \dots\right)x$$

$$= \text{Real part } e^{ix}\left(\frac{1}{2iD} - \frac{1}{4i^2} + \frac{D}{8i^3}\right) x$$

$$= \text{Real part } e^{ix}\left(\frac{x^2}{2i\times 2} - \frac{1}{4}x - \frac{1}{8i}\right)$$

$$= \text{Real part } e^{ix}\left(\frac{ix^2}{-4} + \frac{x}{4} + \frac{i}{8}\right)$$

$$= \text{Real part } (\cos x + i\sin x)\left(\frac{-ix^2}{4} + \frac{x}{4} + \frac{i}{8}\right)$$

$$P.I_2 = \text{Real part}\left[\frac{-ix^2\cos x}{4} + \frac{x\cos x}{4} + \frac{i\cos x}{8} - \frac{i^2 x^2 \sin x}{4} + \frac{ix\sin x}{4} + \frac{i^2 \sin x}{8}\right]$$

$$= \text{Real part}\left[\frac{-ix^2\cos x}{4} + \frac{x\cos x}{4} + \frac{i\cos x}{8} + \frac{x^2\sin x}{4} + \frac{ix\sin x}{4} - \frac{\sin x}{8}\right]$$

$$P.I_2 = \frac{x\cos x}{4} + \frac{x^2\sin x}{4} - \frac{\sin x}{8}$$

$\therefore$ The solution of the given equ is

$$y = C.F + P.I_1 + P.I_1$$

$$y = A\cos x + B\sin x + \frac{e^{2x}}{125}(25x^2 - 40x - 22) + \frac{x\cos x}{4} + \frac{x^2\sin x}{4} - \frac{\sin x}{8}$$

Linear equations with variable co-efficien

Form:

$$\frac{d^n y}{dx^n} + P_1 x^{n-1} \frac{d^{n-1}}{dx^n} y + \dots + P_n y = X$$

where, $P_1, P_2, P_3, P_4, \dots, P_n$ are constants

and X is a function of x.

By putting $z = \log x$ i.e: $x = e^z$

This equation can be transformed into one with constant co-efficients.

Let $\frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx}$

$$= \frac{1}{x} \cdot \frac{dy}{dz}$$

$$x \frac{dy}{dx} = \frac{dy}{dz} = Dy \quad \text{if} \quad D = \frac{d}{dz}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$$

$$= \frac{d}{dx}\left(\frac{1}{x}\frac{dy}{dz}\right)$$

$$\frac{d^2y}{dx^2} = \frac{1}{x}\frac{d^2y}{dz^2}\frac{dz}{dx} - \frac{1}{x^2}\frac{dy}{dz}$$

$$\frac{d^2y}{dx^2} = \frac{1}{x^2}\left(\frac{d^2y}{dz^2} - \frac{dy}{dz}\right)$$

$$\frac{d^2y}{dx^2} = \frac{1}{x^2}(D^2 - D)$$

$$x^2\frac{d^2y}{dx^2} = D^2 - D = D(D-1)$$

Ily

$$x^m \frac{d^m y}{dx^n} = D(D-1)(D-2)\dots(D-m+1)y$$

1) solve : $(D^2-4D+3)y = \sin 3x \cos 2x$

soln :

Given equ is

$(D^2-4D+3)y = \sin 3x \cos 2x$

Auxillary equ is,

$m^2-4m+3=0$ $\qquad \frac{-1}{m}\Big|\frac{-3}{m}$

$(m-1)(m-3)=0$

$m_1 = 1 \; ; \; m_2 = 3$

$C.F = Ae^{m_1x} + Be^{m_2x}$

$C.F = Ae^{x} + Be^{3x}$

$\sin 3x \cos 2x = \frac{\sin(3x+2x)+\sin(3x-2x)}{2}$

$\sin 3x \cos 2x = \frac{\sin 5x + \sin x}{2}$

$P.I = \frac{1}{D^2-4D+3} \sin 3x \cos 2x$

$= \frac{1}{D^2-4D+3}\left(\frac{\sin 5x + \sin x}{2}\right)$

$= \frac{1}{D^2-4D+3}\left(\frac{\sin 5x}{2}\right) + \frac{1}{D^2-4D+3}\left(\frac{\sin x}{2}\right)$

$= \frac{1}{-25-4D+3}\left(\frac{\sin 5x}{2}\right) + \frac{1}{-1-4D+3}\left(\frac{\sin x}{2}\right)$

$= \frac{1}{-22-4D}\left(\frac{\sin 5x}{2}\right) + \frac{1}{2-4D}\left(\frac{\sin x}{2}\right)$

$= \frac{1}{-2(11+2D)}\left(\frac{\sin 5x}{2}\right) + \frac{1}{+2(-2D+1)}\left(\frac{\sin x}{(2)}\right)$

$= \frac{(2D-11)(\sin 5x)}{-4(4D^2-121)} + \frac{(-2D-1)(\sin x)}{4(-2D+1)(-2D-1)}$

$= -\frac{(2D-11)\sin 5x}{4(4D^2-11^2)} - \frac{(2D+1)+\sin x}{4(4(-1)-1)}$

$$= \frac{-2D(\sin 5x) + 11\sin 5x}{4(4(-25)-121)} - \frac{2D(\sin x) + \sin x}{4(4(-1)-1)}$$

$$= \frac{-2\cos 5x \times 5 + 11\sin 5x}{4(-100-121)} - \frac{2\cos x + \sin x}{4(-4-1)}$$

$$= \frac{-10\cos 5x + 11\sin 5x}{-884} + \frac{2\cos x + \sin x}{20}$$

$$P.I = \frac{1}{884}(10\cos 5x - 11\sin 5x) + \frac{1}{20}(2\cos x + \sin x)$$

$\therefore$ The solution of the given equ is

$$y = C.F + P.I$$

$$y = Ae^{2x} + Be^{3x} + \frac{1}{884}(10\cos 5x - 11\sin 5x) + \frac{1}{20}(2\cos x + \sin x)$$

---

2) $(D^2 - 6D + 13)y = 5e^{2x}$

Soln :

Given equ is,

$$(D^2 - 6D + 13)y = 5e^{2x}$$

Auxillary equ

$$m^2 - 6m + 13 = 0$$

Here $a = 1$; $b = -6$; $c = 13$

$$m = \frac{6 \pm \sqrt{36 - 52}}{2}$$

$$= \frac{6 \pm \sqrt{-16}}{2}$$

$$= \frac{6 \pm i4}{2} = 3 \pm i2$$

$$m = 3 \pm i2$$

Here m is complex

$\alpha = 3, \beta = 2$

$C.F = e^{\alpha x}[A\cos\beta x + B\sin\beta x]$

$C.F = e^{3x}[A\cos 2x + B\sin 2x]$

$P.I = \frac{1}{D^2-6D+13} 5e^{2x}$

$D = \alpha, \alpha = 2$

$= \frac{1}{(2)^2-6(2)+13} 5e^{2x}$

$= \frac{1}{4-12+13} 5e^{2x}$

$= \frac{1}{4+1} 5e^{2x} = \frac{5e^{2x}}{5}$

$P.I = e^{2x}$

$\therefore$ The solution for the given equ is,

$y = C.F + P.I$

$y = e^{3x}[A\cos 2x + B\sin 2x] + e^{2x}$

---

3) Solve : $(D^2+3D+2)y = \sin x + x^2$

Soln :

Given equ is,

$(D^2+3D+2)y = \sin x + x^2$

Auxillary equ is,

$D = m$ $\qquad \frac{1}{m} \Big| \frac{2}{m}$

$m^2+3m+2=0$

$(m+1)(m+2)=0$

$m = -1, -2$

$m_1 = -1, m_2 = -2$

Here m_1 & m_2 are real and distinct

$C.F = Ae^{m_1 x} + Be^{m_2 x}$

$C.F = Ae^{-x} + Be^{-2x}$

$$P.I = \frac{1}{(D^2+3D+2)} \sin x + x^2$$

$$= \frac{1}{(D^2+3D+2)} \sin x + \frac{1}{(D^2+3D+2)} x^2$$

$$= \frac{1}{-1^2+3D+2} \sin x + \frac{1}{2\left(\frac{D^2+3D}{2}+1\right)} x^2$$

$$= \frac{1}{3D+1} \sin x + \frac{1}{2}\left(\frac{D^2+3D}{2}+1\right)^{-1} x^2$$

$$= \frac{3D-1}{(3D+1)(3D-1)} \sin x + \frac{1}{2}\left[1+\left(\frac{D^2+3D}{2}\right)\right]^{-1} x^2$$

$$= \frac{3D-1}{(9D^2-1)} \sin x + \frac{1}{2}\left[1-\left(\frac{D^2+3D}{2}\right)+\left(\frac{D^2+3D}{2}\right)^2 \cdots\right] x^2$$

$$= \frac{3D(\sin x)-\sin x}{9(-1)-1} + \frac{1}{2}\left[1-\left(\frac{D^2+3D}{2}\right)+\frac{D^4+9D^2+\cdots}{4}\right.$$

$$= \frac{3\cos x-\sin x}{-10} + \frac{1}{2}\left[x^2-\left(\frac{D^2(x^2)+3D(x^2)}{2}\right)+\frac{9D^2 x^2}{2}\right.$$

$$= \frac{-3\cos x+\sin x}{10} + \frac{1}{2}\left[x^2-\left(\frac{2+6x}{2}\right)+\frac{18}{4}\right]$$

$$= \frac{-3\cos x+\sin x}{10} + \frac{1}{2}\left[x^2-(1+3x)+9/2\right]$$

$$= \frac{-3\cos x+\sin x}{10} + \frac{1}{2}\left[x^2-3x+\frac{9}{2}-1\right]$$

$$= \frac{-3\cos x+\sin x}{10} + \frac{1}{2}\left[x^2-3x+7/2\right]$$

$$P.I = \frac{\sin x-3\cos x}{10} + \frac{1}{2}\left[x^2-3x+7/2\right]$$

$\therefore$ The solution for the given equ is,

$$y = C.F + P.I$$

$$y = Ae^{-x} + Be^{-2x} + \frac{\sin x - 3\cos x}{10} + \frac{1}{2}\left[x^2 - 3x + \frac{7}{2}\right]$$

1) solve : $x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} - 3y = x^2$

Soln :

Given equ is,

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} - 3y = x^2 \rightarrow (1)$$

let $z = \log x$ (or) $x = e^z$

$$x\frac{d}{dx} = D \rightarrow (2)$$

$$x^2\frac{d^2}{dx^2} = D(D-1) \rightarrow (3)$$

Sub (2) and (3) in (1), we get

$$(D(D-1) + D - 3)y = e^z$$

$$(D^2 - \cancel{D} + \cancel{D} - 3)y = e^z$$

$$(D^2 - 3)y = e^z$$

Auxillary equ is

$$D = m$$

$$m^2 - 3 = 0$$

$$m^2 = 3$$

$$m = \pm\sqrt{3}$$

m_1 & m_2 are real and distinct

$$C.F = Ae^{m_1 z} + Be^{m_2 z}$$

$$C.F = Ae^{\sqrt{3}z} + Be^{-\sqrt{3}z}$$

$$C.F = Ae^{\sqrt{3}\log x} + Be^{-\sqrt{3}\log x}$$

$$= Ae^{\sqrt{3}\log x} + Be^{-\sqrt{3}\log x}$$

$C.F = Ax^{\sqrt{3}} + Bx^{-\sqrt{3}}$

$P.I = \frac{1}{D^2-3} e^{z}$

$= \frac{1}{1^2-3} e^{z}$ $\therefore D = 1$

$P.I = -\frac{1}{2} e^{z}$

$= -\frac{1}{2} x$

$y = Ax^{\sqrt{3}} + Bx^{-\sqrt{3}} - \frac{x}{2}$

---

5) $x^2 \frac{d^2y}{dx^2} + 8x\frac{dy}{dx} + 12y = x^4$

Soln :

Given equ is,

$x^2 \frac{d^2y}{dx^2} + 8x\frac{dy}{dx} + 12y = x^4 \rightarrow (1)$

Let, $z = \log x$ (or) $x^4 = e^{4z}$

$x \cdot \frac{d}{dx} = D \rightarrow (2)$

$x^2 \cdot \frac{d^2}{dx^2} = D(D-1) \rightarrow (3)$

sub (2) & (3) in (1), we get

$(D(D-1) + 8D + 12)y = e^{4z}$ $\frac{4}{D} \Big| \frac{3}{D}$

$(D^2 - D + 8D + 12)y = e^{4z}$

$(D^2 + 7D + 12)y = e^{4z}$

Auxillary equ is

$D = m$

$m^2 + 7m + 12 = 0$

$(m+4)(m+3) = 0$

$m_1 = -4$; $m_2 = -3$

Here m_1 & m_2 are real & distinct

$$C.F = Ae^{m_1 x} + Be^{m_2 x}$$

$$C.F = Ae^{-4z} + Be^{-3z}$$

$$= Ae^{-4\log x} + Be^{-3\log x}$$

$$= Ae^{\log x^{-4}} + Be^{\log x^{-3}}$$

$$C.F = Ax^{-4} + Bx^{-3}$$

$$P.I = \frac{1}{D^2+7D+12} e^{4z}$$

$$= \frac{1}{4^2+7(4)+12} e^{4z}$$

$$= \frac{1}{16+28+12} e^{4z}$$

$$= \frac{1}{56} e^{4z}$$

$$= \frac{1}{56} e^{4\log x}$$

$$= \frac{1}{56} e^{\log x^4}$$

$$P.I = \frac{1}{56} x^4$$

$\therefore$ The solution of the given equ is,

$$y = C.F + P.I$$

$$y = Ax^{-4} + Be^{-3} + \frac{x^4}{56}$$

6) Solve : $x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = x \log x$

Soln :

Given equ is,

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = x \log x \rightarrow (1)$$

let $z = \log x$ (or) $x = e^z$

$x \cdot \frac{d}{dx} = D \rightarrow (2)$

$x^2 \cdot \frac{d^2}{dx^2} = D(D-1) \rightarrow (3)$

sub (2) & (3) in (1), weget

$(D(D-1) + D + 1)y = e^z \cdot z$

$(D^2 - D + D + 1)y = e^z \cdot z$

$(D^2+1)y = e^z \cdot z$

$m^2+1 = 0$

$m^2 = -1$

$m = \pm i$

Here m is complex

$C.F = e^{\alpha x}(A\cos\beta z + B\sin\beta z)$

$\alpha = 0 \; ; \; \beta = 1$

$C.F = e^{0x}(A\cos z + B\sin z$

$C.F = A\cos z + B\sin z$

$P.I = \frac{1}{D^2+1} z e^z$

$D = D +$

$= \frac{e^z \cdot 1 \cdot z}{(D+1)^2+1}$

$= e^z \cdot \frac{1}{D^2+1+2D+1} \cdot z$

$= e^z \cdot \frac{1}{D^2+2D+2} \cdot z$

$= e^z \frac{1}{2\left(\frac{D^2}{2} + D + 1\right)} \cdot z$

$$= \frac{e^z}{2}\left[\underbrace{\frac{D^2}{2}+D}_{x}+1\right]^{-1} z \qquad (1+x)^{-1} = 1-x+x^2-x^3\ldots$$

$$= \frac{e^z}{2}\left[1-\underbrace{\left(D+\frac{D^2}{2}\right)}_{x}\right] z$$

$$= \frac{e^z}{2}(1-D)\,z$$

$$= \frac{e^z}{2}(z-1)$$

$$P.I = \frac{e^{\log x}}{2}(\log x - 1)$$

$$= \frac{x}{2}(\log x - 1)$$

$\therefore$ The solution of the given equ is,

$$y = C.F + P.I$$

$$y = A\cos(\log x) + B\sin(\log x) + \frac{x}{2}(\log x - 1)$$

---

7) Solve :

$$x^2\frac{d^2y}{dx^2} - 3x\frac{dy}{dx} - 5y = \sin(\log x)$$

soln :

Given equ is.

$$x^2\frac{d^2y}{dx^2} - 3x\frac{dy}{dx} - 5y = \sin(\log x) \rightarrow (1) \qquad D^2 = -a^2$$

let $z = \log x$ (Or) $x = e^z$

$$x\cdot\frac{d}{dx} = D \rightarrow (2)$$

$$x^2\frac{d^2}{dx^2} = D(D-1) \rightarrow (3)$$

sub (2) & (3) in (1), we get

$$(D(D-1) - 3D - 5)y = \sin z$$

$$(D^2 \underbrace{- D - 3D}_{-4D} - 5)y = \sin z$$

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$(D^2-4D-5)y = \sin z$ $\quad \frac{1}{D} \Big| \frac{-5}{D}$

Auxillary equ is

$(D+1)(D-5)=0$

$D=m$

$(m+1)(m-5)=0$

$m=-1, 5$

Here m_1, m_2 are real & distinct

$C.F = Ae^{m_1 z} + Be^{m_2 z}$

$C.F = Ae^{-z} + Be^{5z}$

$C.F = Ae^{-\log x} + Be^{5\log x}$

$= Ae^{\log x^{-1}} + Be^{\log x^5}$

$C.F = Ax^{-1} + Bx^5$

$P.I = \frac{1}{D^2-4D-5} \sin z$ $\quad (D^2 = -a^2$

$= \frac{1}{-1-4D-5} \sin z$

$= \frac{1}{-4D-6} \sin z$

$= \frac{-1}{2} \frac{1}{(2D+3)} \sin z$

$= -\frac{1}{2} \frac{(2D-3)}{(2D+3)(2D-3)} \sin z$

$= -\frac{1}{2} \frac{(2D-3)}{4D^2-6D+6D-9} \sin z$

$= -\frac{1}{2} \frac{(2D-3)}{4D^2-9} \sin z$

$= -\frac{1}{2} \frac{(2D-3)}{-4-9} \sin z$

$= -\frac{1}{2} \frac{(2D-3)\sin z}{-13}$

$$= \frac{1}{26}(2\cos z - 3\sin z)$$

$$P.I = \frac{1}{26}[2\cos(\log x) - 3\sin(\log x)]$$

$\therefore$ The solution for the given equ is,

$$y = C.F + P.I$$

$$y = Ax^{-1} + Bx^{5} + \frac{1}{26}[2\cos(\log x) - 3\sin(\log x)]$$

8) $$x^2\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} = 6x^2 + 2x + 1$$

Soln :

Given equ is

$$x^2\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} = 6x^2 + 2x + 1 \rightarrow (1)$$

$$z = \log x \quad (\text{or}) \quad x = e^z$$

$$x\frac{d}{dx} = D \rightarrow (2)$$

$$x^2.\frac{d^2}{dx^2} = D(D-1) \rightarrow (3)$$

sub (2) & (3) in (1), we get

$$[D(D-1) + 2D]y = 6e^{2z} + 2e^{z} + e^{0}$$

$$(D^2 - D + 2D)y = 6e^{2z} + 2e^{z} + 1$$

$$(D^2 + D)y = 6e^{2z} + 2e^{z} + 1$$

Auxiliary equ is

$$m^2 + m = 0$$

$$m(m+1) = 0$$

$$m_1 = 0 \quad ; \quad m+1 = 0$$

$$m_2 = -1$$

Here m_1 & m_2 are real and distinct

$$C.F = Ae^{m_1 z} + Be^{m_2 z}$$

$$C.F = Ae^{0z} + Be^{-z}$$

$$C.F = A + Be^{-z}$$

$$P.I = \frac{1}{D^2+D}(6e^{2z} + 2e^{z} + e^{0z}$$

$$= \frac{1}{D^2+D} 6e^{2z} + \frac{1}{D^2+D} 2e^{z} + \frac{1}{D^2+D} e$$

$$= \frac{1}{6} 6e^{2z} + \frac{1}{2} 2e^{z} + \frac{1}{D(D+1)} e$$

$$= e^{2z} + e^{z} + \frac{z}{1} e^{0z}$$

$$= e^{2z} + e^{z} + ze^{0z}$$

$$= e^{2\log x} + e^{\log x} + \log x (1)$$

$$= e^{\log x^2} + e^{\log x} + \log x$$

$$P.I = x^2 + x + \log x$$

$\therefore$ The solution for the given equ

$$y = C.F + P.I$$

$$y = A + Be^{-z} + x^2 + x + \log x$$

---

9) Solve : $x^3 \frac{d^3y}{dx^3} + 2x^2 \frac{d^2y}{dx^2} + 2y = 10(x + 1/x)$

Soln :

Given equ is,

$$x^3 \frac{d^3y}{dx^3} + 2x^2 \frac{d^2y}{dx^2} + 2y = 10(x + 1/x) \rightarrow$$

$$z = \log x \quad (\text{or}) \quad x = e^{z}$$

$$x\frac{d}{dx} = D \rightarrow ⓐ$$

$x^2\frac{d^2y}{dx^2} = D(D-1)$; $x^3\frac{d^3y}{dx^3} = D(D-1)(D-2)$

sub ⑤ & ④ in ①

$(D(D-1)(D-2)+2D(D-1)+2)y = 10(e^z + \frac{1}{e^z})$

$(D(D^2-D-2D+2)+2D^2-2D+2)y = 10(e^z + \frac{1}{e^z})$

$(D^3-3D^2+2D^2+2D^2-2D+2)y = 10(e^z+\frac{1}{e^z})$

$(D^3-D^2+2)y = 10(e^z+\frac{1}{e^z})$

$(D^3-D^2+2)y = 10(e^z+\frac{1}{e^z})$

Auxiliary equ is

$m^3-m^2+2=0$

$$\begin{array}{r|rrrr} -1 & 1 & -1 & 0 & 2 \\ & 0 & -1 & 2 & -2 \\ \hline & 1 & -2 & 2 & \boxed{0} \end{array}$$

$(m+1)(m^2-2m+2)=0$

$m=-1$, $m = \frac{2 \pm \sqrt{4-8}}{2} = \frac{2 \pm \sqrt{-4}}{2} = \frac{2 \pm i2}{2}$

$m = 1 \pm i$; $m_1 = -1$

C.F $= Ae^{mx} + e^{\alpha x}(B\cos\beta x + C\sin\beta x)$ $= Ae^{-z} + e^{z}$

$= Ae^{-\log x} + e^{\log x}(B\cos z + C\sin z)$

$= Ae^{-\log x} + e^{\log x}(B\cos(\log x) + C\sin(\log x))$

$= Ae^{\log x^{-1}} + e^{\log x}(B\cos\log x + C\sin\log x)$

$= Ax^{-1} + x(B\cos(\log x) + C\sin(\log x))$

P.I $= \frac{1}{D^3-D^2+2} 10e^{z} + 10e^{-z}$

$= \frac{1}{D^3-D^2+2} 10e^{z} + \frac{1}{D^3-D^2+2} 10e^{-z}$

10) Solve : $x^3\frac{d^3y}{dx^3}+3x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+y=x+\log$

Soln : The given eqn is

$$x^3\frac{d^3y}{dx^3}+3x^2\frac{d^2y}{dx^2}+\frac{x\,dy}{dx}+y=x\log x$$

let $z=\log x$ (or) $x=e^z$

$$x\frac{d}{dz}=D \quad ; \quad x^2\frac{d^2}{dx^2}=D(D-1) \ ;$$

$$x^3\frac{d^3}{dx^3}=D(D-1)(D-2)$$

Sub ② in ①

$$(D(D-1)(D-2)+3D(D-1)(1)+D+1)y=e^{z}$$

$$(D(D^2-3D+2)+3D^2+3D^2+D+1)y=e^{z}$$

$$(D^3-3D^2+2D+3D^2-3D+D+1)y$$

$$e^z+z$$

$(D^3+1)y = e^z + z$

Auxiliary equ is

$m^3+1=0 \quad (m+1)(m^2-m+1)=0$

$m=-1 \quad ; \quad m = \frac{1 \pm \sqrt{1-4}}{2} = \frac{1 \pm i\sqrt{3}}{2}$

$C.F = Ae^{-z} + e^{1/2 z}\left(A\cos\frac{\sqrt{3}}{2}z + B\sin\frac{\sqrt{3}}{2}z\right)$

$= Ae^{-\log x} + e^{1/2\log x}\left(B\cos\frac{\sqrt{3}}{2}\log x + c\sin\frac{\sqrt{3}}{2}\log x\right)$

$= Ae^{\log x^{-1}} + e\log x^{1/2}\left(B\cos\frac{\sqrt{3}}{2}(\log x) + c\sin\frac{\sqrt{3}}{2}(\log x)\right)$

$= Ax^{-1} + \sqrt{x}\left(B\cos\frac{\sqrt{3}}{2}\log x + c\sin\frac{\sqrt{3}}{2}(\log x)\right)$

$P.I = \frac{1}{D^3+1}(e^z+z) = \frac{1}{D^3+1}e^z + \frac{1}{D^3+1}z$

$= \frac{1}{D^3+1}e^z + \frac{1}{(D^3+1)}z = \frac{1}{1+1}e^z + (1+D^3)^{-1}z$

$= \frac{1}{2}e^{\log x} + (1-D^3)z = \frac{x}{2} + z - 0$

$= \frac{x}{2} + \log x$

$\therefore$ The solution of given equ is

$y = Ae^{-x} + \sqrt{x}\left(B\cos\frac{\sqrt{3}}{2}\log x + c\sin\frac{\sqrt{3}}{2}\log x\right) + \frac{x}{2} + \log x$

Solve : $x^2\frac{d^2y}{dx^2} - x\frac{dy}{dx} + y = \frac{\log x \sin(\log x) + 1}{x}$

Soln :

The given equ is

$$x^2\frac{d^2y}{dx^2} - x\frac{dy}{dx} + y = \frac{\log x \sin(\log x) + 1}{x}$$

$z = \log x$ (or) $x = e^z$

$x\frac{d}{dx} = D$; $x^2\frac{d^2}{dx^2} = D(D-1)$

sub ② in ①

$$(D(D-1) + \bar{(}D+1))y = \frac{z\sin z + 1}{e^z}$$

$$(D(D-1) - D + 1)y = \frac{z\sin z + 1}{e^z}$$

$$(D^2 - D - D + 1)y = \frac{z\sin z + 1}{e^z}$$

$$(D^2 - 2D + 1)y = \frac{z\sin z + 1}{e^z}$$

Auxillary equ is

$(m-1)(m-1) \Rightarrow m_1 = 1$; $m_2 = 1$

$$C.F = e^{\alpha x}(Az + B)$$

$$= e^z(Az + B).$$

$$C.F = e^{\log x}(A\log x + B)$$

$$C.F = x(A\log x + B)$$

$$P.I = \frac{1}{D^2 - 2D + 1}\left(\frac{z\sin z + 1}{e^z}\right)$$

$$= \frac{1}{(D-1)^2}(z\sin z+1)e^{z}$$

$$= \frac{1}{(D-1)^2}\ z e^{-z}\sin z + e^{-z}$$

$$= \text{imaginary part of } \frac{1}{(D-1)^2}\ z e^{-z} e^{iz} + \frac{e^{-z}}{(D-1)^2}$$

$$= \text{imaginary part of } \frac{1}{(D-1)^2}\ z e^{z(i-1)} + \frac{1}{(-1-1)^2}\ e^{-z}$$

$$= \text{imaginary part of } e^{z(i-1)} \cdot \frac{1}{[(D+i-1)-1]^2} + \frac{1}{(-2)^2} e^{-z}$$

$$= \text{imaginary part of } e^{iz} e^{-z} \frac{1}{(D+i-2)^2} z + \frac{1}{4} e^{-z}$$

$$= \text{imaginary part of } e^{iz} \cdot e^{-z} \cdot \frac{1}{(i-2)^2\left(1+\frac{D}{i-2}\right)} z + \frac{e^{-z}}{4}$$

$$= \text{imaginary part of } e^{iz} \cdot e^{-z} \frac{\left(1+\frac{D}{i-2}\right)^{-2}}{(i-2)^2} z + \frac{e^{-z}}{4}$$

$$(1+x)^{-2} = 1-2x+3$$

$$= \text{imaginary part of } e^{iz} \cdot e^{-z}\ \frac{\left[1-2\left(\frac{D}{i-2}\right)+3\left(\frac{D}{i-2}\right)^2\right] z}{i^2-4i+4} + \frac{1}{4} e^{-z}$$

$$= \text{imaginary part of } e^{iz} \cdot e^{-z} \left(\frac{z - 2\frac{D(z)}{i-2}}{-1-4i+4}\right) + \frac{e^{-z}}{4}$$

$$P.I = \text{imaginary part of } e^{iz} \cdot e^{-z} \left(\frac{z - \frac{2}{i-2}}{3-4i}\right) + \frac{e^{-z}}{4}$$

$$= \text{imaginary part of } e^{-z}(\cos z + i\sin z) \left(\frac{\frac{iz-2z-2}{i-2}}{3-4i}\right) + \frac{e^{-z}}{4}$$

$$= \text{imaginary part of } e^{-z}(\cos z + i\sin z)\ \frac{iz-2z-2}{(3-4i)(i-2)} + \frac{e^{-z}}{4}$$

$$= \text{imaginary part of } e^{-z}(\cos z + i\sin z)\ \frac{iz-2z-2}{-2+11i} + \frac{e^{-z}}{4}$$

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$= \text{imaginary part of } e^{-z} \dfrac{(iz - 2z - 2)(-2-11i)}{(-2+11i)(-2+11i)} + \dfrac{e^{-z}}{4}$

$= \text{imaginary part of } e^{-z} \dfrac{(iz - 2z - 2)(-2-11i)}{(-2)^2 - (11i)^2} + \dfrac{e^{-z}}{4}$

$= \text{imaginary part of } e^{-z} \dfrac{(-2iz + 4z + 4 + 11z + 22iz + 22i)}{4+121} + \dfrac{e^{-z}}{4}$

$= \text{imaginary part of } e^{-z}(\cos z + i \sin z)\left(\dfrac{4 + 15z + 20iz + 22i}{125}\right) + \dfrac{e^{-z}}{4}$

$= \dfrac{e^{-z}}{1} \text{ imaginary part of } (\cos z + i \sin z)\left(\dfrac{(4+15z) + (20iz + 22i)}{125}\right) + \dfrac{e^{-z}}{4}$

$= \dfrac{e^{-z}}{125}(4\sin z + 15z\sin z + 20z\cos z + 22\cos z) + \dfrac{e^{-z}}{4}$

$P.I = \dfrac{e^{-z}}{125}\, 4\sin z + 22\cos z + 5z(3\sin z + 4\cos z) + \dfrac{e^{-z}}{4}$

$= \dfrac{e^{-\log x}}{125}\, 4\sin(\log x) + 22\cos(\log x) + 5(\log x)[3\sin(\log x) + 4\cos(\log x)] + \dfrac{e^{-\log x}}{4}$

$P.I = \dfrac{x^{-1}}{125}\, 4\sin\log x + 22\cos\log x + 5\log x [3\sin\log x + 4\cos\log x] + \dfrac{x^{-1}}{4}$

$\therefore$ The solution of given equ is

$y = C.F + P.I$

$y = x(A\log x + B) + \dfrac{x^{-1}}{125}\, 4\sin\log x + 22\cos\log x + 5\log x[3\sin\log x + 4\cos\log x] + x^{-1}/4$

$y = x(A(\log x) + B) + \dfrac{1}{125x}\, 4\sin\log x + 22\cos\log x + 5\log x[3\sin\log x + 4\cos\log x] + \dfrac{1}{4x}.$

To find P.I for $\frac{x}{\theta-\alpha}$

i.e $P.I = x^{\alpha}\int x^{-\alpha-1} x \, dx$

i) solve : $x^2\frac{d^2y}{dx^2} + 3x\frac{dy}{dx} + y = \frac{1}{(1-x^2)^2}$

soln :

Given equ is

$$x^2\frac{d^2y}{dx^2} + 3x\frac{dy}{dx} + y = \frac{1}{(1-x^2)^2}$$

$$x^2\frac{d^2}{dx^2} = D(D-1) \; ; \quad x\frac{d}{dx} = D$$

Auxillary equ is

$$D=m$$

$$(D(D-1)+3D)y = \frac{1}{(1-x^2)}$$

$$(D^2-D+3D+1)y = \frac{1}{(1-x^2)}$$

$$(D^2+2D+1)y = \frac{1}{(1-x^2)}$$

$$(m^2-2m+1)=0$$

$$(m+1)(m+1)=0$$

$$m=-1,-1$$

$$m=-1$$

Here m are real and equal

$$C.F = e^{m_1 z}(Az+B)$$

$$= e^{-z}(Az+B)$$

$$C.F = e^{-\log x}(A(\log x)+B)$$

$$= e^{\log x^{-1}}[A\log x+B]$$

$$= x^{-1}(A\log x+B)$$

$$P.I = \frac{1}{(D+1)^2} \cdot \frac{1}{(1-x^2)^2}$$

$$= \frac{1}{(\theta+1)^2} \cdot \frac{1}{(1-x)^2} \qquad [\because D=\theta]$$

$$= \frac{1}{(\theta+1)} \left\{ \frac{1}{(\theta+1)} \cdot \frac{1}{(1-x)^2} \right\}$$

$$= \frac{1}{\theta+1} \left[x^{-1} \int \underbrace{x^{-(-1)-1}}_{1} \frac{1}{(1-x)^2} dx \right]$$

$$P.I = \frac{1}{\theta+1} \left[\frac{1}{x} \int \frac{1}{(1-x)^2} dx \right]$$

$$= \frac{1}{\theta+1} \left[\frac{1}{x} \left(\frac{-1}{1-x} \right) \right]$$

$$= -\left[\frac{1}{\theta+1} \frac{1}{x(1-x)} \right]$$

$$= x^{-1} \int x^{-(-1)-1} \frac{1}{x(1-x)} dx$$

$$= \frac{-1}{x} \int \frac{1}{x(1-x)} dx$$

$$\Rightarrow \frac{1}{x(1-x)} = \frac{A}{x} + \frac{B}{1-x}$$

Pe) $1 = A(1-x) + Bx$

let $x = 0$

$1 = A$

let $x = 1$

$B = 1$

$$\frac{1}{x(x-1)} = \frac{1}{x} + \frac{1}{1-x}$$

$$P.I = \frac{-1}{x} \int \left(\frac{1}{x} + \frac{1}{1-x} \right) dx$$

$$= \frac{-1}{x} [\log x - \log(1-x)]$$

$$= \frac{1}{x} [\log(1-x) - \log x]$$

$$= \frac{1}{x} \log\left(\frac{1-x}{x} \right)$$

$\therefore$ The solution of the given equ...

$$y = C.F + P.I$$

$$y = x^{-1}(A \log x + B) + \frac{1}{x} \log\left(\frac{1-x}{x} \right)$$

solve : $x^2\frac{d^2y}{dx^2} + 4x\cdot\frac{dy}{dx} + 2y = e^{x}$

soln :

The given eqn is,

$$x^2\frac{d^2y}{dx^2} + 4x\cdot\frac{dy}{dx} + 2y = e^{x} \rightarrow (1)$$

$z = \log x$ (or) $x = e^z$

$x\cdot\frac{d}{dx} = D \rightarrow (2)$, $x^2\frac{d^2}{dx^2} = D(D-1) \rightarrow (3)$

sub (2) & (3) in (1)

$$(D(D-1)+4D+2)y = e^{e^z}$$

$$(D^2-D+4D+2)y = e^{e^z}$$

$$(D^2+3D+2)y = e^{e^z}\ e^{x}$$

$\frac{1}{D}\Big|\frac{2}{D}$

Auxillary equ is,

$$m^2+3m+2=0$$

$$(m+1)(m+2)=0$$

$$m_1=-1,\ m_2=-2$$

Here m_1 & m_2 are real & distinct

$$C.F = Ae^{m_1 z} + Be^{m_2 z}$$

$$= Ae^{-z} + Be^{-2z}$$

$$= Ae^{-\log x} + Be^{-2\log x}$$

$$= Ae^{\log x^{-1}} + Be^{\log x^{-2}}$$

$$C.F = Ax^{-1} + Bx^{-2}$$

$$P.I = \frac{1}{(D+1)(D+2)}e^{x} \qquad D=0$$

$$= \frac{1}{(0+1)(0+2)}e^{x}$$

$$= \left[\frac{1}{0+1} - \frac{1}{0+2}\right]e^{x}$$

$$= \frac{1}{0+1}e^{x} - \frac{1}{0+2}e^{x}$$

$\alpha = -1,\ \alpha = -2$

$$P.I = x^{-1}\int x^{1-1}e^{x}dx - x^{-2}\int x^{2-1}e^{x}dx$$

$$= x^{-1}\int e^{x}dx - x^{-2}\int xe^{x}dx$$

$$= \frac{1}{x}e^{x} - \frac{1}{x^2}\left[xe^{x} - \int e^{x}dx\right]$$

$$= \frac{1}{x}e^{x} - \frac{1}{x^2}[xe^{x} - e^{x}]$$

$$= e^{x}\left[\frac{1}{x} - \frac{1}{x^2}(x-1)\right]$$

$$P.I = e^{x}\left[\frac{1}{x^2}\right]$$

$\therefore$ The solution of the given equ is,

$$y = C.F + P.I$$

$$y = Ax^{-1} + Bx^{-2} + e^{x}\left[\frac{1}{x^2}\right].$$

---

3) Show that the solution of the differential equ $\frac{d^2y}{dx^2} + 4y = A\sin pt$, which is such that $y = 0$ & $\frac{dy}{dx} = 0$, when $t = 0$ show that $y = \frac{A(\sin pt - \frac{1}{2}p\sin 2t)}{4-p^2}$ if $p \neq 2$; $y = \frac{A(\sin pt - 2t\cos 2t)}{8}$ if $p = 2$.

Soln :

Given equ is

$$\frac{d^2y}{dx^2} + 4y = A\sin pt$$

$y = 1$

$\frac{dy}{dx} = 1 = m$

$\frac{d^2y}{dx^2} = m^2$

Auxiliary equ is

$$m^2 + 4 = 0$$

$$\cancel{m(m+4)=0}$$

$$m^2 = -4$$

$$m = -\sqrt{4}$$

$$m = \pm i2$$

Here m is complex

$$C.F = e^{\alpha t}(B\cos\beta t + C\cos\beta t)$$

$$= e^{0t}(B\cos 2t + C\cos 2t)$$

$$C.F = B\cos 2t + C\cos 2t$$

$$P.I = \frac{1}{D^2+4}A\sin pt$$

$$P.I = \frac{A \sin pt}{4-p^2} \quad \text{if } p \neq 2$$

$\therefore$ The solution is

$$y = C.F + P.I$$

$$y = B\cos 2t + C \sin 2t + \frac{A \sin pt}{4-p^2} \rightarrow (1)$$

Given that

$$y(0) = 0, \quad \frac{dy}{dx} = 0$$

$$t = 0, \ y = 0$$

Here,

$$(1) \rightarrow B \cos 2(0) + C \sin(2(0)) + \frac{A \overset{\sin p(0)}{\cancel{\cos pt . p}}}{4-p^2} = 0$$

$$\Rightarrow B = 0$$

Diff (1) w.r.to 't'

$$\frac{dy}{dx} = -B \sin 2t(2) + 2C \cos 2t + \frac{A \cos pt . P}{4-p^2} \quad (1)$$

$$\frac{dy}{dx} = 0 \Rightarrow -2B \sin 2(0) + 2C \frac{\cos 2(0)}{(1)} + \frac{A \cos(0) P}{4-p^2} = 0$$

$$2C + \frac{AP}{4-p^2} = 0$$

$$2C = \frac{-AP}{4-p^2}$$

$$C = \frac{-AP}{2(4-p^2)}$$

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sub B & C values in ① we get,

$$y = 0 \sin 2t + \left(\frac{-AP}{2(4-P^2)}\right)\sin 2t + \frac{A\sin pt}{4-p^2}$$

$$y = \frac{-AP}{2(4-P^2)} \sin 2t + \frac{A\sin pt}{4-P^2}$$

$$= \frac{A}{4-P^2}\left(\frac{-P}{2}\sin 2t + \sin pt\right)$$

$$y = \frac{A}{4-P^2}\left(\sin pt - P/2 \sin 2t\right)$$

case (ii); $P = 2$

ipt

$i2t$

$$P.I = \frac{1}{D^2+4} \sin 2t$$

$$= \text{imaginary part of } \frac{A}{D^2+4} e^{i2t}$$

$D^2 = -2^2$

$$= \text{imaginary part of } \frac{A}{(2i)^2+4} e^{i2t}$$

$$= \text{imaginary part of } \frac{A}{(D+2i)(D-2i)} e^{i2t}$$

$$= \text{imaginary part of } \frac{A}{(2i+2i)(2i-2i)} e^{i2t}$$

$$= \text{imaginary part of } \frac{At}{4i} e^{i2t}$$

$$= \text{imaginary part of } \frac{Ait}{-4}(\cos 2t + i\sin$$

$$= \text{imaginary part of } \left(\frac{-tA\cos 2t i}{4} + \frac{At\sin}{4}\right.$$

$$P.I = \frac{-At \cos 2t}{4}$$

∴ The solution of is,

$$y = B\cos 2t + C\sin 2t - \frac{At\cos 2t}{4} + 6$$

$y = 0 \Rightarrow$

$$B\cos 2(0) + C\sin 2(0) - \frac{A(0)\cos 2(0)}{4} = 0$$

$B=0$

Diff ⑤ w.r.to 'x'

$$\frac{dy}{dx} = -2B\sin 2t + 2C\cos 2t - \frac{A}{4}\left[t(-\sin 2t)2 + \cos 2t(1)\right]$$

$$= -2B\sin 2t + 2C\cos 2t - \frac{A}{4}\left[-t2\sin 2t + \cos 2t\right]$$

$\frac{dy}{dx}=0$

$$\Rightarrow -2B\sin 2(0) + 2C\cos 2(0) - \frac{A}{4}\left[-2(0)\sin 2(0) + \cos 2t\right]=0$$

$$2C - \frac{A}{4} = 0$$

$$2C = A/4$$

$$C = A/8$$

sub B & C in equ ⑤

$$y = 0\cos 2t + \frac{A}{8}\sin 2t - \frac{At\cos 2t}{4}$$

$$= \frac{A}{8}\sin 2t - \frac{At\cos 2t}{4} \qquad \frac{A}{8}\sin 2t - \frac{2At\cos 2t}{8}$$

$$y = \frac{A}{8}\left[\sin 2t - 2t\cos 2t\right].$$

---

method of variation of parameter:

working rule for $y_1 + Py = Q$, where

$y_1 = \frac{dy}{dx}$

step : 1

rewrite the given equ in the standard form

$$\frac{dy}{dx} + Py = Q \rightarrow ①$$

coefficient of y, must be unity

step : 2 put $Q = 0$

we have

$$① \Rightarrow \frac{dy}{dx} + Py = 0 \rightarrow ②$$

The general solution of ⑤ is

$$y = cu \rightarrow ⑥$$

Here c - arbitary constant

step : 3

The general solution of ① is

$$y = cu(x) + u(x)\int (Q/u)\, dx$$

c is arbitary constant

1) Solve : $(x+4)\frac{dy}{dx} + 3y = 3$ by method of variation of parameter.

Soln :

Given equ is,

$$(x+4)\frac{dy}{dx} + 3y = 3$$

$\div (x+4)$,

$$\frac{dy}{dx} + \frac{3y}{x+4} = \frac{3}{x+4} \rightarrow ①$$

comparing ① with $\frac{dy}{dx} + Py = Q$

$$P = \frac{3}{x+4}, \quad Q = \frac{3}{x+4}$$

Put $Q = 0$ in ①

$$\frac{dy}{dx} + \frac{3y}{x+4} = 0$$

$$\frac{dy}{dx} = \frac{-3y}{x+4}$$

$$\frac{dy}{y} = \frac{-3}{x+4}\, dx$$

$\int$ing on both sides we get,

$$\int \frac{1}{y}\, dy = -3\int \frac{1}{x+4}\, dx$$

$$\log y = -3\log(x+4) + \log c$$

$$\log y = \log(x+4)^{-3} + \log c$$

$$\log y = \log c(x+4)^{-3}$$

$$y = c(x+4)^{-3}$$

(i.e);

where c is arbitrary constant

Here $u = (x+4)^{-3}$

$\therefore$ The general soln of (1) is

$$y = cu + u\int (Q/u)\,dx$$

$$y = c(x+4)^{-3} + (x+4)^{-3}\int \left(\frac{\frac{3}{x+4}}{(x+4)^{-3}}\right)dx$$

$$= c(x+4)^{-3} + (x+4)^{-3}\int \frac{3}{(x+4)(x+4)^{-3}}\,dx$$

$$= c(x+4)^{-3} + (x+4)^{-3}\int \frac{3}{(x+4)^{-2}}\,dx$$

$$= c(x+4)^{-3} + (x+4)^{-3}\int 3(x+4)^{2}\,dx$$

$$= c(x+4)^{-3} + \cancel{(x+4)^{-3}} \cdot \frac{\cancel{3}\cancel{(x+4)^{3}}}{\cancel{3}}$$

$$y = c(x+4)^{-3} + 1$$

Working rule for solving:

$$y_2 + Py_1 + Qy = R$$

Step : 1

In order to make the coefficient of y unity by divide the given equ by the coefficient of y_2 throughout & obtained it in the standard form.

$$y_2 + Py_1 + Qy = R \rightarrow (1)$$

Step : 2

consider $y_2 + Py_1 + Qy = 0 \rightarrow (2)$ which is obtained by taking $R = 0$ solve (2) by any of the previous method.

let $y = Au + Bv \rightarrow (3)$ by be the general solution of (1).

Step : 3

Then A & B are function of x to be determined.

Step : 4

Differentiating (3) with respect to x.

1) using the methods of variation of parameters to solve $y_2 + n^2 y = \sec nx$.

soln :

Given equ is

$$y_2 + n^2 y = \sec nx$$

(i.e);

$$\frac{d^2y}{dx^2} + n^2 y = \sec nx \rightarrow (1)$$

$$(D^2+n^2)y = \sec nx$$

consider $(D^2+n^2)y = 0$

Auxiliary equ is,

$$m^2+n^2 = 0$$

$$m^2 = -n^2$$

$$m = \pm in$$

The solution is

$$y = e^{0x}(A\cos nx + B\sin nx)$$

$y = A\cos nx + B\sin nx \rightarrow (2)$

choose $A_1 = \frac{dA}{dx}$, $B_1 = \frac{dB}{dx}$

which satisfies

$A_1 \cos nx + B_1 \sin nx = 0 \rightarrow (3)$

Diff (2). w. r. to 'x' $uv' - vu'$

$y_1 = \frac{dy}{dx} = A[-\sin nx(n)] + \cos nx \frac{dA}{dx} +$

$B[\cos nx(n)] + \sin nx \frac{dB}{dx}$

$= -nA\sin nx + A_1\cos nx + nB\cos nx + B_1 \sin nx$

using (3) we have,

$\frac{dy}{dx} = -nA\sin nx + nB\cos nx \rightarrow (4)$

Diff (4). w. r to 'x'

$y_2 = \frac{d^2y}{dx^2} = -n^2A\cos nx - n\sin nx\, A_1 +$
$- n^2B\sin nx + n\cos nx\, B_1$

$= -n^2(A\cos nx + B\sin nx) - nA_1 \sin nx$
$+ nB_1 \cos nx$

$= -n^2y - nA_1 \sin nx + nB_1 \cos nx$

[$\because$ by 2]

$\frac{d^2y}{dx^2} + n^2y = -nA_1 \sin nx + nB_1 \cos nx$

$\sec nx = -nA_1 \sin nx + nB_1 \cos nx \rightarrow (5)$

$(3) \times n\sin nx \Rightarrow A_1 n\cos nx \sin nx + nB_1 \sin^2 nx = 0$

$(5) \times \cos nx \Rightarrow -nA_1 \cos nx \sin nx + nB_1 \cos^2 nx = \frac{\sec nx}{\cos nx}$

$nB_1[\sin^2 nx + \cos^2 nx] = 1$

$nB_1 = 1$

$B_1 = \frac{1}{n}$

$\frac{n\,dB}{dx} = 1$

$n\,dB = dx$

$dB = \frac{dx}{n}$

$\int$ing $B = \frac{x}{n} + c_1$

Sub : $B_1 = \frac{1}{n}$ in (3)

$A_1 \cos nx + \frac{1}{n}\sin nx = 0$

$$A_1 \cos nx = \frac{-\sin nx}{n}$$

$$A_1 = \frac{-1}{n} \frac{\sin nx}{\cos n}$$

$$= \frac{-1}{n} \tan nx$$

$$\frac{dA}{dx} = \frac{-1}{n} \tan nx$$

$$dA = \frac{-1}{n} \tan nx \, dx$$

$\int$ ing $\quad A = \frac{-1}{n}\left[(\log(\sec x))\frac{1}{n}\right] + c_2$

$$= \frac{1}{n^2}\left[\log(\sec x)^{-1}\right] + c_2$$

$$A = \frac{1}{n^2} \log(\cos nx) + c_2$$

sub A & B in ②

$$y = \left(\frac{1}{n^2}\log(\cos nx) + c_2\right)\cos nx + \frac{x}{n} + c_1(\sin nx)$$

where,

c_1 & c_2 are orbitary constant.

---

solve : $y_2 + a^2 y = \operatorname{cosec} ax$

soln :

Given equ is

$$y_2 + a^2 y = \operatorname{cosec} ax$$

(i.e);

$$\frac{d^2y}{dx^2} + a^2 y = \operatorname{cosec} ax \rightarrow ①$$

$$(D^2 + a^2) y = \operatorname{cosec} ax$$

consider $(D^2 + a^2) y = 0$

Auxillary equ is,

$$m^2 + a^2 = 0$$

$$m^2 = -a^2$$

$$m = \pm ia$$

The solution equ is

$$y = e^{0x}(A\cos ax + B\sin ax)$$

$$y = A\cos ax + B\sin ax \rightarrow ②$$

choose $A_1 = \frac{dA}{dx}$, $B_1 = \frac{dB}{dx}$ which satisfies,

$$A_1 \cos ax + B_1 \sin ax = 0 \rightarrow (3)$$

Diff (2) w.r.to 'x'

$$y_1 = \frac{dy}{dx} = A[-\sin ax(a)] + \cos ax \frac{dA}{dx} + B[\cos ax(a)] + \sin ax \frac{dB}{dx}$$

$$= -Aa \sin ax + A_1 \cos ax + Ba \cos ax + B_1 \sin ax$$

using (3) we have,

$$\frac{dy}{dx} = -Aa \sin ax + Ba \cos ax \rightarrow (4)$$

Diff (4). w.r to 'x'

$$y_2 = -aA\cos ax(a) - a\sin ax \frac{dA}{dx} + Ba(-\sin ax).a + a\cos ax \frac{dB}{dx}$$

$$= -Aa^2 \cos ax - A_1 a \sin ax - a^2 B \sin ax + B_1 a \cos ax$$

$$= a^2 (A\cos ax + B\sin ax) - A_1 a \sin ax + B_1 a\cos ax$$

$$= -a^2 y - A_1 a \sin ax + B_1 a \cos ax$$

$$y_2 = -a^2 y - A_1 a \sin ax + B_1 a \cos ax$$

$$y_2 + a^2 y = -A_1 a \sin ax + B_1 a \cos ax$$

$$\operatorname{cosec} ax = -A_1 a \sin ax + B_1 a \cos ax$$

(3) x $a \sin ax \Rightarrow$

$$aA_1 \cos ax \sin ax + aB_1 \sin ax \sin ax = 0$$

(4) x $\cos ax \Rightarrow$

$$-A_1 a \sin ax \cos ax + B_1 a \cos^2 ax = \operatorname{cosec} ax \cos ax$$

$$aB_1 (\sin^2 ax + \cos^2 ax) = \cot ax$$

$$aB_1 = \cot ax$$

$$B_1 = \frac{\cot x}{a}$$

$$\frac{dB}{dx} = \cot ax / a$$

$$dB = \frac{\cot ax}{a}\, dx$$

$\int$ ing

$$\int dB = \frac{1}{a}\int \cot ax\, dx$$

$$B = \frac{1}{a} \log(\sin ax)\frac{1}{a} + c_1$$

$$B = \frac{1}{a^2} \log(\sin ax) + c_1$$

sub $B_1 = \frac{\cot ax}{a}$ in ③

$$A_1 \cos ax + \frac{\cot ax}{a} \sin ax = 0$$

$$A_1 \cos ax + \frac{1}{a}\frac{\cos ax}{\sin ax} \sin ax = 0$$

$$A_1 \cos ax + \frac{1}{a}\cos ax = 0$$

$$A_1 \cos ax = \frac{-1}{a}\cos ax$$

$$\frac{dA}{dx}\cos ax = \frac{-1}{a}\cos ax$$

$$\frac{dA}{dx} = \frac{-1}{a}\frac{\cos ax}{\cos ax}$$

$$dA = \frac{-1}{a}dx$$

$\int$ ing

$$\int dA = \frac{-1}{a}\int dx$$

$$A = \frac{-1}{a}x + c_2$$

$$A = \frac{-x}{a} + c_2$$

sub A & B in ②

$$y = \left(\frac{-x}{a} + c_2\right)\cos ax + \left(\frac{1}{a^2}\log(\sin ax) + c_1\right)\sin ax$$

where c_1 & c_2 are orbitary constant.

---

3) $y'' + 4y = \sec 2x$

Soln: Given equ is

$$y'' + 4y = \sec 2x$$

(i.e);

$$\frac{d^2y}{dx^2}+4y = \sec 2x \rightarrow (1)$$

$$(D^2+4)y = \sec 2x$$

consider $(D^2+4)y=0$

Auxiliary equ is

$$m^2+4=0$$

$$m^2=-4$$

$$m=\pm i2$$

Here m is complex roots

$$C.F = e^{\alpha x}(A\cos\beta x + B\sin\beta x)$$

$$y = e^{0x}(A\cos 2x + B\sin 2x)$$

$$y = A\cos 2x + B\sin 2x \rightarrow (2)$$

(i.e);

choose,

$A_1 = \frac{dA}{dx}$, $B_1 = \frac{dB}{dx}$ which satisfies

$$A_1\cos 2x + B_1\sin 2x = 0 \rightarrow (3)$$

Diff (2) w.r.to 'x'

$$\frac{dy}{dx} = A(-\sin 2x)(2) + \cos 2x\cdot\frac{dA}{dx} + B\cos 2x.2 + \sin 2x\frac{dB}{dx}$$

$$= -2A\sin 2x + \underbrace{A_1\cos 2x + B_1\sin 2x}_{0 \text{ by } (3)} + 2B\cos 2x$$

$$\frac{dy}{dx} = -2A\sin 2x + 2B\cos 2x \rightarrow (4)$$

Diff (4). w.r.to 'x'

$$y_2 = -2A\cos 2x(2) + (-2\sin 2x)\frac{dA}{dx} + 2B(-\sin 2x(2)) + 2\cos 2x\cdot\frac{dB}{dx}$$

$$= -4A\cos 2x - A_1 2\sin 2x - 4B\sin 2x + B_1 2\cos 2x$$

$$= -4(A\cos 2x + B\sin 2x) - A_1 2\sin 2x + B_1 2\cos 2x$$

$$y_2 = -4y - 2A_1\sin 2x + 2B_1\cos 2x$$

$$y_2 + 4y = -2A_1\sin 2x + 2B_1\cos 2x$$

$$\sec 2x = -2A_1\sin 2x + 2B_1\cos 2x \rightarrow (5)$$

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$2x$

③ $\times 2\sin 2x \Rightarrow$
$2A_1 \cos 2x \sin 2x + 2B_1 \sin^2 2x = 0$

⑤ $\times \cos 2x \Rightarrow$

$-2A_1 \sin 2x \cos 2x + 2B_1 \cos^2 2x = \sec 2x \cos 2x$

$2B_1 (\sin^2 2x + \cos^2 2x) = 1$

$2B_1 = 1$

$B_1 = 1/2$

$\frac{dB}{dx} = \frac{1}{2}$

$dB = \frac{1}{2} dx$

$\int$ ing $\int dB = \frac{1}{2} \int dx$

$B = \frac{x}{2} + C_1$

sub $B_1 = \frac{1}{2}$ in ③

$A_1 \cos 2x + \frac{1}{2} \sin 2x = 0$

$A_1 \cos 2x = -\frac{1}{2} \sin 2x$

$A_1 = -\frac{1}{2} \frac{\sin 2x}{\cos 2x}$

$A_1 = -\frac{1}{2} \tan 2x$

$\frac{dA}{dx} = -\frac{1}{2} \tan 2x$

$dA = -\frac{1}{2} \tan 2x \, dx$

$\int$ ing

$\int dA = -\frac{1}{2} \int \tan 2A \, dx$

$A = -\frac{1}{2} \log(\sec 2x) \cdot \frac{1}{2} + C_2$

$= -\frac{1}{4} \log(\sec 2x) + C_2$

$= \frac{1}{4} \log(\sec 2x)^{-1} + C_2$

$A = 1/4 \log(\cos 2x) + C_2$

sub A & B in (2)

$$y = \left(\frac{1}{4}\log(\cos 2x) + c_2\right)\cos 2x + \left(\frac{x}{2} + c_1\right)\sin 2x$$

where c_1 & c_2 are orbitary constant.

Partial differential equ of the first order :

We now consider equs in which the no of independent variables is two or more and only one dependent variable. We usually denote this by z and the independent variable by x & y. If there be two if there be n independent variable we shall call them $x_1, x_2, x_3, \ldots x_n$

The partial deviation $\frac{\partial z}{\partial x}$, $\frac{\partial z}{\partial y}$ are denoted by p and q, while the letter case $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial x_2}, \ldots \frac{\partial z}{\partial x_n}$ are represented by $P_1, P_2, \ldots P_n$ respectively.

Partial differential equ are those which involve are or more partial derivations

Classification of Integral :

Let the partial differential equ (Pde) be

$$F(x, y, z, p, q) = 0 \rightarrow (1)$$

let the soln of this equ(1) be

$$\phi(x, y, z, a, b) = 0 \rightarrow (2)$$

where a, b are arbitrary constants.

complete integral :

The soln (2) which contains as many arbitrary constants as there are independent variables is called the complete integral of (1).

Partial integral :

A point of (1) is that get by giving particular values to a and b in (2)

Singular integral:

The eliminant of a and b b/w

$\phi(x, y, z, a, b) = 0$

$\frac{\partial \phi}{\partial a} = 0$

$\frac{\partial \phi}{\partial b} = 0$

when it exists is called the singular integral.

General integral:

In (2), we shall assume an arbitrary relation b/w a & b of the form $b = f(a)$

$\therefore (2) \Rightarrow \phi(x, y, z, a, f(a)) = 0 \rightarrow (3)$

Diff. p. w. r to 'a' we get

$$\frac{\partial \phi}{\partial a} + \frac{\partial \phi}{\partial b} f'(a) = 0 \rightarrow (4)$$

The eliminant of a b/w (3) & (4) is called the general integral.

The envelope of the family of the surface touches them along this curve which is called the characteristic of the envelope.

Thus G.I represents the envelope of a family of surfaces considered as composed of its characteristics.

Note:

When the singular integral is formed, it is necessary to verify whether the eliminant of a & b between

$$\phi = 0, \quad \frac{\partial \phi}{\partial a} = 0, \quad \frac{\partial \phi}{\partial b} = 0$$ satisfies (1).

This eliminating may be include extraneous loci such locus of conical pts & double lines which are not solns in general of (1).

Derivation of P.D.E:

(i) By elimination of constants:

let $\phi(x, y, z, a, b) = 0 \rightarrow (1)$

be a relation b/w x, y, z involving two arbitrary constants a & b.

Diff. p. (1). w.r. to 'x & y', we get

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial z} p = 0 \rightarrow (2)$$

$$\frac{\partial \phi}{\partial y} + \frac{\partial \phi}{\partial z} q = 0 \rightarrow (3)$$

Eliminating a, b we have a p.d.e of first order of the form. $F(x, y, z, p, q)$

Notations:

$$p = \frac{\partial z}{\partial x}, \quad q = \frac{\partial z}{\partial y}$$

$$r = \frac{\partial^2 z}{\partial x^2}, \quad s = \frac{\partial^2 z}{\partial y^2}$$

$$t = \frac{\partial^2 z}{\partial y^2}$$

1) Eliminate a and b form:

Given equ; $z = (x+a)(y+b) \rightarrow (1)$

Soln: Diff partially (1). w.r to 'x'

$(1+0)(y+b)$

$$\frac{\partial z}{\partial x} = y+b$$

$$p = y+b \rightarrow (2)$$

Diff p(1). w.r to 'y'

$$\frac{\partial z}{\partial y} = x+a$$

$$q = x+a \rightarrow (3)$$

sub (2) & (3) in (1) we get

$$z = pq$$

---

2) Eliminate h and k from the relation:

$$(x-h)^2 + (y-k)^2 + z^2 = r^2 \text{ (or)}$$

Ques: Find the PDE of all spheres of radius r having centre in xy plane. $(h, k, 0)$

Soln: Given equ is

$$(x-h)^2 + (y-k)^2 + z^2 = r^2 \rightarrow (1)$$

Diff (1). partially with. r to 'x'

$$\frac{\partial z}{\partial x} \Rightarrow 2(x-h) + 2z \cdot \frac{\partial z}{\partial x} = 0$$

p

$$\text{(i.e)} \quad (x-h) + zp = 0$$

Diff (1). par. w. to 'y', $x-h = -pz$

$$2(y-k) + 2z\frac{\partial z}{\partial y} = 0$$

By Eliminating arbitrary functions:

let $\phi(u,v)=0$ be given

$$\phi(u,v)=0 \rightarrow (1)$$

u, v are functions of x and y

1) Eliminate arbitrary function from $z=f(x^2+y^2)$

Soln:

Given equ is

$$z=f(x^2+y^2) \rightarrow (1)$$

Diff p.w.r. to 'x'

$$\frac{\partial z}{\partial x}=f'(x^2+y^2)\,2x$$

$$p=2xf'(x^2+y^2) \rightarrow (2)$$

Diff.p.w.r.to 'y'

$$\frac{\partial z}{\partial y}=f'(x^2+y^2)\,2y$$

$$q=2yf'(x^2+y^2) \rightarrow (3)$$

$$\frac{(2)}{(3)} \Rightarrow \frac{p}{q}=\frac{\cancel{2}x\cancel{f'(x^2+y^2)}}{\cancel{2}y\cancel{f'(x^2+y^2)}}=\frac{x}{y}$$

$$py=qx$$

2) Eliminating the arbitrary function f from $f(x^2+y^2+z^2,\ z^2-2xy)=0$

Soln:

Given equ is

$$f(\underbrace{x^2+y^2+z^2}_{u}, \underbrace{z^2-2xy}_{v}) = 0$$

Here

$$u = x^2+y^2+z^2 , \quad v = z^2 - 2xy$$

$$u_x = 2x + 2z \underbrace{\frac{\partial z}{\partial x}}_{p} , \quad v_x = 2z \overbrace{\frac{\partial z}{\partial x}}^{p} - 2y$$

$$u_x = 2x + 2pz , \quad v_x = 2pz - 2y$$

$$u_y = 2y + 2z\frac{\partial z}{\partial y} , \quad v_y = 2z\frac{\partial z}{\partial y} - 2x$$

$$u_y = 2y + 2qz , \quad v_y = 2qz - 2x$$

consider

$$\begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = 0$$

$$\begin{vmatrix} 2x+2pz & 2y+2qz \\ 2pz-2y & 2qz-2x \end{vmatrix} = 0$$

$$4qxz + \cancel{4pqz^2} - 4x^2 - 4pxz - 4yzp - 4y^2 - \cancel{4pqz^2} + 4qyz =$$

$$qxz - x^2 - pxz - yzp + y^2 + qyz = 0$$

$$qz(x+y) - pz(x+y) - (x^2-y^2) = 0$$

$$qz(x+y) - pz(x+y) - (x+y)(x-y) = 0$$

$$x+y(qz - pz - (x-y)) = 0$$

$$qz - pz - (x-y) = 0$$

$$qz - pz - (x-y) = 0$$

$$z(q-p) = x-y$$

Ex :

3) Eliminating f and ϕ from the relation

$$z = f(x+ay) + \phi(x-ay).$$

Soln :

Given equ is ,

$$z = f(x+ay) + \phi(x-ay) \rightarrow ①$$

① Diff . p . w . r . to 'x'

$$\frac{\partial z}{\partial x} = f'(x+ay) + \phi'(x-ay) \rightarrow ②$$

$$p = f'(x+ay) + \phi'(x-ay)$$

① Diff p.w.r to 'y'

$$\frac{\partial z}{\partial y} = f'(x+ay).a + \phi'(x-ay)(\cdot$$

$$q = af'(x+ay) - \phi' a(x-ay) \rightarrow$$

Diff ③. p.w.r to 'x'

$$\frac{\partial^2 z}{\partial x^2} = f''(x+ay) + \phi''(x-ay)$$

$$r = f''(x+ay) + \phi''(x-ay) \rightarrow ($$

Diff ③. w.p. to 'y'

$$\frac{\partial^2 z}{\partial y^2} = af''(x+ay)(a) - a\phi'(x-$$
$$(-a$$

$$t = a^2 f''(x+ay) + a^2\phi''(x-ay$$
$$\rightarrow ⑤$$

$$\frac{④}{⑤} \Rightarrow \frac{r}{t} = \frac{1}{a^2}$$

(i.e) ; $ra^2 = t$

$$ra^2 - t = 0$$

---

4) Eliminating the arbitary function from

$f(x^2+y^2, z-xy) = 0$

Soln:

Given equ is

$$f(x^2+y^2, z-xy) = 0$$

let $x^2+y^2 = F(z-xy)$ (or) $z-xy = F(x^2+y^2) \rightarrow ①$

Diff. p.w.r to 'x'

$$2x = F'(z-xy)\left(\frac{\partial z}{\partial x} - y\right)$$

$$2x = F'(z-xy)(p-y)$$

$$\frac{2x}{p-y} = F'(z-xy) \rightarrow ②$$

Diff. p.w.r to 'y'

$$2y = F'(z-xy)\left(\frac{\partial z}{\partial y} - x\right)$$

$$= F'(z-xy)(q-z)$$

$$\frac{2y}{q-z} = F'(z-xy) \rightarrow ③$$

$$\frac{②}{③} \quad \frac{\frac{\partial x}{p-y}}{\frac{\partial y}{q-x}} = \frac{F'(z-xy)}{F'(z-xy)}$$

$$\frac{\partial x / p-y}{\partial y / q-x} = 1$$

$$\frac{\partial x}{p-y} \times \frac{q-x}{\partial y} = 1$$

$$\frac{qx-x^2}{py-y^2} = 1$$

$$qx-x^2 = py-y^2$$

$$x^2-y^2+py-qx = 0$$

Home work sum :

1) $\frac{x^2+y^2}{a^2}+\frac{z^2}{b^2} = 1$

soln : Given equ is

$$\frac{x^2+y^2}{a^2}+\frac{z^2}{b^2} = 1 \rightarrow ①$$

Diff. p. w. r to 'x'

$$\frac{b^2(x^2+y^2)+a^2z^2}{a^2b^2} = 1$$

$$b^2(x^2+y^2)+a^2z^2 = a^2b^2$$

Diff. p. w. r to 'x'

$$b^2(2x)+a^2 2z\frac{\partial z}{\partial x} = 0$$

$$2(b^2x+a^2zp) = 0$$

$$bx^2+a^2zp = 0$$

$$a^2zp = -bx^2$$

$$p = \frac{-b^2x}{a^2z}$$

Diff. p. w. r to 'y'

$$b^2(2y)+a^2 2z\frac{\partial z}{\partial y} = 0$$

$$b^2y+a^2zq = 0$$

$$q = -\frac{b^2y}{a^2z} \rightarrow ④$$

$$\frac{③}{④} \Rightarrow \frac{p}{q} = \frac{\cancel{-b^2}x/\cancel{a^2z}}{\cancel{-b^2}y/\cancel{a^2z}} = \frac{x}{y}$$

$$py = qx$$

2) Eliminating a and b from $z = ax + by + a$.

Soln :

Given equ is

$$z = ax + by + a \rightarrow (1)$$

Diff p.w.r.to 'x'

$$\frac{\partial z}{\partial x} = a \Rightarrow p = a \rightarrow (2)$$

Diff p.w.r.to 'y'

$$\frac{\partial z}{\partial y} = b \Rightarrow q = b \rightarrow (3)$$

sub (2) & (3) in (1)

$$z = px + qy + p$$

$$= p(x+1) + qy$$

which is the required PDE.

---

3) $z = e^{y} f(x+y)$

soln :

Given equ is

$$z = e^{y} f(x+y)$$

Diff.p.w.r.to 'x'

$$\frac{\partial z}{\partial x} = e^{y} f'(x+y) \Rightarrow p = e^{y} f'(x+y) \rightarrow (2)$$

Diff.p.w.r.to 'y'

$$\frac{\partial z}{\partial x} = e^{y} f'(x+y) + f(x+y) . e^{y}$$

$$q = e^{y} f'(x+y) + f(x+y) e^{y} \rightarrow (3)$$

using (2) & (3) in (1), we get

$q = p + z$ is the required PDE

---

4) $z = f\left(\frac{xy}{z}\right)$

Soln :

Given equ is

$$z = f\left(\frac{xy}{z}\right) \rightarrow (1)$$

Diff.p.w.r.to 'x'

$$\frac{\partial z}{\partial x} = f'\left(\frac{xy}{z}\right)\left(\frac{z(y) - xy\frac{\partial z}{\partial x}}{z^2}\right)$$

$$p = f'\left(\frac{xy}{z}\right)\left(\frac{yz - xyp}{z^2}\right) \rightarrow ②$$

Diff. p.w.r to 'y'

$$\frac{\partial z}{\partial y} = f'\left(\frac{xy}{z}\right)\left[\frac{z(x) - xy \cdot \frac{\partial z}{\partial y}}{x^2}\right]$$

$$q = f'\left(\frac{xy}{z}\right)\left[\frac{zx - xyq}{z^2}\right] \rightarrow ③$$

$$\frac{②}{③} \Rightarrow \frac{p}{q} = \frac{f'(xy/z)\left(\frac{yz - xyp}{z^2}\right)}{f'\left(\frac{xy}{z}\right)\left(\frac{zx - xyq}{z^2}\right)}$$

$$\frac{p}{q} = \frac{yz - xyp}{xz - xyq}$$

$$p(xz - xyq) = q(yz - xyp)$$

$$pxz - pqxy = qyz - qpxy$$

$pxz = qyz$ is the required PDE

---

5) 8) $2z = (ax+y)^2 + b$

Soln: Given equ is

$$2z = (ax+y)^2 + b \rightarrow ①$$

Diff. p.w.r to 'x'

$$2 \cdot \frac{\partial z}{\partial x} = 2(ax+y)(a)$$

$$\frac{\partial z}{\partial x} = a(ax+y)$$

$$p = a(ax+y) \rightarrow ②$$

Diff. p.w.r to 'y'

$$2 \cdot \frac{\partial z}{\partial y} = 2(ax+y)$$

$$\frac{\partial z}{\partial y} = (ax+y)$$

$$q = ax+y \rightarrow ③$$

sub ③ in ②

$$p = aq$$

$$a = p/q \rightarrow ④$$

sub ④ in ③

$$q = p/q\, x + y$$

$$q = \frac{px + qy}{q}$$

(ii) $ax^2+by^2+z^2=1$

soln:

Given equ is

$$ax^2+by^2+z^2=1 \rightarrow (1)$$

Diff. p.w.r.to 'x'

$$2ax+2z\cdot\frac{\partial z}{\partial x}=0$$

$$ax+pz=0$$

$$ax=-pz$$

$$a=\frac{-pz}{x} \rightarrow (2)$$

Diff p.w.r.to 'y'

$$2by+2z\cdot\frac{\partial z}{\partial y}=0$$

$$by+qz=0$$

$$by=-qz$$

$$y=-qz/b$$

$$b=\frac{-qz}{y} \rightarrow (3)$$

sub (2) & (3) in (1)

$$\frac{-pz}{x}\times x^2-\frac{qz}{y}\times y^2+z^2=1$$

$$-pxz-qyz+z^2=1$$

$$z(z-px-qy)=1$$

6) $ax+by+cz=f(x^2+y^2+z^2)$

Soln:

Given equ is

$$ax+by+cz=f(x^2+y^2+z^2) \rightarrow (*)$$

Diff. p.w.r.to 'x'

$$a+c\cdot\frac{\partial z}{\partial x}=f'[x^2+y^2+z^2][2x+2z\cdot\partial z/\partial x]$$

$$a+cp=f'[x^2+y^2+z^2]\,2[x+zp]$$

$$\frac{a+cp}{x+zp}=2f'[x^2+y^2+z^2] \rightarrow (1)$$

Diff. p.w.r.to 'y'

$$b + c\cdot\frac{\partial z}{\partial y} = f'[x^2+y^2+z^2][2y+2z\cdot \partial z/\partial y]$$

$$b+cq = f'[x^2+y^2+z^2]\cdot 2[y+zq]$$

$$\frac{b+cq}{y+zq} = 2f'[x^2+y^2+z^2] \rightarrow (2)$$

$(1) = (2)$

$$\frac{a+cp}{x+zp} = \frac{b+cq}{y+zq}$$

$$(a+cp)(y+zq) = (b+cq)(x+zp)$$

$$ay + zaq + cpy + cpzq = bx + bzp + cqx + cqzp$$

$$ay + aqz + pcy + pqcz - bx - bpz - qcx - pqcz =$$

$$ay - bx + (p-q)cx + (aq-bp)z = 0$$

$$(a+pc)y - (b+qc)x + (aq-bp)z = 0$$

$$\therefore (a+pc)y + (aq-bp)z = (b+qc)x$$

---

7) $x = f(y) + \phi(z)$

Soln :

Given equ is,

$$x = f(y) + \phi(z) \rightarrow (1)$$

$p = \frac{\partial z}{\partial x}$

$q = \frac{\partial z}{\partial y}$

Diff (1) p. w. r. to 'x'

$$1 = 0 + \phi'(z)\cdot\frac{\partial z}{\partial x}$$

$$1 = p\phi'(z) \rightarrow (2)$$

Diff (1) p. w. r. to 'y'

$$0 = f'(y) + \phi'(z)\cdot\frac{\partial z}{\partial y}$$

$$0 = f'(y) + q\phi'(z) \rightarrow (3)$$

Diff (2) p. w. r. to 'x'

$$0 = p\cdot\phi''(z)\cdot\frac{\partial z}{\partial x} + \phi'(z)\cdot\frac{\partial^2 z}{\partial x^2}$$

$$0 = p^2\phi''(z) + r\phi'(z) \rightarrow (4)$$

Diff (3) p. w. r. to 'x'

$$0 = 0 + pq\,\phi''(z) + \phi'(z)\cdot\frac{\partial^2 z}{\partial x \partial y}$$

$$0 = pq\,\phi''(z) + s\phi'(z)$$

$$0 = pq\,\phi''(z) + s/p$$

$$0 = pq\,\phi''(z) + s/p$$

$pq\phi''(z) = -s/p$

$\phi''(z) = -s/p^2q \rightarrow (5)$

sub (3) & (5) in (4)

$0 = \frac{p^2 \times -s}{p^2 q} + r \times 1/p$

$= -s/q + r/p$

$0 = -ps + qr/qp$

$0 = -ps + qr$

$ps = qr$.

8) $z = (x+y)f(x^2-y^2)$

soln:

Given equ is

$z = (x+y)f(x^2-y^2) \rightarrow (1)$

diff (1) p.w.r.to 'x'

$\frac{\partial z}{\partial x} = (x+y)f'(x^2-y^2)(2x) + f(x^2-y^2)(1)$

$p = 2x(x+y)f'(x^2-y^2) + f(x^2-y^2)$

$p - f(x^2-y^2) = 2x(x+y)f'(x^2-y^2) \rightarrow (2)$

diff (1) p.w.r.to 'y'

$\frac{\partial z}{\partial y} = (x+y)f'(x^2-y^2)(-2y) + f(x^2-y^2)$

$q = -2y(x+y)f'(x^2-y^2) + f(x^2-y^2)$

$q - f(x^2-y^2) = -2y(x+y)f'(x^2-y^2) \rightarrow (3)$

from (1) $\Rightarrow \frac{z}{x+y} = f(x^2-y^2) \rightarrow (4)$

Using (4) in (2) & (3), weget

(2) $\Rightarrow p - \frac{z}{x+y} = 2x(x+y)f'(x^2-y^2) \rightarrow (5)$

(3) $\Rightarrow q - \frac{z}{x+y} = -2y(x+y)f'(x^2-y^2) \rightarrow (6)$

$$\frac{⑤}{⑥} \Rightarrow \frac{p-\frac{z}{x+y}}{q-\frac{z}{x+y}} = \frac{2x(x+y)f'(x^2-y^2)}{-2y(x+y)f'(x^2-y^2)}$$

$$\frac{(x+y)p-z/x+y}{(x+y)q-z/x+y} = \frac{-x}{y}$$

$$[(x+y)p-z]y = -x[q(x+y)-z]$$

$$(px+py-z)y = -x(qx+qy-z)$$

$$pxy+py^2-zy = -x^2q - xqy + zx$$

$$pxy+py^2-zy-xz+qx^2+qxy=0$$

$$py(x+y)-z(y+x)+qx(x+y)=0$$

$$(x+y)(py-z+qx)=0$$

$$py-z+qx=0$$

$\therefore z = py+qx$ is the P.D.E.

---

9) $f(x+y+z) = xyz$.

Soln: Given equ is ; $f(x+y+z) = xyz \rightarrow ①$

D. ① P.w.r.to 'x'

$$f'(x+y+z)\left(1+\frac{\partial z}{\partial x}\right) = y\left(x\cdot\frac{\partial z}{\partial x}+z(1)\right)$$

$$f'(x+y+z)(1+p) = y(xp+z)$$

$$f'(x+y+z) = \frac{y(xp+z)}{1+p} \rightarrow ②$$

Diff ① P.w.r.to 'y'

$$f'(x+y+z)\left(1+\frac{\partial z}{\partial y}\right) = x\left(y\frac{\partial z}{\partial y}+z(1)\right)$$

$$f'(x+y+z) = \frac{x(yq+z)}{(1+q)} \rightarrow ③$$

② = ③

$$\frac{y(xp+z)}{1+p} = \frac{x(yq+z)}{1+q}$$

$$(1+q)y(xp+z) = (1+p)x(yq+z)$$

$$(y+qy)(xp+z) = (x+px)(yq+z)$$

$$xyp + yz + xyqp + qyz = xyq + xz + xypq + zqx$$

$$px(y-z)+qy(z-x) = z(x-y).$$

Lagrange's Method of solving the linear Equations

consider the equs $u=a$ and $v=b$ where a and b are arbitary constants. Eliminating a and b, we form the differentequ corresponding to them

$$\frac{\partial u}{\partial x}dx+\frac{\partial u}{\partial y}dy+\frac{\partial u}{\partial z}dz=0$$

and

$$\frac{\partial v}{\partial x}dx+\frac{\partial v}{\partial y}dy+\frac{\partial v}{\partial z}dz=0$$

$$\therefore \frac{dx}{u_y v_z-u_z v_y}=\frac{dy}{u_z v_x-u_x v_z}=\frac{dz}{u_x v_y-u_y v_x}$$

(ie) $\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}$, the elimination of the arbitary function ϕ from $\phi(u,v)=0$ lead to the linear partial differential equ.

$$Pp+Qq=R$$

Thus lagrange's method of solving the linear equ $Pp+Qq=R$ is as follows,

write down the subsidiary equ, $\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}$ let the two independent integrals of these ordinary differential equation be $u=a$ & $v=b$ from then the solution of the given equ is $\phi(u,v)=0$ where ϕ is an arbitary function

Partial differential equ of the first order:

$\phi(u,v)=0$ is called the general integral of lagrange's linear equation.

Corollary:1 This method canbe extended to the case of the linear equation n of n independent variables consider the equ $P_1p_1+P_2p_2+P_3p_3+\dots+P_np_n=R$

From the subsidiary equs

$$\frac{dx_1}{P_1}=\frac{dx_2}{P_2}=\frac{dx_3}{P_3}=\dots\frac{dx_n}{P_n}=\frac{dz}{R}$$

Let n independent integrals of these be $u_1 = a_1$, $u_2 = a_2$, $u_3 = a_3$ .... $u_n = a_n$.

Then $\phi(u_1, u_2, u_3, \ldots, u_n) = 0$ is a solution of the given equ, where ϕ denotes an arbitary function.

Note:

The above relation $\phi(u, v) = 0$ (or) $\phi(u_1, \ldots u_n) = 0$ contains all the integrals of the equ which are not of the type called singular.

corollary : 2

when either $u = a$ or $v = b$ involves z, it is an integral of the differential equ.

$\phi(u, v) = 0$ can be written as $u = f(v)$ where f is arbitary. we can take $f(v) = av$ where a is an arbitary constant thus the solution reduces to $u = a$.

Lagrange's method of solving the linear equations:

Ex:

1) $a(p + qx) = 1$ $\qquad Pp + Qq = R$

Soln: Given equ is

$$a(p + qx) = 1$$

$$ap + aqx = 1$$

The auxillary equ is

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

Here, $P = a$, $Q = ax$, $R = 1$

(i.e); $\frac{dx}{a} = \frac{dy}{ax} = \frac{dz}{1}$

Taking first ratios

$$\frac{dx}{a} = \frac{dy}{ax}$$

$$dx = \frac{dy}{x}$$

$$x\,dx = dy$$

$\int$ing,

$$\frac{x^2}{2} = y + c_1$$

$$\frac{x^2}{2} - y = c_1 = u \text{ (say)}$$

Taking $\frac{dx}{a} = \frac{dz}{1}$

$$dx = a\,dz$$

$\int$ing $x = az + c_2$

$$x - az = c_2 = v \text{ (say)}$$

The solution of given equ is

$$\phi(u, v) = 0$$

(i.e) $\phi\left(\frac{x^2}{2} - y,\ x - az\right) = 0$

---

2) Solve: $x^2p + y^2q = z^2$

Soln :

Given equ is

$$x^2p + y^2p = z^2$$

The auxillary equ is

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

Here, $P = x^2$; $Q = y^2$; $R = z^2$.

(i.e); $\frac{dx}{x^2} = \frac{dy}{y^2} = \frac{dz}{z^2}$

Taking first ratios

$$\frac{dx}{x^2} = \frac{dy}{y^2}$$

$$y^2dx = x^2dy$$

$\int$ing, $x^{-2}dx = y^{-2}dy$

$$\frac{x^{-1}}{-1} = \frac{y^{-1}}{-1} + c_1$$

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$$\frac{-1}{x} + \frac{1}{y} = c_1$$

$$\frac{1}{y} - \frac{1}{x} = c_1 = u \text{ (say)}$$

Taking $\frac{dx}{x^2} = \frac{dz}{z^2}$

$\int$ing, $\frac{x^{-1}}{-1} = \frac{z^{-1}}{-1} + c_2$

$$-\frac{1}{x} + \frac{1}{z} = c_2$$

$$\frac{1}{z} - \frac{1}{x} = c_2 = v \text{ (say)}$$

The solution of the given equ is

$$\phi(u, v) = 0$$

(i.e) $\phi\left(\frac{1}{y} - \frac{1}{x}, \frac{1}{z} - \frac{1}{x}\right) = 0$

$a(p+q) = 1$

soln :

Given equ is

$$a(p+q) = 1$$

$$ap + aq = 1$$

The auxillary equ is

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

Here, $P = a$; $Q = a$; $R = 1$

(i.e) ; $\frac{dx}{a} = \frac{dy}{a} = \frac{dz}{1}$

Taking first ratios

$$\frac{dx}{\cancel{a}} = \frac{dy}{\cancel{a}}$$

$$dx = dy$$

$\int$ing, $x = y + c_1$

$$x - y = c_1 = u \text{ (say)}$$

Taking $\frac{dx}{a} = \frac{dz}{1}$

$\int$ing ; $dx = a\,dz$

$$x = az + c_2$$

$$x - az = c_2 = v \text{(sa}$$

The solution of given equ is

$$\phi(u, v) = 0$$

(i.e) $\phi(x - y, x - az) = 0$

Ex:

1) solve : $(y+z)p+(z+x)q = x+y$

Soln :

Given equ is

$$(y+z)p + (z+x)q = x+y$$

The auxillary equ is

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

Here,

$$P = y+z \; ; \; Q = z+x \; ; \; R = x+y$$

$$\text{(i.e)} \quad \frac{dx}{y+z} = \frac{dy}{z+x} = \frac{dz}{x+y}$$

It can be written as

$$\frac{dx-dy}{y-x} = \frac{dy-dz}{z-y} = \frac{dz-dx}{x-z} = \frac{dx+dy+dz}{2(x+y+z)}$$

Taking the first two ratios

$$\frac{dx-dy}{y-x} = \frac{dy-dz}{-y+z}$$

$$\frac{-d(x-y)}{(x-y)} = \frac{-d(y-z)}{(y-z)}$$

$\int$ing

$$-\log(x-y) = -\log(y-z) + \log c$$

$$-\log(x-y) + \log(y-z) = \log c_1$$

$$\log\left(\frac{y-z}{x-y}\right) = \log c_1$$

(i.e) :

$$\frac{y-z}{x-y} = c_1 = u$$

Taking the second two ratios

$$\frac{dy-dz}{z-y} = \frac{dz-dx}{x-z}$$

$$\frac{-d(y-z)}{(y-z)} = \frac{-d(z-x)}{(z-x)}$$

$\int$ ing $-\log(y-z) = -\log(z+x) + \log c_2$

$-\log(y-z) + \log(z-x) = \log c_2$

$\log\left(\frac{z-x}{y-z}\right) = \log c_2$

(i.e) : $\frac{z-x}{y-z} = c_2 = v$

The solution of the given equation is

$\phi(u, v) = 0$

$\phi\left(\frac{y-z}{x-y}, \frac{z-x}{y-z}\right) = 0$

(X) university Question (8) *

Ex :

Solve : $px(y^2+z) - qy(x^2+z) = z(x^2-y^2)$. Find the surface that contains the st. line $x+y=0, z=1$.

Soln : Given equ is

$px(y^2+z) - qy(x^2+z) = z(x^2-y^2)$

The auxillary equ is

$$\frac{dx}{x(y^2+z)} = \frac{dy}{-y(x^2+z)} = \frac{dz}{z(x^2-y^2)}$$

Taking the first two ratios

$$\frac{xdx+ydy}{x^2(y^2+z) - y^2(x^2+z)} = \frac{dz}{z(x^2-y^2)}$$

$$\frac{xdx+ydy}{\cancel{x^2y^2} + x^2z - \cancel{x^2y^2} + y^2z} = \frac{dz}{z(x^2-y^2)}$$

$$\frac{xdx+ydy}{z(x^2+y^2)} = \frac{dz}{\cancel{z}(x^2+y^2)}$$

$$xdx + ydy = dz$$

$\int$ ing $\frac{x^2}{2} + \frac{y^2}{2} = z + \frac{c_1}{2}$

$x^2+y^2 = 2z + c_1$

$x^2+y^2-2z = c_1 = u$

$$\frac{dx/x + dy/y}{\frac{\cancel{x}(y^2+z)}{x} - \frac{y(x^2+z)}{y}} = \frac{dz}{z(x^2-y^2)}$$

$$y^2+z-x^2-z \qquad x^2-y^2$$

$$\frac{\frac{dx}{x}+\frac{dy}{y}}{-(x^2-y^2)} = dz/z$$

$\int$ing

$$-(\log x+\log y) = \log z - \log c_2$$

$$-\log x - \log y - \log z = -\log c_2$$

$$\log (xyz) = \log c_2$$

$$xyz = c_2 = v$$

The solution of given equ is

$$\phi(x^2+y^2-2z, xyz) = 0$$

let, $x^2+y^2+\cancel{\ }-2z = f(xyz) \rightarrow (2)$

$-2(xy+1)$

$$x^2+y^2+2xy-2xy-2z = f(xyz)$$

$$(x+y)^2-2xy-2z = f(xyz)$$

Given that $x+y=0, z=1$

$$\therefore 0-2xy-2(1) = f(xyz)$$

$$-2(xy+1) = f(xyz) \rightarrow (3)$$

sub (3) in (2), weget

$$x^2+y^2-2z = -2(xy+1)$$

$\therefore$ The required surface is

$$x^2+y^2-2z = -2(xy+1).$$

1) Find the integral surface of $x^2p+y^2q+z^2=0$ which passes through the hyperbola $xy=x+y$, $z=1$.

Soln :

Given equ is

$$xy = x+y, z=1$$

Auxillary equation is

$$\frac{dx}{x^2} = \frac{dy}{y^2} = \frac{dz}{-z^2}$$

Taking first two ratios,

$$\frac{dx}{x^2} = \frac{dy}{y^2}$$

$\int$ing, $\int\frac{dx}{x^2} = \int\frac{dy}{y^2}$

$$-\frac{1}{x} = -\frac{1}{y}+c_1$$

$$\frac{1}{y}-\frac{1}{x} = c_1 = u \text{ (say)}$$

Taking second and third ratios

$\frac{dy}{y^2} = \frac{dz}{-z^2}$; $\int$ing, $\int\frac{dy}{y^2} = \int\frac{dz}{-z^2}$

$$-\frac{1}{y} = \frac{1}{z}+c_2$$

$$\frac{1}{y}+\frac{1}{z} = c_2$$

Taking first and third ratios

$$\frac{dx}{x^2} = \frac{dz}{-z^2}$$

$\int$ing, $\int\frac{dx}{x^2} = \frac{dz}{-z^2}$

$$-\frac{1}{x} = \frac{1}{z}+c_3$$

$$\frac{1}{x}+\frac{1}{z} = c_3 = v \text{ (say)}$$

The general solution is

$$\frac{1}{x}+\frac{1}{z} = f\left(\frac{1}{y}+\frac{1}{z}\right) \rightarrow (1)$$

The passes through $xy = x+y$ and $z=1$

$$\frac{1}{x}+1 = f\left(\frac{1}{y}+1\right)$$

$$xy = x+y$$

$$1 = \frac{x+y}{xy}$$

$$1 = \frac{1}{y}+\frac{1}{x}$$

$$1+\frac{1}{x} = 1-\frac{1}{y}+1$$

$$= 1-\frac{1}{y}+1$$

$$f(1+\tfrac{1}{y}) = 2-1/y$$
$$= 3-1-\tfrac{1}{y}$$
$$f(1+\tfrac{1}{y}) = 3-(1+1/y)$$
$$① \Rightarrow \frac{1}{x}+\frac{1}{z} = 3-\left(\frac{1}{y}+\frac{1}{z}\right)$$

★ (S) 10m

Ex :

Determined the surface which satisfied $(x^2-a^2)p + (xy-az\tan\alpha)q = xz - ay\cot\alpha \rightarrow ①$ and passes through the curve $x^2+y^2=a^2$, $z=0$.

Soln:

The auxillary equ

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

(i.e); $$\frac{dx}{x^2-a^2} = \frac{dy}{xy-az\tan\alpha} = \frac{dz}{xz-ay\cot\alpha}$$

consider

$$\frac{dx}{x^2-a^2} = \frac{z\,dy - y\,dz}{\cancel{xyz}-az^2\tan\alpha-\cancel{xyz}+ay^2\cot\alpha}$$

$$\frac{dx}{x^2-a^2} = \frac{z\,dy-y\,dz}{a(y^2\cot\alpha - z^2\tan\alpha)}$$

$$= \frac{\dfrac{z\,dy-y\,dz}{z^2}}{a\left(\dfrac{y^2\cot\alpha}{z^2}-\tan\alpha\right)}$$

$$\frac{dx}{x^2-a^2} = \frac{d(y/z)}{a\cot\alpha\left(y^2/z^2 - \tan^2\alpha\right)}$$

$$\frac{\tan\alpha}{\cot\alpha} = \tan^2\alpha$$

$x \qquad a$

$\int$ing,

$$\frac{1}{2a}\log\left(\frac{x-a}{x+a}\right) = \frac{1}{\cancel{a}\cot\alpha}\;\frac{1}{\cancel{2}\tan\alpha}\log\left(\frac{\frac{y}{z}-\tan\alpha}{y/z+\tan\alpha}\right) + \log c_1$$

$$\log\left(\frac{x-a}{x+a}\right) = \log\left(\frac{\frac{y}{z}-\tan\alpha}{y/z+\tan\alpha}\right)+\log c_1$$

$$\log\left(\frac{x-a}{x+a}\right)-\log\left(\frac{y/z-\tan\alpha}{y/z+\tan\alpha}\right)=\log c_1$$

$$\log\left(\frac{\left(\frac{x-a}{x+a}\right)}{\frac{y/z-\tan\alpha}{y/z+\tan\alpha}}\right)=\log c_1$$

$$\left(\frac{x-a}{x+a}\right)\left(\frac{\frac{y}{z}+\tan\alpha}{y/z-\tan\alpha}\right)=c_1=u$$

consider

$$\frac{dx}{x^2-a^2}=\frac{y\cot\alpha\,dy-z\tan\alpha\,dz}{xy^2\cot\alpha-ayz-xz^2\tan\alpha+ayz}$$

$$=\frac{y\cot\alpha\,dy-z\tan\alpha\,dz}{xy^2\cot\alpha-xz^2\tan\alpha}$$

$$=\frac{\cot\alpha\,y\,dy-\tan\alpha\,z\,dz}{x(y^2\cot\alpha-z^2\tan\alpha)}$$

$$\frac{x\,dx}{x^2-a^2}=\frac{\cot\alpha\,y\,dy-\tan\alpha\,z\,dz}{y^2\cot\alpha-z^2\tan\alpha}$$

$\int$ing, $\frac{1}{2}\log(x^2-a^2)=\frac{1}{2}\log(y^2\cot\alpha-z^2\tan\alpha)+\frac{1}{2}\log c_2$

$$\log(x^2-a^2)-\log(y^2\cot\alpha-z^2\tan\alpha)=\log c_2$$

$$\log\left(\frac{x^2-a^2}{y^2\cot\alpha-z^2\tan\alpha}\right)=\log c_2$$

(i.e); $\dfrac{x^2-a^2}{y^2\cot\alpha-z^2\tan\alpha}=c_2=v$

The solution of given eqn is

$$\phi\left[\left(\frac{x-a}{x+a}\right)\left(\frac{y/z+\tan\alpha}{y/z-\tan\alpha}\right),\ \frac{x^2-a^2}{y^2\cot\alpha-z^2\tan\alpha}\right]=0$$

let $\dfrac{x^2-a^2}{y^2\cot\alpha-z^2\tan\alpha}=f\left[\left(\frac{x-a}{x+a}\right)\cdot\left(\frac{y/z+\tan\alpha}{y/z-\tan\alpha}\right)\right]$

$$=f\left[\frac{x-a}{x+a}\times\frac{(y+z\tan\alpha)/z}{(y-z\tan\alpha)/z}\right]$$

$$\frac{x^2-a^2}{y^2\cot\alpha - z^2\tan\alpha} = f\left[\frac{x-a}{x+a} \times \frac{y+z\tan\alpha}{y-z\tan\alpha}\right] \rightarrow ($$

Given that

$$x^2+y^2 = a^2, \quad z=0$$

$$x^2-a^2 = -y^2, \quad z=0 \rightarrow (3)$$

sub (3) in (2); we get

$$\frac{-y^2}{y^2\cot\alpha - 0} = f\left(\frac{x-a}{x+a} \times \frac{\cancel{y}}{\cancel{y}}\right)$$

$$\frac{-1}{\cot\alpha} = f\left(\frac{x-a}{x+a}\right)$$

$$-\tan\alpha = f\left(\frac{x-a}{x+a}\right) \rightarrow (4)$$

sub (4) in (2); we get

$$\frac{x^2-a^2}{y^2\cot\alpha - z^2\tan\alpha} = -\tan\alpha$$

$$x^2-a^2 = -y^2\tan\alpha\underbrace{\cot\alpha}_{\frac{1}{\tan\alpha}} + z^2\tan^2$$

$$x^2+y^2 = a^2+z^2\tan^2\alpha$$

$$x^2-y^2 = -y^2+z^2\tan^2\alpha$$

Hence the required surface is

$$x^2+y^2 = a^2+z^2\tan^2\alpha$$

Special methods : standard forms:

Standard : 1

Equ in which the variable to do occur. explicity can be written in the form

$F(p,q)=0$

A solution of this $z=ax+by+c$ where a,b are connected by $F(a,b)=0$ solving this for b, $b=f(a)$.

hence the complete integral is $z = ax + yf(a) + c$.

The singular integral is obtained by eliminating a, c b/w

$$z = ax + yf(a) + c$$
$$0 = x + yf'(a) \text{ [D.w.r to a]}$$
$$0 = 1 \text{ [D.w.r to c]}$$

The last equ is obsured & shows that these is no singular integral in the case. To obtain the general integral we assume an arbitary relation

$$c = \phi(a) \text{ D.w.r to a}$$
$$0 + x + yf'(a) + \phi'(a)$$

The eliminant of b/w these equ is the general integral.

__Note:__ The singular & general integrals must be indicated in every equ besides the complete integral. Then only the equ is said to be the completly solved.

__standard : 2__

Only one of the variable x, y, z occur explicity. such equs can be written in are of the forms.

$$F(x,p,q) = 0 \; ; \; F(y,p,q) = 0 \; ; \; F(z,p,q) = 0$$

(i) Let us consider the form $F(x,p,q) = 0$ since z is a function of x & y

$$dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy$$
$$= p\,dx + q\,dy$$

let us assume that $q = a$

The equ becomes $F(x,p,a) = 0$ solving this for p we get $p = \phi(x,a)$

$$\therefore dz = \phi(x,a)\,dx + a\,dy$$
$$z = \int \phi(x,a)\,dx + ay + b$$

This contains two arbitrary constants a & b

hence it is a complete integral.

(ii) Let us consider the form $F(y,p,q)=0$

Let us assume that $p=a$

$$F(y,a,q)=0$$

$$q=\phi(y,a)$$

Hence, $dz = a\,dx + \phi(y,a)\,dy$

$z = ax + \int \phi(y,a)\,dy + b$ which is a complete integral.

(iii) Let us consider the equ $F(z,p,q)=0$

let us assume that $q=ap$ then this equ becomes $F(z,p,ap)=0$ (i.e) $p=\phi(z,a)$

(i.e); $\dfrac{dz}{\phi(z,a)} = dx + a\,dy$

(i.e); $\int \dfrac{dz}{\phi(z,a)} = x+ay+b$ which is a complete integral.

Standard : 3

Equs of the form $f_1(x,p)=f_2(y,q)$. In

This form the equ is of the 1st order & the variables are separable in this equ z does not appear we shall assume a tentative solution that each of these eq is equal to a.

Solving $f_1(x,p)=a$, $p=\phi_1(a,x)$

Solving $f_2(y,q)=a$, $q=\phi_2(a,y)$

Hence $dz=\phi_1(a,x)\,dx+\phi_2(a,y)\,dy$

$z=\int\phi_1(a,x)\,dx+\int\phi_2(a,y)\,dy+b$

which is a complete integral.

A general method of solving of differential equs is due to charpit.

consider $F(x, y, z, p, q) = 0 \rightarrow (1)$

It can be we find another relation b/w the variables & the differential co-efficient the two can be regarded as simultaneous equ is p & q & their values obtained in terms of x, y, z explicity these values when substituted in $dz = pdx + qdy$ will render it easily integrable the integral thus obtained is a solution of (1).

let the other relation sought be

$f(x, y, z, p, q) = A \rightarrow (2)$

where f is arbitrary & A is an arbitrary constant.

Differenting (1) & (2) partially with respect to x and y.

$$Fx + Fz\,p + Fp \cdot \frac{\delta z}{\delta x} + Fq \cdot \frac{\delta q}{\delta x} = 0 \rightarrow (i)$$

$$fx + fz\,p + fp \frac{\delta p}{\delta x} + fq \cdot \frac{\delta q}{\delta x} = 0 \rightarrow (ii)$$

$$Fy + F_z q + Fp \frac{\delta p}{\delta y} + Fq \frac{\delta q}{\delta y} = 0 \rightarrow (iii)$$

$$fy + f_z q + fa \frac{\delta p}{\delta y} = f q \frac{\delta q}{\delta y} = 0 \rightarrow (iv)$$

Eliminating $\frac{\delta p}{\delta x}$ b/w (i) & (ii), we have

$$(Fxfp - Fpfx) + p(Fpfp - Fpfz) + \frac{\delta q}{\delta x}(Fqfp - Fpfq) = 0 \rightarrow (v)$$

Eliminating $\frac{\delta q}{\delta y}$ b/w (iii) & (iv) we get

$$(F_y f_q - F_q f_y) + q(F_z f_q - F_q f_z) + \frac{\delta p}{\delta y}(F_p f_q - F_q f_p) = 0$$

As.

$$\frac{\delta p}{\delta y} = \frac{\delta}{\delta y}\left(\frac{\delta z}{\delta x}\right) = \frac{\delta}{\delta x}\left(\frac{\delta z}{\delta y}\right) = \frac{\delta q}{\delta x}$$, we find

adding (v) & (vi) the terms involving the equs cancel out, hence we get on rearang

$$f_p(F_x + pF_z) + f_q(F_y + qF_z) + f_z(-pF_p - qF_q) + f_x(-F_p) + f_y(-F_q) = 0$$

This is a linear differential equ of the 1st order &nf : by lagrange's method, we write down the auxillary equs

(i.e); $$\frac{dp}{F_x + pF_z} = \frac{dq}{F_y + qF_z} = \frac{dz}{-pF_p - qF_q} = \frac{dx}{-F_p} = \frac{dy}{-F_q}$$

Integrating these equs we get $a = B$ (a constant).

As $df = 0$, $f = A$, an arbitrary const

$\therefore a = B$ is a solution of the differentia equ determining f. The simpler one relatio u is b/w p & q the better for us to solve for p & q b/w $f = 0$ & $F = 0$ u must involve either p or q or both.

Ex:
solving : $p^2+q^2 = npq$

soln:

Given equ is

$$p^2+q^2 = npq \rightarrow (1)$$

This equ is of the form $F(p,q)=0$

The complete soln of the (1) is in the form

$$z = ax+by+c \rightarrow (2)$$

let $p=a, q=b \rightarrow (3)$

sub (3) in (1) we get

$$a^2+b^2 = nab$$

$$a^2+b^2-nab=0$$

$$b^2-nab+a^2=0$$

$$b = \frac{na \pm \sqrt{n^2a^2-4a^2(1)}}{2(1)}$$

$$= \frac{na \pm a\sqrt{n^2-4}}{2}$$

$\therefore$ The complete integral of (1) is

$$z = ax + \left(\frac{na \pm a\sqrt{n^2-4}}{2}\right) y + c \rightarrow (4)$$

Singular integral :

Diff (4). w.r to a.c. we have no solution

There is no singular integral

~~Diff (5) w.r~~

General integral :

Put $c = f(a)$

$$(4) \Rightarrow z = ax + \left(\frac{na \pm a\sqrt{n^2-4}}{2}\right) y + f(a) \rightarrow (5)$$

Diff (5). w.r to a

$$0 = x + \left(\frac{n \pm \sqrt{n^2-4}}{2}\right) y + f'(a) \rightarrow (6)$$

Eliminating a b/w (5) & (6) we get the general integral.

2) $pq = k$

soln:

given equ is $pq = k \rightarrow (1)$

This equ is of the form $F(p,q)=0$

The complete solution of (1) is in the form

$$z = ax + by + c \rightarrow (2)$$

let $p = a, q = b \rightarrow (3)$

sub (3) in (2), we get

$$ab = k$$

$$b = k/a$$

The complete integral of (1) is

$$z = ax + \left(\frac{k}{a}\right)y + c \rightarrow (4)$$

Singular integral:

Diff (4) w.r.to a & c, we get no solution. There is no singular integral.

General integral:

$$c = f(a)$$

$$z = ax + \left(\frac{k}{a}\right)y + f'(a) \rightarrow (5)$$

Diff w.r.to 'a'

$$0(z) = x + k\left(\frac{-1}{a^2}\right)y + f'(a)$$

$$0(z) = x - \frac{ky}{a^2} + f'(a) \rightarrow (6)$$

Eliminating a b/w (5) & (6) we get the general equ.

---

3) $p^2+q^2 = 4$

soln:

Given equ is

$$p^2+q^2 = 4 \rightarrow (1)$$

This equ is of the form $F(p,q)=0$

The complete solution of (1) is in the form

$$z = ax + by + c \rightarrow (2)$$

let $p = a$ & $q = b \rightarrow (3)$

sub (3) in (2), we get

$$a^2 + b^2 = 4$$

$$a^2 + b^2 - 4 = 0$$

$$b^2 + a^2 - 4 = 0$$

$$b^2 = 4 - a^2$$

$$b = \pm\sqrt{4-a^2}$$

The complete integral of (1) is

$$z = ax \pm \sqrt{4-a^2}\, y + c \rightarrow (4)$$

singular integral:

Diff (4) w.r.to a & c, we get no solution. There is no singular Integral

General integral:

$$c = f(a)$$

$$z = ax \pm \sqrt{4-a^2}\, y + f(a) \rightarrow (5)$$

Diff (5) w.r.to 'a'

$$0 = x \pm \frac{1}{2\sqrt{4-a^2}}(-2a)\, y + f'(a)$$

$$0 = x \pm \frac{ay}{\sqrt{4-a^2}} + f'(a) \rightarrow (6)$$

Eliminating a b/w (5) & (6), we get the general integral.

---

4) $\sqrt{p} + \sqrt{q} = 1$

soln: Given equ is

$$\sqrt{p} + \sqrt{q} = 1 \rightarrow (1)$$

The equ is of the form $F(p,q) = 0$

The complete solution of (1) is in the form

$$z = ax + by + c \rightarrow (2)$$

let $p = a$, $q = b \rightarrow$ (3)

$$\sqrt{a} + \sqrt{b} = 1$$

$$\sqrt{b} = 1 - \sqrt{a}$$

$$b = (1 - \sqrt{a})^2$$

The complete integral is of (1) is

$$z = ax + (1-\sqrt{a})^2 y + c \rightarrow (4)$$

Singular integral :

Diff (4). w.r to a & c. We have no solution. There is no singular integral.

General integral :

put $c = f(a)$

$$(4) \Rightarrow z = ax + (1 - \sqrt{a})^2 y + f(a) \rightarrow (5)$$

Diff (5). w.r to 'a'

$$0 = x + 2(1 - \sqrt{a}) y \frac{1}{(-2\sqrt{a})} + f'(a)$$

$$0 = x - \frac{(1 - \sqrt{a})}{\sqrt{a}} y + f'(a) \rightarrow (6)$$

Eliminating a b/w (5) & (6) we get the general integral.

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Ex:

1) Prove that the characterstics of $q = 3p^2$ that pass through the point $(-1,0,0)$ generate the cone $(x+1)^2 + 12yz = 0$.

soln: Given equ is

$$q = 3p^2 \rightarrow (1)$$

This equ is of the form $F(p,q) = 0$

The complete soln of the (1) is in the form

$$z = ax + by + c \rightarrow (2)$$

$$\text{let } p = a,\ q = b \rightarrow (3)$$

sub (3) in (1) we get

$$b = 3p^2$$

$$b = 3a^2$$

$\therefore$ The complete integral of (1) is

$$z = ax + y3a^2 + c \rightarrow (4)$$

Singular integral:

Diff (4). w.r.to a.c. we have no solution, There is no singular integral.

General integral:

$$\text{put } c = f(a)$$

$$(4) \Rightarrow z = ax + y\,3a^2 + f(a) \rightarrow (5)$$

If this passes through $(-1,0,0)$

$$0 = a(-1) + 3a^2(0) + f(a)$$

$$f(a) - a = 0 \qquad f(a) = +a$$

$$f(a) = a$$

$$z = ax + 3a^2y + a \rightarrow (6)$$

Diff .p. w. r.to 'a'

$$0 = x + 6ay + 1$$

$$6ay = -(x+1)$$

$$a = \frac{-(x+1)}{6y} \rightarrow (7)$$

sub (7) in (6), we get

$$z = \frac{-(x+1)}{6y}x + \frac{3(x+1)^2}{36y^2}y + \frac{(-[x+1])}{6y}$$

$$= \frac{-(x+1)}{6y}(x+1) + \frac{(x+1)^2}{12y}$$

$$= \frac{-(x+1)^2}{6y} + \frac{(x+1)^2}{12y}$$

$$= \frac{-2(x+1)^2}{12y} + \frac{(x+1)^2}{12y}$$

$$z = \frac{-(x+1)^2}{12y}$$

$$12yz = -(x+1)^2$$

$$(x+1)^2 + 12yz = 0$$

Hence, proved

The locus of the characteristic is the cone.

Standart - II :

P) $F(x, p, q) = 0$

Ex : 1 solve $q = xp + p^2$

soln :

Given equ

$$q = xp + p^2 \rightarrow (1)$$

This PDE is of the form $F(x, p, q) = 0$

The general solution is,

$$\int dz = \int (p\,dx + q\,dy) \rightarrow (2)$$

Put $q = a$ in (1)

(1) $\Rightarrow a = xp + p^2$

$$p^2 + xp - a = 0 \qquad \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$p = \frac{-x \pm \sqrt{x^2 - 4(a)}}{2}$$

$$p = \frac{-x \pm \sqrt{x^2 + 4a}}{2}$$

(2) $\Rightarrow \int dz = \int \left[\left(\frac{-x \pm \sqrt{x^2 + 4a}}{2} \right) dx + a\,dy \right]$ $(2\sqrt{a})^2$

$$z = \int \frac{-x\,dx}{2} \pm \frac{1}{2} \int \sqrt{x^2 + 4a}\,dx + \int a\,dy$$

$(2\sqrt{a})^2$

$$= \frac{-x^2}{4} \pm \frac{1}{2} \left[\frac{x}{2} \sqrt{x^2 + 4a} + \cancel{4}a \sinh^{-1}\left(\frac{x}{2\sqrt{a}}\right) \right]$$

$\frac{+}{a}$ $\frac{x}{\sqrt{a}}$

$$z = -\frac{x^2}{4} \pm \frac{x}{4}\sqrt{x^2+4a} + a\sinh^{-1}\left[\frac{x}{2\sqrt{a}}\right] + ay + b$$

where a, b are arbitary constants.

ii) $F(y, p, q)$

Ex : solve : $p = y^2q^2$

soln :

Given $p = y^2q^2 \rightarrow (1)$

This PDE is of the form $F(y, p, q)$

Hence the soln,

$dz = p\,dx + q\,dy \rightarrow (2)$

Put $p = a^2$

$(1) \Rightarrow a^2 = y^2q^2$

$q^2 = a^2/y^2$

$q = \pm a/y$

$\frac{1}{y}dy = \log y$

$(2) \Rightarrow dz = a^2 dy \pm a/y\, dy$

$\int$ing, $z = a^2x \pm a\log y + b$

where a, b are arbitary constants.

iii) $F(z, p, q)$

Ex: $p(1+q^2) = q(z-1)$

soln:

Given $p(1+q^2) = q(z-1) \rightarrow (1)$

This PDE is the form.

$F(z, p, q)$

$dz = p\,dx + q\,dy \rightarrow (2)$

put $q = ap \rightarrow (3)$

$(1) \Rightarrow \cancel{p}(1+a^2p^2) = a\cancel{p}(z-1)$

$1 + a^2p^2 = az - a$

$$p^2 = \frac{az-a-1}{a^2}$$

$$p = \pm\frac{\sqrt{az-a-1}}{a}$$

$$(3) \Rightarrow q = \pm \cancel{a}\left[\frac{\sqrt{az-a-1}}{\cancel{a}}\right]$$

$$q = \pm\sqrt{az-a-1}$$

The solution is,

$$② \Rightarrow dz = \pm \frac{\sqrt{az-a-1}}{a} dx \pm \sqrt{az-a-1}\, dy$$

$$dz = \pm \frac{\sqrt{az-a-1}}{a} \left[dx + a dy \right]$$

$$\pm \frac{a\, dz}{\sqrt{az-a-1}} = dx \pm a dy$$

$\int$ing

$$\pm \int a (az-a-1)^{-1/2} dz = \int dx \pm \int a dy$$

$$\pm a \left[\frac{(az-a-1)^{1/2}}{1/2\, a} \right] = x \pm ay + b$$

$$\pm 2\sqrt{az-a-1} = x \pm ay + b$$

where a, b are arbitrary constants

Standard - III :

$$F_1(x, P) = F_2(y, q)$$

Ex : Solve : $P + q = x + y$

Soln : Given,

$$P + q = x + y$$

$$P - x = q - y$$

This PDE is of the form

$$F_1(x, P) = F_2(y, q)$$

Hence the solution is,

$$Pz = p dx + q dy \rightarrow ①$$

put, $p - x = \cancel{q - y}\; y - q = a$

$p - x = a \qquad \cancel{y - q}\; y - q = a \qquad -q = a - y$

$p = x + a \qquad q = y - a \qquad q = y - a$

$$① \Rightarrow dz = (a + x) dx + (y - a) dy$$

$$\int \text{ing},\ z = \frac{(a+x)^2}{2} + \frac{(y-a)^2}{2} + b \rightarrow ②$$

which the required solution is,

There is no singular integral general

integral

put $b = f(a)$

$$② \Rightarrow z = \frac{(a+x)^2}{2} + \frac{(y-a)^2}{2} + f(a) \rightarrow ③$$

$$\frac{\partial z}{\partial a} = 0 = (a+x) - (y-a) + f'(a) \rightarrow ④$$

Eliminating a b/w ③ & ④ we get general integral.

---

1) solve $q(p - \sin x) = \cos y$

Soln: Given, $q(p - \sin x)$

$$p - \sin x = \frac{\cos y}{q}$$

This is the form.

$$f_1(x, p) = f_2(y, q)$$

Hence put, $p - \sin x = \frac{\cos y}{q} = a$

$$p - \sin x = a$$

$$p = a + \sin x$$

$$\frac{\cos y}{q} = a$$

$$q = \frac{\cos y}{a}$$

Hence the soln is

$$dz = p\,dx + q\,dy$$

$$dz = (a + \sin x)dx + (\cos y/a)\,dy$$

$\int$ing, $z = ax - \cos x + \frac{\sin y}{a} + b \rightarrow ①$

which is the required complete integral. There is no singular integral.

General integral:

put $b = f(a)$

$$① \Rightarrow z = ax - \cos x + \frac{\sin y}{a} + f(a) \rightarrow ②$$

$$\frac{\partial z}{\partial a} = 0 = x - \frac{\sin y}{a^2} + f'(a) \rightarrow ③$$

Eliminating a b/w ② & ③ we get general integral.

Example :

1) solve : $z = px + qy + \sqrt{1+p^2+q^2}$

soln :

Given equ is

$$z = px + qy + \sqrt{1+p^2+q^2} \rightarrow (1)$$

(1) is in clairaut's form

put $p = a, q = b$

$$(1) \Rightarrow z = ax + by + \sqrt{1+a^2+b^2} \rightarrow (2)$$

To find singular integral

Diff (2) partially w.r.to 'a'

$$\frac{\partial z}{\partial a} = x + \frac{1}{2}(1+a^2+b^2)^{-1/2} \cdot 2a = 0$$

i.e $\quad x + a(1+a^2+b^2)^{-1/2} = 0$

$$x + \frac{a}{\sqrt{1+a^2+b^2}} = 0$$

$$a = -x\sqrt{1+a^2+b^2}$$

Diff (2) p. w.r.to 'b'

$$\frac{\partial z}{\partial b} = y + \frac{1}{2}(1+a^2+b^2)^{-1/2} \cdot 2b = 0$$

$$y + \frac{b}{\sqrt{1+a^2+b^2}} = 0$$

$$b = -y\sqrt{1+a^2+b^2} \rightarrow (4)$$

$$(3) \Rightarrow x = \frac{-a}{\sqrt{a^2+b^2+1}} \Rightarrow x^2 = \frac{a^2}{1+a^2+b^2}$$

$$(4) \Rightarrow y = \frac{-b}{\sqrt{1+a^2+b^2}} \Rightarrow y^2 = \frac{b^2}{1+a^2+b^2}$$

$$1-(x^2+y^2) = 1-\left(\frac{a^2}{1+a^2+b^2} + \frac{b^2}{1+a^2+b^2}\right)$$

$$= \frac{1+a^2+b^2-a^2-b^2}{1+a^2+b^2}$$

$$= \frac{1}{a^2+b^2+1}$$

i.e $1+a^2+b^2 = \frac{1}{1-x^2-y^2}$

$$\sqrt{1+a^2+b^2} = \frac{1}{\sqrt{1-x^2-y^2}}$$

$$\left.\begin{aligned} a &= \frac{-x}{\sqrt{1-x^2-y^2}} \\ b &= \frac{-y}{\sqrt{1-x^2-y^2}} \end{aligned}\right\} \rightarrow (5)$$

sub (5) in (2), we get

$$z = \left(\frac{-x}{\sqrt{1-x^2-y^2}}\right)x + \left(\frac{-y}{\sqrt{1-x^2-y^2}}\right)y + \left(\frac{1}{\sqrt{1-x^2-y^2}}\right)$$

$$z = \frac{1}{\sqrt{1-x^2-y^2}}(-x^2-y^2+1)$$

$$z = \frac{1}{\sqrt{1-x^2-y^2}}(1-x^2-y^2)$$

$$z = \sqrt{1-x^2-y^2}$$

i.e $z^2 = 1-x^2-y^2$

$x^2+y^2+z^2 = 1$

The singular integral is $x^2+y^2+z^2=1$

To find general integral :

put $b=f(a)$ in (2)

$$(2) \Rightarrow z = ax + f(a)y + \sqrt{1+a^2+[f(a)]^2} \rightarrow (6)$$

Diff (6) w.r to 'a' eliminating a b/w (2) & (6) we get general integral.

---

2) **Solve : $z = px+qy+pq$**

soln :

Given eqn is

$z = px+qy+pq \rightarrow (1)$

(1) is in clairaut's form

put $p=a$, $q=b$

The complete integral of ① is

$$z = ax + by + ab \rightarrow ②$$

To find singular integral

diff ② w.r.to 'a'

$$\frac{\partial z}{\partial a} = x + b$$

$$x + b = 0$$

$$b = -x \rightarrow ③$$

Diff ② w.r.to 'b'

$$\frac{\partial z}{\partial b} = y + a$$

$$y + a = 0$$

$$a = -y \rightarrow ④$$

sub ③ & ④ in ②

$$z = -xy + (-xy) + xy$$

$$z = -xy$$

$$z + xy = 0$$

The singular integral is $z + xy = 0$

To find general integral:

put $b = f(a)$ in ②

$$② \Rightarrow z = ax + f(a)y + af(a) \rightarrow ⑤$$

Diff ⑤ w.r to 'a' & eliminating a b/w ② & ⑤ we get general integral.

---

2) Solve: $pxy + pq + qy = yz$.

soln: Given equ is

$$pxy + pq + qy = yz \rightarrow ①$$

Let $F = pxy + pq + qy - yz = 0$

By charpits auxillary equsare

$$\frac{dp}{F_x + pF_z} = \frac{dq}{F_y + qF_z} = \frac{dz}{-pF_p - qF_q} = \frac{dx}{-pF_p - qF_q} = \frac{dx}{-F_p} = \frac{dy}{-F_q}$$

$$\rightarrow ②$$

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$F_x = py \quad , \quad F_p = xy + q$

$F_y = px + q \quad , \quad F_q = p + y$

$F_z = -y$

$$② \Rightarrow \frac{dp}{py + p(-y)} = \frac{dq}{px + q + q(-y)} = \frac{dz}{-p(xy+q) - q(p+y)} = \frac{dx}{-(xy+q)} = \frac{dy}{-(p+y)}$$

Taking first two ratios we get

$$\frac{dp}{py - py} = \frac{dq}{px + q - qy}$$

sing. $\log p = \log a$

$p = a$

sub $p = a$ in ①, we get

$axy + aq + qy = yz$

$axy + aq + qy - yz = 0$

$q(a+y) + y(ax - z) = 0$

$(a+y)q = -(ax - z)y$

$q = \frac{(z - ax)y}{a+y}$

Hence the soln is,

$dz = pdx + q\,dy$

$dz = a\,dx + \frac{(z - ax)y\,dy}{a+y}$

$dz - a\,dx = \frac{(z-ax)y}{a+y}\,dy$

$$\frac{d(z-ax)}{(z-ax)} = \frac{y+a-a}{a+y}\,dy$$

$$\frac{d(z-ax)}{z-ax} = \frac{y+a}{a+y}\,dy - \frac{a}{a+y}\,dy$$

$$\frac{d(z-ax)}{z-ax} = dy - \frac{a}{a+y}\,dy$$

sing

$\log(z - ax) = y - a\log(a+y) + b \rightarrow ③$

which is the required complete integral

There is no singular integral

General integral :

Put $b = f(a)$ in ③ where F is arbitary is differential ③ with respect to 'a' eliminating a b/w these equs we get general equs.

2) $xp^2 - ypq - y^3q - y^2z = 0$

soln:

The given equ is

$$xp^2 - ypq - y^3q - y^2z = 0 \rightarrow (1)$$

$$\text{let } F = xp^2 - ypq + y^3q - y^2z = 0$$

By charpits auxillary equ are

$$\frac{dp}{Fx+pFz} = \frac{dq}{Fy+qFz} = \frac{dz}{-pFp-qFq} = \frac{dx}{-Fp} = \frac{dy}{-Fq} \rightarrow (3)$$

$$FX = p^2$$

$$Fy = -pq + 3y^2q - 2yz$$

$$Fz = -y^2, \quad Fp = -2px - yq, \quad Fq = -yp + y^2$$

$$(3) \Rightarrow \frac{dp}{p^2+p(-y^2)} = \frac{dq}{-pq+3y^2q-2yz+q(-y^2)} = \frac{dz}{-p^3+qyp-qy^3}$$

$$= \frac{dx}{-(2px-yq)} = \frac{dy}{-(yp+y^3)}$$

$$\frac{dp}{p-py^2} = \frac{dq}{-pq+3y^2q-2yz-qy^2} = \frac{dz}{-p^3+qyp-qy^3}$$

$$= \frac{dx}{-2px+yq} = \frac{dy}{yp-y^3}$$

solving first & last eqes

$$\frac{dp}{p(p-y^2)} = \frac{dy}{y(p-y^2)} \Rightarrow \frac{dp}{p} = dy/y$$

$\int$ ing $\log p = \log y + \log a \Rightarrow p = ay$

Put $p = ay$ in given equ, we get (1)

$$(1) \Rightarrow xa^2y^2 - yayq + y^3q - y^2z = 0$$

$$y^2q(-a+y) = y^2z - xa^2y^2$$

$$q = \frac{y^2(z-a^2x)}{y^2(y-a)} = \frac{z-a^2x}{y-a}$$

Hence the soln is

$dz = p\,dx + q\,dy$

$dz = ay\,dx + \left(\frac{z-a^2x}{y-a}\right)dy \rightarrow (2)$

$dz - ay\,dx = \frac{z-a^2x}{y-a}\,dy$

Treating this as a total differential equ neglect by the present

$\int$ing (2) $\Rightarrow z = ayx + f(y) \rightarrow (3)$

diff (3) totally

$dz = ax\,dx + ay\,dx + df$

$= ay\,dx + ax\,dy + df \rightarrow (4)$

comparing (2) & (4) we get

$ax\,dy + df = \left(\frac{z-a^2x}{y-a}\right)dy$

$ax + \frac{df}{dy} = \frac{z-a^2x}{y-a}$

$\frac{df}{dy} = \frac{z-a^2x}{y-a} - ax = \frac{z-a^2x-ax(y-a)}{y-a}$

$= \frac{z-a^2x-axy+a^2x}{y-a} = \frac{z-axy}{y-a}$ [by (2) $z - axy - f(y)$]

$= \frac{f(y)}{y-a} \Rightarrow \frac{df}{f} = \frac{dy}{y-a}$

$\int$ing $\log f = \log(y-a) + \log b$

$\log f = \log[b(y-a)] \Rightarrow f = b(y-a)$

(2) $\Rightarrow z = axy + b(y-a)$

which is the required complete integral

singular integral.

There is no singular integral G.I

put $p = f(a) \rightarrow (3)$

put f is an arbitary & eliminate b/w two equ we get general equ.

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3) $p^2+qy=z$

soln: The given equ is $p^2+qy=z \rightarrow (1)$

Let $F=p^2+qy-z=0 \rightarrow (2)$

By charpits auxillary equ

$$\frac{dp}{F_x+pF_z}=\frac{dq}{F_y+qF_z}=\frac{dz}{-pF_p-qF_q}=\frac{dx}{-F_p}=\frac{dy}{-F_q} \rightarrow (3)$$

$F_x=0,\ F_y=q,\ F_z=-1,\ F_p=2p,\ F_q=y$

$$(2) \Rightarrow \frac{dp}{0+p(-1)}=\frac{dq}{q+q(-1)}=\frac{dz}{-p(2p)-q(y)}=\frac{dx}{-2p}=\frac{dy}{-y}$$

solving first & last equ

$$\frac{dp}{-p}=dy/-y$$

$\int$ing $\log p=\log y+\log a$

$p=ay$

put $p=ay$ equ in (1)

$(1) \Rightarrow a^2y^2+qy-z=0$

$qy=z-a^2y^2 \Rightarrow q=z/y-\frac{a^2y^2}{y}=\frac{z}{y}-a^2y$

Hence the soln is

$dz=pdx+qdy=aydx+\left(\frac{z}{y}-a^2y\right)dy$

$\int$ing $z=axy+z\log y-\frac{a^2y^2}{2}+b \rightarrow (3)$

which is the required complete integral

singular integral.

There is no singular integral general integral.

There is

put $b=f(a) \rightarrow (3)$ where F is arbitary & diff (3) with respect to 'a'.

Eliminating a b/w there equ we get general integral.

$(3) \Rightarrow z=axy+z\log y-\frac{a^2y^2}{2}+f(a)$

4) $p^2+q^2=xz$: solve

soln : The given equ is $p^2+q^2=xz \rightarrow (1)$

$$F=p^2+q^2-xz=0$$

By charpits auxi liary equ are

$$\frac{dp}{Fx+pFz}=\frac{dq}{Fy+pFz}=\frac{dz}{-pFp-qFq}=\frac{dx}{-Fp}=\frac{dy}{-Fq} \rightarrow (2)$$

$Fx=-z, Fy=0, Fz=-x, Fp=2p, Fq=2q$

$$\frac{dp}{-z-px}=\frac{dq}{0-qx}=\frac{dz}{-2p^2-2q^2}=\frac{dx}{-2p}=\frac{dy}{-2q}$$

$$\frac{dp}{-z-px}=\frac{dq}{-qx}=\frac{dz}{-2(p^2+q^2)}=\frac{dx}{-2p}=\frac{dy}{-2q}$$

$$\frac{dp}{-z-px}=\frac{dq}{-qx}=\frac{dz}{-2xz}=\frac{dx}{-2p}=\frac{dy}{-2q}$$

solving second & third equs

$$\frac{dq}{-qx}=\frac{dz}{2xz}$$

$\int$ing $2\log q=\log z+\log a$

$$\log q^2=\log(za) \Rightarrow q^2=za \Rightarrow q=\sqrt{za}$$

put, $q=\sqrt{za}$ in (1)

$$p^2+za-xz=0$$

$$p^2=-z(a+x) \Rightarrow p=\sqrt{z(x-a)}$$

Hence the soln is $dz=pdx+qdy$

$$dz=\sqrt{z(x-a)}\,dx+\sqrt{za}\,dy$$

$$dz=\sqrt{z}\left[(x-a)dx+\sqrt{a}\,dy\right]$$

$$\frac{dz}{z^{1/2}}=xdx-adx+a^{1/2}y+b$$

$\int$ing, $\frac{z^{1/2}}{1/2}=x^2/2-ax+a^{1/2}y+b$

$$2z^{1/2}=\frac{x^2}{2}-ax+a^{1/2}y+b \rightarrow (3)$$

which is the required complete integral

singular integral.

there is no singular integral in general integral

General integral.

put $b=f(a)$ where f is arbitrary constant

5) $z^2(p^2+q^2+1)=a^2$

soln :

Given equ is

$$z^2(p^2+q^2+1)=a^2 \rightarrow (1)$$

This P.D.E is of the form $F(z,p,q)$

Hence soln is,

$$dz = p\,dx + q\,dy \rightarrow (2)$$

put $q = ap \rightarrow (3)$

$$(1) \Rightarrow z^2(p^2+a^2p^2+1) = a^2$$

$$p^2+a^2p^2+1 = a^2/z^2$$

$$p^2+a^2p^2 = a^2/z^2 - 1$$

$$p^2(1+a^2) = a^2/z^2 - 1$$

$$p^2 = \frac{a^2/z^2 - 1}{(1+a^2)}$$

$$p = \pm\sqrt{\frac{a^2/z^2-1}{(1+a^2)}} = \pm\sqrt{\frac{a^2-z^2}{z^2(a^2+1)}}$$

$$(3) \Rightarrow \therefore q = ap$$

$$q = \pm a\sqrt{\frac{a^2-z^2}{z^2(1+a^2)}}$$

The soln is,

$$(2) \Rightarrow dz = \pm\sqrt{\frac{a^2-z^2}{z^2(a^2+1)}}\,dx \pm a\sqrt{\frac{a^2-z^2}{z^2(a^2+1)}}\,dy$$

$$dz = \pm\sqrt{\frac{a^2-z^2}{z^2(a^2+1)}}\,[dx + a\,dy]$$

$$\pm\frac{\sqrt{z^2(a^2+1)}\,dz}{\sqrt{a^2-z^2}} = dx + a\,dy$$

$$\frac{(\sqrt{a^2+1}\,\sqrt{z^2})\,dz}{\sqrt{a^2-z^2}} = dx + a\,dy$$

$\int$ing,

$$\sqrt{a^2+1}\int\frac{z\,dz}{\sqrt{a^2-z^2}} = \int dx + a\int dy \rightarrow (4)$$

$$\text{Let } u = \sqrt{a^2-z^2} = (a^2-z^2)^{1/2}$$

$$du = \tfrac{1}{2}(a^2-z^2)^{-1/2}(-2z)\,dz$$

$$du = \frac{-z\,dz}{\sqrt{a^2-z^2}}$$

$$④ \Rightarrow -\sqrt{a^2+1}\int du = \int dx + a\int dy$$

$$-\sqrt{a^2+1}\,(u) = x+ay+b$$

$$-(\sqrt{a^2+1})(\sqrt{a^2-z^2}) = x+ay+b$$

Taking of squares on both sides, we get

$$(a^2+1)(a^2-z^2) = (x+ay+b)^2$$

---

6) Solve : $pq = z$

soln: Given equ is

$$pq = z \rightarrow ①$$

This P.D.E is of the form $F(z,p,q)$

Hence soln is,

$$d = pdx + qdy \rightarrow ②$$

$$\text{put } q = ap \rightarrow ③$$

$$① \Rightarrow p(ap) = z$$

$$ap^2 = z$$

$$p^2 = z/a$$

$$p = \pm\sqrt{z/a}$$

$$q = ap$$

$$q = \pm a\sqrt{z/a}$$

The soln is,

$$② \Rightarrow dz = \pm\sqrt{z/a}\,dx \pm a\sqrt{z/a}\,dy$$

$$= \pm\sqrt{z/a}\,[dx + ady]$$

$$\Rightarrow \sqrt{a/z}\,dz = dx + ady \;;\quad \frac{\sqrt{a}}{\sqrt{z}}dz = dx + ady$$

Sing,

$$\sqrt{a}\int\frac{1}{\sqrt{z}}\cdot dz = \int dx + a\int dy$$

$$\sqrt{a}\int (z)^{-1/2}dz = \int dx + a\int dy$$

$$\sqrt{a}\left(\frac{z^{1/2}}{1/2}\right) = x+ay+b$$

$$2\sqrt{a}\sqrt{z} = x+ay+b$$

$$2\sqrt{z} = \frac{x+ay}{\sqrt{a}} + b$$

$$2\sqrt{az} = x+ay+b.$$

solve : $p^2+q^2 = x-y$

soln: Given equ is

$$p^2+q^2 = x-y \rightarrow (1)$$

$$q^2+y = x-p^2$$

$$y+q^2 = x-p^2$$

This P.D.E is of the form

$$F_1(x,p) = F_2(y,q)$$

Hence soln is,

$$dz = p\,dx + q\,dy \Rightarrow (2)$$

put $y+q^2 = x-p^2 = a$

$y+q^2=a$	$x-p^2=a$
$q^2=-y+a$	$x-a=p^2$
$q=\pm\sqrt{(-y+a)}$	$p^2=x-a$
	$p=\pm\sqrt{x-a}$

(2) $\Rightarrow$ $dz = \sqrt{x-a}\,dx + (-y+a)dy$

Sing, $\int dz = \int (x-a)^{1/2}dx + \int(-y+a)^{1/2}dy$

$$z = \frac{(x-a)^{3/2}}{3/2} + \frac{(a-y)^{3/2}}{3/2}(-1)+b$$

$$z = \frac{(x-a)^{3/2}}{3/2} - \frac{(a-y)^{3/2}}{3/2} + b$$

$$z = \frac{2}{3}(x-a)^{3/2} - \frac{(a-y)^{3/2}}{3/2} + b$$

$$z = \frac{2}{3}(x-a)^{3/2} - 2/3\,(a-y)^{3/2} + b \rightarrow (3)$$

which is the required soln is,

There is no singular integral general integral put $b = f(a)$

(3) $\Rightarrow$ $z = 2/3\,(x-a)^{3/2} - \frac{2}{3}(a-y)^{3/2} + f(a) \rightarrow (4)$

$$\frac{\partial z}{\partial a} = 0 \Rightarrow = \frac{2}{3}\times\frac{3}{2}(x-a)^{3/2-1} - \frac{2}{3}\times\frac{3}{2}(a-y)^{3/2-1} + f'(a)$$

$$= (x-a)^{1/2} - (a-y)^{1/2} + f'(a) \rightarrow (5)$$

Eliminating a b/w (4) & (5), we get the G.I.

8) solve : $pq = xy$

soln: Given equ is

$$pq = xy \rightarrow (1)$$

$$\frac{p}{x} = y/q$$

This P.D.E is of the form

$$F_1(x, p) = F_2(y, q)$$

Hence the soln is,

$$dz = p\,dx + q\,dy \rightarrow (2)$$

put $p/x = y/q = a$

$p/x = a$ | $y/q = a$

$p = ax$ | $q = y/a$

$(2) \Rightarrow dz = ax\,dx + y/a\,dy$

Sing, $\int dz = a\int x\,dx + 1/a \int y\,dy$

$$z = \frac{ax^2}{2} + \frac{1}{a} \cdot \frac{y^2}{2} \cdot b$$

$$z = ax^2/2 + y^2/2a + b$$

$$2z = ax^2 + y^2/a + b \rightarrow (3)$$

which is the required soln is,

There is no singular integral general integral

put $b = f(a)$

$$(3) \Rightarrow 2z = ax^2 + y^2/a + f(a) \rightarrow (4)$$

$$\frac{\partial z}{\partial a} = 0 = \frac{a^2}{2}x^2 - \frac{y^2}{a^2} + f'(a) \rightarrow (5)$$

Eliminating a b/w (4) & (5) we get G.I

Unit - IV

Homogenous linear partial differential equations with constant co.efficients :

PDE of higher order

let $D = \frac{\partial}{\partial x}$, $D' = \frac{\partial}{\partial y}$

Homogenous : $f(D, D')z = F(x,y)$

Ex:

$3D^2 + 2DD' + 4D'^2 = 0$

soln:

To find C.F : $F(D,D') = 0$

Replace $D = m$, $D' = 1$

(i.e); $f(m,1) = 0$

solving these we'ue roots

case(i):

If $m_1, m_2, \ldots, m_n$ are distinct roots.

$C.F = \phi_1(y+m_1x) + \phi_2(y+m_2x) + \ldots + \phi_n(y+m_nx)$

suppose,

$m_1 = \frac{a_1}{b_1}$, $m_2 = \frac{a_2}{b_2}$

$C.F = \phi_1(b_1y + a_1x) + \phi_2(b_2y + a_2x)$

case(ii):

If $m_1 = m_2 = \ldots m_r = m$, $m_{r+} \ldots m_n$

then,

$C.F = \phi_1(y+mx) + \phi_2(y+mx) + \ldots + x^{r-1}\phi_r(y+mx) + \phi_{r+1}(y+m_{r+1}x) + \ldots + \phi_n(y+m_nx)$

$p = \frac{\partial z}{\partial x}$, $q = \frac{\partial z}{\partial y}$

$r = \frac{\partial^2 z}{\partial x^2}$, $t = \frac{\partial^2 z}{\partial y^2}$, $s = \frac{\partial^2 z}{\partial x \partial y}$

$p = Dz$, $q = D'z$, $r = D^2z$, $t = D'^2z$, $s = DD'z$

Ex : 1

1) solve : $r = a^2 t$

soln : given $r = a^2 t \rightarrow (1)$

Here $r = \frac{\partial^2 z}{\partial x^2}$, $t = \frac{\partial^2 z}{\partial y^2}$

$(1) \Rightarrow \frac{\partial^2 z}{\partial x^2} = a^2 \frac{\partial^2 z}{\partial y^2} \Rightarrow \frac{\partial^2 z}{\partial x^2} - a^2 \frac{\partial^2 z}{\partial y^2} = 0$

$(D^2 - a^2 D'^2) z = 0 \Rightarrow D = m, D' = 1$

The auxillary eqn is,

$(m^2 - a^2) = 0 \Rightarrow m^2 = a^2 \Rightarrow m = \pm a$

Roots are distinct

$\therefore C.F = \phi_1(y + m_1 x) + \phi_2(y + m_2 x)$

$= \phi(y + ax) + \phi_2(y - ax)$

The general solution is $z = C.F = \phi_1(y + ax) + \phi_2(y - ax)$

where ϕ_1, ϕ_2 are arbitrary function.

---

2) solve : $(D^2 - 6D^2D' + 11D'D^2 - 6D'^3)z = 0$

soln : Given $(D^2 - 6D^2D' + 11D'D^2 - 6D'^3)z = 0$

$D = m$, $D' = 1$

The auxillary equ is $m^3 - 6m^2 + 11m - 6 = 0$

$(m-1)(m^2 - 5m + 6) = 0$

$m - 1 = 0$; $m = 1$

The roots are distinct

1	1	-6	11	-6
	0	1	-5	6
	1	-5	6	0

$C.F = \phi_1(y + x) + \phi_2(y + 2x) + \phi_3(y + 3x)$

The general solution is,

$z = C.F = \phi_1(y + x) + \phi_2(y + 2x) + \phi_3(y + 3x)$

where ϕ_1, ϕ_2 are the arbitrary function.

---

3) solve : $(2r + 5s + 2t) = 0$

soln : Given $(2r + 5s + 2t) = 0$

$r = \frac{\partial^2 z}{\partial x^2}$, $s = \frac{\partial^2 z}{\partial x \partial y}$, $t = \frac{\partial^2 z}{\partial y^2}$

$2\frac{\partial^2 z}{\partial x^2} + 5\frac{\partial^2 z}{\partial x \partial y} + 2\frac{\partial^2 z}{\partial y^2} = 0$

$(2D^2 + 5DD' + 2D'^2) z = 0$

The auxillary equ is,

$2m^2+5m+2=0$; $2m^2+m+4m+2=0$

$m(2m+1)+2(2m+1)=0$; $(2m+1)(m+2)=0$

$m=-1/2$; $m=-2$

$\therefore C.F = \phi_1(y-\frac{1}{2}x)+\phi_2(y-2x)$

where ϕ_1, ϕ_2 are arbitrary function.

---

7) solve: $(4D^2+12DD'+9D'^2)z=0$

soln: Given $(4D^2+12DD'+9D'^2)z=0$

$D'=1$; $D=m$

The auxillary equ is, $4m^2+12m+9=0$

$4m^2+6m+6m+9=0$

$2m(2m+3)+3(2m+3)=0$

$(2m+3)(2m+3)=0$

$2m+3=0$; $2m+3=0$

$m=-3/2$; $m=-3/2$

The roots are eqn,

$C.F=\phi_1(y-3/2x)+\phi_2(y-3/2x)$

The general solution is $z = C.F=\phi_1(2y-3x)+x\phi_2(2y-3x)$.

where ϕ_1 & ϕ_2 are arbitrary function.

---

8) solve: $(D^4-2D^3D'+2DD'^3+D'^4)z=0$

soln: given $(D^4-2D^3D'+2DD'^3+D'^4)z=0$

The auxillary eqn is

$m^4-2m^3+2m-1=0$

$(m-1)(m-1)(m^2-1)=0 \Rightarrow m^2=1$; $m^2=\pm 1$

$m_1=1$; $m_2=1$; $m_3=1$; $m_4=-1$

$C.F=\phi_1(y+x)+\phi_2(y-x)+\phi_3(y+ix)+\phi_4(y-ix)$

The general solution

$z=C.F=\phi_1(y+x)+\phi_2(y-x)+\phi_3(y+ix)+\phi_4(y-ix)$

where ϕ_1, ϕ_2 are arbitrary function.

6) solve: $\frac{\partial^4 z}{\partial x^4} + \frac{\partial^4 z}{\partial y^4} = 0$

soln:

given $\frac{\partial^4 z}{\partial x^4} + \frac{\partial^4 z}{\partial y^4} = 0$

$(D^4 + D'^4)z = 0$

The auxillary equ is, $m^4 + 1 = 0 \Rightarrow (m^2)^2 + 2m^2 - 2m^2 + 1 = 0$

$(m^2+1)^2 - 2m^2 = 0$

$(m^2 + 1 + \sqrt{2}m)(m^2 + 1 - 2\sqrt{m}) = 0 \Rightarrow m^2 - \sqrt{2}m + 1 = 0$

$m^2 + 1 + \sqrt{2}m = 0$

$m = \frac{-\sqrt{2} \pm \sqrt{2 - 4(1)(1)}}{2} = \frac{-\sqrt{2} \pm \sqrt{-2}}{2} = \frac{-\sqrt{2} \pm i\sqrt{2}}{2}$

$= \frac{\sqrt{2}(-1 \pm i)}{2} = \frac{-1 \pm i}{\sqrt{2}}.$

$m = \frac{\pm\sqrt{2} \pm \sqrt{2 - 4(1)(1)}}{2} = \frac{\sqrt{2} \pm \sqrt{-2}}{2} = \frac{\sqrt{2}(1 \pm i)}{2} = \frac{1 \pm i}{\sqrt{2}}$

$m = \frac{-1 \pm i}{\sqrt{2}}$

let $z_1 = \frac{-1 \pm i}{\sqrt{2}}$; $z_2 = \frac{1 \pm i}{\sqrt{2}}$

$\bar{z}_1 = \frac{-1 - i}{\sqrt{2}}$; $\bar{z}_2 = \frac{1 - i}{+2}$

where $\bar{z}_1, \bar{z}_2$ are complete conjugates of z_1 & z_2 respectively.

$C.F = \phi_1(y + z_1 x) + \phi_2(y + \bar{z}_1 x) + \phi_3(y + z_2 x) + \phi_4(y + \bar{z}_2 x)$

The general solution

$z = \phi_1(y + z_1 x) + \phi_2(y + \bar{z}_1 x) + \phi_3(y + z_2 x) + \phi_4(y + \bar{z}_2 x)$

7) solve: $\frac{\partial^4 z}{\partial x^4} - \frac{\partial^4 z}{\partial y^4} = 0$

soln:

Given $\frac{\partial^4 z}{\partial x^4} - \frac{\partial^4 z}{\partial y^4} = 0$

$(D^4 - D'^4) = 0$

The auxillary equ is

$m^4 - 1 = 0$; $(m^2)^2 - 1^2 = 0$

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$(m^2-1)(m^2+1)=0$

$m^2=1$; $m^2+1=0 \Rightarrow m^2=-i$; $m=\pm i$

$m=\pm 1$

$m_1=1, m_2=-1, m_3=+i, m_4=-i$

$C.F = \phi_1(y+x)+\phi_2(y-x)+\phi_3(y+ix)+\phi_4(y-ix)$

The general equ is,

$z = C.F = \phi_1(y+x)+\phi_2(y-x)+\phi_2(y+ix)+\phi_4(y-ix)$

where ϕ_1, ϕ_2, ϕ_3 are arbitrary constant.

P.I of homogenous linear PDE:

$$F(D, D') = f(x,y)$$

Then $P.I = \dfrac{f(x,y)}{F(D,D')}$

Note :

$$F(D,D')\left[\frac{F(x,y)}{F(D,D')}\right] = f(x,y)$$

where D is only partially with respect to x

D' is only partially with respect to y alone.

$\frac{1}{D}$ is integration w.r to alone.

$\frac{1}{D'}$ is integration w.r to y alone.

Method I :

$$F(D,D') = \phi(ax+by)$$

case (i) :

$F(a,b) \neq 0$ and $F(D,D')$ is a homogenous function degree n

$$P.I = \frac{\phi(ax+by)}{F(D,D')}$$

$$= \frac{1}{F(a,b)} \iiint \cdots \int v\,dv\,dv \ldots dv$$

where $v = ax+by$

1) solve : $(D^2+3DD'+2D'^2)z = x+y$

Soln :

Given $(D^2+3DD'+2D'^2)z = x+y$

The auxillary equ is

$$m^2+3m+2=0$$

$$(m+1)(m+2)=0$$

$$m_1=-1;\ m_2=-2$$

$$C.F = \phi_1(y-x)+\phi_2(y-2x)$$

To find P.I :

$$P.I = \frac{(x+y)}{D^2+3DD'+2D'^2}$$

$$= \frac{1}{1^2+3(1)(1)+2(1)^2}\iint v\,dv\,dv$$

where $v = x+y$

$$= \frac{1}{1+3+2}\int \frac{v^2}{2}\,dv$$

$$= \frac{1}{6}\left(\frac{v^3}{6}\right)$$

$$= \frac{v^3}{36}$$

$$P.I = \frac{(x+y)^3}{36}$$

The general solution is,

$$z = C.F + P.I$$

$$z = \phi_1(y-x)+\phi_2(y-2x)+\frac{(x+y)^3}{36}$$

2) $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 12(x+y)$.

Soln: Given $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 12(x+y)$

The auxillary equation is,

$$m^2+1=0$$
$$m^2=-1$$
$$m=\pm i$$
$$m_1=i,\ m_2=-i$$
$$C.F = \phi_1(y+ix)+\phi_2(y-ix)$$

where ϕ_1, ϕ_2 are arbitrary function.

To find P.I:

$$P.I = \frac{12(x+y)}{D^2+D'^2}$$
$$= \frac{12}{1^2+1^2}\left[\iint v\,dv\,dv\right] \text{ where } v=x+y$$
$$= \frac{12}{2}\left[\int \frac{v^2}{2}dv\right]$$
$$= 6\left(\frac{1}{2}\frac{v^3}{3}\right)$$
$$= v^3$$
$$P.I = (x+y)^3$$

The general solution is,

$$z = C.F + P.I$$
$$= \phi_1(y+ix)+\phi_2(y-ix)+(x+y)^3$$

---

3) $\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial y^2} = 0$

Soln: Given $\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial y^2} = 0$

$$(D^2-D'^2)=0$$

The auxillary equation is,

$$m^2-1=0$$
$$m^2=1$$
$$m=\pm 1$$
$$m_1=1,\ m_2=-1$$
$$C.F = \phi_1(y+m_1x)+\phi_2(y+m_2x)$$

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$C.F = \phi_1(y+x) + \phi_2(y-x)$

The general solution is,

$z = C.F = \phi_1(y+x) + \phi_2(y-x)$

where ϕ_1, ϕ_2 are arbitrary function.

4) $(D^3 - 3D^2D' + 2DD'^2)z = 0$

soln :

Given $(D^3 - 3D^2D' + 2DD'^2)z = 0$

The auxiliary equation is

$m^3 - 3m^2 + 2m = 0$

$m(m^2 - 3m + 2) = 0$

$m = 0 \quad , m^2 - 3m + 2 = 0$

$\frac{-1}{m} \Big| \frac{-2}{m}$

$m_1 = 0, \ m_2 = 1, \ m_3 = 2$

The roots are distinct

$C.F = \phi_1(y + 0x) + \phi_2(y+x) + \phi_3(y+2x)$

$= \phi_1(y) + \phi_2(y+x) + \phi_3(y+2x)$

The general solution is,

$z = C.F = \phi_1 y + \phi_2(y+x) + \phi_3(y+2x)$

where ϕ_1, ϕ_2, ϕ_3 are arbitrary function.

5) Solve :

$(D^3 - 4D^2D' + 4DD')z = 0$

soln :

Given $(D^3 - 4D^2D' + 4DD')z = 0$

The auxiliary equ is

$m^3 - 4m^2 + 4m = 0$

$m(m^2 - 4m + 4) = 0$

$m = 0 \ , \ m^2 - 4m + 4 = 0$

$(m-2)(m-2) = 0$

$\frac{-2}{m} \Big| \frac{-2}{m}$

$m_1 = 0; \ m_2 = 2 \ ; \ m_3 = 2$

The roots are real and equal

$C.F = \phi_1(y + m_1 x) + \phi_2(y + m_2 x) + \phi_3(y + m_3 x$

$= \phi_1(y + 0x) + \phi(y+2x) + \phi_3(y+2x)$

$C.z = \phi_1 y + \phi_2(y+2x) + \phi_3(y+2x)$

The general solution is,

$Z = C.F = \phi_1 y + \phi_2 (y+2x) + \phi_3 (y+3x)$

where ϕ_1, ϕ_2, ϕ_3 are arbitrary function.

---

6) solve : $(2D^2 - 5D'D + 2D'^2) = 24(y-x)$

soln : Given $(2D^2 - 5D'D + 2D'^2) = 24(y-x)$

The auxillary equ is

$2m^2 - 5m + 2 = 0$

$(2m-1)(m-2) = 0$

$2m-1 = 0 \; ; m-2 = 0$

$\frac{-1}{m} \Big| \frac{-4}{m}$

$m = 1/2 \; ; \; m = 2$

$F(a,b) = F(D,D')$

The roots are distinct $F(-1,1) = F(-1,1)$

$C.F = \phi_1 (y + 1/2 x) + x \phi_2 (y+2x)$

To find P.I :

$$P.I = \frac{24(y-x)}{2D^2 - 5DD' + 2D'^2}$$

where $v = y - x$

$$= \frac{24}{2(-1)^2 - 5(-1)(1) + 2(1)^2}$$

$$= \frac{24}{2+5+2} \left[\int \frac{v^2}{2} dv \right]$$

$$= \frac{24}{9} \left[\frac{v^3}{6} \right]$$

$$= \frac{24}{54} [y-x]^3$$

$$P.I = \frac{12}{27} (y-x)^3$$

The general soln is,

$z = C.F + P.I$

$$z = \phi_1 (y + 1/2 x) + x\phi_2 (y+2x) + \frac{12}{27}(y-x)^3$$

where ϕ_1 & ϕ_2 are arbitrary function.

---

7) solve : $(D - 3D')^2 (D + 3D') z = e^{3x+y}$

soln : Given equ

$(D-3D')^2 (D+3D') z = e^{3x+y}$

The Auxillary equ is

$D = m \; ; \; D' = 1$

$(m-3)^2 (m+3) = 0$

$(m-3)(m-3)=0$; $m+3=0$

$m=3,3,-3$

The roots are distinct

$C.F = \phi_1(y+3x) + x\phi_2(y+3x) + \phi_3(y-3x)$

To find P.I:

$$P.I = \frac{1}{(D-3D')^2(D+3D')} e^{3x+y}$$ $a=3, b=1$, $D=3$ $D'=1$

$$= \frac{1}{(D-3D')^2}\left[\frac{1}{6}\int e^v dv\right]$$ $P.I = \frac{1}{(bD-aD')}$

$$= \frac{1}{(D-3D')^2}\left[\frac{1}{6}e^v\right]$$ $= \frac{x^m}{b^m m!}$

$$= \frac{x^2}{2!\,1^2}\left(\frac{1}{6}e^v\right)$$

$$P.I = \frac{x^2}{12} e^{3x+y}$$

$$\text{soln} = C.F + P.I$$

$$= \phi_1(y+3x) + x\phi_2(y+3x) + \phi_3(y-3x) + \frac{x^2}{12}e^{3x+y}$$

---

solve: $r-2s+t = \sin(2x+3y)$

Soln: Given equ is

$$r-2s+t = \sin(2x+3y) \rightarrow (1)$$

Here, $r = \frac{\partial^2 z}{\partial x^2}$, $s = \frac{\partial^2 z}{\partial x \partial y}$, $t = \frac{\partial^2 z}{\partial y^2}$

sub (1) $\Rightarrow$

$$\frac{\partial^2 z}{\partial x^2} - 2\frac{\partial^2 z}{\partial x \partial y} + \frac{\partial^2 z}{\partial y^2} = \sin(2x+3y)$$

$$(D^2 - 2DD' + D'^2)z = \sin(2x+3y)$$

$$D=m, \; D'=1$$

$$m^2 - 2m + 1 = 0$$ $\frac{-1}{m}\Big|\frac{-1}{m}$

$$m = 1, 1$$

$C.F = \phi_1(y+x) + x\phi_2(y+x)$

$$P.I = \frac{1}{D^2-2DD'+D'^2} \sin(\underset{a}{2}x+\underset{b}{3}y)$$

$$= \frac{1}{4-12+9} \iint \sin v\, dv\, dv \quad \begin{bmatrix} a=2, b=3 \\ v = 2x+3y \end{bmatrix}$$

not zero, so sub $D \& D'$ values $\quad D=2, D'=3$

$$= \frac{1}{1} \int -\cos v\, dv$$

$$= -\sin v$$

$P.I = -\sin(2x+3y)$

The soln of equ is,

$y = C.F + P.I$

$y = \phi_1(y+x) + x\phi_2(y+1) - \sin(2x+3y)$

---

9) solve : $(D^2-2DD'+D'^2)z = \tan(y+x)$

soln : $\quad a^2-2ab+b^2$
$(a-b)^2$

Given equ is

$(D^2-2DD'+D'^2)z = \tan(y+x)$

Auxillary equ is

$D = m \;; D' = 1$

$m^2 - 2m + 1 = 0$

$(m-1)(m-1) = 0$

$m = 1, 1$

$\begin{array}{c|c} -1 & -1 \\ \hline m & m \end{array}$

$C.F = \phi_1(y+x) + x\phi_2(y+x)$

$$P.I = \frac{1}{D^2-DD'+D'^2} \tan(y+x)$$

$$= \frac{1}{D-(D-D')^2} \tan(\underset{1}{y}+\underset{1}{x})$$

$\hookrightarrow (bD-aD')^2$

power = 2
$m = 2$

$$= \frac{x^m}{b^m m!}$$

$$= \frac{x^2}{1^2 2!} \iint \tan v\, dv\, dv \qquad a=1, b=1$$
$v = y+x$

$$= \frac{x^2}{2} \int + \log(\sec v)\, dv$$

$$P.I = \frac{x^2}{2!} \tan(y+x)$$

The solution for the equ is

$y = C.F + P.I$

$$y = \phi_1(y+x) + x\phi_2(y+x) + \frac{x^2}{2!}\tan(y+x).$$

10) solve: $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = \cos mx . \cos ny$.

soln:

Given equ is

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = \cos mx \cos ny$$

The auxillary equ is

$$D^2 + D'^2 = 0$$

$$m^2 + 1 = 0$$

$$m^2 = -1$$

$$m = \pm i$$

equal roots
$C.F = \phi_1 + \phi_2 x$

The roots are complex and distinct

$$C.F = \phi_1 (y + ix) + \phi_2 (y - ix)$$

$$P.I = \frac{1}{\underset{A}{D^2} + \underset{B}{D'^2}} (\underbrace{\cos mx}_{A} \underbrace{\cos ny}_{B})$$

$\cos(A+B) + \cos(A-B) = 2\cos A \cos B$

$$= \frac{1}{D^2+D'^2} \left[\frac{\cos(mx+ny) + \cos(mx-ny)}{2} \right]$$

$$= \frac{1}{2D^2+D'^2} \left[\cos(mx+ny) + \cos(mx-ny) \right]$$

$$= \frac{1}{2D^2+D'^2} \cos(mx+ny) + \frac{1}{2D^2+D'^2} \cos(mx-ny)$$

$$= \frac{1}{2} \left(\frac{1}{m^2+n^2} \cos(\underbrace{mx+ny}_{v}) + \frac{1}{m^2+n^2} \cos(\underbrace{mx-ny}_{v}) \right)$$

$$= \frac{1}{2(m^2+n^2)} \iint \cos v \, dv dv + \frac{1}{2(m^2+n^2)} \iint \cos v \, dv dv$$

$$= \frac{1}{2(m^2+n^2)} \int \sin v \, dv + \frac{1}{2(m^2+n^2)} \int \sin v \, dv$$

$$= \frac{-\cos v}{2(m^2+n^2)} + \frac{(-\cos v)}{2(m^2+n^2)}$$

$$= \frac{-1}{2(m^2+n^2)} \left[\frac{\cos(mx+ny)+\cos(mx-ny)}{2}\right]$$

$$P.I = \frac{-1}{m^2+n^2} (\cos mx \cos ny)$$

The soln of equ is

$$y = C.F + P.I$$

$$y = \phi_1(y+ix) + \phi_2(y-ix) - \frac{1}{m^2+n^2}(\cos mx \cos ny).$$

Method II :

When $f(x,y)$ is of the form $x^m y^n$ or a rational integral algebraic function of x and y. i) If $n<m$, $\frac{1}{f(D,D')}$ should be expanded in power of $\frac{D'}{D}$

ii) If $n>m$, $\frac{1}{f(D,D')}$ should be expanded in power of $\frac{D}{D'}$

1) solve : $(D^2-a^2D'^2)z = x$ (or) $\frac{\partial^2 z}{\partial x^2} - a^2\frac{\partial^2 z}{\partial y^2} = x$

soln :

Given equ is

$$(D^2-a^2D'^2)z = x$$

Auxillary equ is

$$m^2-a^2=0$$

$$m^2=a^2$$

$$m=\pm a$$

$$m=a, \quad m=-a$$

The roots are distinct

$$C.F = \phi_1(y+ax)+\phi_2(y-ax)$$

$$P.I = \frac{1}{D^2-a^2D'^2} x^1 y^0 \quad (1)$$

$$= \frac{1}{D^2\left(1-\frac{a^2D'^2}{D^2}\right)} x$$

$$= \frac{1}{D^2}\left(1 - \frac{a^2 D'^2}{D^2}\right)^{-1} x$$

$$= \frac{1}{D^2}\left(1 + \frac{a^2 D'^2}{D^2} + \cdots\right) x$$

$$= \frac{1}{D^2}(x)$$

$D =$ Differentiate

$\frac{1}{D} =$ integrate

$\frac{1}{D^2} = \int\int$

$$P.I = \frac{1}{D}\left[\frac{x^2}{2}\right]$$

$$P.I = \frac{x^3}{6}$$

The solnof the given equ is

$$y = \phi_1(y + ax) + \phi_2(y - ax) + \frac{x^3}{6}$$

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General method of finding the particular integral of linear homogenous equ with constant coefficients.

Working rule for finding P.I

$$F(D,D')z = f(x,y)$$

$$P.I = \frac{1}{(D-m_1D')(D-m_2D')\ldots(D-m_nD')} f(x,y)$$

Formula : 1

$$\frac{1}{D-m_1D'} f(x,y) = \int f(x, c-mx)\,dx \qquad [c = y+m \quad y = c\ldots$$

Formula : 2

$$\frac{1}{D+m_1D'} f(x,y) = \int f(x, \underset{y}{\overset{\text{constant}}{c}}+\overset{\text{coefficient term}}{m}x)\,dx \qquad \left[\begin{array}{l} c = y-mx \\ y = c+mx \end{array}\right.$$

1) solve: $(D^2-DD'-2D'^2)z = (y-1)e^x$

soln :

Given equ is

$$(D^2-DD'-2D'^2)z = (y-1)e^x$$

Auxillary equ is

$$m^2-m-2=0 \qquad \frac{1}{m}\Big|\frac{2}{m}$$

$$(m+1)(m-2)=0$$

$$m_1=-1 \,;\, m_2=2$$

The roots are real and distinct

$$C.F = \phi_1(y-x)+\phi_2(y+2x)$$

$$P.I = \frac{1}{D^2-DD'-2D'^2}(y-1)e^x$$

$$= \frac{1}{(D-2D')(D+D')}(y-1)e^x$$

$$= \frac{1}{(D-2D')}\left[\frac{1}{(D+D')}(y-1)e^x\right]$$

$u = x \,;\, dv = e^x dx$

$du = dx \,;\, v = e^x$

$$= \frac{1}{(D-2D')}\int e^x(c+x-1)\,dx$$

$$= \frac{1}{D-2D'}\int (ce^x+xe^x-e^x)\,dx$$

$$= \frac{1}{D-2D'}\left[ce^x+xe^x-\int e^x dx-e\ldots\right.$$

$$= \frac{1}{D-2D'}\left[ce^x+xe^x-e^x-e^x\right]$$

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$$P.I = \frac{1}{D-2D'}\left[ce^{x}+xe^{x}-2e^{x}\right]$$

$$= \frac{1}{D-2D'}\left[e^{x}(c+x-2)\right]$$

$\int u\,dv = uv - \int v\,du$

$= uv - u'v_1 + u''v_2 - u'''v_3$

$$= \frac{1}{D-2D'}\left[e^{x}(y-2)\right]$$

$$= \int e^{x}(c_1-2x-2)\,dx$$

$y = c - mx$

$= c_1 - 2x$

$$= c_1e^{x} - 2\int xe^{x} - 2e^{x}$$

$$= c_1e^{x} - 2xe^{x} + \cancel{2e^{x}} - \cancel{2e^{x}}$$

$$P.I = c_1e^{x} - 2xe^{x}$$

$$= e^{x}(c_1-2x)$$

$$P.I = e^{x}y$$

The general solution is

$$z = C.F + P.I$$

$$z = \phi_1(y-x) + \phi_2(y+2x) + e^{x}y$$

---

2) solve: $(D^3+D^2D'-DD'^2-D'^3)z = e^{y}\cos 2x$

soln:

The given equ is

$$(D^3+D^2D'-DD'^2-D'^3)z = e^{y}\cos 2x$$

The auxiliary equ is

$$m^3+m^2-m-1 = 0$$

$$\begin{array}{r|rrrr} +1 & 1 & 1 & -1 & -1 \\ & 0 & +1 & 2 & +1 \\ \hline & 1 & 2 & +1 & \boxed{0} \end{array}$$

$$(m-1)(m^2+2m+1) = 0$$

$$m = 1 \;;\; (m+1)^2 = 0$$

$$(m+1)(m+1) = 0$$

$$\frac{1}{m}\Big|\frac{1}{m}$$

$$m_1 = 1 \;;\; m_2 = -1 \;;\; m_3 = -1$$

The roots are equal and distinct

$$C.F = \phi_1(y+x) + \phi_2(y-x) + \phi_3 x(y-x)$$

$$P.I = \frac{1}{(D^3+D^2D'-DD'^2-D'^3)}e^{y}\cos 2x$$

Unit - V

The Laplace Transforms

Definition :

If a function $f(t)$ is defined for all positive value of the variable t & if $\int_0^{\infty} e^{-st} f(t)dt$ transform of $f(t)$ & is equal to $f(s)$, then $F(s)$ is called by the symbol $L\{f(t)\}$.

Hence, $L\{f(t)\} = \int_0^{\infty} e^{-st} f(t)dt = F(s)$. The operator L that transform $f(t)$ into $F(s)$ is called the laplace transform operation.

Note :

$$\underset{s\to\infty}{Lt} f(s) = 0$$

Definitions piecewise continuity:

A function $f(t)$ is said to be piecewise continuous in a closed interval. (a,b) if it is defined on that interval & is such that the interval can be broken up into a finite number of sub-interval in each of which $f(t)$ is continuous $f(t)$ can have only ordinary finite discontinuedties in the interval.

Exponential order :

A function $f(t)$ is said to be of exponential order if $L\, te^{-st} f(t) = 0$ or for some number s_0, the product $e^{-s_0 t}|f(t)| < M$ for $t > T$, (i.e); $e^{-s_0 t}|f(t)|$ is bounded for large values of t, say for $t > T$.

Sufficient conditions for the existance of the laplace Transforms :

2m (i) $f(t)$ is continuous or piecewise continuous in the closed interval (a,b) where $a > 0$.

(ii) It is of exponential order.

(iii) $t^n f(t)$ is bounded near to $t=0$ for some number $n > 1$) 2m

From the definition the following results can easily be proved.

i) $L\{f(t)+\phi(t)\} = L\{f(t)\}+L\{\phi(t)\}$

we have,

$$L\{f(t)+\phi(t)\} = \int_0^{\infty} e^{-st}[f(t)+\phi(t)]\,dt$$

$$= \int_0^{\infty} e^{-st} f(t)dt + \int_0^{\infty} e^{-st}\phi(t)dt$$

$$= L\{f(t)\} + L\{\phi(t)\}$$

ii) $L\{f(t)\} = cL\{f(t)\}$, we have c is a constant.
we have

$$L\{cf(t)\} = \int_0^\infty e^{-st} cf(t)dt$$
$$= c\int_0^\infty e^{-st} f(t)dt$$
$$= cL\{f(t)\}$$

iii) $L\{f(t)\} = sL\{f(t)\} - f(0)$
we have $L\{f'(t)\} = \int_0^\infty e^{-st} f'(t)dt$

$$= f(t)[e^{-st}]_0^\infty - \int_0^\infty f(t)(-s)e^{-st}dt$$
$$= -f(0) + s\int_0^\infty f(t)e^{-st}dt$$
$$L\{f(t)\} = sL\{f(t)\} - f(0) \text{ formula}$$

iv) $L\{f''(t)\} = s^2 t\{f(t). sf(0) - f'(0)\}$

$$L\{ff''(t)\} = L\{F'(t)\} \text{ where } F(E) - f'(t)$$
$$= sL\{F(t)\} - F(0)$$
$$= sL\{f'(t)\} - f'(0)$$
$$= s[sL\{f(t)\} - f(0)] - f'(0)$$
$$= s^2 L\{f(t)\} - sf(0) = f'(0)$$

v) By extending the previous result, we get

$$L\{f^n(t)\} = s^n L\{f(t)\} - s^{n-1} f(0) - s^{n-2} f(0) \dots f^{n-1}(0)$$

vi) If $\{f(t)\} = F(s)$

a) $\lim_{t\to 0} f(t) = \lim_{s\to\infty} sF(s)$

b) $\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$

$$L\{f'(t)\} = sL\{f(t)\} - f(0)$$
$$= sL(s) - f(0)$$

Taking limits as $s\to\infty$ on both sides we get,

$$\lim_{s\to\infty} [sF(s) - f(0)] = \lim_{s\to\infty} L\{f'(t)\}$$
$$= \lim_{s\to\infty} \int_0^\infty e^{-st} f'(t)\, dt$$
$$= 0$$
$$\therefore \lim_{s\to\infty} sF(s) = f(0)$$
$$= \lim_{t\to 0} f(t)$$

This result is known as initial value theorem Taking limitss $\to 0$ on both sides of $L\{f'(t)$

$$\lim_{s\to 0} [sF(s) - f(0)] = \lim_{s\to 0} \int_0^\infty e^{-st} f(t)\, dt$$
$$= \int_0^\infty f'(t)\, dt$$

$= f[(t)]_0^\infty$

$= \lim_{t\to\infty} f(t) - f(0)$

$\lim_{s\to 0} sF(s) = \lim_{t\to\infty} f(t)$

This result is known as final value theorem.

viii) $L(e^{-at}) = \frac{1}{s+a}$ provided $s+a>0$

$L(e^{-at}) = \int_0^\infty e^{-st} e^{-at} dt$

$= \int_0^\infty e^{-(s+a)t} dt$

$= \left[\frac{e^{-(s+a)t}}{-(s+a)}\right]_0^\infty$

$= \frac{1}{s+a}$

similarly $L(e^{at}) = \frac{1}{s-a}$ provided $s-a>0$

Corollary:

$L[\cosh at] = L\left[\frac{e^{at}+e^{-at}}{2}\right]$

$= 1/2\, L(e^{at}) + 1/2\, L(e^{-at})$

$= 1/2 \frac{1}{s-a} + 1/2 \frac{1}{s+a}$

$= \frac{s}{s^2-a^2}$

similarly $L(\sinh at) = a/s^2-a^2$

viii) $L(\cos at) = s/s^2+a^2$

$L(\cos at)$ real part of $\int_0^\infty e^{-st} e^{ait} dt$

$=$ real part of $L(e^{ait})$

$=$ real part of $\frac{1}{s-ai}$

$=$ real part of $\frac{s+ai}{s^2+a^2}$

$= s/s^2+a^2$

ix) $L(\sin at) = a[s^2+a^2]$

$L(\sin at) =$ Imaginary part of $\frac{s+ai}{s-ai}$

$= \frac{a}{s^2+a^2}$

x) $L(t^n) = \frac{\sqrt{(n+1)}}{s^{n+1}}$

we have,

$L(t^n) = \int_0^\infty e^{-st} t^n dt$

put $st = x$, Then $dt = 1/s\, dx$

$L(t^n) = \int_0^\infty \left(\frac{x}{s}\right)^n e^{-x} \frac{1}{s} dx$

$= \frac{1}{s^{n+1}} \int_0^\infty x^n e^{-x} dx$

$$= \frac{\sqrt{(n+1)}}{s^{n+1}}$$

when n is a +ve integral

$$\sqrt{(n+1)} = n!$$

$$\therefore L(t^n) = \frac{n!}{s^{n+1}}$$ when n is a +ve integral

corollary:

$$L(1) = \frac{1}{s}, \quad L(t) = 1/s^2$$

$$L(t^2) = 2/s^3$$

$$L(t^{1/2}) = \frac{\sqrt{3/2}}{s^{3/2}} = \frac{1/2\sqrt{(1/2)}}{s^{3/2}}$$

$$= \frac{\sqrt{\pi}}{2s^{3/2}}$$

$$L(t^{1/2}) = \frac{\sqrt{(1/2)}}{s^{1/2}}$$

$$= \sqrt{\pi}/s^{1/2}$$

---

1) Find $L(t^2+2t+3)$

Soln:

$$L(t^2+2t+3) = L(t^2) + L(2t) + L(3)$$

$$= L(t^2) + 2L(t) + 3L(1)$$

W.K.T $L(t^n) = \frac{n!}{s^{n+1}}$

$$= \frac{2}{s^3} + 2\left(\frac{1}{s^2}\right) + 3\left(\frac{1}{s}\right)$$

$$L(t^2+2t+3) = \frac{2}{s^3} + \frac{2}{s^2} + \frac{3}{s}$$

---

2) $L(\sin^2 2t)$

Soln: given $\sin^2 2t = \frac{1-\cos 2(2t)}{2}$

W.K.T

$$= \frac{1}{2} - \frac{\cos 4t}{2}$$

$$L(t^n) = \frac{n!}{s^{n+1}}$$

$$L(\cos at) = \frac{s}{s^2+a^2}$$

$$L(\sin^2 2t) = L\left[\frac{1}{2} - \frac{\cos 4t}{2}\right]$$

$$= \frac{1}{2}\left[L(1) - L(\cos 4t)\right]$$

$$= \frac{1}{2}\left[\frac{1}{s} - \frac{s}{s^2+4^2}\right]$$

$$= \frac{1}{2}\left[\frac{1}{s} - \frac{s}{s^2+16}\right]$$

$$= \frac{1}{\cancel{2}1}\left[\frac{\cancel{s^2}+\cancel{16}^{8}-\cancel{s^2}}{s(s^2+16)}\right]$$

$$L(\sin^2 2t) = \frac{8}{s(s^2+16)}$$

3) Find $L(\sin^3 2t)$

Soln:

$$\sin^3 2t = \frac{1}{4}[3\sin 2t - \sin 6t]$$

W.K.T

$$L(\sin at) = a/s^2+a^2$$

$$\therefore L(\sin^3 2t) = L[\tfrac{1}{4}(3\sin 2t - \sin 6t)]$$

$$= \frac{3}{4}[L(\sin 2t)] - \frac{1}{4}L[\sin 6t]$$

$$= \frac{3}{4}[2/s^2+2^2] - \frac{1}{4}[6/s^2+6^2]$$

$$= \frac{1}{4}\left[\frac{6}{s^2+4} - \frac{6}{s^2+36}\right] \Rightarrow = \frac{6}{4}\left[\frac{s^2-36-s^2-4}{(s^2+4)(s^2+36)}\right]$$

$$= \frac{6}{4}\left[\frac{32}{(s^2+4)(s^2+36)}\right]$$

$$L(\sin^3 2t) = \frac{48}{(s^2+4)(s^2+36)}$$

---

4) Find $L\{f(t)\}$, where $f(t)=0$, when $0<t<2$
3, when $t>2$.

Soln:

W.K.T

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$$

$$= \int_0^2 e^{-st} f(t)\,dt + \int_2^\infty e^{-st} f(t)\,dt$$

$$= 0 + \int_2^\infty e^{-st}(3)\,dt$$

$$= 3\left[\int_2^\infty e^{-st}(1)\,dt\right]$$

$$= 3\left[\frac{e^{-st}}{-s}\right]_2^\infty$$

$$= \frac{3}{-s}[e^{-\infty} - e^{-2s}]$$

$$= \frac{3}{-s}[0 - e^{-2s}]$$

$$L\{f(t)\} = \frac{3e^{-2s}}{s}$$

Laplace transform of periodic function:

Let $f(t)$ be a function with period a

Then $f(t) = f(a+t) = f(2a+t) \ldots$

$$= f(na+t)$$

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$$

$$= \int_0^a e^{-st} f(t)\,dt + \int_a^{2a} e^{-st} f(t)\,dt + \int_{2a}^{3a} e^{-st} f(t) + \ldots + \int_{(n-1)a}^{na} e^{-st} f(t)\,dt$$

In the second integral $t = T+a$

In the third integral put $t = T+2a$

In the Fourth integral put $t = T+(n-1)a$

and so on

Hence $L\{f(t)\} = \int\limits_0 e^{-st} f(t)\,dt + \int\limits_0 e^{-s(T+a)} f(T+a)\,dt + \int\limits_0^a e^{-s(T+2a)} f(T+2a)\,dt + \cdots$

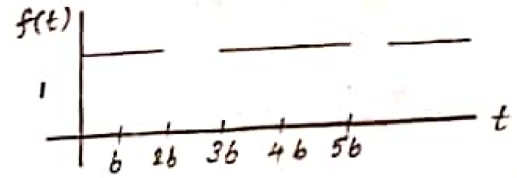
$= \int\limits_0^a e^{-st} f(t)\,dt + e^{-sa}\int\limits_0^a e^{-st} f(t)\,dt + e^{-2sa}\int\limits_0^a e^{st} f(t)\,dt$

$= (1+e^{-sa}+e^{-2sa})\int\limits_0^a e^{-st} f(t)\,dt$

$= \frac{1}{1-e^{-sa}} \int\limits_0^a e^{-st} f(t)\,dt.$

Examples:

1) Find the transform of the rectangular wave shown below:



Here $f(t) = 1$ when $0 < t < b$
$= -1$ when $b < t < 2b$

Soln: W.K.T $f(t)$ be a fn with period a, then

$$L\{f(t)\} = \frac{1}{1-e^{-sa}}\int\limits_0^a e^{-st} f(t)\,dt$$

Here $a = 2b$

$$\therefore L\{f(t)\} = \frac{1}{1-e^{-s2b}}\left[\int\limits_0^{2b} e^{-st} f(t)\,dt\right]$$

$$= \frac{1}{1-e^{-2sb}}\left[\int\limits_0^{b} e^{-st} f(t)\,dt + \int\limits_0^{2b} e^{-st} f(t)\,dt\right]$$

$$= \frac{1}{1-e^{-2sb}}\left[\int\limits_0^{b} e^{-st}(1)\,dt + \int\limits_b^{2b} e^{-st}(-1)\,dt\right]$$

$$= \frac{1}{1-e^{-2sb}}\left[\left(\frac{e^{-st}}{-s}\right)_0^b - \left[\frac{e^{-st}}{-s}\right]_b^{2b}\right]$$

$$= \frac{1}{1-e^{-2sb}}\left[\left(\frac{e^{-3b}}{-s} + \frac{e^{0}}{s}\right) + \left(\frac{e^{-2sb}}{s} - \frac{e^{-sb}}{s}\right)\right]$$

$$= \frac{1}{s(1-e^{-2sb})}\left[-e^{-sb} + 1 + e^{-2sb} - e^{-sb}\right]$$

$$= \frac{1}{s(1-e^{2sb})}\left[1 - 2e^{-sb} + e^{-2sb}\right]$$

$$= \frac{(1-e^{-sb})^2}{s[1^2 - (e^{-sb})^2]}$$

$$= (1-e^{-sb})^2 / s(1-e^{-sb})(1+e^{-sb})$$

$$= (1-e^{-sb}) / s(1+e^{-sb})$$

a) what is the transform of the fn - shown below.

The function $f(t) = \begin{cases} t, & 0 < t < b \\ 2b - t, & b < t < 2b. \end{cases}$

soln:

$$L\{f(t)\} = \frac{1}{1-e^{-2sb}}\left[\int_0^{2b} e^{-st} f(t)\,dt\right]$$

$$= \frac{1}{1-e^{-2sb}}\left[\int_0^{b} e^{-st}\, t\,dt + \int_b^{2b} e^{-st}(2b-t)\,dt\right]$$

$u = t$ | $dv = e^{-st}dt$

$du = dt$ | $v = \frac{e^{-st}}{-s}$

$u = 2b - t$

$du = -dt$

$$= \frac{1}{1-e^{-2sb}}\left[\left(\left(t\frac{e^{-st}}{s}\right)_0^b - \int_0^b \frac{e^{-st}}{-s}dt\right) + \left(\left((2b-t)\left(\frac{e^{-st}}{-s}\right)\right)_b^{2b} - \int_b^{2b} \frac{e^{-st}}{-s}(-dt)\right)\right]$$

$$= \frac{1}{1-s^{-2bs}}\left[\left[\left(\frac{be^{-sb}}{-s}\right) + \frac{1}{s}\left(\frac{e^{-st}}{-s}\right)_0^b\right] + \left[0 + \left(\frac{be^{-sb}}{s}\right) - \frac{1}{s}\left(\frac{e^{-st}}{-s}\right)_b^{2b}\right]\right] + 0$$

$$= \frac{1}{1-e^{-2bs}}\left[\left[\frac{-be^{-bs}}{s} - \frac{1}{s^2}\left[e^{-sb} - e^0\right] + \left[\frac{be^{-sb}}{s} + \frac{1}{s^2}\left(e^{-2bs} - e^{-sb}\right)\right]\right]\right]$$

$$= \frac{1}{1-e^{-2bs}}\left[\frac{1}{s^2}\left[-e^{-sb} + 1 + e^{-2bs} - e^{-sb}\right]\right]$$

$$= \frac{1}{1-e^{-2bs}}\left[\frac{1}{s^2}\left(1 - 2e^{-sb} + e^{-2bs}\right)\right]$$

$$= \frac{1}{s^2}\left[\frac{(1-e^{-bs})^2}{(1-e^{-bs})(1+e^{-bs})}\right] \Rightarrow = \frac{1}{s^2}\left[\frac{1-e^{-bs}}{1+e^{-bs}}\right]$$

$$L\{f(t)\} = \frac{1}{s^2}\tanh\left(\frac{bs}{2}\right)$$

Some general theorems :

i) If $L\{f(t)\} = F(s)$, then

$$L\{f(at)\} = \frac{1}{a}F(s/a)$$

ii) $L\{e^{-at}f(t)\} = F(s+a)$

$$= \int_0^\infty e^{-(s+a)} f(t)\,dt$$

iii) $L\{t\,f(t)\} = \frac{-d}{dt}F(s)$

proof: $L\{f(t)\} = \int_0^\infty e^{-st} f(t)dt$

$$\frac{d}{ds}F(s) = \frac{d}{ds}\int_0^\infty e^{-st} f(t)dt$$

$$= \int_0^\infty (e^{-st})(-t)f(t)dt$$

$$= -\int_0^\infty e^{-st}(t f(t))\, dt$$

$$= L\{t\, f(t)\}$$

$$L\{f(t)\} = \frac{-d}{ds} = F(s)$$

corollary:

$$L\{t^n f(t) = (-1)^n \frac{d^n}{ds^n}[L\{f(t)\}]$$

we have,

$$L\{t\, f(t)\} = \frac{-d}{ds} L\{f(t)\}$$

$$\therefore \{t^2 f(t)\} = L\{t \cdot t\, f(t)\}$$

$$= \frac{-d}{ds} L\{t f(t)\}$$

$$= \frac{d}{ds}\left[\frac{-d}{ds} L\{f(t)\}\right]$$

$$= (-1)^2 \frac{d^2}{ds^2} L\{f(t)\}$$

containing the process weget the result. This value result canbe written as follows.

If $L[f(t)] = f(s)$, then

$$F'(s) = L[(-t) f(t)]$$

$$F^n(s) = L[(-t)^2 f(t)]$$

$$F^n(s) = L[(-t^n) f(t)]$$

iv) If $L\{f(t)\} = F(s)$ & if $\frac{f(t)}{t}$ has limit as $t \to 0$

iv) If $L\{f(t)\} = F(s)$ & if $\frac{f(t)}{t}$ has limit as $t \to 0$, then

$$L\left\{\frac{f(t)}{t}\right\} = \int_0^\infty F(s)ds$$

$$F(s) = L\{f(t)\}$$

$$= \int_0^\infty e^{-st} f(t)\, dt$$

$$\int_s^\infty F(s)ds = \int_s^\infty \int_0^\infty e^{-st} f(t)\, dt\, ds$$

$$= \int_0^\infty \int_s^\infty e^{-st} f(t)\, ds\, dt$$

on interchanging the order of integration

$$= \int_0^\infty f(t) \left[\frac{e^{-st}}{-t} \right]_s^\infty dt$$

$$= \int_0^\infty \frac{f(t)}{t} e^{-st}\, dt$$

$$= L\left\{ \frac{f(t)}{t} \right\}$$

The inverse transform:

Let the symbol $L^{-1}\{F(S)\}$ denote a function whose laplace transform is $F(S)$.

Thus if $L\{f(t)\} = F(S)$, Then $f(t) = L^{-1}\{F(S)\}$

The most obvious way of finding the inverse transform of a given fuchtion whose laplace transform is the given function.

We compile the table of transforms from the known results.

Ex: 1 Find (cosat)

soln: given cosat

W.K.T

By them $L\{f(at)\} = \frac{1}{a} F(S/a)$

$L(\cos t) = S/S^2+1$

$$L(\cos at) = 1/a \left[\frac{S/a}{(S/a)^2+1} \right]$$

$$= \frac{1}{a} \left(\frac{S/a}{S^2/a^2+1} \right)$$

$$= \frac{1}{a} \left[\frac{S}{a\left(\frac{S^2+a^2}{a^2}\right)} \right]$$

$$= \frac{1}{a} \left(\frac{S}{S^2+a^2} \right)$$

$$L(\cos at) = S/S^2+a^2$$

Ex: 2 Find $L\{e^{-at} \cos bt\}$

soln: W.K.T By theorem

$$L\{e^{-at} f(t)\} = F(S+a)$$

$$= \int_0^\infty e^{-(S+a)} f(t)\, dt$$

Also,

$$L\{\cos bt\} = \frac{S}{S^2+b^2}$$

$$L\{e^{-at} \cos bt\} = \frac{S+a}{(S+a)^2+b^2}$$

	$f(t)$	$F(S)$
1)	e^{at}	$1/S-a$
2)	coshat	$1/S^2-a^2$
3)	sinhat	a/S^2-a^2
4)	cosat	S/S^2+a^2
5)	sinat	a/S^2+a^2
6)	1	$1/S$
7)	t	$1/S^2$
8)	t^2	$n!/S^{n+1}$
9)	te^{at}	(n is a -ve integer)
10)	$t^2 e^{at}$	$1/(S-a)^2$
11)	$t^n e^{at}$	$2/(S-a)^3$
12)	$e^{-at} \sin bt$	n is a +ve integer
13)	$e^{-at} \cos bt$	$S+a/(S+a)^2+b^2$
14)	$t \sin at$	$2as/(S^2+a^2)^2$
15)	$t \cos at$	$S^2a^2/(S^2+a^2)^2$

Ex: 3 Find $L\{e^{-at}\sin bt\}$

soln:

W.K.T By thm

$L\{\sin bt\} = b/s^2+b^2$

$L\{e^{-at}\sin bt\} = b/(s+a)^2+b^2$

Ex: 4 Find $L\{e^{at}t^n\}$

soln:

W.K.T $L\{t^n\} = \dfrac{n!}{s^{n+1}}$

$L\{e^{at}t^n\} = \dfrac{n!}{(s-a)^{n+1}}$

Ex: 5 Find $L\{e^{-at}t^n\}$

soln:

W.K.T

$L(t^n) = \dfrac{n!}{s^{n+1}}$

$L\{e^{-at}t^n\} = \dfrac{n!}{(s+a)^{n+1}}$

Ex: 6 Find $L\{te^{-at}\}$

soln: By thm

$L\{tf(t)\} = \dfrac{-d}{ds}F(s)$

given $f(t) = e^{-at}$

W.K.T

$L\{e^{-at}\} = \dfrac{1}{s+a}$

$L\{tf(t)\} = \dfrac{-d}{ds}[1/s+a]$

$= -d/ds[(s+a)^{-1}]$

$= -[-(s+a)^{-2}]$

$L\{tf(t)\} = 1/(s+a)^2.$

Ex: 7 Find $L\{t^2e^{-3t}\}$

soln: W.K.T $L\{t^n f(t)\} = (-1)^n \dfrac{d^n}{ds^n} L\{f(t)\}$

given $f(t) = e^{-3t}$

$L\{f(t)\} = 1/st^3$

$L\{t^2e^{-3t}\} = (-1)^2 \dfrac{d^2}{ds^2}[1/st^3]$

$= \dfrac{d}{ds}\left[\dfrac{d}{ds}[(s+3)^{-1}]\right]$

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$$= \frac{d}{ds}\left[-(s+3)^{-2}\right]$$

$$= (-1)(-2)(s+3)^{-3}$$

$$L\{t^2 e^{-3t}\} = \frac{2}{(s+3)^3}$$

8) Find $L(t \sin at)$

soln:

W.K.T

$$L\{t\, f(t)\} = -\frac{d}{ds} L\{f(t)\}$$

given $f(t) = \sin at$

$$L\{f(t)\} = a/s^2+a^2$$

$$L\{t \sin at\} = \frac{-d}{ds}\left[\frac{a}{s^2+a^2}\right]$$

$$= -a \frac{d}{ds}\left[(s^2+a^2)^{-1}\right]$$

$$= -a\left[(-1)(s^2+a^2)^{-2}(2s)\right]$$

$$L\{t \sin at\} = \frac{2as}{(s^2+a^2)^2}$$

9) Find $L\{te^{-t}\sin t\}$

soln: W.K.T $L\{t\, f(t)\} = \frac{-d}{ds} L\{f(t)\} \to ①$

Here $f(t) = e^{-t}\sin t$

$$L\{f(t)\} = L\{e^{-t}\sin t\}$$

W.K.T

$$L\{e^{-at} f(t)\} = F(s+a)$$

$$L\{\sin t\} = 1/s^2+1$$

$$L\{e^{-t}\sin t\} = \frac{1}{(s+1)^2+1}$$

$$= \frac{1}{s^2+2s+1+1} \Rightarrow = \frac{1}{s^2+2s+2}$$

$$\therefore (1) \Rightarrow L\{tf(t)\} = \frac{-d}{ds}\left[\frac{1}{s^2+2s+2}\right]$$

$$= \frac{-d}{ds}\left[(s^2+2s+2)^{-1}\right]$$

$$= (-1)(-1)\left[(s^2+2s+2)^{-2}(2s+2)\right]$$

$$L\{te^{-t}\sin t\} = \frac{2(s+1)}{(s^2+2s+2)^2}$$

10) Find $L\left\{\frac{1-e^t}{t}\right\}$

soln: W.K.T By thm

$$L\left\{\frac{f(t)}{t}\right\} = \int_s^\infty F(s)\, ds,$$

if $\lim_{t\to 0}\left\{\frac{f(t)}{t}\right\}$ exists

11) Find $L\left\{\frac{\sin at}{t}\right\}$

soln:

By thm

$$L\left\{\frac{f(t)}{t}\right\} = \int_s^\infty F(s)ds, \text{ if } \lim_{t\to 0} \frac{f(t)}{t} \text{ exists.}$$

Here, $\lim_{t\to 0} \frac{\sin at}{t} = a$

$\therefore$ Here $f(t) = \sin at$

$L\{f(t)\} = a/s^2+a^2$

W.K.T $\lim_{\theta\to 0} \frac{\sin\theta}{2} = 1$

$$L\left\{\frac{\sin at}{t}\right\} = \int_s^\infty \frac{a}{s^2+a^2} ds$$

$$= [\tan^{-1}(s/a)]_s^\infty$$

$$= \tan^{-1}(\infty - \tan^{-1}(s/a))$$

$\therefore$ Here $f(t) = \sin at$

$L\{f(t)\} = a/s^2+a^2$

W.K.T $\lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1$

$$L\left\{\frac{\sin at}{t}\right\} = \int_s^\infty \frac{a}{s^2+a^2} ds$$

$$= (\tan^{-1}(s/a))_s^\infty$$

$$= \tan^{-1}(\infty) - \tan^{-1}(s/a)$$

$$= \pi/2 - \tan^{-1}(s/a)$$

$$L\left\{\frac{\sin at}{t}\right\} = \cot^{-1}(s/a).$$

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Evaluation of integrals :

1) Evaluate $\int_0^\infty e^{-2t} \sin 3t \, dt$

soln : W.K.T

$$\int_0^\infty e^{-st} f(t)\,dt = L\{f(t)\}$$

Here, $f(t) = \sin 3t$, $s = -2$

$$L\{f(t)\} = L\{\sin 3t\}$$

$$= \frac{3}{s^2+3^2}$$

$$\int_0^\infty e^{-2t} \sin 3t\,dt = \frac{3}{(-2)^2+9} = \frac{3}{4+9}$$

$$\int_0^\infty e^{-2t} \sin 3t\,dt = 3/13$$

2) Evaluate $\int_0^\infty t e^{-3t} \cos t\,dt$

Soln :

W.K.T $\int_0^\infty e^{-st} f(t)\,dt = L\{f(t)\}$

$\therefore$ Here $f(t) = t\cos t$

$$L\{f(t)\} = L\{t\cos t\}$$

$$= \frac{-d}{ds} L\{\cos t\}$$

$$= \frac{-d}{ds}\left[\frac{s}{s^2+1}\right]$$

$$= -\left[\frac{(s^2+1)(1) - s(2s)}{(s^2+1)^2}\right]$$

$$= -\left(\frac{s^2+1-2s^2}{(s^2+1)^2}\right)$$

$$= \frac{s^2-1}{(s^2+1)^2}$$

Hence, put $s = -3$

$$\int_0^\infty e^{-3t}\, t\cos t\,dt = \frac{(-3)^2-1}{((-3)^2+1)^2}$$

$$= \frac{9-1}{(9+1)^2}$$

$$= \frac{8}{(10)^2}$$

$$= \frac{8}{100}$$

$$\int_0^\infty e^{-3t}\, t\cos t\,dt = 2/25$$

3) Evaluate $\int_0^\infty \frac{e^{-t}-e^{-2t}}{t}\,dt$

Soln

W.K.T $\int_0^\infty e^{-st} f(t)\,dt = L\{f(t)\}$

$$\int_0^\infty e^{-st}\left[\frac{e^{-t}-e^{-2t}}{t}\right]dt = L\left\{\frac{e^{-t}-e^{-2t}}{t}\right\} \rightarrow (1)$$

W.K.T

$$L\left\{\frac{f(t)}{t}\right\} = \int_s^\infty F(s)\,ds$$

Here, $f(t) = e^{-t} - e^{-2t}$

$$L\{f(t)\} = L\{e^{-t} - e^{-2t}\}$$
$$= L\{e^{-t}\} - L\{e^{-2t}\}$$
$$= \frac{1}{s+1} - \frac{1}{s+2}$$

$$\therefore (1) \Rightarrow L\left\{\frac{e^{-t}-e^{-2t}}{t}\right\} = \int_0 \left[\frac{1}{s+1} - \frac{1}{s+2}\right]ds$$
$$= [\log(s+1) - \log(s+2)]_s^\infty$$
$$= 0 - [\log(s+1) - \log(s+2)]$$
$$= \log(s+2)\log(s+1)$$
$$= \log\left(\frac{s+2}{s+1}\right)$$

$$\int_0^\infty e^{-st}\left[\frac{e^{-t}-e^{-2t}}{t}\right]dt = \log\left(\frac{s+2}{s+1}\right)$$

Put $s=0$

$$\int_0^\infty \left(\frac{e^{-t}-e^{-2t}}{t}\right)dt = \log(2/1) = \log_e 2.$$

---

4) $L\{te^{-5t}\}$

Soln: By thm $L\{t f(t)\} = \frac{-d}{ds} F(s)$

given $f(t) = e^{-5t}$

W.K.T $L\{e^{-at}\} = \frac{1}{s+a} = \frac{1}{s+5}$

$$L\{te^{-5t}\} = \frac{-d}{ds}\left(1/s+5\right)$$
$$= \frac{-d}{ds}\left[(s+5)^{-1}\right]$$
$$= -(-1)(s+5)^{-2}$$
$$= (s+5)^{-2}$$
$$L\{te^{-5t}\} = 1/(s+5)^2$$

---

5) $\int_0^\infty te^{-3t}\sin t\,dt$

Soln: W.K.T $\int_0^\infty e^{-st} f(t)\,dt = L\{f(t)\}$

Here, $f(t) = t\sin t$

$$= \frac{-d}{ds} L\{\cos t\}$$

$$= \frac{-d}{ds}\left[1/s^2+1\right]$$

$$= -\left[\frac{(s^2+1)(0)-(1)(2s)}{(s^2+1)^2}\right]$$

$$= -\left[\frac{-2s}{(s^2+1)^2}\right] = \frac{2s}{(s^2+1)^2}$$

Hence put $s=-3$

$$\int_0^\infty e^{-3t}\, t\sin t\,dt = \frac{-[(-3)^2+1-2(-3)]}{((-3)^2+1)^2}$$

$$= -\left[\frac{9+1+6}{(9+1)^2}\right] = \frac{-16}{(10)^2}$$

$$= -16/100 \Rightarrow = -3/50$$

$$\int_0^\infty t e^{-3t}\sin t\,dt = -3/50$$

The inverse transform:

Rule I: If $L\{f(t)\} = F(s)$, then

$$L\{e^{-at}f(t)\} = F(s+a)$$

Hence $L^{-1}\{F(s+a)\} = e^{-at}f(t)$, where $f(t) = L^{-1}[F(s)]$

EX:

1) Find $L^{-1}[1/(s+a)^2]$

Soln:

$$L^{-1}\left[\frac{1}{(s+a)^2}\right] = e^{-at}L^{-1}(1/s^2)$$

$$\left(L\{t^n\} = \frac{n!}{s^{n+1}} \Rightarrow L^{-1}\left(\frac{n!}{s^{n+1}}\right) = t^n\right)$$

put $n=1$

$$L^{-1}[1/s^2] = t$$

$$(1) \Rightarrow L^{-1}(1/(s+a)^2) = e^{-at}t$$

---

2) Find $L^{-1}\left(\frac{1}{(s+2)^2+16}\right)$

Soln:

$$L^{-1}\left(\frac{1}{(s+2)^2+16}\right) = e^{-2t}L^{-1}\left(\frac{1}{s^2+4^2}\right)$$

$$= e^{-2t}\frac{1}{4}\left(L^{-1}\left(\frac{3}{s^2+a^2}\right)\right) \rightarrow \sin at$$

$$= \frac{e^{-2t}}{4}(\sin 4t)$$

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3) Find $L^{-1}\left[\frac{s-3}{(s-3)^2+4}\right]$

soln:

$$L^{-1}\left[\frac{s-3}{(s-3)^2+4}\right] = e^{3t}\left(L^{-1}\left(\frac{s}{s^2+2^2}\right)\right) \quad \cos at$$

$$= e^{3t}\cos 2t$$

4) Find $L^{-1}\left[\frac{s}{s^2+2s+5}\right]$

soln: given

$$\frac{s}{s^2+2s+5} = \frac{s}{s^2+2s+1^2-1^2+5}$$

$$= \frac{s}{(s+1)^2+4}$$

$$= \frac{(s+1)-1}{(s+1)^2+4} = \frac{s+1}{(s+1)^2+4} - \frac{1}{(s+1)^2+4}$$

$$L^{-1}\left[\frac{s}{s^2+2s+5}\right] = L^{-1}\left(\frac{s+1}{(s+1)^2+4}\right) - L^{-1}\left(\frac{1}{(s+1)^2+4}\right)$$

$$= e^{-t}L^{-1}\left(\frac{s}{s^2+2^2}\right) - e^{-t}\left(L^{-1}\left(\frac{1}{s^2+2^2}\right)\right)$$

$$= e^{-t}[\cos 2t] - \frac{e^{-t}}{2}L^{-1}\left(\frac{2}{s^2+2^2}\right)$$

$$= e^{-t}(\cos 2t) - \frac{1}{2}(\sin 2t)$$

Rule II :

If $L\{f(t)\} = F(s)$, then

$$L\{f(at)\} = \frac{1}{a}F(s)$$

$$L^{-1}\left\{\frac{1}{a}F(s/a)\right\} = f(at)$$

put $\frac{1}{a} = k$,

$$L^{-1}[F(ks)] = \frac{1}{k}f(t/k)$$

$$F(s) = \frac{-1}{2}\left[\frac{1}{s^2+a^2}\right]$$

$$L^{-1}\left[\frac{s}{(s^2+a^2)^2}\right] = +tL^{-1}\left(\frac{1}{2}\left(1/s^2+a^2\right)\right)$$

$$= \frac{t}{2}L^{-1}\left(\frac{1}{s^2+a^2}\right)$$

$$= \frac{t}{2a}L^{-1}\left(\frac{a}{s^2+a^2}\right)$$

$$= \frac{t}{2a}L^{-1}\left[a/s^2+a^2\right]$$

$$= \frac{t}{2a}\sin at$$

2) Find $L^{-1}\left(s/(s^2-1)^2\right)$

soln: given $L^{-1}\left(s/(s^2-1)^2\right)$

Here

$$F'(s) = s/(s^2-1)^2$$

$$F(s) = \int\left(s/(s^2-1)^2\right)ds$$

$$= \int d - 1/2\left[1/(s^2-1)\right]$$

$$F(s) = -1/2\left(1/s^2-1\right)$$

$$L^{-1}\left(s/(s^2-1)^2\right) = -t L^{-1}\left[\frac{-1}{2}\left(1/s^2+1^2\right)\right]$$

$$= \frac{-t}{2} L^{-1}\left(1/s^2+1^2\right)$$

$$= t/2a\, L^{-1}\left(1/s^2+1^2\right)$$

$$= t/2 \sin t$$

3) Find $L^{-1}\left[\frac{s+2}{(s^2+4s+5)^2}\right]$

soln:

$$\text{given } \frac{s+2}{(s^2+4s+5)^2} = \frac{s+2}{((s^2+4s+2^2)+1)^2}$$

$$= (s+2)/((s+2)^2+1)^2$$

$$= s+2/((s+2)^2+1)^2$$

Here

$$F'(s) = \frac{s+2}{((s+2)^2+1)^2}$$

$$F(s) = \int\frac{s+2}{((s+2)^2+1)^2}ds$$

$$= \int d\left[-1/2\left(-1/(s+2)^2+(1)\right)\right]$$

$$L\{f(t)\} = F(s) = \frac{-1}{2}\left(\frac{1}{(s+2)^2+1}\right)$$

$$L^{-1}\left(\frac{s+2}{(s^2+4s+5)^2}\right) = -t f(t)$$

$$= -t\left[L^{-1}\left[\frac{1}{2((s+2)^2+1)}\right]\right]$$

$$= \frac{t}{2} L^{-1}\left[\frac{1}{(s+2)^2+1}\right]$$

$$= \frac{t}{2} e^{-2t} L^{-1}\left(\frac{1}{s^2+1^2}\right)$$

$$L^{-1}\left(\frac{s+2}{(s^2+4s+5)^2}\right) = \frac{t}{2} e^{-2t}(\sin t)$$

Rule IV :

If $L\{f(t)\} = F(s)$, then $L\{t f(t)\} = -F'(s)$

Ex :

1) Find $L^{-1}\left[\log\left(\frac{s+1}{s-1}\right)\right]$

Soln :

Let

$$f(t) = L^{-1}\left[\log\left(\frac{s+1}{s-1}\right)\right]$$

$$L\{f(t)\} = \log\left(\frac{s+1}{s-1}\right)$$

$$L\{t f(t)\} = \frac{-d}{ds}\left[L\{f(t)\}\right]$$

$$= \frac{-d}{ds}\left[\log\left(\frac{s+1}{s-1}\right)\right]$$

$$= \frac{-d}{ds}\left[\log(s+1) - \log(s-1)\right]$$

$$= -\left[\frac{1}{s+1} - \frac{1}{s-1}\right]$$

$$L\{t f(t)\} = \frac{1}{s-1} - \frac{1}{s+1}$$

$$t f(t) = L^{-1}\left[\frac{1}{s-1}\right] - L^{-1}\left[\frac{1}{s+1}\right]$$

$$t f(t) = e^{t} - e^{-t}$$

$\sinh t = \frac{e^{t} - e^{-t}}{2}$

$$t f(t) = 2\sinh t$$

$2\sinh t = e^{t} - e^{-t}$

$$f(t) = \frac{2\sinh t}{t}$$

$$\therefore L^{-1}\left[\log\left(\frac{s+1}{s-1}\right)\right] = \frac{2\sinh t}{t}$$

Rule V :

If $L\{f(t)\} = sF(s)$, then

$$L^{-1}[sF(s)] = f(t)$$

$$= \frac{d}{dt}\phi(t)$$

$$= \frac{d}{dt} L^{-1}[F'(s)]$$

provided $L^{-1}[F(s)] = 0$, when $t = 0$

Ex :

1) Find $L^{-1}\left[\frac{s}{s^2+k^2}\right]$

Soln :

$$L^{-1}\left[\frac{s}{s^2+k^2}\right] = \frac{d}{dt} L^{-1}\left[\frac{1}{s^2+k^2}\right]$$

$$= \frac{d}{dt}\left[\frac{1}{k}\sin kt\right]$$

Note:

$\left[\text{Here } \lim_{t\to 0} \frac{\sin kt}{t} = 0\right]$

2) Find $L^{-1}\left[\frac{s}{(s+3)^2+4}\right]$

soln:

$$L^{-1}\left[\frac{s}{(s+3)^2+4}\right] = \frac{d}{dt}\left[L^{-1}\left(1/(s+3)^2+4\right)\right]$$

$$= \frac{d}{dt}\left[e^{-3t} L^{-1}\left(1/s^2+2^2\right)\right]$$

$$= \frac{d}{dt}\left(\frac{e^{-3t}}{2} L^{-1}\left(2/s^2+2^2\right)\right]$$

$$= \frac{1}{2}\frac{d}{dt}\left(e^{-3t}\sin 2t\right]$$

$$= \frac{1}{2}\left[e^{-2t}\cos 2t(2) + \sin 2t\, e^{-3t}(3)\right]$$

$$L^{-1}\left[\frac{s}{(s+3)^2+4}\right] = \frac{e^{-3t}}{2}\left[2\cos 2t - 3\sin 2t\right]$$

---

5) Find $L^{-1}\left[\frac{s-3}{s^2+4s+13}\right]$

soln:

$$L^{-1}\left[\frac{s-3}{s^2+4s+13}\right] = L^{-1}\left[\frac{s}{s^2+4s+13}\right] - 3L^{-1}\left[\frac{1}{s^2+4s+13}\right]$$

$$= \frac{d}{dt}\left[L^{-1}\left(\frac{1}{s^2+4s+13}\right)\right] - 3L^{-1}\left[\frac{1}{s^2+4s+2^2-2^2+13}\right]$$

$$= \frac{d}{dt}\left[L^{-1}\left(\frac{1}{(s+2)^2+9}\right] - 3L^{-1}\left[\frac{1}{(s+2)^2+9}\right)\right]$$

$$= \frac{d}{dt}\left(e^{-2t} L^{-1}\left[\frac{1}{s^2+3^2}\right]\right] - 3e^{-2t}\left[L^{-1}\left(\frac{1}{s^2+3^2}\right)\right]$$

$$= \frac{d}{dt}\left[\frac{e^{-2t}}{3} L^{-1}\left(\frac{3}{s^2+3^2}\right)\right] - e^{-2t} L^{-1}\left(\frac{3}{s^2+3^2}\right)$$

$$= \frac{d}{dt}\left[\frac{e^{-2t}}{3}\sin 3t\right] - e^{-2t}\sin 3t$$

$$= \frac{1}{3}\left[e^{-2t}\cos 3t(3) + \sin 3t\, e^{-2t}(-2)\right] \cdot e^{-2t}\sin 3t$$

$$= \frac{e^{-2t}}{3}\left[3\cos 3t - 2\sin 3t - 3\sin 3t\right]$$

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4) Find $L^{-1}\left[\frac{s}{(s+2)^2}\right]$

soln :

$$L^{-1}\left[\frac{s}{(s+2)^2}\right] = \frac{d}{dt} L^{-1}\left[\frac{1}{(s+2)^2}\right]$$

$$= \frac{d}{dt}\left[e^{-2t} L^{-1}(1/s^2)\right]$$

$$= \frac{d}{dt}\left[e^{-2t}(t)\right]$$

$$= e^{-2t}(1) + e^{-2t} t(-2)$$

$$L^{-1}\left[\frac{s}{(s+2)^2}\right] = e^{-2t}\left[1-2t\right]$$

5) Find $L^{-1}\left[\frac{s^2}{(s-3)^2}\right]$

soln :

$$L^{-1}\left[\frac{s^2}{(s-3)^2}\right] = \frac{d^2}{dt^2}\left[1/(s-3)^3\right]$$

$$= \frac{d^2}{dt^2}\left(e^{3t} L^{-1}(1/s^3)\right)$$

$$= \frac{d^2}{dt^2}\left[\frac{e^{3t}}{2}\left[L^{-1}\left(\frac{2}{s^3}\right)\right]\right]$$

$$= \frac{d^2}{dt^2}\left[\frac{e^{3t}}{2}(t^2)\right]$$

$$= \frac{1}{2}\frac{d}{dt}\left[e^{3t}(2t) + t^2 e^{3t}(3)\right]$$

$$= \frac{1}{2}\left[2t\, e^{3t}(3) + e^{3t}(2) + 3t^2 e^{3t}(3) + e^{3t}(3)\cdot 2t\right]$$

$$= \frac{e^{3t}}{2}\left[6t + 2 + 9t^2 + 6t\right]$$

$$L^{-1}\left(\frac{s^2}{(s-3)^3}\right) = \frac{e^{3t}}{2}(9t^2 + 12t + 2)$$

Rule $\overline{VI}$:

$$L\left[\int_0^t f(x)dx\right] = \frac{1}{s} L\{f(t)\}$$

$$\int_0^t f(x)dx = L^{-1}\left[\frac{1}{s} L\{f(t)\}\right]$$

$$= L^{-1}\left[\frac{1}{s} F(s)\right]$$

EX :

1) Find $L^{-1}\left[\frac{1}{s(s+a)}\right]$

Soln :

$$L^{-1}\left[\frac{1}{s(s+a)}\right] = \int_0^t L^{-1}\left[\frac{1}{s+a}\right]dt$$

$$= \int_0^t e^{-at}dt$$

$$= \left[\frac{e^{-at}}{-a}\right]_0^t$$

$$= \frac{1}{-a}\left[e^{-at} - e^0\right]$$

$$= \frac{+1}{-a}\left[e^{-at} - 1\right]$$

$$L^{-1}\left[\frac{1}{s(s+a)}\right] = \frac{1-e^{-at}}{a}$$

---

2) Find $L^{-1}\left[s/s^2a^2+b^2\right]$

Soln :

$$\frac{s}{s^2a^2+b^2} = \frac{sa}{a[s^2a^2+b^2]}$$

$$= \frac{1}{a}\left[\frac{sa}{(sa)^2+b^2}\right]$$

$$L^{-1}\left[\frac{s}{s^2a^2+b^2}\right] = \frac{1}{a}\left[\frac{1}{a}f(t/a)\right]$$

where $f(t) = L^{-1}\left(\frac{s}{s^2+b^2}\right)$

$f(t) = \cos bt$

$f(t/a) = \frac{\cos bt}{a}$

$$(1) \Rightarrow L^{-1}\left(\frac{s}{s^2a^2+b^2}\right) = \frac{1}{a^2}\cos(bt/a)$$

Rule III :

If $L\{f(t)\} = F(s)$

then $L\{t f(t)\} = -F'(s)$

$L^{-1}[F'(s)] = \pm f(t)$

$= -t L^{-1}[F(s)]$

EX :

1) Find $L^{-1}\left[\frac{s}{(s^2+a^2)^2}\right]$

Soln :

given $L^{-1}\left[\frac{s}{(s^2+a^2)^2}\right]$

Here

$$F'(s) = \frac{s}{(s^2+a^2)^2}$$

$$F(s) = \int \frac{s}{(s^2+a^2)^2}ds$$

$$= \int d\left(-1/2\left(1/s^2+a^2\right)\right).$$

3) Find $L^{-1}\left[\frac{1}{(s^2+a^2)^2}\right]$

Soln:

By Rule III

$$L^{-1}\left[\frac{1}{(s^2+a^2)^2}\right] = L^{-1}\left[\frac{s}{s(s^2+a^2)^2}\right]$$

$$= \int_0^t L^{-1}\left[\frac{s}{(s^2+a^2)^2}\right]dt \rightarrow (1)$$

To find:

$$L^{-1}\left[\frac{s}{(s^2+a^2)^2}\right]:$$

Here

$$F'(s) = \frac{s}{(s^2+a^2)^2}$$

$$F(s) = \int \frac{s}{(s^2+a^2)^2}\,ds$$

$$= \int d\left(-\tfrac{1}{2}\left(1/s^2+a^2\right)\right)$$

$$= \frac{-1}{2}\left[\frac{1}{s^2+a^2}\right]$$

$$L^{1}[F'(s)] = -tf(t)$$

$$\therefore \left[\frac{s}{(s^2+a^2)^2}\right] = -t\left[L^{-1}\left(\frac{-1}{2(s^2+a^2)}\right)\right]$$

$$= \frac{1}{2}t\left[L^{-1}\left(1/s^2+a^2\right)\right]$$

$$= \frac{t}{2a}(\sin at)$$

$$\therefore (1) \Rightarrow L^{-1}\left[\frac{1}{(s^2+a^2)^2}\right] = \int_0^t \frac{t}{2a}\sin at\,dt$$

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$[\int u\,dv = uv - \int v\,du]$

let $u = t$ $\quad$ $dv = \int \sin at\,dt$

$du = dt$ $\quad$ $v = -\frac{\cos at}{a}$

$$L^{-1}\left[\frac{1}{(s^2+a^2)^2}\right] = \frac{1}{2a}\left[\frac{-t\cos at}{a}\right]_0^t - \int_0^t \frac{-\cos at}{a}\,dt$$

$$= \frac{1}{2a}\left[\left(-\frac{t\cos at}{a}\right) - 0 + \frac{1}{a}\left(\frac{\sin at}{a}\right)_0^t\right]$$

$$= \frac{1}{2a}\left[-\frac{t\cos at}{a} + \frac{1}{a^2}(\sin at - 0)\right]$$

$$= \frac{1}{2a^3}(-at\cos at + \sin at)$$

$$L^{-1}\left(1/(s^2+a^2)^2\right) = \frac{1}{2a^3}(\sin at - at\cos at)$$

The methode of Partial fraction canbe used to find the inverse laplace transforms:

Ex : 1

1) Find $L^{-1}\left[\frac{1}{s(s+1)(s+2)}\right]$.

soln:

given $\frac{1}{s(s+1)(s+2)} \rightarrow (1)$

We can split (1) into partial fractions,

$$\frac{1}{s(s+1)(s+2)} = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+2}$$

$$1 = A(s+1)(s+2) + Bs(s+2) + C(s)(s+1)$$

put $s = -1$ $\quad$ | $\quad$ $s = 0$

$1 = 0 + B(-1)(-1+2) + 0$ $\quad$ | $\quad$ $1 = A(1)(2) + 0$

$1 = -B$ $\quad$ | $\quad$ $A = 1/2$

$B = -1$

put $s = -2$

$1 = A(0) + B(0) + C(-2)(-2+1)$

$1 = 2C$

$C = 1/2$

$$\therefore \frac{1}{s(s+1)(s+2)} = \frac{1}{2s} - \frac{1}{s+1} + \frac{1}{2(s+2)}$$

Applying I.L.T on both sides

$$L^{-1}\left[\frac{1}{s(s+1)(s+2)}\right] = L^{-1}\left(1/2s\right) - L^{-1}\left(1/s+1\right) + L^{-1}\left(1/2(s+1)\right)$$

$$= \frac{1}{2}L^{-1}(1/s) - L^{-1}(1/s+1) + \frac{1}{2}L^{-1}\left(\frac{1}{s(s+1)}\right)$$

2) Find $L^{-1}\left[\frac{1}{(s+1)(s^2+2s+2)}\right]$

soln :

given $\frac{1}{(s+1)(s^2+2s+2)} \rightarrow (1)$

We can split (1) into Partial fractions,

$$\frac{1}{s(s+1)(s+2)} = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+2}$$

$$1 = A(s+1)(s+2) + Bs(s+2) + C(s)(s+1)$$

put $s=-1$

$$1 = 0 + B(-1)(-1+2) + 0$$

$$1 = -B$$

$$B = -1$$

$$\frac{1}{(s+1)(s^2+2s+2)} = \frac{A}{s+1} + \frac{Bs+C}{s^2+2s+2}$$

$$1 = A(s^2+2s+2) + (Bs+C)(s+1)$$

Put $s=-1$

$$1 = A(1-2+2)+0$$

$$A = 1$$

$s=0$

$$1 = 1(0+2)+(0+C)(0+1)$$

$$C=-1$$

Put $s=1$

$$1 = 1(1+2+2) + (B(1)-1)(2)$$

$$1 = 5+2B-2$$

$$1 = 3+2B$$

$$B=-1$$

$$\frac{1}{(s+1)(s^2+2s+2)} = \frac{1}{s+1} + \frac{-s-1}{s^2+2s+2}$$

$$= \frac{1}{s+1} - \frac{(s+1)}{s^2+2s+1^2-1^2+2}$$

$$= \frac{1}{s+1} - \frac{s+1}{(s+1)^2+1}$$

Appling I.L.T we've

$$L^{-1}\left[\frac{1}{(s+1)(s^2+2s+2)}\right] = L^{-1}\left(1/s+1\right) - L^{-1}\left(\frac{s+1}{(s+1)^2+1}\right)$$

$$= e^{-t} - e^{-t}L^{-1}\left[\frac{s}{s^2+1^2}\right]$$

$$= e^{-t} - e^{-t}\cos t \Rightarrow = e^{-t}(1-\cos t)$$

3) Find $L^{-1}\left[\frac{1+2s}{(s+2)^2(s-1)^2}\right]$

soln:

given $\frac{1+2s}{(s+2)^2(s-1)^2} \rightarrow (1)$

we can split (1) into partial fractions,

$$\frac{1+2s}{(s+2)^2(s-1)^2} = \frac{A}{(s+2)^2} + \frac{B}{(s-1)^2}$$

$$1+2s = A(s-1)^2 + B(s+2)^2$$

put $s=1$

$$1+2 = A(0)+B(1+2)^2$$

$$3 = 9B$$

$$B = 1/3$$

put $s=-2$

$$1+2(-2) = A(-2-1)^2+0$$

$$-3 = 9A$$

$$A = -3/9$$

$$A = -1/3$$

$$\frac{1+2s}{(s+2)^2(s-1)^2} = \frac{-1}{3(s+2)^2} + \frac{1}{3(s-1)^2}$$

Applying I.L.T we've

$$L^{-1}\left(\frac{1+2s}{(s+2)^2(s-1)^2}\right) = L^{-1}\left(\frac{-1}{3(s+2)^2}\right) + L^{-1}\left[\frac{1}{3(s-1)^2}\right]$$

$$= \frac{-1}{3}L^{-1}\left[\frac{1}{(s+2)^2}\right] + \frac{1}{3}L^{-1}\left(\frac{1}{(s-1)^2}\right)$$

$$= \frac{-1}{3}e^{-2t}L^{-1}\left(\frac{1}{s^2}\right) + \frac{1}{3}e^{t}L^{-1}\left(1/s^2\right)$$

$$= \frac{-1}{3}e^{-2t}t + \frac{1}{3}e^{t}t$$

$$= \frac{t}{3}\left[e^{t}-e^{-2t}\right]$$

4) $L^{+1}\left\{\frac{\sin^2 t}{t}\right\}$

soln:

given $\frac{\sin^2 t}{t}$

$$L\left\{\frac{\sin^2 t}{t}\right\} = \int_s^\infty L(\sin^2 t)\,dt$$

$$= \int_s^\infty L\left(\frac{1-\cos 2t}{2}\right)dt$$

$$= \int_S^{\infty} L(1/2) - L\left(\frac{\cos 2t}{2}\right) dt$$

$$= \int_S^{\infty} 1/2\, L(1) - 1/2\, L(\cos 2t) dt$$

$$= \int_S^{\infty} 1/2 \cdot 1/s - 1/2 \cdot \frac{s}{s^2+2^2} dt$$

$$= \frac{1}{2}(\log s)_S^{\infty} - \frac{1}{2} \cdot \frac{1}{2}\left(\log(s^2+4)\right)_S^{\infty}$$

$$= \frac{1}{2}(0 - \log s) - \frac{1}{4}\, 0 - \log(s^2+4)$$

$$= \frac{-1}{2} \log s + \frac{1}{4} \log(s^2+4)$$

$$= \frac{1}{4}\left[-2\log s + \log(s^2+4)\right]$$

$$= \frac{1}{4}\left[\log(s^2+4) - \log s^2\right]$$

$$L\left\{\frac{\sin^2 t}{t}\right\} = \frac{1}{4} \cdot \log\left(\frac{s^2+4}{s^2}\right)$$

5) $L\{e^{-t} \cos 2t\}$

Soln :

$$L\{e^{-t} \cos 2t\}$$

$$L\{\cos 2t\} = \frac{s}{s^2+2^2}$$

Put $s = s+1$

$$L\{e^{-t} \cos 2t\} = \frac{s+1}{(s+1)^2+4}$$

$$L\{e^{-t} \cos 2t\} = \frac{s+1}{s^2+2s+5}$$

6) $\int_0^{\infty} \frac{e^{-3t} - e^{6t}}{t} dt$

Soln :

$$\int_0^{\infty} e^{-st}\left[\frac{e^{-3t} - e^{6t}}{t}\right] dt = L\left\{\frac{e^{-3t} - e^{6t}}{t}\right\}$$

$$= \int_S^{\infty} L\{e^{3t}\} - L\{e^{6t}\} ds$$

$$= \int_S^{\infty} \frac{1}{s+3} - \frac{1}{s-6} ds$$

$$= \left[\log(s+3) - \log(s-6)\right]_S^{\infty}$$

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$$= 0 - \left(\log(s+3) - \log(s-6)\right)_s^{\infty}$$

$$= \log\left(\frac{s-6}{s+3}\right)$$

put $s=0$

$$\int_0^{\infty} \frac{e^{-3t} - e^{6t}}{t}\,dt = \log 2$$

1) $t^3 - 3t^2 + 2$

Soln :

given $t^3 - 3t^2 + 2 \rightarrow (1)$

$$L(t^3 - 3t^2 + 2) = L(t^3) - L(3t^2) + L(2)$$

$$= L(t^3) - 3L(t^2) + 2L(1)$$

$$\therefore L(t^n) = \frac{n!}{s^{n+1}}$$

$$L(t^3 - 3t^2 + 2) = \frac{3!}{s^4} - 3\frac{2!}{s^3} + 2\cdot\frac{1}{s}$$

$$L(t^3 - 3t^2 + 2) = \frac{6}{s^4} - \frac{6}{s^3} + \frac{2}{s}$$

2) $\cos^2 3t$

Soln : given $\cos^2 3t \rightarrow (1)$

$$\cos^2 3t = \frac{1 + \cos 2(3t)}{2}$$

$$\cos^2 3t = \frac{1}{2}(1 + \cos 6t)$$

$$L(\cos^2 3t) = \frac{1}{2} L(1 + \cos 6t)$$

$$= \frac{1}{2}\left[L(1) + L(\cos 6t)\right]$$

$$\therefore L(t^n) = \frac{n!}{s^{n+1}}, \quad L(\cos at) = \frac{s}{s^2+a^2}$$

$$L(\cos^2 3t) = \frac{1}{2}\left[\frac{1}{s} + \frac{s}{s^2+6^2}\right]$$

$$= \frac{1}{2}\left[\frac{s^2+6^2+s^2}{s(s^2+6^2)}\right]$$

$$L(\cos^2 3t) = \frac{1}{2}\left[\frac{36+2s^2}{s(s^2+36)}\right]$$

$$L(\cos^2 3t) = \frac{2}{2}\left[\frac{s^2+18}{s(s^2+36)}\right]$$

$$L(\cos^2 3t) = \frac{s^2+18}{s(s^2+36)}$$

3) Find $L(\cos t \cdot \cos 2t)$

Soln :

given $L(\cos 2t \cdot \cos t) \to (1)$

$\cos t \cos 2t = \frac{1}{2}[\cos 3t + \cos t]$

$L[\cos t \cdot \cos 2t] = \frac{1}{2} L[\cos 3t + \cos t]$

$= \frac{1}{2}[L(\cos 3t) + L(\cos t)]$

$L(\cos at) = s/s^2+a^2$

$L(\cos t \cdot \cos 2t) = \frac{1}{2}\left[\frac{s}{s^2+3^2} + \frac{s}{s^2+1^2}\right]$

$= \frac{1}{2}\left[\frac{s(s^2+1) + s(s^2+9)}{(s^2+9)(s^2+1)}\right]$

$= \frac{s}{2}\left[\frac{s^2+1+s^2+9}{(s^2+9)(s^2+1)}\right]$

$= \frac{2s}{2}\left[\frac{s^2+5}{(s+9)(s^2+1)}\right]$

$L(\cos t \cdot \cos 2t) = \frac{s(s^2+5)}{(s^2+9)(s^2+1)}$

1) $\frac{1}{(s-3)^5}$

Soln :

$L^{-1}\left[\frac{1}{(s-3)^5}\right] = e^{3t} L^{-1}\left[\frac{1}{s^5}\right]$

$= \frac{e^{3t}}{24} L^{-1}\left[\frac{24}{s^5}\right]$

$L^{-1}\left[\frac{1}{(s-3)^5}\right] = \frac{e^{3t} \cdot 6^4}{24}$

2) $\frac{1}{(s^2+4)s}$

Soln :

$L^{-1}\left[\frac{1}{(s^2+4)s}\right] = \int_0^t L^{-1}\left[\frac{1}{s^2+2^2}\right] dt$

$= +\frac{1}{2} \int_0^t L^{-1}\left[\frac{2}{s^2+2^2}\right] dt$

$= \frac{1}{2} \int_0^t L^{-1}[\sin 2t \, dt]$

$$= \frac{1}{2}\left[\frac{-\cos 2t}{2}\right]_0^t$$

$$= \frac{1}{4}\left[-\cos 2t - (-\cos 0)\right]$$

$$L^{-1}\left[\frac{1}{(s^2+4)s}\right] = \frac{1}{4}(1-\cos 2t)$$

---

3) $L^{-1}\left[\frac{1}{(s^2+9)^2}\right]$

Soln :

$$L^{-1}\left[\frac{s}{s(s^2+9)^2}\right] = \int_0^t L^{-1}\left[\frac{3}{(s^2+9)^2}\right]dt$$

$$L^{-1}\left[\frac{s}{(s^2+9)^2}\right]$$

$$F'(s) = \frac{s}{(s^2+a)^2}$$

$$F(s) = \frac{-1}{2(s^2+a)}$$

$$L^{-1}[F'(s)] = -t L^{-1}F(s)$$

$$= -t\, L^{-1}\frac{-1}{2(s^2+9)}$$

$$= \frac{t}{2} L^{-1}\frac{1}{s^2+3^2}$$

$$= \frac{t}{2}\left[\frac{\sin 3t}{3}\right]$$

$$= \frac{1}{6}\int_0^t t\sin 3t\,dt$$

$$= \frac{1}{6}\left[\left(-t\frac{\cos 3t}{3}\right]_0^t - \int_0^t -\frac{\cos 3t}{3}dt\right]$$

$$= \frac{1}{6}\left[\left[\frac{-t\cos 3t}{3}\right] + \left[\frac{\sin 3t}{9}\right]_0^t\right]$$

$$= \frac{1}{6}\left[\frac{-t\cos 3t}{3} + \frac{\sin 3t}{9}\right]$$

$$= \frac{1}{54}\left[-3t\cos 3t + \sin 3t\right]$$

Ex :

solve the eqn $\frac{d^2y}{dt^2}+2\frac{dy}{dt}-3y=\sin t$ given that $y \cdot \frac{dy}{dt}=0$ when $t=0$.

soln: given $\frac{d^2y}{dt^2}+2\frac{dy}{dt}-3y \ \sin t \rightarrow (1)$

with $y(0)=0$, $y'(0)=0$

Applying L.T on both sides, we've,

$L(y'')+2L(y')-3L(y)=L(\sin t) \rightarrow (2)$

W.K.T

$L(y'')=s^2L(y)-s(y)0-y'(0)$

$L(y')=sL(y)-y(0)$

$(2) \Rightarrow s^2L(y)-sy(0)-y'(0)+2[sL(y)-y(0)]-3L(y) = L(\sin t)$

$L(y)[s^2+2s-3]+y(0)[-s-2]-y'(0)=L(\sin t)$

$L(y)[s^2+2s-3]+0 = \frac{1}{s^2+1^2}$

$L(y)=\frac{1}{(s^2+1)(s^2+2s-3)}$

$y=L^{-1}\frac{1}{(s^2+2s-3)(s^2+1)}$

$=L^{-1}\left[\frac{1}{(s-1)(s+3)(s^2+1)}\right] \rightarrow (3)$

Applying partial fraction, we've

$\frac{1}{(s-1)(s+3)(s^2+1)}=\frac{A}{s-1}+\frac{B}{s+3}+\frac{Cs+D}{s^2+1}$

$1=A(s+3)(s^2+1)+B(s-1)(s^2+1)+(Cs+D)(s-1)(s+3)$

Put $s=1$

$1=A(4)(2)+0$

$A=1/8$

Put $s=-3$,

$1=0+B(-4)(10)$

$B=-1/40$

put $S=0$

$$1 = \frac{1}{8}(5)(1) + (-1)(1)\left(\frac{1}{40}\right) + D(-1)(3)$$

$$1 - \frac{15}{40} + \frac{1}{40} - 3D$$

$$\Rightarrow 1 - \frac{16}{40} = -3D$$

$$\frac{3}{5} = -3D$$

$$D = -1/5$$

put $S=-1$,

$$1 = \frac{1}{8}(2)(2) - \frac{1}{40}(2)(2) + \left(C(-1) - \frac{1}{5}\right)(2)(2)$$

$$1 = \frac{1}{2} + \frac{1}{10} + 4C + 4/5$$

$$1 = \frac{5+1+8}{10} + 4C$$

$$1 = \frac{14}{10} + 4C$$

$$-\frac{4}{10} = 4C$$

$$C = -1/10$$

$$\left[\frac{1}{(S-1)(S+3)(S^2+1)}\right] = \frac{1}{2(S-1)} - \frac{1}{40(S+3)} + \frac{-\frac{1}{10}S - \frac{1}{5}}{S^2+1}$$

Applying I.L.T,

$$L^{-1}\left[\frac{1}{(S-1)(S+3)(S^2+1)}\right] = L^{-1}\left[\frac{1}{2(S-1)}\right] - L^{-1}\left[\frac{1}{40(S+3)}\right] - L^{-1}\left[\frac{S}{10(S^2+1)}\right] - L^{-1}\left[\frac{1}{5(S^2+1)}\right]$$

$$= \frac{1}{8}L^{-1}\left[\frac{1}{S-1}\right] - \frac{1}{40}\left[L^{-1}\left(\frac{1}{S+3}\right) - \frac{1}{10}L^{-1}\left(\frac{S}{S^2+1}\right) - \frac{1}{5}L^{-1}\left(1/S^2+1\right)\right]$$

$$y(t) = \frac{1}{8}e^{t} - \frac{1}{40}e^{-3t} - \frac{1}{10}\cos t - \frac{1}{5}\sin t.$$

---

2) S.T the solution of the differential equ $\frac{d^2y}{dt^2} + 4y = A\sin kt$ which is such that the $y=0$ & $\frac{dy}{dt}=0$ when $\frac{dy}{dt}=0$ when $t=0$ is $y = \frac{A\sin kt - k/2 \sin 2t}{4-k^2}$ if $k \neq 2$

$$= \frac{A(\sin 2t - 2t\cos 2t)}{8} \text{ if } k=2.$$

Soln:

given $\frac{d^2y}{dt^2} + 4y = A\sin kt$

It can be written as,

Applying on both sides

$L(y'') + 4L(y) = L(A\sin kt) \to (1)$

W.K.T

$L(y'') = s^2 L(y) - sy(0) - y'(0)$

$(1) \Rightarrow s^2 L(y) - sy(0) - y'(0) + 4L(y) = L(A\sin kt)$

$L(y)(s^2+4) = A(k/s^2+k^2) \quad \because (y(0) = y'(0) \ldots$

$$L(y) = \frac{Ak}{(s^2+k^2)(s^2+4)}$$

$$y(t) = L^{-1}\left[\frac{Ak}{(s^2+k^2)(s^2+4)}\right]$$

$$y(t) = Ak\, L^{-1}\left[\frac{1}{(s^2+k^2)(s^2+4)}\right]$$

case(i): If $k \neq 2$

$$y = Ak\, L^{-1}\left[\frac{1}{(s^2+k^2)(s^2+4)}\right]$$

Applying partial fractions

$$= AkL^{-1}\left[\frac{\frac{1}{s^2+k} - \frac{1}{s^2+k^2}}{k^2-4}\right]$$

$$= \frac{Ak}{k^2-4} L^{-1}\left[\frac{1}{s^2+4} - \frac{1}{s^2+k^2}\right]$$

$$= \frac{Ak}{k^2-4}\left[L^{-1}\left(\frac{1}{s^2+4}\right) - L^{-1}\left(\frac{1}{s^2+k^2}\right)\right]$$

$$= \frac{Ak}{4-k^2}\left[\frac{\sin kt}{k} - \frac{\sin 2k}{2}\right]$$

$$y = \frac{A}{4-k^2}\left[\sin kt - k/2 \sin 2t\right], \text{ if } k \neq 2$$

case(ii): if $k = 2$

$$y = 2A\, L^{-1}\left[\frac{1}{(s^2+0)(s^2+4)}\right]$$

$$= 2AL^{-1}\left[\frac{1}{(s^2+4)(s^2+4)}\right]$$

$$= 2AL^{-1}\left[\frac{1}{(s^2+4)^2}\right]$$

$$= 2A L^{-1}\left[\frac{1}{s}\cdot\frac{s}{(s^2+4)^2}\right]$$

$$= 2A L^{-1}\int_0^t L^{-1}\frac{s}{(s^2+4)^2}\,dt$$

$$= 2A\int_0^t t\, L^{-1}\left[\frac{1}{2(s^2+4)}\right]dt$$

$$= \frac{2A}{2}\int_0^t t\ \frac{\sin 2t}{2}\,dt$$

$$= \frac{A}{2}\int_0^t t\sin 2t\,dt$$

$$= \frac{A}{2}\int_0^t t\sin 2t\,dt$$

$$= \frac{A}{2}\left[\left(-\frac{t\cos 2t}{2}\right)_0^t - \int_0^t \frac{\cos 2t}{2}\,dt\right]$$

$$= \frac{A}{2}\left[-t\ \frac{\cos 2t}{2} + \frac{\sin 2t}{4}\right]$$

$$= \frac{A}{2}\left[\frac{-2t\cos 2t + \sin 2t}{4}\right]$$

$$= \frac{A}{8}\left[\sin 2t - 2t\cos 2t\right] \text{ if } \quad K = 2.$$