Vector Calculus and Fourier Series

Unit-I

Vector differentiation - velocity & accelaration - Vector & scalar fields - Gradient of a vector - Directional derivative - divergence & curl of a vector solinoidal & irrational vectors - Laplacian double operation - simple problems

Unit - II

Vector integration - Tangential line integral conservative force field - scalar Potential - work done by a force - Normal Surface integral - volume integral - Simple Problems

Unit - III

Gauss divergence Theorem - stoke's Theorem - Green's Theorem - Simple problems & Verification of the theorem for simple Problems

Unit - IV

Fourier Series - definition - Fourier Series expension of Periodic functions with Period 2π and Period $2a$ - use of odd & even functions in Fourier series.

Unit - V

Half-range Fourier Series - definitio[n]

Development in cosine series & in sine series

change of integral - combination of series

Text Book(s)

1) M.L.Khanna, vector calculus, Jai Prakash Nath & co., 8th edition, 1986.

2) S. Narayanan, T.K. Manicavasac[hagom] Pillai calculus, vol - III S. Viswanath PVt Vijay Nichole PVt Ltd 2004.

Unit-I

12/19

∴ VECTOR DIFFERENTIATION

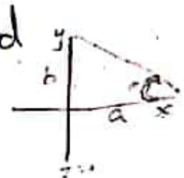
Vector valued functions of a single scalar variable:

W.K.T the eqn of an ellipse and Parabola are,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 ; \quad z=0 \quad \text{and} \quad y^2 = 4ax ; z=0.$$

In Parametric form these can be written as,

$x = a\cos t$, $y = b\sin t$, $z=0$ and



$x = at^2, y = 2at, z=0$.

Any vector $\vec{r}$ can be written as,

$$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k} \qquad \vec{i}.\vec{i}=1, \vec{j}.\vec{j}=1, \vec{k}.\vec{k}=1$$

where $\vec{i}, \vec{j}, \vec{k}$ are mutually $\perp^r$ unit vector

$\therefore \vec{r} = a\cos t\vec{i} + b\sin t\vec{j} + 0\vec{k} = f(t)$ for an ellipse

$\vec{r} = at^2\vec{i} + 2at\vec{j} + 0\vec{k} = f(t)$ for a Parabola

we say that $\vec{r}$ is a vector function of a scalar variable

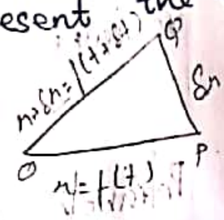
Differentiation of a vector function of single variable 't':

Let $\vec{r} = f(t)$ be a single value of d and continuous vector function of a scalar variable t.

corresponding to any value of t

scalar variable t let $\vec{OP}$ represent the v

$\vec{r}$ w.r to origin O.



$$\therefore \vec{r} = \vec{f}(t) \rightarrow ①$$

Again corresponding to the value t

where δt is small let $\vec{OQ}$ represent the

vector $\vec{r} + \delta\vec{r}$

$$\vec{r} + \delta\vec{r} = \vec{f}(t + \delta t) \rightarrow ②$$

Thus corresponding to small increm

δt in t there is corresponding increme

$\delta\vec{r}$ in $\vec{r}$ where

$$\delta\vec{r} = \vec{OQ} - \vec{OP} = \vec{PQ}$$

① - ② we get,

$$(\vec{r} + \delta\vec{r}) - \vec{r} = \vec{f}(t+\delta t) - \vec{f}(t)$$

$$\frac{\delta\vec{r}}{\delta t} = \frac{\vec{f}(t + \delta t) - \vec{f}(t)}{\delta t}$$

$\div$ by δt on both side.,

$$\frac{\delta\vec{r}}{\delta t} = \frac{\vec{f}(t+\delta t) - \vec{f}(t)}{\delta t}$$

By taking $\lim_{\delta t \to 0}$ on both side.,

$$\lim_{\delta t \to 0} \frac{\delta\vec{r}}{\delta t} = \lim_{\delta t \to 0} \frac{\vec{f}(t+\delta t) - \dots}{\delta t}$$

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$$\frac{d\vec{r}}{dt} = \frac{d\vec{f}}{dt} = \vec{f}(t)$$

$\frac{d\vec{r}}{dt}$ is called the differential co-efficient of $\vec{r}$. w.r to t.

Again $\frac{d\vec{r}}{dt}$ is also a vector function of scalar variable t and we can find its differential co-efficient w.r to 't'.

If this derivative exists then it will we denoted by $\frac{d^2\vec{r}}{dt^2}$

$||^{ly}$ we can find differential co-efficients of higher order.

velocity :

If the scalar variable t stands for time then,

$\vec{PQ} = \delta\vec{r}$ gives the displacement of the point p in the time δt (or)

$\frac{\delta r}{\delta t}$ gives the average velocity during the interval δt.

Taking the limit $\delta t \to 0$ (ie) $Q \to P$ and chord PQ becomes tangent at P.

∴ we get velocity at p be,

$$\vec{v} = \lim_{\delta t \to 0} \frac{\delta r}{\delta t} = \frac{dr}{dt}$$

$\vec{v} = \frac{d\vec{r}}{dt}$ is the velocity which is a scalar variable of 't'.

Accelaration

Accelaration is the rate of change of velocity

$$a = \frac{dv}{dt} = \frac{d}{dt}\left(\frac{dr}{dt}\right)$$

$a = \frac{d^2 r}{dt^2}$ is the accelaration which is a vector function of scalar variable of 't'.

Theorems on differentiation

1) P.T $d/dt(A \pm B) = d/dt(A) \pm d/dt(B)$

1)

let $f = A \pm B$

$$f(t) = A(t) \pm B(t) \quad -①$$

$$f(t+\delta t) = A(t+\delta t) \pm B(t+\delta t) \quad ②$$

② − ①

$$f(t+\delta t) - f(t) = A(t+\delta t) \pm B(t+\delta t) - [A(t) \pm B(t)]$$

$$= A(t+\delta t) - A(t) \pm [B(t+\delta t) - B(t)]$$

$\div$ by δt & $\lim_{\delta t \to 0}$

$$\lim_{\delta t \to 0} \frac{f(t+\delta t) - f(t)}{\delta t} = \lim_{\delta t \to 0} \frac{A(t+\delta t) - A(t)}{\delta t} \pm \lim_{\delta t \to 0} \frac{B(t+\delta t) - B(t)}{\delta t}$$

$$\therefore \frac{df}{dt} = \frac{dA}{dt} \pm \frac{dB}{dt}$$

$$\frac{d}{dt}(f) = d/dt\,(A) \pm d/dt\,(B)$$

$$d/dt\,(A \pm B) = d/dt\,(A) \pm d/dt\,(B)$$

2) P.T. $d/dt\,(A \cdot B) = A\frac{d}{dt}(B) + B\frac{d}{dt}(A)$

2) let $f = A \cdot B$

$$f(t) = A(t) \cdot B(t)$$

$$f(t+\delta t) = A(t+\delta t) \cdot B(t+\delta t)$$

$$f(t+\delta t) - f(t) = A(t+\delta t) \cdot B(t+\delta t) - A(t) \cdot B(t)$$

Add & subtract $A(t+\delta t) \cdot B(t)$ we get.

$$f(t+\delta t) - f(t) = A(t+\delta t) \cdot B(t+\delta t) - A(t) \cdot B(t) + A(t+\delta t) \cdot B(t) - A(t+\delta t) \cdot B(t)$$

$$= A(t+\delta t)\left[B(t+\delta t) - B(t)\right] + B(t)\left[A(t+\delta t) - A(t)\right]$$

$\div \delta t$ & taking $\lim_{\delta t \to 0}$ on both side.,

$$\lim_{\delta t \to 0} \frac{f(t+\delta t) - f(t)}{\delta t} = \lim_{\delta t \to 0} A(t+\delta t)\left[\frac{B(t+\delta t) - B(t)}{\delta t}\right] + \lim_{\delta t \to 0} B(t)\left[\frac{A(t+\delta t) - A(t)}{\delta t}\right]$$

$\delta t \to 0$, $\delta 0$, $\delta t = 0$, $A(t)$

$$\frac{df}{dt} = A(t)\frac{dB}{dt} + B(t)\frac{dA}{dt}$$

$$\frac{d}{dt}f = A(t)\frac{d}{dt}B + B(t)\frac{d}{dt}A$$

$$\frac{d}{dt}(A \cdot B) = A\frac{d}{dt}(B) + B$$

3) P.T $\frac{d}{dt}(A \times B) = A \times \frac{dB}{dt} + \frac{dA}{dt} \times$

let $A = A_1\vec{i} + A_2\vec{j} + A_3\vec{k}$.

$$\frac{dA}{dt} = \frac{dA_1}{dt}\vec{i} + \frac{dA_2}{dt}\vec{j}$$

compain in $\frac{dA}{dt} \times B$

$B = B_1\vec{i} + B_2\vec{j} + B_3\vec{k}$.

$$\frac{dB}{dt} = \frac{dB_1}{dt}\vec{i} + \frac{dB_2}{dt}\vec{j}$$

w.k.t

$$A \times B = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{vmatrix}$$

On diff w.k.t on both side.

$$\frac{d}{dt} A \times B = \frac{d}{dt}\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{vmatrix}$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{dA_1}{dt} & \frac{dA_2}{dt} & \frac{dA_3}{dt} \\ B_1 & B_2 & B_3 \end{vmatrix}$$

$$\begin{vmatrix} \vec{i} & \vec{j} \\ A_1 & A_2 \\ \frac{dB_1}{dt} & \frac{dB_2}{dt} \end{vmatrix}$$

$$\frac{d}{dt}(A \times B) = \frac{dA}{dt} \times B + A \times \frac{dB}{dt}$$

$\frac{d}{dt}(\phi A) = \phi \frac{dA}{dt} + \frac{d\phi}{dA} A$ where ϕ is a scalar function of 't'.

P.T $\frac{d}{dt}[A\ B\ C] = \frac{dA}{dt}(B\times C) + A\left(\frac{dB}{dt}\times C\right) + A\left(B\times\frac{dC}{dt}\right)$ or

$$= \left[\frac{dA}{dt}\ B\ C\right] + \left[A\ \frac{dB}{dt}\ C\right] + \left[A\ B\ \frac{dC}{dt}\right]$$

$$[A\ B\ C] = A\cdot(B\times C)$$

on diff w.r to 't'.

$$\frac{d}{dt}[A\ B\ C] = \frac{d}{dt} A\cdot(B\times C)$$

$$= A\cdot\frac{d}{dt}(B\times C) + (B\times C)\cdot\frac{dA}{dt}$$

$$= A\cdot\left[\frac{dB}{dt}\times C + B\times\frac{dC}{dt}\right] + (B\times C)\frac{dA}{dt}$$

$$= A\cdot\left(\frac{dB}{dt}\times C\right) + \left(B\times\frac{dC}{dt}\right)\cdot A + (B\times C)\frac{dA}{dt}$$

$$= \frac{dA}{dt}(B\times C) + A\left(\frac{dB}{dt}\times C\right) + A\left(B\times\frac{dC}{dt}\right)$$

$$= \left[\frac{dA}{dt}\ B\ C\right] + \left[A\ \frac{dB}{dt}\ C\right] + \left[A\ B\ \frac{dC}{dt}\right]$$

Function of a function

let f be a vector function of scalar variable t and t be a continous function of another scalar variable u

i.e) $t = \phi(u)$ then f is a derivable function of

i.e) $$\frac{df}{du} = \frac{df}{dt} \cdot \frac{dt}{du}$$

$$= \frac{df}{dt} \cdot \frac{d\phi}{du} \quad \text{where } t = \phi(u)$$

let δt be a small increment in t which produces corresponding increment δf and δu in f and u respectively and also when $\delta t \to$ both δf & $\delta u \to 0$

Also, $$\frac{\delta f}{\delta u} = \frac{\delta f}{\delta t} \cdot \frac{\delta t}{\delta u}$$

Proceeding to limit when $\delta t \to 0$, consequently $\delta u \to 0$ we get

$$\frac{df}{du} = \frac{df}{dt} \cdot \frac{d\phi}{du}$$

constant vector:

we know that a vector has be magnitude and direction. Hence a vector will change when either its magnitude changes or its direction change or both changed but when a vector has both its magnitude and direction constant, we will say that the vector is constant.

III'y a vector may have only constant magnitude or only constant direction.

Derivative of a constant vector function is zero.

let f be a constant vector function of scalar variable t,

As t changes from t to $t+\delta t$, where is no change in f,

$$\delta f = 0$$

$\div$ by δt and $\lim_{\delta t \to 0}$; we get

$$\lim_{\delta t \to 0} \frac{\delta f}{\delta t} = 0$$

$$\frac{df}{dt} = 0.$$

Note:

i) condition for a vector function f(t) to be a constant magnitude then $f \cdot \frac{df}{dt} = 0$

ii) if f(t) is a constant vector function of constant magnitude then f and $\frac{df}{dt}$ are ope $\perp^r$ to each other

ii) condition for a vector funtion

$f(t)$ to have constant direction

then $f \times \frac{df}{dt} = 0$

1) A Particle moves along the curve,

6) a) $x = e^{-t}$, $y = 2\cos 3t$, $z = 2\sin 3t$

b) $x = 4\cos t$, $y = 4\sin t$, $z = 6t$.

c) $x = a\cos t$, $y = a\sin t$, $z = at\tan$

Determine the velocity & accelarati at any time t & their magnitude at t

let r be the position vector of any pt $p(x, y, z)$ on the curve

$$\therefore \vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$$

where $\vec{i}, \vec{j}, \vec{k}$ mutually $\perp^r$ unit vector.

a) $\vec{r} = e^{-t}\vec{i} + 2\cos 3t\,\vec{j} + 2\sin 3t\,\vec{k}$.

velocity $\Rightarrow \vec{v} = \frac{d\vec{r}}{dt} = -e^{-t}\vec{i} - 2\sin 3t\,(3)\vec{j} + 2\cos 3t\,(3)\vec{k}$.

$= -e^{-t}\vec{i} - 6\sin 3t\,\vec{j} + 6\cos 3t\,\vec{k}$

$\vec{v}$ $t=0 \Rightarrow -\vec{i} - 0\vec{j} + 6\vec{k}$.

$|\vec{v}| = \sqrt{1^2+6^2} = \sqrt{37}$.

Accelaration

$\vec{a} = \frac{d\vec{v}}{dt} = e^{-t}\vec{i} - 6\cos 3t\,(3)\vec{j} -$

$6\sin 3t\,(3)\,\vec{k}$

$$= e^{-t}\vec{i} - 18\cos 3t\,\vec{j} - 18\sin 3t\,\vec{k}$$

$$= \sqrt{1^2 + 18^2}$$

$$= \sqrt{1+341} = \sqrt{342}$$

b) $$\vec{r} = 4\cos t\,\vec{i} + 4\sin t\,\vec{j} + 6t\vec{k}$$

velocity $\Rightarrow \vec{v} = \dfrac{d\vec{r}}{d\vec{t}}$

$$= -4\sin t\,\vec{i} + 4\cos t\,\vec{j} + 6\vec{k}$$

$$\vec{v} \Rightarrow t = 0 \Rightarrow -4(\sin 0)\vec{i} + 4(\cos 0)\vec{j} + 6\vec{k}$$

$$= -4(0)\vec{i} + 4(1)\vec{j} + 6\vec{k}$$

$$|\vec{v}| = \sqrt{4^2+6^2} = \sqrt{16+36}$$

$$= \sqrt{52} \Rightarrow \sqrt{13\times 4} \Rightarrow 2\sqrt{13}$$

Accelaration

$$\vec{a} = \frac{d\vec{v}}{dt} = -4\sin t\,\vec{i} + 4\cos t\,\vec{j} + 6\vec{k}$$

$$= -4\cos t\,\vec{i} - 4\sin t\,\vec{j} + 0$$

$t=0$

$$= -4(1)\vec{i} - 4(0) + 0$$

$$= \sqrt{-4^2}$$

$$= \sqrt{16}$$

accelaration $= 4$

c) $$\vec{r} = a\cos t\,\vec{i} + a\sin t\,\vec{j} + at\tan\alpha\,\vec{k}$$

velocity $\Rightarrow \vec{v} = \dfrac{d\vec{r}}{dt} = -a\sin t\,\vec{i} + a\cos t\,\vec{j} + a\tan\alpha\,\vec{k}$

$$\vec{v} \Rightarrow t=0 \Rightarrow -a(0)\vec{i} + a(1)\vec{j} + a\tan\alpha\,\vec{k}$$

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$$|\vec{v}| = \sqrt{0^2+a^2+a^2\tan^2\alpha}$$
$$= a\sqrt{1+\tan^2\alpha}$$
$$= a\sqrt{\sec^2\alpha}$$
$$|\vec{v}| = a\sec\alpha$$

Accelaration

$$\vec{a} = \frac{d\vec{v}}{dt} = -a\cos t\,\vec{i} - a\sin t\,\vec{j} + 0$$

$$t=0 \quad = -a(1)\vec{i} - a(0)\vec{j} + 0$$
$$= -a\vec{i}$$
$$\vec{a} = \sqrt{a^2+0^2+0^2}$$
$$= \sqrt{a^2}$$

accelaration $= a$

6) P.T $\frac{d}{dt}(\underbrace{A}_{A} \times \underbrace{(B\times C)}_{B}) = \frac{dA}{dt} \times (B\times C) + A \times \frac{d}{dt}$

$+ A \times \left(B \times \frac{dC}{dt}\right)$

12/12/19 A particle moves along the curve $x=2t^2$,

1)

7) $y=t^2-4t$, $z=3t-5$ where t is a time find the component of its velocity & acc accelaration at time $t=1$ in the direction $\vec{i}-3\vec{j}+2\vec{k}$.

$$\vec{v} = 4t\,\vec{i} + (2t-4)\vec{j} + 3\vec{k}.$$

$$\vec{v} \text{ at } t=1 \Rightarrow 4\vec{i} + (-2)\vec{j} + 3\vec{k}$$

Hence the component of velocity vector in the direction of $\vec{i}-3\vec{j}+2\vec{k}$.

$$= \frac{(4\vec{i}-2\vec{j}+3\vec{k})\cdot(\vec{i}-3\vec{j}+2\vec{k})}{\sqrt{1^2+3^2+2^2}}$$

Projection of B in the direction by A $= \frac{A \cdot B}{|B|}$

$$= \frac{4+6+6}{\sqrt{1+9+4}}$$

$$= \frac{16}{\sqrt{14}}$$

accelaration:

$$\vec{a} = \frac{d\vec{v}}{dt}$$

$$= \frac{d}{dt}\left(4t\vec{i}+(2t-4)\vec{j}+3\vec{k}\right)$$

$$\vec{a} \text{ at } t=1 = 4\vec{i}+2\vec{j}$$

Hence the component of accelaration vector in the direction $\vec{i}-3\vec{j}+2\vec{k}$

$$\frac{(\vec{i}-3\vec{j}+2\vec{k})\;(4\vec{i}+2\vec{j})}{\sqrt{1+3^2+2^2}}$$

$$= \frac{4-6}{\sqrt{14}} = \frac{-2}{\sqrt{14}}.$$

8) A particle moves along the curve $x=3t^2$, $y=t^2-2t$, $z=t^3$ find the velocity & accelariation at $t=1$

$$\vec{r} = 3t^2\vec{i}+(t^2-2t)\vec{j}+t^3\vec{k}$$

$$\vec{v} = \frac{d\vec{r}}{dt}$$

$$= 6t\vec{i}+(2t-2)\vec{j}+3t^2\vec{k}$$

$$\vec{v} = t=1 \Rightarrow 6\vec{i}+(2(1)-2)\vec{j}+3(1)^2\vec{k}$$

$$= 6\vec{i}+0\vec{j}+3\vec{k}$$

$= 6\vec{i} + 3\vec{k}$

$|\vec{v}| = \sqrt{6^2+3^2}$

$= \sqrt{36+9}$

$= \sqrt{45} \Rightarrow \sqrt{5\times 9} \Rightarrow 3\sqrt{5}$

accelaration $\vec{a} = \dfrac{d\vec{v}}{dt}$

$= 6\vec{i} + 2\vec{j} + 6t\,\vec{k}$

$t=1 \Rightarrow 6(1)\vec{k} + 2\vec{j} + 6\vec{i}$

$\vec{a} = \sqrt{6^2+2^2+6^2}$

$= \sqrt{36+4+36}$

$\vec{a} = \cancel{\sqrt{456}}\ \sqrt{76} = 2\sqrt{19}$

1) 9) $\dfrac{d}{dt}[A \times (B \times C)] = \dfrac{dA}{dt} \times (B\times C) + A \times \left(\dfrac{dB}{dt} \times C\right)$
$+ A \times \left(B \times \dfrac{dC}{dt}\right)$

Let $A = A_1\vec{i} + A_2\vec{j} + A_3\vec{k}$

$\dfrac{dA}{dt} = \dfrac{dA_1}{dt}\vec{i} + \dfrac{dA_2}{dt}\vec{j} + \dfrac{dA_3}{dt}\vec{k}$

let, $B = B_1\vec{i} + B_2\vec{j} + B_3\vec{k}$

$\dfrac{dB}{dt} = \dfrac{dB_1}{dt}\vec{i} + \dfrac{dB_2}{dt}\vec{j} + \dfrac{dB_3}{dt}\vec{k}$

let $C = C_1\vec{i} + C_2\vec{j} + C_3\vec{k}$

$\dfrac{dC}{dt} = \dfrac{dC_1}{dt}\vec{i} + \dfrac{dC_2}{dt}\vec{j} + \dfrac{dC_3}{dt}\vec{k}$

W.K.T

$[A \times (B \times C)] = \begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix}$

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$$\frac{d}{dt}[A\times(B\times C)] = \frac{d}{dt}\begin{bmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{bmatrix}$$

$$= \begin{vmatrix} \frac{dA_1}{dt} & \frac{dA_2}{dt} & \frac{dA_3}{dt} \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix} + \begin{vmatrix} A_1 & A_2 & A_3 \\ \frac{dB_1}{dt} & \frac{dB_2}{dt} & \frac{dB_3}{dt} \\ C_1 & C_2 & C_3 \end{vmatrix}$$

$$+ \begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ \frac{dC_1}{dt} & \frac{dC_2}{dt} & \frac{dC_3}{dt} \end{vmatrix}$$

$$\frac{d}{dt}[A\times(B\times C)] = \frac{dA}{dt}\times(B\times C) + A\times\left(\frac{dB}{dt}\times C\right) + A\times\left(B\times\frac{dC}{dt}\right)$$

$\therefore$ Hence proved.

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10) If $A = \sin t\,\vec{i} + \cos t\,\vec{j} + t\vec{k}$.

$B = \cos t\,\vec{i} - \sin t\,\vec{j} - 3\vec{k}$.

$C = 2\vec{i} + 3\vec{j} - \vec{k}$.

Find $\frac{d}{dt}[A\times(B\times C)]$ at $t=0$

$$B\times C = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \cos t & -\sin t & -3 \\ 2 & 3 & -1 \end{vmatrix}$$

$$= \vec{i}(\sin t+9) + -\vec{j}(-\cos t+6) + \vec{k}(3\cos t+2\sin t)$$

$$A\times(B\times C) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \sin t & \cos t & t \\ \sin t+9 & \cos t-6 & 3\cos t+2\sin t \end{vmatrix}$$

$$= \vec{i}\,(3\cos^2 t + 2\sin t\cos t) - \vec{j}\,(t\cos t - 6t)$$

$= -\vec{j}\,[3\sin t\cos t + 2\sin \ldots -9t]$

$+\vec{K}\,[\sin t\,\cancel{\cos t} - 6\sin t - \sin t\cos t + \ldots \cos t$

$= \vec{i}\,[3\cos^2 t + \sin 2t \quad \overset{u\ v}{-t\cos t} + 6t] - \vec{j}$

$[3/2\sin 2t + 2\sin^2 t - t\sin t - 9t]$

$+\vec{K}\,(9\cos t - 6\sin t)$

$3\times 2\cos t\sin t$

$\frac{d}{dt}\,[A\times(B\times C)] = \vec{i}\,[6\cos t\sin t + \cos 2t \ldots$

$-\cos t + t\sin t + 6]$

$-\vec{j}\,[3/2\cos 2t\,(2) + 4\sin t\cos t - \sin t$

$-t\cos t - 9] + \vec{K}\,(-9\sin t - 6\cos t$

$= \vec{i}\,[-3\sin 2t + 2\cos 2t - \cos t + t\sin t + 6]$

$-\vec{j}\,[3\cos 2t + 2\sin 2t - \sin t - t\cos t - 9]$

$+\vec{K}\,(-9\sin t - 6\cos t)$

$t = 0$

$= \vec{i}\,[-3\sin 2(0) + 2\cos 2(0) - \cos(0) + 0\sin(0) + 6]$

$-\vec{j}\,[3\cos 2(0) + 2\sin 2(0) - \sin 0 - 0\cos 0 - 9]$

$+\vec{K}\,(-9\sin 0 - 6\cos 0)$

$= \vec{i}\,[2(1) - 1 + 6] - \vec{j}\,[3 - 9] + \vec{K}\,[-6]$

$= 7\vec{i} + 6\vec{j} - 6\vec{K}$

2) If $\vec{r} = a\cos t\,\vec{i} + a\sin t\,\vec{j} + at\tan\alpha\,\vec{k}$

ii) Find $\left|\frac{dr}{dt} \times \frac{d^2r}{dt^2}\right|$ & $\left[\frac{dr}{dt}\ \frac{d^2r}{dt^2}\ \frac{d^3r}{dt^3}\right]$

$\left|\left[\dot{\vec{r}} \times \ddot{\vec{r}}\right]\right|$ & $\left[\dot{\vec{r}}\ \ddot{\vec{r}}\ \dddot{\vec{r}}\right] \rightarrow$ Box Product

$$\vec{r} = a\cos t\,\vec{i} + a\sin t\,\vec{j} + at\tan\alpha\,\vec{k}.$$

$$\dot{\vec{r}} = -a\sin t\,\vec{i} + a\cos t\,\vec{j} + a\tan\alpha\,\vec{k}$$

$$\ddot{\vec{r}} = -a\cos t\,\vec{i} - a\sin t\,\vec{j}$$

$$\dddot{\vec{r}} = a\sin t\,\vec{i} - a\cos t\,\vec{j}$$

$$\dot{\vec{r}} \times \ddot{\vec{r}} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -a\sin t & a\cos t & a\tan\alpha \\ -a\cos t & -a\sin t & 0 \end{vmatrix}$$

$$\dot{\vec{r}} \times \ddot{\vec{r}} = \vec{i}\left[0 + a^2\sin t\tan\alpha\right] - \vec{j}\,(0 + a^2\cos t\tan\alpha) + \vec{k}\,(a^2\sin^2 t + a^2\cos^2 t)$$

$$= a^2\tan\alpha\sin t\,\vec{i} - a^2\tan\alpha\cos t\,\vec{j} + a^2\vec{k}$$

$$|\dot{\vec{r}} \times \ddot{\vec{r}}| = \sqrt{a^4\tan^2\alpha\sin^2 t + a^4\tan^2\alpha\cos^2 t + a^4}$$

$$= \sqrt{a^4\tan^2\alpha\,(\sin^2 t + \cos^2 t) + a^4}$$

$$= \sqrt{a^4(\tan^2\alpha + 1)}$$

$$= \sqrt{a^4\sec^2\alpha}$$

$$= a^2\sec\alpha.$$

$$\left[\dot{\vec{r}}\ \ddot{\vec{r}}\ \dddot{\vec{r}}\right] = \begin{vmatrix} -a\sin t & a\cos t & a\tan\alpha \\ -a\cos t & -a\sin t & 0 \\ a\sin t & -a\cos t & 0 \end{vmatrix}$$

$= -a\sin t\,(0-0) - a\cos t\,(0-0) +$
$a\,\tan\alpha\,(a^2\cos^2 t + a^2\sin^2 t)$

$= 0 + 0 + a\tan\alpha.\,(a^2(\cos^2 t + \sin^2 t)$

$= a^3\tan\alpha.$

1) Scalar point function

If corresponding to each point P of the regan ~~reject~~ R of space there correspondes a scalar denoted by $\phi(P)$ then ϕ is said to be a scalar function for the regan R. example length, only no direction

2) vector point function

If corresponding to each point P of a regan R of space there corresponds a vector defined by $f(p)$ then f is called a vector point function for the regan R. example velocity

16/12/19 vector differential operation (∇)

$$\nabla \vec{i}\frac{\partial\phi}{\partial x}+\vec{j}\frac{\partial\phi}{\partial y}+\vec{k}\frac{\partial\phi}{\partial z}=\Sigma\vec{i}\frac{\partial}{\partial x}$$

Gradient of a scalar function

If $f(x,y,z)$ be scalar

Point function and continous differentiable,

then

$$\text{Grad }\phi=\nabla\phi=\vec{i}\frac{\partial\phi}{\partial x}+\vec{j}\frac{\partial\phi}{\partial y}+\vec{k}\frac{\partial\phi}{\partial z}$$

$$\text{Grad }\phi=\Sigma\vec{i}\frac{\partial\phi}{\partial x}$$

$$\nabla(\phi\pm\psi)=\nabla\phi\pm\nabla\psi$$

Sum P.T $\text{Grad }(\phi\pm\psi)=\text{Grad }\phi\pm\text{Grad }\psi$

$$\text{Grad }(\phi\pm\psi)=\Sigma\vec{i}\frac{\partial}{\partial x}(\phi\pm\psi)$$

$$=\Sigma\vec{i}\left(\frac{\partial\phi}{\partial x}\pm\frac{\partial\psi}{\partial x}\right)$$

$$=\Sigma\vec{i}\frac{\partial\phi}{\partial x}\pm\Sigma\vec{i}\frac{\partial\psi}{\partial x}$$

$$=\text{Grad }\phi\pm\text{Grad }\psi$$

$$\nabla(\phi\pm\psi)=\phi\,\nabla\psi+\psi\,\nabla(\phi)$$

Sum : P.T $\text{Grad }(\phi\psi)=\phi\text{ Grad}(\psi)+\psi\text{ Grad}\phi$

$$=\Sigma\vec{i}\,\partial/\partial x\,(\phi\psi)$$

$$=\Sigma\vec{i}\left(\partial\phi/\partial x\,\psi+\phi\frac{\partial\psi}{\partial x}\right)$$

$$=\Sigma\vec{i}\phi\frac{\partial\psi}{\partial x}+\Sigma\vec{i}\psi\frac{\partial\phi}{\partial x}$$

$$= \phi \left(\Sigma \vec{i} \frac{\partial \psi}{\partial x} \right) + \psi \left(\Sigma \vec{i} \frac{\partial \phi}{\partial x} \right)$$

$$= \phi \text{ Grad } \psi + \psi \text{ Grad } \phi$$

P.T $\text{Grad} \left(\frac{\phi}{\psi} \right) \frac{u}{v} = \frac{\overset{vu'+uv'}{\psi \text{ Grad } \phi} - \phi \text{ Grad } \psi}{\psi^2}$

$$\text{Grad} \left(\frac{\phi}{\psi} \right) = \Sigma \vec{i} \frac{\partial}{\partial x} \left(\frac{\phi}{\psi} \right)$$

$$= \Sigma \vec{i} \left[\frac{\psi \frac{\partial \phi}{\partial x} - \phi \frac{\partial \psi}{\partial x}}{\psi^2} \right]$$

$$= \frac{1}{\psi^2} \left[\psi \Sigma \vec{i} \frac{\partial \phi}{\partial x} - \phi \Sigma \frac{\partial \psi}{\partial x} \right]$$

$$= \frac{1}{\psi^2} \left[\psi \text{ grad } \phi - \phi \text{ grad } \right.$$

11) Find grad (r^n) If $r^2 = x^2+y^2+z^2$.

12) P.T $\nabla (r^n) = n r^{n-2} \vec{r}$.

Gin

$$r^2 = x^2+y^2+z^2 \rightarrow (1)$$

p. Diff w.r to 'x'.

$$2r \frac{\partial r}{\partial x} = 2x \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r}$$

$||^{ly}$ P. diff w.r to 'y'.

$$2r \frac{\partial r}{\partial y} = 2y \Rightarrow \frac{\partial r}{\partial y} = \frac{y}{r}$$

P. diff w.r to 'z'

$$2r \frac{\partial r}{\partial z} = 2z \Rightarrow \frac{\partial r}{\partial z} = \frac{z}{r}$$

$$\nabla (r^n) = \Sigma \vec{i} \frac{\partial}{\partial x} (r^n)$$

$$= \Sigma \vec{i} \, n r^{n-1} \frac{\partial r}{\partial x}$$

$$= n r^{n-1} \sum \vec{i} \frac{\partial r}{\partial x}$$

$$= n r^{n-1} \sum \vec{i} \left(\frac{x}{r}\right)$$

$$= n r^{n-1} \left(x/r \, \vec{i} + y/r \, \vec{j} + z/r \, \vec{k} \right)$$

$$= \frac{n r^{n-1}}{r} \left(x\vec{i} + y\vec{j} + z\vec{k} \right)$$

$$\left[\vec{r} = x\vec{i} + y\vec{j} + z\vec{k} \right]$$

$$\nabla (r^n) = n r^{n-2} \vec{r}.$$

12) 13) If $\vec{a}$ is a constant vector & $\vec{r}$ is a point function then prove that $\nabla(\vec{a} \cdot \vec{r}) = \vec{a}$.

let $\vec{a} = a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$.

and also $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$.

$$\vec{a} \cdot \vec{r} = a_1 x + a_2 y + a_3 z.$$

$$\nabla(\vec{a} \cdot \vec{r}) = \sum \vec{i} \frac{\partial}{\partial x} (a_1 x + a_2 y + a_3 z)$$

$$= \vec{i} \frac{\partial}{\partial x} (a_1 x + a_2 y + a_3 z) + \vec{j} \frac{\partial}{\partial y} (a_1 x + a_2 y + a_3 z) + \vec{k} \frac{\partial}{\partial z} (a_1 x + a_2 y + a_3 z)$$

$$= \vec{i}(a_1) + \vec{j}(a_2) + \vec{k}(a_3)$$

$$= a_1\vec{i} + a_2\vec{j} + a_3\vec{k}.$$

$$\nabla(\vec{a} \cdot \vec{r}) = \vec{a}$$

13) Given $f(t) = (5t^2-3t)\vec{i} + 6t^3\vec{j} - 7t\vec{k}$

Evaluate $\int_{t=2}^{t=4} f(t)\,dt$

14) $\int_{t=2}^{t=4} f(t)\,dt = \int_{t=2}^{t=4} [(5t^2-3t)\vec{i} + 6t^3\vec{j} - 7t\vec{k}]$

$= \left[\left(\frac{5t^3}{3} - \frac{3t^2}{2}\right)\vec{i} + \frac{6t^4}{4}\vec{j} - \frac{7t^2}{2}\vec{k}\right]_2^4$

$= \left[\left(\frac{5(4)^3}{3} - \frac{3(4)^2}{2}\right)\vec{i} + \frac{6(4)^4}{4}\vec{j} - \frac{7(4)^2}{2}\right]$

$- \left[\left(\frac{5(2)^3}{3} - \frac{3(2)^2}{2}\right)\vec{i} + \frac{6(2^4)}{4}\vec{j} - \frac{7(2)^2}{2}\vec{k}\right]$

$= \left[\left(\frac{5\times 64}{3} - 24\right)\vec{i} + 6\times 64\vec{j} - 56\vec{k}\right] -$

$\left[\left(\frac{40}{3} - 6\right)\vec{i} + 24\vec{j} - 14\vec{k}\right]$

$= \vec{i}\left(\frac{320}{3} - 24 - \frac{40}{3} + 6\right) + \vec{j}(384 - 24)\vec{j} +$

$\vec{k}(-56 + 14)$

$= \left(\frac{320}{3} - \frac{40}{3} - 18\right)\vec{i} + 360\vec{j} -$

$42\vec{k}$

$= \left(\frac{320 - 40 - 54}{3}\right)\vec{i} + 360\vec{j} - 42\vec{k}$

$= \left(\frac{320 - 94}{3}\right)\vec{i} + 360\vec{j} - 42\vec{k}$

$= \left(\frac{226}{3}\right)\vec{i} + 360\vec{j} - 42\vec{k}.$

14) P.T $(\vec{a}\cdot\nabla)\,1/r = -\frac{\vec{a}\cdot\vec{r}}{r^3}$.

let $\vec{a} = a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$.

$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$.

15) $(\vec{a}\cdot\nabla) = \vec{a}\cdot\left(\sum \vec{i}\frac{\partial}{\partial x}\right)$

$= \cancel{\vec{a}\,\vec{i}}\,(a_1\vec{i} + a_2\vec{j} + a_3\vec{k})\cdot\left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)$

$= a_1\frac{\partial}{\partial x} + a_2\frac{\partial}{\partial y} + a_3\frac{\partial}{\partial z} = \sum \vec{a}\frac{\partial}{\partial x}$.

$(\vec{a}\cdot\nabla)\,1/r = \sum \vec{a}\frac{\partial}{\partial x}(1/r)$

$= \sum \vec{a}\left(-r^{-2}\frac{\partial r}{\partial x}\right)$

$= \sum \vec{a}\left(-\frac{1}{r^2}\frac{x}{r}\right)$ $\left[\because \frac{\partial r}{\partial x} = \frac{x}{r}\right]$

$= -\frac{1}{r^3}\sum \vec{a}\cdot x$ $\quad \sum x\vec{i} = x\vec{i} + y\vec{j} + z\vec{k}$

$= -1/r^3\,\vec{a}\cdot\vec{r}$

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15) Find grad r if $r^2 = x^2 + y^2 + z^2$.

16) grad $r = \nabla\cdot r$

$= \sum \vec{i}\frac{\partial}{\partial x}(r)$

$= \sum \vec{i}\;x/r$

$= x/r\,\vec{i} + y/r\,\vec{j} + z/r\,\vec{k}$

$= 1/r\,(x\vec{i} + y\vec{j} + z\vec{k})$

$= 1/r\,\vec{r}$

$= \frac{\vec{r}}{|r|}$

$= \hat{r}$

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Result :-

$$\left.\begin{array}{l}\text{Normal vector}\\ \text{(tangent vector)}\end{array}\right\} = \text{grad}\ \phi = \nabla\phi.$$

$$\left.\begin{array}{l}\text{Unit Normal vector}\\ \text{(unit tangent vector)}\end{array}\right\} = \frac{\text{grad}\phi}{|\text{grad}\phi|} = \frac{\nabla\phi}{|\nabla\phi|}$$

16) Find the Gradient and unit normal to the level surface $x^2+y-z=1$ at the point $(1,0,0)$.

Gn $\phi = x^2+y-z-1$

$$\text{grad}\cdot\phi = \nabla\phi$$

$$= \sum \vec{i}\,\frac{\partial}{\partial x}(x^2+y-z-1)$$

$$= \vec{i}\,\partial/\partial x\,(x^2+y-z-1) + \vec{j}\,\partial/\partial y\,(x^2+y-z-1) + \vec{z}\,\partial/\partial z\,(x^2+y-z-1)$$

$$= \vec{i}(2x) + \vec{j}(1) + \vec{k}(-1)$$

$$\text{grad}\ \phi_{at(1,0,0)} = 2\vec{i} + \vec{j} - \vec{k}$$

$$|\text{grad}\ \phi| = \sqrt{2^2+1^2+1^2}$$

$$= \sqrt{4+1+1} = \sqrt{6}$$

$$\text{Unit Normal vector (unit tangent vector)} = \frac{\text{grad}\phi}{|\text{grad}\phi|}$$

$$= \frac{2\vec{i} + \vec{j} - \vec{k}}{\sqrt{6}}.$$

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Directional derivative:

Directional derivative of a scalar point function $\phi(x,y,z)$ in the direction of unit vector is $\hat{a}\,\nabla\phi$ (or) $\hat{a}$ grad ϕ

$$\boxed{\text{Maximum directional derivative}} = |\text{grad}\,\phi|$$

1) Find the directional derivative of $\phi = (xy+yz+zx)$ in the direction of a vector $\vec{i}+2\vec{j}+2\vec{k}$ at the pt $(1,2,0)$

Gn $\phi = xy+yz+zx$ $\hat{a}$ = grad

$$\text{grad}\,\phi = \nabla\phi$$

$$= \sum \vec{i}\,\frac{\partial}{\partial x}(xy+yz+zx)$$

$$= \vec{i}\,\partial/\partial x\,(xy+yz+zx) + \vec{j}\,\frac{\partial}{\partial y}(xy+yz+zx) + \vec{k}\,\partial/\partial z\,(xy+yz+zx)$$

D. D $= \vec{i}(y+z) + \vec{j}(x+z) + \vec{k}(y+x)$

Gn $\vec{a} = \vec{i}+2\vec{j}+2\vec{k}$.

$$|\vec{a}| = \sqrt{1^2+2^2+2^2}$$

$$= \sqrt{1+4+4} = \sqrt{9}$$

$$= 3$$

$$\hat{a} = \frac{\vec{a}}{|\vec{a}|} = \frac{\vec{i}+2\vec{j}+2\vec{k}}{3}$$

Directional ~~derivative~~ of a vector $= \hat{a}$ grad ϕ

$$= \left(\frac{\vec{i}+2\vec{j}+2\vec{k}}{3}\right)\left(\vec{i}(y+z)+\vec{j}(x+z)+\vec{k}(y+x)\right)$$

$= \frac{1}{3}\left[(y+z)+2(x+z)+2(y+x)\right]$

$= \frac{1}{3}\left[3y+3z+4x\right]$

At pt $=(1,2,0)$

$\hat{a} \cdot \text{grad}\,\phi = 10/3.$

18) Find D.D of $f(x,y,z) = \log\sqrt{x^2+y^2+z^2} + xy$ in the direction of the normal to the surface $\phi(x,y,z) = xy+yz+zx$ at $(1,1$

To Find $\vec{a}$:

let $\vec{a}$ be the normal to

$\nabla = \sum \vec{i}\,\frac{\partial}{\partial x}$ $\quad \cancel{\phi(x,y}$,

$\boxed{\frac{\partial x}{\partial x} = x/x}$ $\quad \phi = xy+yz+zx$

Normal Vector $= \nabla\phi$

$= \sum \vec{i}\,\partial/\partial x\,(xy+yz+zx)$

$(x,y,z)=(1,1,1)$ $\quad = \vec{i}(y+z)+\vec{j}(x+z)+\vec{k}(y+x)$

$\text{grad}\,\phi_{at}(1,1,1) = 2\vec{i}+2\vec{j}+2\vec{k}.$

$\vec{a} = 2\vec{i}+2\vec{j}+2\vec{k}.$

$|\vec{a}| = \sqrt{2^2+2^2+2^2}.$

$= \sqrt{4+4+4} = \sqrt{12}.$

$= 2\sqrt{3}.$

$\hat{a} = \frac{\vec{a}}{|\vec{a}|} = \frac{2\vec{i}+2\vec{j}+2\vec{k}}{2\sqrt{3}}.$

$\hat{a} = \frac{\vec{i}+\vec{j}+\vec{k}}{\sqrt{3}}.$

$$f(x,y,z) = \log\sqrt{x^2+y^2+z^2} + xyz$$

$$\nabla f = \sum \vec{i}\,\frac{\partial}{\partial x}\left(\log\sqrt{x^2+y^2+z^2} + xyz\right)$$

$$\text{grad} f = \vec{i}\left(\frac{\partial}{\partial x}\log\sqrt{x^2+y^2+z^2} + xyz\right) + \vec{j}\left(\frac{\partial}{\partial y}\log\sqrt{x^2+y^2+z^2} + xyz\right) + \vec{k}\left(\frac{\partial}{\partial z}\log\sqrt{x^2+y^2+z^2} + xyz\right)$$

$$\text{grad} f = \vec{i}\left[\left(\frac{1}{\sqrt{x^2+y^2+z^2}}\,\tfrac{1}{\cancel{2}}(x^2+y^2+z^2)^{-1/2}\,\cancel{2}x\right) + yz\right]$$
$$+ \vec{j}\left[\left(\frac{1}{\sqrt{x^2+y^2+z^2}}\,\tfrac{1}{\cancel{2}}(x^2+y^2+z^2)^{-1/2}\,\cancel{2}y\right) + xz\right]$$
$$+ \vec{k}\left[\left(\frac{1}{\sqrt{x^2+y^2+z^2}}\,\tfrac{1}{\cancel{2}}(x^2+y^2+z^2)^{-1/2}\,\cancel{2}z\right) + xy\right]$$

$$\text{grad} f = \vec{i}\left[\frac{x}{x^2+y^2+z^2} + yz\right] + \vec{j}\left[\frac{y}{x^2+y^2+z^2} + xz\right] + \vec{k}\left[\frac{z}{x^2+y^2+z^2} + xy\right]$$

$$\text{grad} f(x,y,z) \text{ at } (1,1,1) = \vec{i}\left[\frac{1}{1^2+1^2+1^2} + (1)(1)\right] + \vec{j}\left[\frac{1}{1^2+1^2+1^2} + (1)(1)\right] + \vec{k}\left[\frac{1}{1^2+1^2+1^2} + (1)(1)\right]$$

$$= \vec{i}\left[\frac{1}{3} + 1\right] + \vec{j}\left[\frac{1}{3} + 1\right] + \vec{k}\left[\tfrac{1}{3} + 1\right]$$

$$\text{grad} f = \tfrac{4}{3}\vec{i} + \tfrac{4}{3}\vec{j} + \tfrac{4}{3}\vec{k}.$$

$$\hat{a}\ \text{grad} f = \left(\frac{\vec{i}+\vec{j}+\vec{k}}{\sqrt{3}}\right)$$

$$\left(\frac{4}{\sqrt{3}}\vec{i}+\frac{4}{\sqrt{3}}\vec{j}+\frac{4}{\sqrt{3}}\vec{k}\right)$$

$$= \frac{1}{\sqrt{3}}\left[\frac{4}{\sqrt{3}}+\frac{4}{\sqrt{3}}+\frac{4}{\sqrt{3}}\right]$$

$$= \frac{1}{\sqrt{3}}(4) = 4/\sqrt{3}.$$

19) If $\vec{F} = (t^3-t+3)\vec{i} + (1-2t^4)\vec{j} + (t-4\sin t)\vec{k}$

Find $\frac{d\vec{F}}{dt}$ & $\frac{d^2\vec{F}}{dt^2}$

$$\vec{F} = (t^3-t+3)\vec{i} + (1-2t^4)\vec{j} + (t-4\sin t)\vec{k}.$$

$$\frac{d\vec{F}}{dt} = \frac{d}{dt}\left[(t^3-t+3)\vec{i} + (1-2t^4)\vec{j} + (t-4\sin t)\vec{k}\right]$$

$$= (3t^2-1)\vec{i} + (-8t^3)\vec{j} + (1-4\cos t)\vec{k}$$

$$\frac{d}{dt}\left(\frac{d\vec{F}}{dt}\right) = d/dt\left[(3t^2-1)\vec{i} + (-8t^3)\vec{j} + (1-4\cos t)\vec{k}\right]$$

$$= 6t\vec{i} - 24t^2\vec{j} + 4\sin t\,\vec{k}$$

20) 1) Find the unit normal vector to the Level surface $x^2-y^2+z=2$ at $(1,-1,2)$

$$\phi = x^2-y^2+z-2$$

$$\text{grad}\,\phi = \nabla\phi$$

$$= \Sigma \vec{i}\, \partial/\partial x\,(x^2 y^2 + z - 2)$$

$$= \vec{i}\, \partial/\partial x\,(x^2 - y^2 + z - 2) + \vec{j}\, \partial/\partial y\,(x^2 - y^2 + z - 2) + \vec{k}\, \partial/\partial z\,(x^2 - y^2 + z - 2)$$

$$= \vec{i}\,(2x) + \vec{j}\,(-2y) + \vec{k}\,(1)$$

$$\text{grad}\,\phi = 2x\vec{i} - 2y\vec{j} + \vec{k}.$$

$$|\text{grad}\,\phi| = |2x\vec{i} - 2y\vec{j} + \vec{k}|$$

$$= \sqrt{2^2 + 2^2 + 1^2}$$

$$= \sqrt{4+4+1} = \sqrt{9}.$$

$$= 3.$$

$$\left.\begin{array}{l}\text{unit normal}\\ \text{vector}\end{array}\right\} = \frac{\text{grad}\,\phi}{|\text{grad}\,\phi|}$$

$$= \frac{2x\vec{i} - 2y\vec{j} + \vec{k}}{3}$$

unit normal vector at $(1,-1,2)$

$$= \frac{2(1)\vec{i} - 2(-1)\vec{j} + \vec{k}}{3}$$

$$= \frac{2\vec{i} + 2\vec{j} + \vec{k}}{3}$$

Find $\nabla\phi$ unit normal vector to $\phi(x,y,z) = x^2y + y^2z + z^2x$ at $(1,2,1)$.

$$\phi = x^2y + y^2z + z^2x$$

$$\text{grad}\,\phi = \nabla\phi$$

$$= \Sigma \vec{i}\, \partial/\partial x\,(x^2y + y^2z + z^2x)$$

$$= \vec{i}\, \partial/\partial x\,(x^2y + y^2z + z^2x)$$

$+\vec{j}\,\partial/\partial y\,(x^2y+y^2z+z^2x)+\vec{k}\,\partial/\partial z\,(x^2y+y^2z+z^2x)$

$\text{grad}\,\phi = \vec{i}\,(2xy+z^2)+\vec{j}\,(x^2+2yz)+\vec{k}\,(y^2+$

$\text{grad}\,\phi$ at $(1,2,1)$ $= \vec{i}\,(2(1)(2)+1^2)+\vec{j}\,(1^2+2(2)(1))+\vec{k}\,(2^2+2(1)(1))$

$= \vec{i}\,(4+1)+\vec{j}\,(1+4)+\vec{k}\,(4+2)$

$= 5\vec{i}+5\vec{j}+6\vec{k}.$

$\left|\text{grad}\,\phi \text{ at } (1,2,1)\right| = \sqrt{5^2+5^2+6^2}$

$= \sqrt{25+25+36}$

$= \sqrt{86}.$

unit normal vector $= \dfrac{\text{grad}\,\phi}{|\text{grad}\,\phi|}$

$= \dfrac{5\vec{i}+5\vec{j}+6\vec{k}}{\sqrt{86}}$

3) Find the D.D of $\phi = xyz$ in the direction of the vector $\vec{i}+\vec{j}+\vec{k}$ at $(1,1,1)$.

Normal vector $\Big\} = \nabla\phi$

$\text{grad}\,\phi = \sum \vec{i}\,\partial/\partial x\,(xyz)$

$= \vec{i}\,\partial/\partial x\,(xyz)+\vec{j}\,\partial/\partial y\,(xyz)+\vec{k}\,\partial/\partial z\,(xyz)$

$$= \vec{i}(yz) + \vec{j}(xz) + \vec{k}(xy)$$

$$= yz\vec{i} + xz\vec{j} + xy\vec{k}$$

grad ϕ at $(1,1,1)$ $= (1)(1)\vec{i} + (1)(1)\vec{j} + (1)(1)\vec{k}$

$$= \vec{i} + \vec{j} + \vec{k}$$

$$\vec{a} = \vec{i} + \vec{j} + \vec{k}$$

$$|\vec{a}| = \sqrt{1^2+1^2+1^2}$$

$$= \sqrt{3}$$

$$\hat{a} = \frac{\vec{a}}{|\vec{a}|} = \frac{\vec{i} + \vec{j} + \vec{k}}{\sqrt{3}}$$

$$\text{grad } \phi = \vec{i} + \vec{j} + \vec{k}$$

$$\text{D.D } \cancel{\text{grad}\,\phi} = \hat{a} \text{ grad } \phi$$

$$= \frac{\vec{i} + \vec{j} + \vec{k}}{\sqrt{3}} (\vec{i} + \vec{j} + \vec{k})$$

$$= \frac{1^2+1^2+1^2}{\sqrt{3}} \Rightarrow \frac{1+1+1}{\sqrt{3}}$$

$$= \frac{3}{\sqrt{3}} = \sqrt{3}.$$

$\hat{a} = \frac{\vec{a}}{|\vec{a}|}$

Find the D.D of $\phi = x^2 - 2y^2 - 4z^2$ in the direction $\hat{a}$ $2\vec{i} - \vec{j} - 2\vec{k}$ at $(1,-2,-1)$

$\vec{n} = \nabla \phi$

$$\left.\begin{matrix}\text{Normal} \\ \text{vector}\end{matrix}\right\} = \nabla \phi$$

$$\text{grad}\,\phi = \Sigma \vec{i}\, \partial/\partial x\, (x^2 - 2y^2 - 4z^2)$$

$$= \vec{i}\, \partial/\partial x\, (x^2 - 2y^2 - 4z^2) + \vec{j}\, \partial/\partial y\, (x^2 - 2y^2 - 4z^2)$$

$$+ \vec{k}\, \partial/\partial z\, (x^2 - 2y^2 - 4z^2)$$

$$= \vec{i}(2x) + \vec{j}(-4y) + \vec{k}(-8z)$$

$\text{grad}\,\phi = 2x\vec{i} - 4y\vec{j} - 8z\vec{k}.$

$\vec{a} = 2\vec{i} - \vec{j} - 2\vec{k}$

$|\vec{a}| = \sqrt{2^2 + (-1)^2 + (-2)^2}.$

$= \sqrt{4+1+4}$

$= \sqrt{9} = 3$

$\hat{a} = \dfrac{\vec{a}}{|\vec{a}|}$

$= \dfrac{2\vec{i} - \vec{j} - 2\vec{k}}{3}$

D.D ~~grad f~~ $= \hat{a}\,\text{grad}\,\phi.$

$= \dfrac{2\vec{i} - \vec{j} - 2\vec{k}}{3}\,\text{grad}\,\phi$

$\text{grad}\,\phi$ at $(1,-2,-1)$ $= 2(1)\vec{i} - 4(-2)\vec{j} - 8(-1)\vec{k}$

$= 2\vec{i} + 8\vec{j} + 8\vec{k}$

$\text{grad}\,f_{D.D} = \hat{a}\cdot\text{grad}\,\phi$

$= \dfrac{2\vec{i} - \vec{j} - 2\vec{k}}{3}\,(2\vec{i} + 8\vec{j} + 8\vec{k})$

$= 1/3\,[(2)(2) - 1(8) - 2(8)]$

$= 1/3\,[4 - 8 - 16] \Rightarrow 1/3\,[4-24]$

$= -20/3$

18/12/19 If $\nabla\phi = 2xyz^3\vec{i} + x^2z^3\vec{j} + 3x^2yz^2\vec{k}$.

24) 1) Find ϕ, if $\phi(1,-2,2) = 4$.

$$\nabla\phi = 2xyz^3\vec{i} + x^2z^3\vec{j} + 3x^2yz^2\vec{k}$$

$$\Sigma\vec{i}\frac{\partial\phi}{\partial x} = 2xyz^3\vec{i} + x^2z^3\vec{j} + 3x^2yz^2\vec{k}$$

$$\vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z} = 2xyz^3\vec{i} + x^2z^3\vec{j} + 3x^2yz^2\vec{k}$$

Equating co-eff of $\vec{i},\vec{j},\vec{k}$ on both side.

$$\frac{\partial\phi}{\partial x} = 2xyz^3 \longrightarrow (1)$$

$$\frac{\partial\phi}{\partial y} = x^2z^3 \longrightarrow (2)$$

$$\frac{\partial\phi}{\partial z} = 3x^2yz^2 \longrightarrow (3)$$

on $\int$ing (1) partially w.r.to x,

$$\int\partial\phi = \int 2xyz^3\,dx$$

$$\phi = \frac{2x^2yz^3}{2} + f(y,z)$$

$$\phi = x^2yz^3 + f(y,z)$$

$\int$ing (2) p.w.r to 'y'

$$\int\partial\phi = \int x^2z^3\,dy.$$

$$\phi = x^2yz^3 + f(x,z)$$

$\int$ing (3) p.w.r to 'z'

$$\int\partial\phi = \int 3x^2yz^2\,dy.$$

$$\phi = \frac{\cancel{3}x^2yz^3}{\cancel{3}} + f(x,y)$$

$$\phi = x^2yz^3 + f(x,y).$$

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$\because \phi = x^2 y z^3 + c$

At $(1,-2,2) = 4$

$1^2(-2)\,2^3 + c = 4$

$-16 + c = 4$

$c = 20$

$\phi(x,y,z) = x^2 y z^3 + 20$

2) If $\nabla\phi = (y^2\cos x + z^3)\vec{i} + (2y\sin x - 4)\vec{j} + (3xy^2 + 2)\vec{k}$.

25) and $\phi(0,1,-1) = -6$ Then find ϕ.

$\nabla\phi = (y^2\cos x + z^3)\vec{i} + (2y\sin x - 4)\vec{j} + (3xy^2 + 2)\vec{k}$.

$\vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z} = (y^2\cos x + z^3)\vec{i} + (2y\sin x - 4)\vec{j} + (3xy^2+2)\vec{k}$.

$\frac{\partial\phi}{\partial x} = y^2\cos x + z^3 \rightarrow (1)$

$\frac{\partial\phi}{\partial y} = 2y\sin x - 4 \rightarrow (2)$

$\frac{\partial\phi}{\partial z} = 3xy^2 + 2 \rightarrow (3)$

(1) $\Rightarrow$ P.w.r to 'x'

$\int \partial\phi = \int y^2\cos x\,\partial x + \int z^3\,\partial x$

$\phi = +y^2\sin x + z^3 x + f(y,z) \rightarrow (4)$

(2) $\Rightarrow$ P.w.r to 'y'

$\int\partial\phi = \int (2y\sin x)\partial y - \int 4\,\partial y$

$= y^2 \sin x - 4y + f(x,z) \rightarrow (5)$

(3) $\Rightarrow$ p.w.r to 'z'

$$\int \partial\phi = \int 3xy^2 \partial z + \int 2 \partial z$$

$$\phi = \frac{3zy^2x}{1} + 2z$$

$$= 3xy^2z + 2z + f(z,x) \rightarrow (6)$$

combining (4), (5) & (6)

$$\phi = y^2 \sin x + z^3x - 4y + 3xy^2z + 2z + c$$

$$\phi = z^3x + 3xy^2z - 4y + 2z + y^2\sin x + c$$

At $\phi(0,1,-1) = -6$

$$0+0-4-2+0+c = -6$$

$$-6\,c = -6$$

$$c = 0$$

$$\phi = z^3x + 3xy^2z - 4y + 2z + y^2\sin x + 0$$

$$\phi = z^3x + 3xy^2z - 4y + 2z + y^2\sin x$$

Divergence of a vector

Let $\vec{F}$ be a vector point function which is continously derivable Then

$$\text{div}\,\vec{F} = \vec{i}\cdot\frac{\partial \vec{F}}{\partial x} + \vec{j}\cdot\frac{\partial \vec{F}}{\partial y} + \vec{k}\cdot\frac{\partial \vec{F}}{\partial z}$$

$$= \left(\vec{i}\,\partial/\partial x + \vec{j}\,\partial/\partial y + \vec{k}\,\partial/\partial z\right)\cdot\vec{F}$$

$$= \nabla\cdot\vec{F}$$

$$\text{div}\,\vec{F} = \sum \vec{i}\cdot\frac{\partial \vec{F}}{\partial x}$$

$$= \nabla\cdot\vec{F}$$

Divergence of $\vec{F}$ in terms of c

of $\vec{F}$.

let $\vec{F} = f_1 \vec{i} + f_2 \vec{j} +$

where f_1, f_2, f_3, are scalar fun

of x, y, z.

$$\therefore \frac{\partial \vec{F}}{\partial x} = \frac{\partial f_1 \vec{i}}{\partial x} + \frac{\partial f_2 \vec{j}}{\partial x}$$

$$\vec{i} \cdot \frac{\partial \vec{F}}{\partial x} = \frac{\partial f_1}{\partial x}$$

$||^{ly}$

$$\vec{j} \cdot \frac{\partial \vec{F}}{\partial y} = \frac{\partial f_2}{\partial y}$$

$$\vec{k} \cdot \frac{\partial \vec{F}}{\partial z} = \frac{\partial f_3}{\partial z}$$

$$\therefore \operatorname{div} \vec{F} = \vec{i} \cdot \frac{\partial \vec{F}}{\partial x} + \vec{j} \cdot \frac{\partial \vec{F}}{\partial y}$$

$$= \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f}{\partial}$$

It is clear that the above

is a scalar function.

2b) 1) Find the unit normal vector to

surface $x^2 + 2y^2 + z^2 = 7$ at $(1, -1, 2)$

$$\phi = x^2 + 2y^2 + z^2 - 7$$

$$\operatorname{grad}\phi = \nabla \phi$$

$$= \sum \vec{i}\, \partial/\partial x \,(x^2+2y^2+z^2-7)$$

$$\text{grad}\,\phi = \vec{i}\, \partial/\partial x (x^2+2y^2+z^2-7) + \vec{j}\, \partial/\partial y (x^2+2y^2+z^2-7)$$
$$+\vec{k}\, \partial/\partial z (x^2+2y^2+z^2-7)$$

$$\text{grad}\,\phi = 2x\vec{i} + 4y\vec{j} + 2z\vec{k}$$

$$\text{grad}\,\phi_{\text{at }(1,-1,2)} = 2(1)\vec{i} + 4(-1)\vec{j} + 2(2)\vec{k}$$
$$= 2\vec{i} - 4\vec{j} + 4\vec{k}$$

$$\left|\text{grad}\,\phi_{\text{at }(1,-1,2)}\right| = \sqrt{2^2+(-4)^2+4^2}$$
$$= \sqrt{4+16+16}$$
$$= \sqrt{36} = 6$$

$\frac{\text{grad}\,\phi}{|\text{grad}\,\phi|}$

Unit normal vector $= \dfrac{\text{grad}\,\phi}{|\text{grad}\,\phi|}$

$$= \frac{2\vec{i} - 4\vec{j} + 4\vec{k}}{6}.$$

Find D.D of $\phi = x^2 - 2y - 4z^2$ in the direction $2\vec{i} + \vec{j} - \vec{k}$ at $(1,1,-1)$ and also find the maximum value of D.D.

$$\phi = x^2 - 2y - 4z^2$$
$$\text{grad}\,\phi = \nabla\phi$$
$$= \sum \vec{i}\, \partial/\partial x \,(x^2-2y-4z^2)$$
$$= \vec{i}\, \partial/\partial x (x^2-2y-4z^2) + \vec{j}\, \partial/\partial y (x^2-2y-4z^2)$$

$+ \vec{k}\, \partial/\partial z\, (x^2 - 2y^2 - 4z^2)$

$\text{grad}\,\phi = 2x\vec{i} - 4y\vec{j} - 8z\vec{k}$

$\text{grad}\,\phi$ at $(1,1,-1)$ $= 2(1)\vec{i} - 4(1)\vec{j} - 8(-1)\vec{k}$

$= 2\vec{i} - 4\vec{j} + 8\vec{k}$

$\vec{a} = 2\vec{i} + \vec{j} - \vec{k}$

$|\vec{a}| = \sqrt{2^2 + 1^2 + (-1)^2}$

$= \sqrt{4+1+1}$

$= \sqrt{6}$

$\hat{a} = \frac{\vec{a}}{|\vec{a}|} = \frac{2\vec{i} + \vec{j} - \vec{k}}{\sqrt{6}}$

~~grad~~

$D.D = \hat{a}\, \text{grad}\,\phi$

$= \frac{2\vec{i} + \vec{j} - \vec{k}}{\sqrt{6}} (2\vec{i} - 4\vec{j} + 8\vec{k})$

$= \frac{1}{\sqrt{6}} (4 - 4 - 8)$

$= \frac{1}{\sqrt{6}} (4 - 12)$

$= \frac{1}{\sqrt{6}} (-8)$

$= \frac{-8}{\sqrt{6}}$

Maximum value of $D.D = |\text{grad}\,\phi|$

$$= \sqrt{2^2+4^2+8^2}$$

$$= \sqrt{4+16+64}$$

$$= \sqrt{84}$$

$$= \sqrt{21 \times 4}$$

maximum D.D. $= 2\sqrt{21}$

Find the unit normal vector to the surface $x^2+y^2-z^2=1$ at $(1,1,1)$

$$\phi = x^2+y^2-z^2-1$$

$$\text{grad } \phi = \nabla \phi$$

$$= \sum \vec{i}\, \partial/\partial x\, (x^2+y^2-z^2-1)$$

$$= \vec{i}\, \partial/\partial x\, (x^2+y^2-z^2-1) + \vec{j}\, \partial/\partial y\, (x^2+y^2-z^2-1) + \vec{k}\, \partial/\partial z\, (x^2+y^2-z^2-1)$$

$$\text{grad } \phi = 2x\vec{i} + 2y\vec{j} - 2z\vec{k}$$

grad ϕ at $(1,1,1)$ $= 2(1)\vec{i} + 2(1)\vec{j} - 2(1)\vec{k}$

$$= 2\vec{i} + 2\vec{j} - 2\vec{k}$$

$$\left| \text{grad } \phi \text{ at } (1,1,1) \right| = \sqrt{2^2+2^2+(-2)^2}$$

$$= \sqrt{4+4+4}$$

$$= \sqrt{12}$$

$$= 2\sqrt{3}$$

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unit normal vector $= \dfrac{\text{grad}\,\phi}{|\text{grad}\,\phi|}$

$= \dfrac{2\vec{i}+2\vec{j}-2\vec{k}}{2\sqrt{3}}$

$= \dfrac{2(\vec{i}+\vec{j}-\vec{k})}{2\sqrt{3}}$

$= \dfrac{\vec{i}+\vec{j}-\vec{k}}{\sqrt{3}}$.

1) If $\nabla f = yz\vec{i}+zx\vec{j}+xy\vec{k}$ Find f

2a)

$\sum \vec{i}\dfrac{\partial f}{\partial x} = yz\vec{i}+zx\vec{j}+xy\vec{k}$

$\vec{i}\dfrac{\partial f}{\partial x}+\vec{j}\dfrac{\partial f}{\partial y}+\vec{k}\dfrac{\partial f}{\partial z} = yz\vec{i}+zx\vec{j}+xy\vec{k}$

$\dfrac{\partial f}{\partial x} = yz \rightarrow (1)$

$\dfrac{\partial f}{\partial y} = zx \rightarrow (2)$

$\dfrac{\partial f}{\partial z} = xy \rightarrow (3)$

on $\int$ing (1) partially w.r to 'x'

$\int \dfrac{\partial f}{\partial x} = \int yz$

$\int \partial f = \int yz\,\partial x$

$f = xyz + f(y,z)$

on $\int$ing ② Partially w.r to 'y'.

$$\int \frac{\partial f}{\partial y} = \int xz$$

$$\int \partial f = \int xz\, \partial y.$$

$$f = xyz + f(x,z)$$

on $\int$ing ③ Partially w.r to 'z'.

$$\int \frac{\partial f}{\partial z} = \int xy$$

$$\int \partial f = \int xy\, \partial z$$

$$f = xyz + f(x,y)$$

$$f = xyz + c$$

2) If $\nabla\phi$ Find the scalar point function whose gradient is $2xyz\vec{i} + x^2z\vec{j} + x^2y\vec{K}$

30)

$$\nabla\phi = 2xyz\vec{i} + x^2z\vec{j} + x^2y\vec{K}$$

$$\Sigma\vec{i}\frac{\partial\phi}{\partial x} = 2xyz\vec{i} + x^2z\vec{j} + x^2y\vec{K}$$

$$\vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z} = 2xyz\vec{i} + x^2z\vec{j} + x^2y\vec{K}$$

$$\frac{\partial\phi}{\partial x} = 2xyz \rightarrow ①$$

$$\frac{\partial\phi}{\partial y} = x^2z \rightarrow ②$$

$$\frac{\partial\phi}{\partial z} = x^2y \rightarrow ③$$

on $\int$ing ① Partially w.r to 'x'

$$\int \frac{\partial\phi}{\partial x} = \int 2xyz$$

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$\int \partial\phi = 2yz\int x\,\partial x$

$\phi = \frac{x^2}{2} 2yz + f(y,z)$

$\phi = x^2yz + f(y,z)$

on $\int$ing ② partially w.r to 'y'.

$\int \frac{\partial\phi}{\partial y} = \int x^2 z$

$\int \partial\phi = x^2 z \int \partial y$

$\phi = x^2yz + f(x,z)$

on $\int$ing ③ partially w.r to 'z'

$\int \frac{\partial\phi}{\partial z} = \int x^2 y$

$\int \partial\phi = x^2 y \int \partial z$

$\phi = x^2yz + f(x,y)$

$\phi = x^2yz + c$

3) If $\nabla\phi = (y^2 - 2xyz^3)\vec{i} + (3 + 2xy - x^2z^3)\vec{j}$
31) $+ (6z^3 - 3x^2yz^3)\vec{k}$ find ϕ.

$\nabla\phi = (y^2 - 2xyz^3)\vec{i} + (3 + 2xy - x^2z^3)\vec{j} + (6z^3 - 3x^2yz^3)\vec{k}$.

$\Sigma \vec{i}\frac{\partial\phi}{\partial x} = (y^2 - 2xyz^3)\vec{i} + (3 + 2xy - x^2z^3)\vec{j} + (6z^3 - 3x^2yz^3)\vec{k}$.

$\vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z} = (y^2 - 2xyz^3)\vec{i} + (3 + 2xy - x^2z^3)\vec{j} + (6z^3 - 3x^2yz^3)\vec{k}$.

$$\frac{\partial \phi}{\partial x} = y^2 - 2xyz^3 \longrightarrow ①$$

$$\frac{\partial \phi}{\partial y} = 3 + 2xy - x^2z^3 \longrightarrow ②$$

$$\frac{\partial \phi}{\partial z} = 6z^3 - 3x^2yz^3 \longrightarrow ③$$

On $\int$ing ① Partially w.r to 'x'

$$\int \frac{\partial \phi}{\partial x} = \int (y^2 - 2xyz^3)$$

$$\int \partial \phi = \int y^2 \partial x - \int 2xyz^3 \partial x$$

$$\phi = xy^2 - x^2yz^3 + f(y, z)$$

On $\int$ing ② Partially w.r to 'y'

$$\int \frac{\partial \phi}{\partial y} = \int (3 + 2xy - x^2z^3)$$

$$\int \partial \phi = \int 3\, \partial y + \int 2xy\, \partial y - x^2z^3 \int \partial y$$

$$\phi = 3y + xy^2 - x^2yz^3 + f(z, x)$$

On $\int$ing ③ Partially w.r to 'z'

$$\int \frac{\partial \phi}{\partial z} = \int 6z^3 - 3x^2yz^3$$

$$\int \partial \phi = 6 \int z^3 \partial z - 3x^2y \int z^3 \partial z$$

$$\phi = 6\frac{z^4}{4} - 3\frac{x^2yz^4}{4} + f(x, y)$$

$$\phi = \frac{3z^4}{2} - \frac{3x^2yz^4}{4} + f(x, y)$$

$$\phi = xy^2 - x^2yz^3 + 3y + 3/2\, z^4 - 3/4\, x^2yz^4 + c$$

4) If $\nabla\phi = (6xy+z^3)\vec{i} + (3x^2-z)\vec{j} + (3xz^2-y)\vec{k}$

32) find ϕ

$$\nabla\phi = (6xy+z^3)\vec{i} + (3x^2-z)\vec{j} + (3xz^2-y)\vec{k}$$

$$\sum \vec{i}\frac{\partial\phi}{\partial x} = (6xy+z^3)\vec{i} + (3x^2-z)\vec{j} + (3xz^2-y)\vec{k}.$$

$$\vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z} = (6xy+z^3)\vec{i} + (3x^2-z)\vec{j} + (3xz^2-y)\vec{k}.$$

$$\frac{\partial\phi}{\partial x} = 6xy+z^3 \rightarrow (1)$$

$$\frac{\partial\phi}{\partial y} = 3x^2-z \rightarrow (2)$$

$$\frac{\partial\phi}{\partial z} = 3xz^2-y \rightarrow (3)$$

On $\int$ing (1) Partially w.r to 'x'

$$\int\frac{\partial\phi}{\partial x} = \int(6xy+z^3)$$

$$\int\partial\phi = 6y\int x\,\partial x + z^3\int\partial x$$

$$\phi = \frac{6x^2y}{2} + z^3x + f(y,z)$$

$$\phi = 3x^2y + xz^3 + f(y,z)$$

On $\int$ing (2) Partially w.r to 'y'

$$\int\frac{\partial\phi}{\partial y} = \int 3x^2 - z$$

$$\int \partial\phi = \int 3x^2 \partial y + \int z \partial y.$$

$$\phi = 3x^2y - zy + f(z,x)$$

On $\int$ing ③ partially w.r to 'z'.

$$\int \frac{\partial\phi}{\partial z} = \int (3xz^2 - y)$$

$$\int \partial\phi = \int 3xz^2 \partial z - \int y \partial z.$$

$$= \frac{3xz^3}{3} - yz + f(x,y)$$

$$\phi = xz^3 - yz + f(x,y)$$

$$\phi = 3x^2y + xz^3 - yz + c$$

5) If $\nabla\phi = (y + y^2 + z^2)\vec{i} + (x + z + 2xy)\vec{j} + (y + 2zx)\vec{k}$

and $\phi(1,1,1) = 3$ then find ϕ.

$$\nabla\phi = (y + y^2 + z^2)\vec{i} + (x + z + 2xy)\vec{j} + (y + 2zx)\vec{k}$$

$$\Sigma \vec{i} \frac{\partial\phi}{\partial x} = (y + y^2 + z^2)\vec{i} + (x + z + 2xy)\vec{j} + (y + 2zx)\vec{k}.$$

$$\vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z} = (y + y^2 + z^2)\vec{i} + (x + z + 2xy)\vec{j} + (y + 2zx)\vec{k}.$$

$$\frac{\partial\phi}{\partial x} = y^2 + y + z^2 \longrightarrow ①$$

$$\frac{\partial\phi}{\partial y} = x + z + 2xy \longrightarrow ②$$

$$\frac{\partial\phi}{\partial z} = y + 2zx \longrightarrow ③$$

On $\int$ing ① Partially w.r.to 'x'.

$$\int \frac{\partial \phi}{\partial x} = \int (y + y^2 + z^2)$$

$$\int \partial \phi = \int y \, \partial x + y^2 \int \partial x + z^2 \int \partial x$$

$$\phi = xy + xy^2 + xz^2 + f(y,z)$$

On $\int$ing ② Partially w.r.to 'y'.

$$\int \frac{\partial \phi}{\partial y} = \int (x + z + 2xy)$$

$$\int \partial \phi = x \int \partial y + z \int \partial y + 2x \int y \, \partial y$$

$$= xy + zy + \frac{2xy^2}{2} + f(z,x)$$

$$\phi = xy + zy + xy^2 + f(z,x)$$

On $\int$ing ③ Partially w.r.to 'z'.

$$\int \frac{\partial \phi}{\partial z} = \int (y + 2zx)$$

$$\int \partial \phi = y \int \partial z + 2x \int z \, \partial z.$$

$$= yz + \frac{2xz^2}{2} + f(x,y)$$

$$\phi = yz + xz^2 + f(x,y)$$

$$\phi = xy + xy^2 + xz^2 + yz + c$$

at $\phi(1,1,1) = 3$

$$\phi = xy + xy^2 + xz^2 + yz + c$$

$$3 = 1 + 1 + 1 + 1 + c$$

$$3 = 4 + c$$

$$3 - 4 = c$$

$$c = -1$$

$$\phi = xy + yz + xy^2 + xz^2 - 1.$$

19/12/19 Solenoidal vector:-

A vector $\vec{F}$ is called solenoidal vector, if div $\vec{F}$ vanishes.

divergent $\vec{F}$

curl $\vec{F}$:

$$\text{curl } \vec{F} = \vec{i} \times \frac{\partial \vec{F}}{\partial x} + \vec{j} \times \frac{\partial \vec{F}}{\partial y} + \vec{k} \times \frac{\partial \vec{F}}{\partial z}$$

$$= \left(\vec{i}\, \partial/\partial x + \vec{j}\, \partial/\partial y + \vec{k}\, \partial/\partial z\right) \times \vec{F}.$$

$$= \nabla \times \vec{F} \qquad \nabla \times \vec{F}$$

$$\text{curl } \vec{F} = \Sigma\, \vec{i} \times \frac{\partial \vec{F}}{\partial x} = \nabla \times \vec{F}$$

curl $\vec{F}$ in terms of components:-

let $\vec{F} = f_1\vec{i} + f_2\vec{j} + f_3\vec{k}$ where f_1, f_2, f_3 are scalar functions of x, y, z.

$$\therefore \frac{\partial \vec{F}}{\partial x} = \frac{\partial f_1}{\partial x}\vec{i} + \frac{\partial f_2}{\partial x}\vec{j} + \frac{\partial f_3}{\partial x}\vec{k}$$

$$\vec{i} \times \frac{\partial \vec{F}}{\partial x} = \vec{i} \times \left(\frac{\partial f_1}{\partial x}\vec{i} + \frac{\partial f_2}{\partial x}\vec{j} + \frac{\partial f_3}{\partial x}\vec{k}\right)$$

$$= \frac{\partial f_2}{\partial x}\vec{k} - \frac{\partial f_3}{\partial x}\vec{j}$$

Similarly,

$$\vec{j} \times \frac{\partial \vec{F}}{\partial y} = \frac{\partial f_3}{\partial y}\vec{i} - \frac{\partial f_1}{\partial y}\vec{k}$$

$$\vec{k} \times \frac{\partial \vec{F}}{\partial z} = \frac{\partial f_1}{\partial z}\vec{j} - \frac{\partial f_2}{\partial z}\vec{i}$$

$$\therefore \text{crul } \vec{F} = \Sigma \vec{i} \times \frac{\partial \vec{F}}{\partial x}$$

$$= \left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right)\vec{i} + \left(\frac{\partial f_1}{\partial z} - \cdots\right)\vec{j}$$

$$+ \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right)\vec{k}$$

$$\text{crul } \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ f_1 & f_2 & f_3 \end{vmatrix}$$

Irrotational vector

A vector $\vec{F}$ is said to be irrotational if curl $\vec{F} = 0$

The laplacian operator ∇^2.

The operator $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$

If ϕ be any scalar function the

$$\nabla^2 \phi = \nabla \cdot (\nabla \phi)$$

$$= (\nabla \cdot \nabla)\phi$$

$$\nabla \cdot \nabla \phi = \left(\vec{i}\,\partial/\partial x + \vec{j}\,\partial/\partial y + \vec{k}\,\partial/\partial z\right)$$

$$\left(\vec{i}\frac{\partial \phi}{\partial x} + \vec{j}\frac{\partial \phi}{\partial y} + \vec{k}\frac{\partial \phi}{\partial z}\right)$$

$$= \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2}$$

$$= \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) \phi$$

$$\nabla^2 \phi = \nabla \cdot (\nabla \phi)$$

$$= (\nabla \cdot \nabla) \phi$$

$$= \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2}.$$

$$\nabla^2 \vec{F} = \frac{\partial^2 \vec{F}}{\partial x^2} + \frac{\partial^2 \vec{F}}{\partial y^2} + \frac{\partial^2 \vec{F}}{\partial z^2}.$$

2/01/2020

1) Prove that $\operatorname{div} (\vec{f} \pm \vec{g}) = \operatorname{div} \vec{f} \pm \operatorname{div} \vec{g}$

34)

$$\operatorname{div} (\vec{f} \pm \vec{g}) = \nabla \cdot (\vec{f} \pm \vec{g}).$$

$$\nabla \cdot (\vec{f} \pm \vec{g}) = \Sigma \vec{i}\, \partial/\partial x \cdot (\vec{f} \pm \vec{g})$$

$\nabla \phi$

$$= \Sigma \vec{i} \cdot \partial/\partial x (\vec{f} \pm \vec{g})$$

Σ

$$= \Sigma \vec{i} \left(\frac{\partial \vec{f}}{\partial x} \pm \frac{\partial \vec{g}}{\partial x} \right)$$

$$= \Sigma \vec{i} \frac{\partial \vec{f}}{\partial x} \pm \Sigma \vec{i} \frac{\partial \vec{g}}{\partial x}$$

$$= \Sigma \vec{i}\, \partial/\partial x \cdot (\vec{f}) \pm \Sigma \vec{i}\, \partial/\partial x \cdot (\vec{g})$$

$$= \nabla \cdot \vec{f} \pm \nabla \cdot \vec{g}$$

$$= \operatorname{div} \vec{f} \pm \operatorname{div} \vec{g}.$$

2) Prove that $\operatorname{div} \vec{r} = 3$

35)

W.K.T

$$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}.$$

$$\operatorname{div} \vec{r} = \nabla \cdot \vec{r}.$$

$$= \Sigma \vec{i}\, \partial/\partial x \cdot (x\vec{i} + y\vec{j} + z\vec{k})$$

$= (\vec{i}\,\partial/\partial x + \vec{j}\,\partial/\partial y + \vec{k}\,\partial/\partial z)\cdot$

$(x\vec{i} + y\vec{j} + \vec{k}z)$

$= 1+1+1$

$= 3.$

3) Prove that $\operatorname{div}(r^n \vec{r}) = (n+3)\, r^n.$

36)

$\operatorname{div}(r^n \vec{r}) = \nabla \cdot (r^n \vec{r})$

$= \sum \vec{i}\, \partial/\partial x\, (r^n \vec{r}).$

$= \sum \vec{i} \cdot \left[n r^{n-1} \frac{\partial r}{\partial x} \vec{r} + r^n \frac{\partial \vec{r}}{\partial x} \right.$

$= \sum \vec{i} \cdot \left[n r^{n-1}\, x/r\, \vec{r} + r^n \vec{i} \right.$

$= \sum \vec{i} \cdot \left[n r^{n-2} x\, \vec{r} + r^n \vec{i} \right.$

$= \sum \vec{i} \cdot \left[r^n \vec{i} \right] + \sum \vec{i}\,(n r^{n-2} x\, \vec{r})$

$= r^n \sum (\vec{i}\cdot\vec{i}) + n r^{n-2} (\sum x\vec{i} \cdot \vec{r})$

$= r^n (\vec{i}\cdot\vec{i} + \vec{j}\cdot\vec{j} + \vec{k}\cdot\vec{k}) + n r^{n-2} (x\vec{i} + y\vec{j} + \vec{k}z)\cdot(x\vec{i} + y\vec{j} + z\vec{k})$

$= r^n (1+1+1) + n r^{n-2} (x^2+y^2+z^2)$

$= 3r^n + n r^{n-2} (r^2)$

$= 3r^n + n r^n$

$= r^n (n+3)$

Prove that $\operatorname{div} \frac{\vec{r}}{r^3} = 0$

$$\operatorname{div}\left(\frac{\vec{r}}{r^3}\right) = \operatorname{div}(r^{-3}\vec{r})$$

W.K.T

$$\operatorname{div} r^n \vec{r} = (n+3) r^n.$$

Put $n = -3$.

$$\operatorname{div} r^{-3} \vec{r} = (-3+3) r^n.$$

$$\operatorname{div} r^{-3}\vec{r} = 0$$

P.T $\operatorname{div}(\operatorname{grad} r^n) = n(n+1) r^{n-2}$. (or).

$$\nabla\cdot(\nabla r^n) = n(n+1) r^{n-2}.$$

$$\nabla\cdot(\nabla r^n) = \nabla\cdot\left(\Sigma \vec{i}\, \partial/\partial x\, r^n\right)$$

$$= \nabla\cdot\left(\Sigma \vec{i}\, n r^{n-1} \frac{\partial r}{\partial x}\right)$$

$$= \nabla\cdot\left(\Sigma \vec{i}\, n r^{n-1} x/r\right)$$

$$= \nabla\cdot\left(\Sigma \vec{i}\, n r^{n-2} x\right)$$

$$= \nabla\cdot n r^{n-2}\, \Sigma x\vec{i}$$

$$= \nabla\cdot n r^{n-2}\, \vec{r}.$$

$$= \Sigma \vec{i}\, \partial/\partial x \cdot (n r^{n-2}\vec{r})$$

$$= n\, \Sigma \vec{i}\, \partial/\partial x \cdot (r^{n-2}\vec{r})$$

$$= n\, \Sigma \vec{i}\cdot \partial/\partial x (r^{n-2}\vec{r}).$$

$$= n\, \Sigma \vec{i}\cdot\left[(n-2) r^{n-3} \frac{\partial r}{\partial x}\vec{r} + r^{n-2} \frac{\partial \vec{r}}{\partial x}\right].$$

$$= n \sum \vec{i} \cdot \left[(n-2)\, r^{n-2} (x/r) \vec{r} + r^{n-2} \vec{i} \right.$$

$$= n \sum \vec{i} \cdot \left[(n-2)\, r^{n-4}\, x\, \vec{r} + r^{n-2} \vec{i} \right]$$

$$= n \cdot \left[(n-2)\, r^{n-4} \left(\sum x \vec{i} \cdot \vec{r} \right) + r^{n-2} \sum \vec{i} \cdot \vec{i} \right]$$

$$= n \left[(n-2)\, r^{n-4} (\vec{r} \cdot \vec{r}) + r^{n-2} (3) \right.$$

$$= n \left[(n-2)\, r^{n-4}\, r^2 + r^{n-2} (3) \right]$$

$$= n \left[(n-2)\, r^{n-2} + 3 r^{n-2} \right]$$

$$= r^{n-2}\, n \left[n-2+3 \right]$$

$$= n (n+1)\, r^{n-2}$$

If $\vec{F} = (y^2+z^2-x^2)\vec{i} + (z^2+x^2-y^2)\vec{j} + (x^2+y^2-z^2)\vec{K}$

find $\operatorname{div} \vec{f}$, curl $\vec{f}$.

$$\operatorname{div} \vec{F} = \nabla \cdot \vec{F}$$

$$= \sum \vec{i}\, \partial/\partial x \cdot \left[(y^2+z^2-x^2)\vec{i} + (z^2+x^2-y^2)\vec{j} + (x^2+y^2-z^2)\vec{K} \right]$$

$$= \left(\vec{i}\, \partial/\partial x + \vec{j}\, \partial/\partial y + \vec{K}\, \partial/\partial z \right) \cdot \left[(y^2+z^2-x^2)\vec{i} + (z^2+x^2-y^2)\vec{j} + (x^2+y^2-z^2)\vec{K} \right]$$

$$= \partial/\partial x\, (y^2+z^2-x^2) + \partial/\partial y\, (z^2+x^2-y^2) +$$

$\partial/...$

$= -2x - 2y - 2z.$

$$\text{curl } \vec{F} = \nabla \times \vec{F}$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ (y^2+z^2-x^2) & (x^2+z^2-y^2) & (x^2+y^2-z^2) \end{vmatrix}$$

$$= \vec{i}\left[\partial/\partial y\,(x^2+y^2-z^2) - \partial/\partial z\,(x^2+\cancel{z^2}-y^2)\right]$$

$$-\vec{j}\left[\partial/\partial x\,(x^2+y^2-z^2) - \partial/\partial z\,(y^2+z^2-x^2)\right]$$

$$+\vec{k}\left[\partial/\partial x\,(x^2+z^2-y^2) - \partial/\partial y\,(y^2+z^2-x^2)\right]$$

$$= \vec{i}\,[2y-2z] - \vec{j}\,[+2x-2z] + \vec{k}\,[2x-2y]$$

$$= 2(y-z)\vec{i} - 2(x-z)\vec{j} + 2(x-y)\vec{k}$$

$$= 2\left[(y-z)\vec{i} + (z-x)\vec{j} + (x-y)\vec{k}\right]$$

2/1/20

40) P.T $\text{div}(\text{grad}\,\phi) = \nabla^2\phi$ (or) $\nabla\cdot(\nabla\phi) = \nabla^2\phi$

$$\nabla\cdot\nabla\phi = \nabla\cdot\left(\sum \vec{i}\frac{\partial\phi}{\partial x}\right)$$

$\text{grad}\,\phi = \nabla\phi = \sum \vec{i}\frac{\partial\phi}{\partial x}$

$\text{div}\,\phi = \nabla\cdot\phi = \sum \vec{i}\,\partial/\partial x\cdot\phi$

$$= \nabla\cdot\left(\vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z}\right)$$

$$= \left(\vec{i}\,\partial/\partial x + \vec{j}\,\partial/\partial y + \vec{k}\,\partial/\partial z\right)\cdot$$

$$\left(\vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z}\right)$$

$$= \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2}.$$

$$= \nabla^2 \phi.$$

1) If $\vec{F} = xy^3\vec{i} - 2x^2yz\vec{j} + 2yz^4\vec{k}$ at $(1,$

11) Find the divergent of $\vec{F}$ and curl (curl $\vec{F}$)

$\nabla \times (\nabla \times \vec{F})$. $\quad \nabla \cdot \vec{F} \quad \nabla \times (\nabla \times \vec{F})$

$$\vec{F} = xy^3\vec{i} - 2x^2yz\vec{j} + 2yz^4\vec{k}$$

$$\text{div } \vec{F} = \nabla \cdot \vec{F}$$

$$= \left(\vec{i}\, \partial/\partial x + \vec{j}\, \partial/\partial y + \vec{k}\, \partial/\partial z\right)$$

$$(xy^3\vec{i} - 2x^2yz\vec{j} + 2yz^4\vec{k})$$

$$= \partial/\partial x\,(xy^3) + \partial/\partial y\,(-2x^2yz) + \partial/\partial z\,(2yz^4)$$

$$= y^3 - 2x^2z + 8yz^3.$$

$$\text{div } \vec{F}_{\text{at }(1,-1,1)} = (-1)^3 - 2(1)^2(1) + 8(-1)(1)^3$$

$$= -1 - 2 - 8$$

$$= -11$$

$$\text{curl } \vec{F} = \nabla \times \vec{F}.$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ xy^3 & -2x^2yz & 2yz^4 \end{vmatrix}$$

$$= \vec{i}(2z^4 + 2x^2y) - \vec{j}(0-0) + \vec{k}(-4xyz - 3xy^2)$$

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$$\text{curl}\,\vec{F}\Big|_{at\,(1,-1,1)} = \vec{i}\,(2(1)^4 + 2(1)^2(-1)) + \vec{k}\,(-4(1)(-1)(1) - 2(1)(-1)^2)$$

$$= \vec{k}$$

$$\text{curl}(\text{curl}\,\vec{F}) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 0 & 0 & 1 \end{vmatrix}$$

$$= \vec{i}\,(0-0) - \vec{j}\,(0-0) + \vec{k}\,(0-0)$$

$$= 0$$

2) Determine the constant a, s.t the vector

$\vec{F} = (x+2)\vec{i} + (3x+ay)\vec{j} + (x-5z)\vec{k}$ is such that its divergence is 0.

$$\vec{F} = (x+2)\vec{i} + (3x+ay)\vec{j} + (x-5z)\vec{k}$$

$$\text{div}\,\vec{F} = 0 \quad \nabla\cdot\vec{F} = 0$$

$$\nabla\cdot\vec{F} = 0 \quad (\vec{i}\,\partial/\partial x + \vec{j}\,\partial/\partial y + \vec{k}\,\partial/\partial z)$$

$$\left(\vec{i}\,\partial/\partial x + \vec{j}\,\partial/\partial y + \vec{k}\,\partial/\partial z\right)\left((x+2)\vec{i} + (3x+ay)\vec{j} + (x-5z)\vec{k}\right) = 0$$

$$\partial/\partial x\,(x+2) + \partial/\partial y\,(3x+ay) + \partial/\partial z\,(x-5z) = 0$$

$$1 + a - 5 = 0$$

$$a - 4 = 0$$

$$a = 4.$$

3) P.T grad $(\operatorname{div}\vec{F}) = \operatorname{curl}(\operatorname{curl}\vec{F}) + \nabla^2 \vec{F}$

43) $\nabla(\nabla\cdot\vec{F}) = \nabla\times(\nabla\times\vec{F}) + \nabla^2\vec{F}$.

$$\nabla(\nabla\cdot\vec{F}) = \nabla\left(\Sigma\vec{i}\cdot\frac{\partial\vec{F}}{\partial x}\right)$$

$$= \nabla\left(\vec{i}\cdot\frac{\partial\vec{F}}{\partial x} + \vec{j}\cdot\frac{\partial\vec{F}}{\partial y} + \vec{k}\cdot\frac{\partial\vec{F}}{\partial z}\right)$$

$$= \Sigma\vec{i}\,\partial/\partial x\left(\vec{i}\cdot\frac{\partial\vec{F}}{\partial x} + \vec{j}\cdot\frac{\partial\vec{F}}{\partial y} + \vec{k}\cdot\frac{\partial\vec{F}}{\partial z}\right)$$

$$= \Sigma\vec{i}\left(\vec{i}\cdot\frac{\partial^2\vec{F}}{\partial x^2} + \vec{j}\cdot\frac{\partial^2\vec{F}}{\partial y\partial x} + \vec{k}\cdot\frac{\partial^2\vec{F}}{\partial z\partial x}\right)$$

$$= \Sigma\left[\vec{i}\left(\vec{i}\cdot\frac{\partial^2\vec{F}}{\partial x^2}\right) + \left(\vec{j}\cdot\frac{\partial^2\vec{F}}{\partial y\partial x}\right) + \left(\vec{k}\cdot\frac{\partial^2\vec{F}}{\partial x\partial z}\right)\right]$$

$$= \Sigma\left[\vec{i}\left(\vec{i}\cdot\frac{\partial^2\vec{F}}{\partial x^2}\right) + \vec{i}\left(\vec{j}\cdot\frac{\partial^2\vec{F}}{\partial x\partial y}\right) + \vec{i}\left(\vec{k}\cdot\frac{\partial^2\vec{F}}{\partial z\partial x}\right)\right] \rightarrow ①$$

$$\operatorname{curl}(\operatorname{curl}\vec{F}) = \nabla\times(\nabla\times\vec{F})$$

$$= \nabla\times\left(\Sigma\vec{i}\times\frac{\partial\vec{F}}{\partial x}\right)$$

$$= \nabla\times\left(\vec{i}\times\frac{\partial\vec{F}}{\partial x} + \vec{j}\times\frac{\partial\vec{F}}{\partial y} + \vec{k}\times\frac{\partial\vec{F}}{\partial z}\right)$$

$$= \Sigma \vec{i} \times \partial/\partial x \left[\vec{i} \times \frac{\partial \vec{F}}{\partial x} + \vec{j} \times \frac{\partial \vec{F}}{\partial y} + \vec{k} \times \frac{\partial \vec{F}}{\partial z} \right]$$

$$= \Sigma \vec{i} \times \left[\vec{i} \times \frac{\partial^2 \vec{F}}{\partial x^2} + \vec{j} \times \frac{\partial^2 \vec{F}}{\partial y \partial x} + \vec{k} \times \frac{\partial^2 \vec{F}}{\partial z \partial x} \right]$$

$$= \Sigma \vec{i} \times \left[\cancel{\vec{i} \times \partial} \right.$$

$$= \Sigma \left[\vec{i} \times \left(\vec{i} \times \frac{\partial^2 \vec{F}}{\partial x^2} \right) + \vec{i} \times \left(\vec{j} \times \frac{\partial^2 \vec{F}}{\partial x \partial y} \right) + \vec{i} \times \left(\vec{k} \times \frac{\partial^2 \vec{F}}{\partial x \partial z} \right) \right]$$

W.K.T

$$\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c}) \cdot \vec{b} - (\vec{a} \cdot \vec{b}) \cdot \vec{c}.$$

$$= \Sigma \left\{ \left[\left(\vec{i} \cdot \frac{\partial^2 \vec{F}}{\partial x^2} \right) \cdot \vec{i} - (\vec{i} \cdot \vec{i}) \cdot \frac{\partial^2 \vec{F}}{\partial x^2} \right] + \left[\left(\vec{i} \cdot \frac{\partial^2 \vec{F}}{\partial x \partial y} \right) \cdot \vec{j} - (\vec{i} \cdot \vec{j}) \cdot \frac{\partial^2 \vec{F}}{\partial x \partial y} \right]. + \left[\left(\vec{i} \cdot \frac{\partial^2 \vec{F}}{\partial x \partial z} \right) \cdot \vec{k} - (\vec{i} \cdot \vec{k}) \cdot \frac{\partial^2 \vec{F}}{\partial x \partial z} \right] \right\}$$

$$= \Sigma \left[\left(\vec{i} \cdot \frac{\partial^2 \vec{F}}{\partial x^2} \right) \cdot \vec{i} - \frac{\partial^2 \vec{F}}{\partial x^2} + \left(\vec{i} \cdot \frac{\partial^2 \vec{F}}{\partial x \partial y} \right) \cdot \vec{j} + \left(\vec{i} \cdot \frac{\partial^2 \vec{F}}{\partial x \partial z} \cdot \vec{k} \right) \right]$$

$$= \Sigma\left[\left(\vec{i}\cdot\frac{\partial^2 \vec{F}}{\partial x^2}\right)\cdot\vec{i} + \left(\vec{j}\cdot\frac{\partial^2 \vec{F}}{\partial x\partial y}\right) + \left(\vec{k}\cdot\frac{\partial^2 \vec{F}}{\partial x\partial z}\right)\vec{i}\right] - \Sigma\frac{\partial^2 \vec{F}}{\partial x^2}$$

$$\text{curl}(\text{curl}\,\vec{F}) = \text{grad}(\text{div}\,\vec{F}) - \nabla^2\vec{F}$$

(By(I)

$$\text{curl}(\text{curl}\,\vec{F}) + \nabla^2\vec{F} = \text{grad}(\text{div}\,\vec{F})$$

1) 44) P.T $\nabla^2(r^n\vec{r}) = n(n+3)\,r^{n-2}\vec{r}$.

$$\nabla^2(r^n\vec{r}) = \nabla\cdot\nabla(r^n\vec{r}) \rightarrow ①$$

We know that,

$$\nabla(r^n\vec{r}) = (n+3)\,r^n.$$

$\nabla\cdot(n+3)r^n$

sub in ① we get,

$$= \nabla\cdot(n+3)\,r^n.$$

$$= \Sigma\vec{i}\cdot\partial/\partial x\,(n+3)\,r^n.$$

$$= \Sigma\vec{i}\,(n+3)\cdot\partial/\partial x\,r^n.$$

$$= \Sigma\vec{i}\,(n+3)\cdot n r^{n-1}\frac{\partial r}{\partial x} \quad \left[\because \frac{\partial r}{\partial x}\right.$$

$$= \Sigma\vec{i}\,(n+3)\cdot n r^{n-1}(x/r)$$

$$= \Sigma\vec{i}\,(n+3)\cdot n r^{n-2}x$$

$$= n(n+3)\,r^{n-2}\,\Sigma x\vec{i}$$

$$= n(n+3)\,r^{n-2}\vec{r}.$$

$$\nabla^2(r^n\vec{r}) = n(n+3)\,r^{n-2}\vec{r}.$$

Hence proved

2) Find div $\vec{F}$ and curl $\vec{F}$ for $\vec{F} = x^2y\vec{i} + xz\vec{j} + 2yz\vec{k}$.

4.5)

$$\text{div}\,\vec{F} = \nabla \cdot \vec{F}$$

$$= \nabla \cdot (x^2y\vec{i} + xz\vec{j} + 2yz\vec{k})$$

$$= \sum \vec{i}\,\partial/\partial x \cdot (x^2y\vec{i} + xz\vec{j} + 2yz\vec{k})$$

$$= \vec{i}\,\partial/\partial x + \vec{j}\,\partial/\partial y + \vec{k}\,\partial/\partial z \cdot (x^2y\vec{i} + xz\vec{j} + 2yz\vec{k})$$

$$= \partial/\partial x\,(x^2y) + \partial/\partial y\,(xz) + \partial/\partial z\,(2yz)$$

$$= 2xy + 2y$$

$$\text{curl}\,\vec{F} = \nabla \times \vec{F}$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^2y & xz & 2yz \end{vmatrix}$$

$$= \vec{i}\left[\partial/\partial y\,(2yz) - \partial/\partial z\,(xz)\right] - \vec{j}\left[\partial/\partial x\,(2yz) - \partial/\partial z\,(x^2y)\right] + \vec{k}\left[\partial/\partial x\,(xz) - \partial/\partial y\,(x^2y)\right]$$

$$= \vec{i}\,(2z - x) - \vec{j}\,(2(0) - 0) + \vec{k}\,[z - x^2]$$

$$= (2z - x)\vec{i} - 0\vec{j} + \vec{k}\,(z - x^2)$$

$$= (2z - x)\vec{i} + (z - x^2)\vec{k}.$$

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$\vec{F} = (x^2+yz)\vec{i} + \vec{j}(y^2+zx) + (z^2+xy)\vec{k}$

find diverg...

$div\vec{F} = \nabla\cdot\vec{F}$

$= \Sigma\vec{i}\,\partial/\partial x\,[(x^2+yz)\vec{i}+(y^2+zx)\vec{j}$
$+(z^2+xy)\vec{k}]$

$=(\vec{i}\,\partial/\partial x+\vec{j}\,\partial/\partial y+\vec{k}\,\partial/\partial z)\cdot$

$[(x^2+yz)\vec{i} + (y^2+xz)\vec{j} +$
$(z^2+xy)\vec{k}]$

$= (x^2+yz)\,\partial/\partial x + (y^2+zx)\,\partial/\partial y$
$+ (z^2+xy)\,\partial/\partial z$

$= 2x+2y+2z.$

$curl\,\vec{F} = \nabla\times\vec{F}$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ (x^2+yz) & (y^2+zx) & (z^2+xy) \end{vmatrix}$$

$=\vec{i}\,[\partial/\partial y\,(z^2+xy) - \partial/\partial z\,(y^2+zx)]$

$-\vec{j}\,[\partial/\partial x\,(z^2+xy) - \partial/\partial z\,(x^2+yz)]$

$+\vec{k}\,[\partial/\partial x\,(y^2+zx) - \partial/\partial y\,(x^2+yz)]$

$= \vec{i}\,(x-x) - \vec{j}\,(y-y) + \vec{k}\,(z-z)$

$= \vec{i}\,(0) - \vec{j}\,(0) + \vec{k}\,(0)$

$=0$

$\vec{F} = x^2y\vec{i} + xz\vec{j} + 2yz\vec{k}$ at $(-1,1,1)$

find $\text{div}\vec{F}$ and $\text{curl}(\text{curl}\vec{F})$.

$$\text{div}\vec{F} = \nabla\cdot\vec{F}$$

$$= \sum \vec{i}\,\partial/\partial x \cdot [x^2y\vec{i} + xz\vec{j} + 2yz\vec{k}]$$

$$= [\partial/\partial x\,\vec{i} + \vec{j}\,\partial/\partial y + \vec{k}\,\partial/\partial z]\cdot$$

$$[x^2y\vec{i} + xz\vec{j} + 2yz\vec{k}]$$

$$= \partial/\partial x\,(x^2y) + \partial/\partial y\,(xz) + \partial/\partial z\,(2yz)$$

$$\text{div}\vec{F} = 2xy + 2y$$

$$\text{div}\vec{F}_{\text{at }(-1,1,1)} = (2(-1)(1) + 2(1))$$

$$= -2+2.$$

$$= 0$$

$$\text{curl}(\text{curl}\vec{F}) = \nabla\times(\nabla\times\vec{F})$$

$$\text{curl}\vec{F} = \nabla\times\vec{F}$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^2y & xz & 2yz \end{vmatrix}$$

$$= \vec{i}\,[\partial/\partial y\,(2yz) - \partial/\partial z\,(xz)]$$

$$-\vec{j}\,[\partial/\partial x\,(2yz) - \partial/\partial z\,(x^2y)]$$

$$+\vec{k}\,[\partial/\partial x\,(xz) - \partial/\partial y\,(x^2y)]$$

$$= \vec{i}\,[2z - x] - \vec{j}\,[0] + \vec{k}\,[z - x^2]$$

$$= (2z - x)\vec{i} + (z - x^2)\vec{k}$$

curl $\vec{F}$ at $(-1, 1, 1)$

$$= (2(1) - (-1))\vec{i} + (1 - (-1)^2)\vec{k}$$

$$= (2+1)\vec{i} + (0)\vec{k}$$

$$= 3\vec{i}.$$

$$\text{curl}(\text{curl}\,\vec{F}) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 3 & 0 & 0 \end{vmatrix}$$

$$= \vec{i}\,(0) - \vec{j}\,(0) + \vec{k}\,(0)$$

$$= 0.$$

) If $\vec{F} = (x+y+1)\vec{i} + \vec{j} - (x+y)\vec{k}$. Prove that $\vec{F} \cdot \text{curl}\,\vec{F} = 0$

$$\text{curl}\,\vec{F} = \nabla \times \vec{F}$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x+y+1 & 1 & -(x+y) \end{vmatrix}$$

$$= \vec{i}\left(\partial/\partial y\,(-(x+y)) - 0\right)$$

$$- \vec{j}\left(\partial/\partial x\,-(x+y) - \partial/\partial z\,(x+y+1)\right)$$

$$+ \vec{k}\left(0 - \partial/\partial y\,(x+y+1)\right)$$

$$= -\vec{i} - \vec{j}(-1-0) + \vec{k}(0-1)$$

$$= -\vec{i} + \vec{j} - \vec{k}$$

$$\vec{F} \cdot \text{curl}\,\vec{F} = \left[(x+y+1)\vec{i} + \vec{j} - (x+y)\vec{k}\right] \cdot \left[-\vec{i} + \vec{j} - \vec{k}\right]$$

$$= -(x+y+1) + 1 + (x+y)$$

$$= -x - y - 1 + 1 + x + y$$

$$= 0$$

$$\vec{F} \cdot \text{curl}\,\vec{F} = 0$$

Hence proved

6/1/2020

1) If $r^2 = x^2+y^2+z^2$, P.T $\nabla^2 f(r) = f''(r) + 2/r\, f'(r)$

(49) and deduce that $\nabla^2 r^2 = 6$

$$\nabla^2 f(r) = \Sigma \frac{\partial^2}{\partial x^2}(f(r))$$

$$= \Sigma\, \partial/\partial x \left[\partial/\partial x (f(r))\right]$$

$$= \Sigma\, \partial/\partial x \left[f'(r) \frac{\partial r}{\partial x}\right]$$

$$= \Sigma\, \partial/\partial x \left[f'(r)\, x/r\right]$$

$$= \Sigma \left[x/r\, f''(r) \frac{\partial r}{\partial x} + \frac{f'(x)}{r} + f'(r)\, x \left(-1/r^2\right) \frac{\partial r}{\partial x}\right]$$

$$= \Sigma \left[\frac{f'(x)}{r} + x/r\, f''(r)\, x/r - f'(r)\, x/r^2 \,(x/r)\right]$$

$$= \Sigma \left[\frac{f'(x)}{r} + \frac{x^2}{r^2} f''(r) - \frac{x^2}{r^3} f'(x)\right]$$

$$= \frac{f'(r)}{r} \Sigma 1 + \frac{\Sigma x^2}{r^2} f''(x) - \frac{\Sigma x^2}{r^3} f'$$

$$= \frac{3 f'(r)}{r} + \frac{\cancel{r^2}}{\cancel{r^2}} f''(r) - \frac{r^2}{r^3} f'($$

$$= f''(r) + \frac{f'(r)}{r}(3-1)$$

$$\nabla^2 f(r) = f''(r) + 2/r\, f'(r)$$

Deduction:-

$$\boxed{f(r) = r^2}$$

$$f'(r) = 2r$$

$$f''(r) = 2$$

$$\nabla^2 f(r) = 2 + 2/r\,(2r)$$

$$= 2 + 2(2)$$

$$= 6$$

$$\nabla^2 f(r) = 6$$

$$\nabla^2 r^2 = 6$$

2) P.T $div(\phi \vec{F}) = \phi\, div \vec{F} + \vec{F}\, grad\, \phi$.

50) $$div(\phi \vec{F}) = \nabla \cdot (\phi \vec{F})$$

$$= \Sigma \vec{i} \cdot \partial/\partial x\,(\phi \vec{F})$$

$$= \Sigma \vec{i} \cdot \left[\phi \frac{\partial \vec{F}}{\partial x} + \vec{F} \frac{\partial \phi}{\partial x}\right]$$

$$= \phi\, \Sigma \vec{i} \cdot \frac{\partial \vec{F}}{\partial x} + \vec{F}\, \Sigma \vec{i} \frac{\partial \phi}{\partial x}$$

$$= \phi\,(\nabla \cdot \vec{F}) + \vec{F}\,(\nabla \phi).$$

$$= \phi\,(div \vec{F}) + \vec{F}\,(grad\, \phi).$$

3) S.T $\vec{f} = (z\cos x + \sin y)\vec{i} + (x\cos y + \sin z)\vec{j} + (y\cos z + \sin x)\vec{k}$

is irrotational & Find a function $\vec{f} = \nabla\phi$.

$$\text{curl } \vec{f} = \nabla \times \vec{f}$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ z\cos x+\sin y & x\cos y+\sin z & y\cos z+\sin x \end{vmatrix}$$

$$= \left[\partial/\partial y\,(y\cos z+\sin x) - \partial/\partial z\,(x\cos y+\sin z)\right]\vec{i}$$

$$-\vec{j}\left[\partial/\partial x\,(y\cos z+\sin x) - \partial/\partial z\,(z\cos x+\sin y)\right]$$

$$+\vec{k}\left[\partial/\partial x\,(x\cos y+\sin z) - \partial/\partial y\,(z\cos x+\sin y)\right]$$

$$= \left[(\cos z + \cancel{y(-\sin z)}) - [x(-\sin y) + \cos y(1) + \cos z\right]\vec{i}$$

$$-\vec{j}\left[y(-\sin z) + \cos z(\right.$$

$$= i\left[\cos z - \cos z\right] - \vec{j}\left[\cos x - \cos x\right] + \vec{k}\left[\cos y - \cos y\right]$$

$$= \vec{i}[0] - \vec{j}[0] + \vec{k}[0]$$

$\text{curl}\,\vec{F} = 0$

$\vec{F}$ is an irrotational.

$$\nabla\phi = (z\cos x + \sin y)\vec{i} + (x\cos y + \sin z)\vec{j} + (y\cos z + \sin x)\vec{k}$$

$$\Sigma\vec{i}\frac{\partial\phi}{\partial x} = (z\cos x + \sin y)\vec{i} + (x\cos y + \sin z)\vec{j} \; (y\cos z + \sin x)\vec{k}$$

$$\vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z} = (z\cos x + \sin y)\vec{i} + (x\cos y + \sin z)\vec{j} + (y\cos z + \sin x)\vec{k}$$

$$\frac{\partial\phi}{\partial x} = z\cos x + \sin y$$

$$\frac{\partial\phi}{\partial y} = x\cos y + \sin z$$

$$\frac{\partial\phi}{\partial z} = y\cos z + \sin x$$

① – partially w.r to 'x'

$$\int\frac{\partial\phi}{\partial x} = z\int\cos x + \int\sin y$$

$$\int\partial\phi = z\int\cos x\,\partial z + \sin y\int\partial x$$

$$= +\sin x(z) + \sin y\,x + f(y,z)$$

② – partially w.r to

$$\int\frac{\partial\phi}{\partial y} = \int x\cos y + \int\sin z$$

$$\int\partial\phi = x\int\cos y\,\partial y + \sin z\int\partial y + f(z,x)$$

$\phi = +\sin y\,(x) + \sin z\; y + f(z,x)$

(3)

$\int \frac{\partial \phi}{\partial z} = y\int \cos z + \int \sin x$

$\int \partial\phi = y\int \cos z\,\partial z + \int \sin x\,\partial z$

$\phi = y + \sin z + \sin x\; z + f(x,y)$

$\phi = \cancel{z\sin x} + \cancel{x\sin y} - x\sin y + y\sin z$

$\cancel{-y\sin z} + z\sin x$

$\phi = x\sin y + y\sin z + z\sin x + c$

5?) S.T the vector function $\vec{f} = x(y-z)\vec{i} + y(z-x)\vec{j} + z(x-y)\vec{k}$

is solenordal.

$\text{div}\,\vec{f} = \text{div}\left[x(y-z)\vec{i} + y(z-x)\vec{j} + z(x-y)\vec{k}\right]$

$= \nabla\cdot\left[x(y-z)\vec{i} + y(z-x)\vec{j} + z(x-y)\vec{k}\right]$

$\cancel{= \Sigma\vec{i}\,\partial}/\partial x\left[(xy - xz\right.$

$= \vec{i}\,\partial/\partial x + \vec{j}\,\partial/\partial y + \vec{k}\,\partial/\partial z\cdot\left[x(y-z)\vec{i} + y(z-x)\vec{j} + (zx-yz)\vec{k}\right]$

$= \partial/\partial x\,(xy - xz) + \partial/\partial y\,(yz - xy) + \partial/\partial z\,(zx - yz)$

$= y - z + z - x + x - y.$

$\operatorname{div} \vec{f} = 0$

So it is solenoidal.

6/1/2020

1) 53) $\operatorname{grad}(\vec{f}\cdot\vec{g}) = \vec{f} \times \operatorname{curl} \vec{g} + (\vec{f}\cdot\nabla)\vec{g} + \vec{g} \times \operatorname{curl} \vec{f} + (\vec{g}\cdot\nabla)\vec{f}.$

$\operatorname{grad}(\vec{f}\cdot\vec{g}) = \nabla(\vec{f}\cdot\vec{g})$

$\operatorname{grad}(\vec{f}\cdot\vec{g}) = \Sigma \vec{i}\, \partial/\partial x\, (\vec{f}\cdot\vec{g})$

$= \Sigma \vec{i} \left[\frac{\partial \vec{f}}{\partial x}\cdot\vec{g} + \vec{f}\cdot\frac{\partial \vec{g}}{\partial x} \right]$

$= \Sigma \vec{i} \left(\frac{\partial \vec{f}}{\partial x}\cdot\vec{g} \right) + \Sigma \vec{i} \left(\vec{f}\cdot\frac{\partial \vec{g}}{\partial x} \right) \rightarrow ①$

We know that $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a}\cdot\vec{c})\vec{b} - (\vec{a}\cdot\vec{b})\vec{c}$

$\vec{f} \times \left(\vec{i} \times \frac{\partial \vec{g}}{\partial x} \right) = \left(\vec{f}\cdot\frac{\partial \vec{g}}{\partial x} \right)\vec{i} - (\vec{f}\cdot\vec{i})\frac{\partial \vec{g}}{\partial x}$

$\left(\vec{f}\cdot\frac{\partial \vec{g}}{\partial x} \right)\vec{i} = \vec{f} \times \left(\vec{i} \times \frac{\partial \vec{g}}{\partial x} \right) + (\vec{f}\cdot\vec{i})\frac{\partial \vec{g}}{\partial x}$

$\Sigma \left(\vec{f}\cdot\frac{\partial \vec{g}}{\partial x} \right)\vec{i} = \Sigma \left(\vec{f} \times \left(\vec{i} \times \frac{\partial \vec{g}}{\partial x} \right) \right) + \Sigma (\vec{f}\cdot\vec{i})\frac{\partial \vec{g}}{\partial x}$

$= \vec{f} \times \left(\Sigma \vec{i}\, \partial/\partial x \times \vec{g} \right) + \vec{f}\, \Sigma \vec{i}\, \frac{\partial}{\partial x}$

$$= \vec{f} \times (\nabla \times \vec{g}) + (\vec{f} \cdot \nabla)\vec{g} \longrightarrow ②$$

$|||^{ly}$

$$\Sigma \left(\vec{g} \cdot \frac{\partial \vec{f}}{\partial x}\right)\vec{i} = \vec{g} \times (\nabla \times \vec{f}) + (\vec{g} \cdot \nabla)\vec{f} \longrightarrow ③$$

using ② x ③ in ①

$$\text{grad}\,(\vec{f} \cdot \vec{g}) = \vec{f} \times (\nabla \times \vec{g}) + (\vec{f} \cdot \nabla)\vec{g} + \vec{g} \times (\nabla \times \vec{f}) + (\vec{g} \cdot \nabla)\vec{f}$$

$$= \vec{f} \times \text{curl}\,\vec{g} + (\vec{f} \cdot \nabla)\vec{g} + \vec{g} \times \text{curl}\,\vec{f} + (\vec{g} \cdot \nabla)\vec{f}$$

Hence proved.

Prove that $\text{div}\,(\vec{f} \times \vec{g}) = \vec{g}\,\text{curl}\,\vec{f} - \vec{f} \cdot \text{curl}\,\vec{g}$

54) $$\text{div}\,(\vec{f} \times \vec{g}) = \nabla \cdot (\vec{f} \times \vec{g})$$

$$= \Sigma \vec{i}\,\partial/\partial x \cdot (\vec{f} \times \vec{g})$$

$$= \Sigma \vec{i} \cdot \left[\frac{\partial \vec{f}}{\partial x} \times \vec{g} + \vec{f} \times \frac{\partial \vec{g}}{\partial x}\right]$$

$$= \Sigma \vec{i} \cdot \left(\frac{\partial \vec{F}}{\partial x} \times \vec{g}\right) + \Sigma \vec{i}\left(\vec{f} \times \frac{\partial \vec{g}}{\partial x}\right)$$

$$= \Sigma \left(\vec{i} \times \frac{\partial \vec{f}}{\partial x}\right) \cdot \vec{g} - \Sigma \left(\vec{i} \times \frac{\partial \vec{g}}{\partial x}\right) \cdot \vec{f}$$

$$= \boxed{\Sigma \left(\vec{i}\,\partial/\partial x \times \vec{f}\right) \cdot \vec{g}} - \boxed{\Sigma \left(\vec{i}\,\partial/\partial x \times \vec{g}\right) \cdot \vec{f}}$$

$$= \text{curl}\,\vec{f} \cdot \vec{g} - \text{curl}\,\vec{g} \cdot \vec{f}$$

$$= \vec{g} \cdot \text{curl}\,\vec{f} - \vec{f}\,\text{curl}\,\vec{g}$$

$\nabla \cdot \vec{r} =$ and $\nabla \times \vec{r} = 0$

55) P.T $\nabla \cdot \vec{r} =$

$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$.

$\nabla \cdot \vec{r} = \Sigma \vec{i}\, \partial/\partial x \cdot (x\vec{i} + y\vec{j} + z\vec{k})$

$= \partial/\partial x \cdot x + \partial/\partial y\, y + \partial/\partial z\, z$

$= 1 + 1 + 1$

$\nabla \cdot \vec{r} = 3$.

$\nabla \times \vec{r} = \Sigma \vec{i}\, \partial/\partial x \times (x\vec{i} + y\vec{j} + z\vec{k})$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x & y & z \end{vmatrix}$$

$= \vec{i}\,(\partial/\partial y\,(z) - \partial/\partial z\,(y)) \mp \vec{j}\,(\partial/\partial x\,(z) - \partial/\partial z\,(x))$

$+ \vec{k}\,(\partial/\partial x\, y - \partial/\partial y\,(x))$

$= \vec{i}\,(0-0) - \vec{j}\,(0-0) + \vec{k}\,(0-0)$

$\nabla \times \vec{r} = 0$

56) P.T curl (grad ϕ) = 0

grad $\phi = \Sigma \vec{i}\, \partial/\partial x\, \phi$.

$= \vec{i}\, \partial\phi/\partial x + \vec{j}\, \partial\phi/\partial y + \partial\phi/\partial z\, \vec{k}$.

$$\text{curl} \times \text{grad}\,\phi = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \frac{\partial\phi}{\partial x} & \frac{\partial\phi}{\partial y} & \frac{\partial\phi}{\partial z} \end{vmatrix}$$

$= \vec{i}\,(\partial/\partial y \ \partial\phi/\partial z - \partial\phi/\partial y \ \ldots$

$\left(\partial/\partial x \frac{\partial\phi}{\partial z} - \frac{\partial\phi}{\partial x}\,\partial/\partial z\right) + \vec{k}$

$\left(\partial/\partial x \frac{\partial\phi}{\partial y} - \frac{\partial\phi}{\partial x}\,\partial/\partial y\right)$

$= \vec{i}\left(\frac{\partial^2\phi}{\partial y\partial z} - \frac{\partial^2\phi}{\partial y\partial z}\right) - \vec{j}\left(\frac{\partial^2\phi}{\partial x\partial z} - \frac{\partial^2\phi}{\partial x\partial z}\right)$

$+ \vec{k}\left(\frac{\partial^2\phi}{\partial x\partial y} - \frac{\partial^2\phi}{\partial x\partial y}\right).$

$= \vec{i}\,(0-0) - \vec{j}\,(0-0) + \vec{k}\,(0-0)$

curl x gradϕ = 0

P.T div (curl $\vec{F}$) = 0

57)

curl $\vec{F} = \nabla \times \vec{F}$

div (curl $\vec{F}$) $= \nabla\cdot(\nabla\times\vec{F}) \Rightarrow a\cdot(a\times b)$

$= [a\ a\ b]$.

$= [\nabla\ \nabla\ \vec{F}]$

$= 0.$

58) P.T curl $(\vec{f}\times\vec{g}) = \vec{f}\,\mathrm{div}\,\vec{g} - \vec{g}\,\mathrm{div}\,\vec{f} + (\vec{g}\cdot\nabla)\vec{f}$

$-(\vec{f}\cdot\nabla)\vec{g}.$

curl $(\vec{f}\times\vec{g}) = \nabla\times(\vec{f}\times\vec{g})$

$= \sum\vec{i}\,\partial/\partial x \times (\vec{f}\times\vec{g})$

$= \sum\vec{i}\times\left[\partial/\partial x\,\vec{f}\times\vec{g} + \vec{f}\times\partial/\partial x\,\vec{g}\right]$

$= \sum\vec{i}\left(\partial/\partial x\,\vec{f}\times\vec{g}\right) + \sum\vec{i}\left(\vec{f}\times\partial/\partial x \ldots\right.$

$$= \Sigma\left[(\vec{i}\cdot\vec{g})\frac{\partial \vec{F}}{\partial x} - (\vec{i}\cdot\frac{\partial \vec{f}}{\partial x})\vec{g}\right] +$$

$$\Sigma\left[(\vec{i}\cdot\frac{\partial \vec{g}}{\partial x})\vec{f} - (\vec{i}\cdot\vec{f})\frac{\partial \vec{g}}{\partial x}\right]$$

$$= \Sigma(\vec{i}\cdot\frac{\partial \vec{g}}{\partial x})\vec{f} - \Sigma(\vec{i}\cdot\vec{f})\frac{\partial \vec{g}}{\partial x}$$

$$+ \Sigma(\vec{i}\cdot\vec{g})\frac{\partial \vec{f}}{\partial x} - \Sigma(\vec{i}\cdot\frac{\partial \vec{f}}{\partial x})\vec{g}$$

$$= \Sigma(\vec{i}\,\partial/\partial x\cdot\vec{g})\vec{f} - (\vec{f}\cdot\nabla)\vec{g} +$$

$$(\vec{g}\cdot\nabla)\vec{f} - \Sigma(\vec{i}\,\partial/\partial x\cdot\vec{f})\vec{g}$$

$$= (\nabla\cdot\vec{g})\vec{f} - (\vec{f}\cdot\nabla)\vec{g} + (\vec{g}\cdot\nabla)\vec{f}$$

$$-(\nabla\cdot\vec{f})\vec{g}$$

$$= \vec{f}\,\mathrm{div}\,\vec{g} - (\vec{f}\cdot\nabla)\vec{g} + (\vec{g}\cdot\nabla)\vec{f}$$

$$-\vec{g}\,\mathrm{div}\,\vec{f}.$$

59) If $\vec{v} = \vec{\omega}\times\vec{r}$, P.T $\vec{\omega} = \frac{1}{2}\,\mathrm{curl}\,\vec{v}$.

Let $\vec{\omega} = w_1\vec{i} + w_2\vec{j} + w_3\vec{k}$.

$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$.

$$\vec{v} = \vec{w}\times\vec{r} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ w_1 & w_2 & w_3 \\ x & y & z \end{vmatrix}$$

$$= \vec{i}\,[zw_2 - yw_3] - \vec{j}\,[zw_1 - w_3 x]$$

$$+ \vec{k}\,[yw_1 - xw_2]$$

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$$\text{curl } \vec{V} = \nabla \times \vec{V}.$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ (z\omega_2 - y\omega_3) & (z\omega_1 - \omega_3 x) & (y\omega_1 - x\omega_2) \end{vmatrix}$$

$$\text{curl } \vec{V} = \vec{i}(\omega_1 + \omega_1) - \vec{j}(\omega_2 + \omega_2) + \vec{k}(\omega_3 + \omega_3)$$

$$= 2\omega_1 \vec{i} + 2\omega_2 \vec{j} + 2\omega_3 \vec{k}.$$

$$= 2(\omega_1 \vec{i} + \omega_2 \vec{j} + \omega_3 \vec{k})$$

$$\text{curl } \vec{V} = 2\vec{\omega}$$

$$\vec{\omega} = 1/2 \text{ curl } \vec{V}.$$

60) Find $\nabla(1/r)$

$$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$$

$$|\vec{r}| = r = \sqrt{x^2 + y^2 + z^2}.$$

$$r^2 = x^2 + y^2 + z^2.$$

Diff p.w.r.t (x)

$$2r \frac{\partial r}{\partial x} = 2x$$

$$\frac{\partial r}{\partial x} = x/r.$$

$|||^{ly}$

$$\frac{\partial r}{\partial y} = y/r$$

$$\frac{\partial r}{\partial z} = z/r.$$

$$\nabla(1/r) = (\vec{i}\,\partial/\partial x + \vec{j}\,\partial/\partial y + \vec{k}\,\partial/\partial z)(1/r)$$

$$= \vec{i}\,\partial/\partial x(1/r) + \vec{j}\,\partial/\partial y(1/r) + \vec{k}\,\partial/\partial z(1/r)$$

$$= \vec{i}(-1/r^2)\frac{\partial r}{\partial x} + \vec{j}(-1/r^2)\frac{\partial r}{\partial y} + \vec{k}(-1/r^2)\frac{\partial r}{\partial z}.$$

$$= -1/r^2\left[\vec{i}\, x/r + \vec{j}\, y/r + \vec{k}\, z/r\right]$$

$$= -1/r^3\left[x\vec{i} + y\vec{j} + z\vec{k}\right]$$

$$\nabla(1/r) = -\frac{\vec{r}}{r^3}$$

61) $\vec{F} = (x^2-yz)\vec{i} + (y^2-zx)\vec{j} + (z^2-xy)\vec{k}$ is irrotational & find ϕ s.t $\vec{f} = \nabla\phi$.

$$\text{curl } \vec{F} = \nabla \times \vec{f}.$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^2-yz & y^2-zx & z^2-xy \end{vmatrix}$$

$$= \vec{i}\left[\partial/\partial y\,(z^2-xy) - \partial/\partial z\,(y^2-zx)\right]$$
$$-\vec{j}\left[\partial/\partial x\,(z^2-xy) - \partial/\partial z\,(x^2-yz)\right]$$
$$+\vec{k}\left[\partial/\partial x\,(y^2-zx) - \partial/\partial y\,(x^2-yz)\right]$$

$$= \vec{i}\,(-x-(-x)) - \vec{j}\,(-y-(-y)) + \vec{k}\,(-z-(-z))$$

$$= \vec{i}(0) - \vec{j}(0) + \vec{k}(0)$$

$$= 0$$

$\vec{f}$ is an irrotational

$$\nabla\phi = (x^2-yz)\vec{i} + (y^2-zx)\vec{j} + (z^2-xy)\vec{k}.$$

$$\Sigma\vec{i}\frac{\partial\phi}{\partial x} = (x^2-yz)\vec{i} + (y^2-zx)\vec{j} + (z^2-xy)\vec{k}.$$

$\vec{i}\dfrac{\partial\phi}{\partial x} + \vec{j}\dfrac{\partial\phi}{\partial y} + \vec{k}\dfrac{\partial\phi}{\partial z} = (x^2-yz)\vec{i} + (y^2-zx)\vec{j} + (z^2-xy)\vec{k}$.

$$\frac{\partial\phi}{\partial x} = x^2 - yz.$$

$$\frac{\partial\phi}{\partial y} = y^2 - zx.$$

$$\frac{\partial\phi}{\partial z} = z^2 - xy.$$

$$\int\frac{\partial\phi}{\partial x} = \int x^2 - \int yz$$

$$\int\partial\phi = \int x^2\partial x - yz\int\partial x$$

$$\phi = x^3/3 - xyz + f(y,z)$$

$$\int\frac{\partial\phi}{\partial y} = \int y^2 - \int zx$$

$$\int\partial\phi = \int y^2\partial y - zx\int\partial y.$$

$$\phi = y^3/3 - xyz + f(z,x)$$

$$\int\frac{\partial\phi}{\partial z} = \int z^2 - \int xy.$$

$$\int\partial\phi = \int z^2\partial z - xy\int\partial z$$

$$\phi = z^3/3 - xyz + f(x,y).$$

$$\phi = x^3/3 + y^3/3 + z^3/3 - xyz + c.$$

b2) $\vec{F} = (x+2y+az)\vec{i} + (bx-3y-z)\vec{j} + (4x+cy+2)\vec{k}$ Find the value of the constants a,b,c so that the vector.

$$\text{curl}\,\vec{F} = \nabla\times\vec{F}$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x+2y+az & bx-3y-z & 4x+cy+2 \end{vmatrix}$$

$$= \vec{i}\left[\partial/\partial y(4x+cy+2) - \partial/\partial z(bx\right.$$

$$-\vec{j}\left[\partial/\partial x\,(4x+cy+2) - \partial/\partial z\,(x+2y+az)\right]$$
$$+\vec{k}\left[\partial/\partial x\,(bx-3y-z) - \partial/\partial y\,(x+2y+az)\right]$$

$$= \vec{i}\,(c-1) - \vec{j}\,(4-a) + \vec{k}\,(b-2)$$
$$= (c-1)\vec{i} - (4-a)\vec{j} + (b-2)\vec{k}.$$

$$c-1=0 \quad , \quad -4+a=0 \quad , \quad b-2=0$$
$$c=1 \quad , \quad a=4 \quad , \quad b=2$$

$$=(1-1)\vec{i} - (4-4)\vec{j} + (2-2)\vec{k}.$$

$$\text{curl}\,\vec{F} = 0$$

7/1/2020 Determine the constant 'a', so that the vector

63) $\vec{f} = (x+3y)\vec{i} + (y-2z)\vec{j} + (x+az)\vec{k}$ is solenoidal.

$$\vec{f} = (x+3y)\vec{i} + (y-2z)\vec{j} + (x+az)\vec{k}$$

$$\text{div}\,\vec{f} = 0$$
$$\nabla\cdot\vec{f} = 0$$

$$\left(\vec{i}\,\partial/\partial x + \vec{j}\,\partial/\partial y + \vec{k}\,\partial/\partial z\right)\cdot\left((x+3y)\vec{i} + (y-2z)\vec{j} + (x+az)\vec{k}\right) = 0$$

$$\partial/\partial x\,(x+3y) + \partial/\partial y\,(y-2z) + \partial/\partial z\,(x+az) = 0$$

$$1+1+a=0$$
$$a+2=0$$
$$a=-2.$$

~~Find the~~

64) Find the value of a, b, c, so that $\vec{F} = (x+2y+az)\vec{i} + (bx-3y-z)\vec{j} + (4x+cy+2)\vec{k}$ is irrotational.

$$\vec{F} = (x+2y+az)\vec{i} + (bx-3y-z)\vec{j} + (4x+cy+2) = 0,$$

$\text{curl}\,\vec{F} = 0$

$\nabla \times \vec{F} = 0$

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ (x+2y+az) & (bx-3y-z) & (4x+cy+2z) \end{vmatrix} = 0$$

$\vec{i}\,(c+1) - \vec{j}\,(4-a) + \vec{k}\,(b-2) = 0\vec{i} + 0\vec{j} + 0\vec{k}$

$c+1 = 0$, $-4+a=0$, $b-2=0$

$c = -1$, $a = 4$, $b = 2$

S.T $f(r)\vec{r}$ is irrotational & hence find $f(r)$ when it is solenoidal.

$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$

$f(r)\vec{r} = x f(r)\vec{i} + y f(r)\vec{j} + z(f(r))\vec{k}$

$$\text{curl}\left[f(r)\vec{r}\right] = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x f(r) & y f(r) & z f(r) \end{vmatrix}$$

$$= f(r) \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x & y & z \end{vmatrix}$$

$$= f(r)\Big[\vec{i}\left[\partial/\partial y\,(z) - \partial/\partial z\,(y)\right]$$

$$-\vec{j}\left[\partial/\partial x\,(z) - \partial/\partial z\,(x)\right]$$

$$+\vec{k}\left[\partial/\partial x\,(y) - \partial/\partial y\,(x)\right]\Big]$$

$= f(r)\,[0]$

$= 0$

$f(r)\vec{r}$ is an irrotational.

To find $f(r)$.

By given $f(r)\vec{r}$ is solenoidal

$$\text{div}\,(f(r)\vec{r}) = 0$$

$$\nabla \cdot (f(r)\vec{r}) = 0$$

$$(\vec{i}x + y\vec{j} + z\vec{k}) \cdot (x f(r)\vec{i} + y f(r)\vec{j} + z f(r)\vec{k}) = 0$$

$$\partial/\partial x\,(x f(r)) + \partial/\partial y\,(y f(r)) + \partial/\partial z\,(\cancel{z} f(r)) = 0$$

$$x f'(r)\frac{\partial r}{\partial x} + f(r) + y f'(r)\frac{\partial r}{\partial y} + f(r)$$

$$+ z f'(r)\frac{\partial r}{\partial z} + f(r) = 0$$

$$3f(r) + x f'(r)\, x/r + y f'(r)\, y/r +$$

$$z f'(r)\, z/r = 0.$$

$$3f(r) + f'(r)\left[x^2/r + y^2/r + z^2/r\right] = 0$$

$$3f(r) + f'(r)\left[\frac{r^2}{r}\right] = 0$$

$$3f(r) + f'(r)\, r = 0$$

~~3f(r~~

$$r f'(r) = -3f(r)$$

$$\frac{f'(r)}{f(r)} = -3/r$$

$$\int \frac{f'(r)}{f(r)}\,dr = -3\int 1/r\,dr$$

$$-\log(f(r)) = 3\log r + \log c$$

$$-\log(f(r)) = \log r^3 + \log c$$

$$-\log(f(r)) = \log r^3 c$$

$$-f(r) = r^3 c$$

$$f(r) = -r^3 c$$

66) If $\vec{A}$ & $\vec{B}$ are irrotational p.T $\vec{A}\times\vec{B}$ is solenoidal.

$$\nabla\times\vec{A}=0$$

$$\nabla\times(\vec{B})=0$$

$$\text{div}(\vec{A}\times\vec{B})=0$$

$$\text{div}(\vec{f}\times\vec{g})=\vec{g}\cdot\text{curl}\,\vec{f}-\vec{f}\,\text{curl}\,\vec{g}.$$

$$\text{div}(\vec{A}\times\vec{B})=\vec{B}\cdot(\nabla\times\vec{A})-\vec{A}\cdot(\nabla\times\vec{B})$$

$$=\vec{B}\cdot(0)-\vec{A}(0)$$

$$\text{div}(\vec{A}\times\vec{B})=0$$

67) If $\vec{A}$ is constant vector, p.T

i) $\nabla\cdot(\vec{A}\times\vec{r})=0$

ii) $\nabla\times(\vec{A}\times\vec{r})=2\vec{A}$

$$\vec{A}=A_1\vec{i}+A_2\vec{j}+A_3\vec{k}$$

$$\vec{r}=x\vec{i}+y\vec{j}+z\vec{k}$$

$$\vec{A}\times\vec{r}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ A_1 & A_2 & A_3\\ x & y & z\end{vmatrix}$$

$$=\vec{i}(A_2z-A_3y)-\vec{j}(A_1z-A_3x)+\vec{k}(A_1y-A_2x)$$

$$\nabla\cdot(\vec{A}\times\vec{r})=(\vec{i}\,\partial/\partial x+\vec{j}\,\partial/\partial y+\vec{k}\,\partial/\partial z)\left[\vec{i}(A_2z-A_3y)-\vec{j}(A_1z-A_3x)+\vec{k}(A_1y-A_2x)\right]$$

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$$= \partial/\partial x\,(A_2 z - A_3 y) - \partial/\partial y\,(A_1 z - A_3 \ldots$$
$$+ \partial/\partial z\,(A_1 y - A_2 x)$$

$$\nabla \cdot (\vec{A} \times \vec{r}) = 0$$

$$\nabla \times (\vec{A} \times \vec{r}) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ A_2 z - A_3 y & A_3 x - A_1 z & A_1 y - A_2 \ldots \end{vmatrix}$$

$$= \vec{i}\,[A_1 + A_1] - \vec{j}\,[-A_2 - A_2] + \vec{k}\,[A_3 + A_3]$$

$$= 2A_1\vec{i} + 2A_2\vec{j} + 2A_3\vec{k}$$

$$= 2\,[A_1\vec{i} + A_2\vec{j} + A_3\vec{k}].$$

$$\nabla \times (\vec{A} \times \vec{r}) = 2\vec{A}.$$

63) Find the value of a, if $\vec{A} = (axy - z^2)\vec{i} + (x^2 + 2yz)\vec{j} + (y^2 - axz)\vec{k}$. is irrotational.

$\vec{A} = (axy - z^2)\vec{i} + (x^2 + 2yz)\vec{j} + (y^2 - axz)\vec{k}$. is irrotational.

$$\text{curl}\,\vec{F} = 0$$

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ axy - z^2 & x^2 + 2yz & y^2 - axz \end{vmatrix} = 0$$

$$\vec{i}\left[\partial/\partial y\,(y^2 - axz) - \partial/\partial z\,(x^2 + 2yz)\right]$$

$-\vec{j}\left[\partial/\partial x\,(y^2-axz)-\partial/\partial x\,(axy-z^2)\right]$

$+\vec{k}\left[\partial/\partial x\,(\cancel{x}^2+2yz)-\partial/\partial y\,(axy-z^2)\right]=0$

$i(2y-2y)-\vec{j}(-az+2z)+\vec{k}(2x-ax)=0$

Each components should be zero

$$-(-az+2z)=0$$
$$az=2z.$$
$$a=2.$$

69) Show that the function $\vec{f}=(y\sin z-\sin x)\vec{i}+(x\sin z+2yz)\vec{j}+(xy\cos z+y^2)\vec{k}$ is an irrotational and find the corresponding scalar function ϕ such that $\vec{f}=\nabla\phi$.

given

$$\vec{f}=(y\sin z-\sin x)\vec{i}+(x\sin z+2yz)\vec{j}+(xy\cos z+y^2)\vec{k}$$

$$\text{curl}\,\vec{f}=0$$

$$\text{curl}\,\vec{f}=0\quad\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ \partial/\partial x & \partial/\partial y & \partial/\partial z\\ y\sin z-\sin x & x\sin z+2yz & xy\cos z+y^2\end{vmatrix}$$

$$=\vec{i}\left[\partial/\partial y\,(xy\cos z+y^2)-\partial/\partial z\,(x\sin z+2yz)\right]$$
$$-\vec{j}\left[\partial/\partial x\,(xy\cos z+y^2)-\partial/\partial z\,(y\sin z-\sin x)\right]$$
$$+\vec{k}\left[\partial/\partial x\,(x\sin z+2yz)-\partial/\partial y\,(y\sin z-\sin x)\right]$$

$$\text{curl}\,\vec{f}=\vec{i}\left[x\cos z+2y-x\cos z-2y\right]-\vec{j}\left[y\cos z-y\cos x\right]$$
$$+\vec{k}\left[\sin z-\sin z\right]$$

$= 0\vec{i} + 0\vec{j} + 0\vec{k}$

$= 0$

$\therefore \vec{f}$ is irrotational vector.

find ϕ.

$$\nabla\phi = (y\sin z - \sin x)\vec{i} + (x\sin z + 2yz)\vec{j} + (xy\cos z + y^2)\vec{k}$$

$$\vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z} = (y\sin z - \sin x)\vec{i} + (x\sin z + 2yz)\vec{j} + (xy\cos z + y^2)\vec{k}.$$

$$\int \partial\phi = \int (y\sin z - \sin x)\,dx.$$

$$\phi = xy\sin z + \cos x + f(y,z)$$

$$\int \partial\phi = \int (x\sin z + 2yz)\,\partial y.$$

$$\phi = xy\sin z + y^2 z + f(x,z)$$

$$\int \partial\phi = \int (xy\cos z + y^2)\,\partial z.$$

$$\phi = xy\sin z + y^2 z + f(x,y)$$

$$\phi = xy\sin z + y^2 z + \cos x + c.$$

70) If $\vec{r} = a\sin\omega t\,\vec{i} + b\cos\omega t\,\vec{j}$ where a, b, ω are constant when i) $\frac{d^2\vec{r}}{dt^2} = -\omega^2\vec{r}$

ii) $\vec{r} \times \frac{d\vec{r}}{dt} = -\omega ab\,\vec{k}$.

$$\vec{r} = a\sin\omega t\,\vec{i} + b\cos\omega t\,\vec{j}$$

$$\frac{d\vec{r}}{dt} = a\cos\omega t\,(\omega)\vec{i} - b\sin\omega t\,(\omega)\vec{j}$$

$= a\omega\cos\omega t\,\vec{i} - b\omega\sin\omega t\,\vec{j}$

$$\frac{d^2\vec{r}}{dt^2} = -a\,\omega\sin\omega t\,(\omega)\,\vec{i} - b\omega\cos\omega t\,(\omega)\,\vec{j}$$

$$= -\omega^2\left[a\sin\omega t\,\vec{i} + b\cos\omega t\,\vec{j}\right]$$

$$\frac{d^2\vec{r}}{dt^2} = -\omega^2\,\vec{r}.$$

ii) $$\vec{r} \times \frac{d\vec{r}}{dt} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a\sin\omega t & b\cos\omega t & 0 \\ a\omega\cos\omega t & -b\omega\sin\omega t & 0 \end{vmatrix}$$

$$= \vec{k}\left[-ab\omega\sin^2\omega t - ab\omega\cos^2\omega t\right]$$

$$= -ab\omega\,\vec{k}\left[\sin^2\omega t + \cos^2\omega t\right]$$

$$\vec{r} \times \frac{d\vec{r}}{dt} = -ab\omega\,\vec{k}$$

Determine the constant a, so that the vector $\vec{f} = (z+3y)\,\vec{i} + (x-2y)\,\vec{j} + (x+az)\,\vec{k}$ is solenoidal.

$$\vec{f} = (z+3y)\,\vec{i} + (x-2y)\,\vec{j} + (x+az)\,\vec{k}$$

$$\operatorname{div}\vec{f} = 0$$

$$\left(\vec{i}\,\partial/\partial x + \vec{j}\,\partial/\partial y + \vec{k}\,\partial/\partial z\right)\cdot\left((z+3y)\,\vec{i} + (x-2y)\,\vec{j} + (x+az)\,\vec{k}\right) = 0$$

$$0 - 2 + a = 0$$

$$a = 2.$$

Show that $\vec{F} = (y+z)\,\vec{i} + (z+x)\,\vec{j} + (x+y)\,\vec{k}$ is solenoidal.

$$\vec{f} = (y+z)\,\vec{i} + (z+x)\,\vec{j} + (x+y)\,\vec{k}$$

$$\text{div} \cdot \vec{f} = 0$$

$$\left(\vec{i}\,\partial/\partial x + \vec{j}\,\partial/\partial y + \vec{k}\,\partial/\partial z\right) \cdot \left((y+z)\vec{i} + (z+x)\vec{j} + (x+y)\vec{k}\right).$$

$$\partial/\partial x\,(y+z) + \partial/\partial y\,(z+x) + \partial/\partial z\,(x+y) = 0$$

$$0 + 0 + 0 = 0$$

$$0 = 0$$

$\vec{f}$ is solenoidal.

73) Show that $\vec{F} = (y-z)\vec{i} + (z-x)\vec{j} + (x-y)\vec{k}$

$$\vec{F} = (y-z)\vec{i} + (z-x)\vec{j} + (x-y)\vec{k}$$

$$\text{div} \cdot \vec{F} = \nabla \cdot \vec{F}$$

$$= \left(\vec{i}\,\partial/\partial x + \vec{j}\,\partial/\partial y + \vec{k}\,\partial/\partial z\right).$$

$$\left((y-z)\vec{i} + (z-x)\vec{j} + (x-y)\vec{k}\right)$$

$$= \partial/\partial x\,(y-z) + \partial/\partial y\,(z-x) + \partial/\partial z\,(x$$

$$= 0 + 0 + 0$$

$$\text{div}\,\vec{f} = 0$$

$\vec{f}$ is solenoidal

74) Prove that the following vectors are solenoidal

i) $z\vec{i} + x\vec{j} + y\vec{k}$

ii) $3y^4z^2\vec{i} + 4x^3z^2\vec{j} - 3x^2y^2\vec{k}$.

given $\vec{F} = z\vec{i} + x\vec{j} + y\vec{k}$.

$$\text{div}\,\vec{F} = \nabla \cdot \vec{F}.$$

$$= \left(\vec{i}\,\partial/\partial x + \vec{j}\,\partial/\partial y + \vec{k}\,\partial/\partial z\right)$$

$$\cdot \left(z\vec{i} + x\vec{j} + y\vec{k}\right).$$

$$= \partial/\partial x\,(z) + \partial/\partial y\,(x) + \partial/\partial z\,(y)$$

$div\vec{F} = 0$

ii) given, $\vec{F} = 3y^4z^2\vec{i} + 4x^3z^2\vec{j} - 3x^2y^2\vec{k}$.

$$div\vec{F} = \nabla\cdot\vec{F}$$

$$= (\vec{i}\,\partial/\partial x + \vec{j}\,\partial/\partial y + \vec{k}\,\partial/\partial z)$$

$$\cdot((3y^4z^2)\vec{i} + 4x^3z^2\vec{j} - 3x^2y^2\vec{k}).$$

$$= \partial/\partial x\,(3y^4z^2) + \partial/\partial y\,(4x^3z^2) + \partial/\partial z\,(3x^2y^2)$$

$$= 0 + 0 + 0$$

$$div\vec{F} = 0$$

If $\vec{u} = y\vec{i} + z\vec{j} + x\vec{k}$ & $\vec{v} = xy\vec{i} + yz\vec{j} + zx\vec{k}$

15) find curl $(\vec{u}\times\vec{v})$ & div $(\vec{u}\times\vec{v})$.

$$\vec{u} = y\vec{i} + z\vec{j} + x\vec{k}$$

$$\vec{v} = xy\vec{i} + yz\vec{j} + zx\vec{k}.$$

$$\vec{u}\times\vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ y & z & x \\ xy & yz & xz \end{vmatrix}$$

$$= \vec{i}(z^2x - xyz) - \vec{j}(xyz - x^2y) + \vec{k}(y^2z - xyz)$$

$$curl(\vec{u}\times\vec{v}) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ z^2x - xyz & xyz - x^2y & y^2z - xyz \end{vmatrix}$$

$$= \vec{i}\left[\partial/\partial y\,(y^2z - xyz) - \partial/\partial z\,(xyz - x^2y)\right]$$

$-\vec{j}\left[\partial/\partial x(y^2z-xyz)-\partial/\partial z(z^2x-xyz)\right]$

$+\vec{k}\left[\partial/\partial x(x^2y-xyz)-\partial/\partial y(z^2x-xyz)\right]$

$=\vec{i}\left[(2yz-xz+xy)\right]-\vec{j}\left[-yz-(2zx-xy)\right]$

$+\vec{k}\left[(2xy-yz)-(-xz)\right]$

$curl(\vec{u}\times\vec{v})=\vec{i}(2yz-xz+xy)+\vec{j}(yz+2zx-xy)$

$+\vec{k}(2xy-yz+xz)$

$div(\vec{u}\times\vec{v})=\left(\vec{i}\,\partial/\partial x+\vec{j}\,\partial/\partial y+\vec{k}\,\partial/\partial z\right)\cdot$

$\left[\vec{i}(z^2x-xyz)-\vec{j}(xyz-x^2y)+\right.$

$\left.\vec{k}(y^2z-xyz)\right]$

$=z^2-xyz-xz+x^2+y^2-xy$

$div(\vec{u}\times\vec{v})=x^2+y^2+z^2-xy-yz-xz.$

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1) Prove that

i) $\nabla^2(1/r)=0$

$\vec{r}=x\vec{i}+y\vec{j}+z\vec{k}$

$\therefore |\vec{r}|=\sqrt{x^2+y^2+z^2}$

$r^2=x^2+y^2+z^2.$

Partially different w.r to x,y,z.

$\frac{\partial r}{\partial x}=x/r, \quad \frac{\partial r}{\partial y}=y/r,$

$\frac{\partial r}{\partial z}=z/r.$

$\nabla^2(1/r)=\sum\frac{\partial^2}{\partial x^2}(1/r)$

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$$= \frac{\partial^2}{\partial x^2}(1/r) + \frac{\partial^2}{\partial y^2}(1/r) + \frac{\partial^2}{\partial z^2}(1/r)$$

$$= \partial/\partial x\left[-1/r^2\left(\frac{\partial r}{\partial x}\right)\right] + \partial/\partial y\left[\left(-1/r^2\frac{\partial r}{\partial y}\right)\right]$$

$$+ \partial/\partial z\left[-1/r^2\frac{\partial r}{\partial z}\right]$$

$$= \partial/\partial x\left(-1/r^2\, x/r\right) + \partial/\partial y\left(-1/r^2\, y/r\right)$$

$$+ \partial/\partial z\left(-1/r^2\, z/r\right)$$

$$\partial/\partial x\left(-x/r^3\right) = -\left[\frac{r^3(1) - x\left(3r^2\frac{\partial r}{\partial x}\right)}{r^6}\right]$$

$$= \frac{-r^3 + 3x\, r^2\, x/r}{r^6}$$

$$\partial/\partial x\left(-x/r^3\right) = \frac{-r^3 + 3x^2 r}{r^6}$$

$$\partial/\partial y\left(-y/r^3\right) = \frac{-r^3 + 3y^2 r}{r^6}$$

$$\partial/\partial z\left(-z/r^3\right) = \frac{-r^3 + 3z^2 r}{r^6}$$

$$= \partial/\partial x\left(-x/r^3\right) + \partial/\partial y\left(-y/r^3\right) +$$

$$\partial/\partial z\left(-z/r^3\right)$$

$$= \frac{-r^3 + 3x^2 r}{r^6} + \frac{-r^3 + 3y^2 r}{r^6} + \frac{-r^3 + 3y^2 r}{r^6}.$$

$$= \frac{-3r^3 + 3r(x^2+y^2+z^2)}{r^6}.$$

$$= \frac{-3r^3 + 3r^3}{r^6}$$

$$\nabla^2(1/r) = 0$$

ii) $\nabla^2(r^m) = m(m+1)r^{m-1}$.

$$\nabla^2(r^m) = \partial^2/\partial x^2 (r^m) + \frac{\partial^2}{\partial y^2}(r^m) + \partial^2/\partial z^2 (r^m)$$

$$\partial^2/\partial x^2 (r^m) = \partial/\partial x \cdot \partial/\partial x (r^m)$$

$$= \partial/\partial x \left(m r^{m-1} \frac{\partial r}{\partial x}\right)$$

$$= \partial/\partial x \left(m r^{m-1} x/r\right)$$

$$= \partial/\partial x \left(m r^{m-2} x\right)$$

$$= m\left[x(m-2) r^{m-3} \frac{\partial r}{\partial x} + r^{m-2}(1)\right]$$

$$= m\left[(m-2) r^{m-3} x\, x/r + r^{m-2}\right]$$

$$= m\left[(m-2) r^{m-4} x^2 + r^{m-2}\right]$$

$$\partial^2/\partial x^2 (r^m) = m(m-2) r^{m-4} x^2 + m r^{m-2}$$

$$\partial^2/\partial y^2 (r^m) = m(m-2) r^{m-4} y^2 + m r^{m-2}$$

$$\partial^2/\partial z^2 (r^m) = m(m-2) r^{m-4} z^2 + m r^{m-2}$$

$$\nabla^2(r^m) = 3m r^{m-2} + m(m-2) r^{m-4} (x^2+y^2+z^2)$$

$$= 3m r^{m-2} + m(m-2) r^{m-4}\, r^2.$$

$$= 3m r^{m-2} + m(m-2) r^{m-2}.$$

$$= m r^{m-2}[3 + m-2].$$

$$= m(m+1) r^{m-2},$$

iii) $\nabla^2(e^r) = e^r(1 + 2/r)$

$$\nabla^2(e^r) = \partial^2/\partial x^2 (e^r) + \partial^2/\partial y^2 (e^r) + \partial^2/\partial z^2 (e^r)$$

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$$\partial^2/\partial x^2 (e^r) = \partial/\partial x \left[\partial/\partial x (e^r)\right]$$

$$= \partial/\partial x \left[e^r \frac{\partial r}{\partial x}\right]$$

$$= \partial/\partial x \left[e^r x/r\right]$$

$$= x/r \left(e^r \partial r/\partial x\right) + \frac{e^r}{1}(1) + x e^r \left(-1/r^2 \frac{\partial r}{\partial x}\right)$$

$$= x/r \cdot e^r (x/r) + e^r + -\frac{x e^r}{r^2} x/r$$

$$\partial^2/\partial x^2 (e^r) = \frac{x^2}{r^2} e^r + e^r - \frac{x^2}{r^3} e^r$$

$$\partial^2/\partial y^2 (e^r) = y^2/r^2 \, e^r + e^r - \frac{y^2}{r^3} e^r.$$

$$\partial^2/\partial z^2 (e^r) = \frac{z^2}{r^2} e^r + e^r - \frac{z^2}{r^3} e^r.$$

$$\nabla^2 (e^r) = \frac{e^r}{r}\left[\frac{x^2}{r} + 1 - \frac{x^2}{r^2} + \frac{y^2}{r} + 1 - \frac{y^2}{r^2} + \frac{z^2}{r} + 1 - \frac{z^2}{r^2}\right]$$

$$= \frac{e^r}{r}\left[3 + \frac{x^2+y^2+z^2}{r} - \frac{x^2+y^2+z^2}{r^2}\right]$$

$$= \frac{e^r}{r}\left[3 + r^2/r - r^2/r^2\right]$$

$$= \frac{e^r}{r}\left[3 + r - 1\right]$$

$$= \frac{e^r}{r}\left[r+2\right]$$

$$= e^r\left[2/r + 1\right]$$

iv) $\nabla^2 (f(r)) = \frac{\partial^2 f}{\partial r^2} + 2/r \frac{\partial f}{\partial r}$.

$$\nabla^2 (f(r)) = \partial^2/\partial x^2 (f(r)) + \partial^2/\partial y^2 (f(r)) \frac{\partial^2}{\partial z^2}(f(r)).$$

$$\frac{\partial^2}{\partial x^2}(f(r)) = \partial/\partial x \cdot \partial/\partial x (f(r))$$

$$= \partial/\partial x \left[f'(r) \frac{\partial r}{\partial x}\right]$$

$$= \partial/\partial x \left[f'(r)\, x/r\right]$$

$$= x/r\, f''(r) \frac{\partial r}{\partial x} + \frac{f'(x)}{r} + 1 + f'(r)\, x(-1/r^2) \frac{\partial r}{\partial x}.$$

$$\frac{\partial^2}{\partial x^2}(f(r)) = \frac{x^2}{r^2} f''(r) + \frac{f'(x)}{r} - \frac{f'(r)\, x^2}{r^3}.$$

$$\partial^2/\partial y^2 (f(r)) = \frac{y^2}{r^2} f''(r) + \frac{f'(r)}{r} - f'(r) \frac{y^2}{r^3}.$$

$$\partial^2/\partial z^2 (f(r)) = \frac{z^2}{r^2} f''(r) + \frac{f'(r)}{r} - f'(r) \frac{z^2}{r^3}.$$

$$\nabla^2 f(r) = \frac{3 \cdot f'(r)}{r} + \frac{f''(r)}{r^2}(x^2+y^2+z^2) - \frac{f'(r)}{r^3}(x^2+y^2+z^2)$$

$$= \frac{3 f'(r)}{r} + \frac{f''(r)}{r^2}(r^2) - \frac{f'(r)}{r^3}($$

$$= \frac{3f'(r)}{r} + f''(r) - \frac{f'(r)}{r}$$

$$= f''(r) - \frac{f'(r)}{r}(3-1)$$

$$= f''(r) - \frac{f'(r)}{r}(2)$$

$$= \frac{\partial^2 f}{\partial r^2} + 2/r \frac{\partial f}{\partial r}$$

P.T $\nabla^2(\log r) = 1/r^2$.

$$\nabla^2(\log r) = \frac{\partial^2}{\partial x^2}(\log r) + \frac{\partial^2}{\partial y^2}(\log r) + \frac{\partial^2}{\partial z^2}(\log r)$$

$$\frac{\partial^2}{\partial x^2}(\log r) = \partial/\partial x \cdot \partial/\partial x (\log r)$$

$$= \partial/\partial x \left(1/r \frac{\partial r}{\partial x}\right)$$

$$= \partial/\partial x \left(1/r \; x/r\right)$$

$$= \partial/\partial x \left(x/r^2\right)$$

$$= \frac{r^2(1) - x(2r)\frac{\partial r}{\partial x}}{r^4}$$

$$\frac{\partial^2}{\partial x^2}(\log r) = \frac{r^2 - 2x^2}{r^4}$$

$$\frac{\partial^2}{\partial y^2}(\log r) = \frac{r^2 - 2y^2}{r^4}$$

$$\frac{\partial^2}{\partial z^2}(\log r) = \frac{r^2 - 2z^2}{r^4}.$$

$$\nabla^2(\log r) = 1/r^4 \left[\frac{3r^2 - 2(x^2+y^2+z^2)}{r^4}\right]$$

$$= \frac{3r^2 - 2r^2}{r^4} = r^2/r^4 = \frac{1}{r^2}.$$

If $\phi = xyz$; $\psi = xy+yz+zx$ find $\nabla\cdot(\nabla\phi \times \nabla\psi)$.

$$\phi = xyz$$
$$\psi = xy+yz+zx$$

$$\nabla\phi = \left(\partial/\partial x\,\vec{i} + \partial/\partial y\,\vec{j} + \partial/\partial z\,\vec{k}\right) xyz$$
$$= \partial/\partial x(xyz)\,\vec{i} + \partial/\partial y\,(xyz)\,\vec{j} + \partial/\partial z\,(xyz)\,\vec{k}$$
$$= yz\,\vec{i} + xz\,\vec{j} + xy\,\vec{k}$$
$$\nabla\phi = yz\,\vec{i} + xz\,\vec{j} + xy\,\vec{k}.$$

$$\nabla\psi = \left(\vec{i}\,\partial/\partial x + \vec{j}\,\partial/\partial y + \vec{k}\,\partial/\partial z\right)(xy+yz+zx)$$
$$= \partial/\partial x(xy+yz+zx)\,\vec{i} + \partial/\partial y\,(xy+yz+zx)\,\vec{j} + \partial/\partial z\,(xy+yz+zx)\,\vec{k}$$
$$\nabla\psi = (y+z)\,\vec{i} + (x+z)\,\vec{j} + (y+x)\,\vec{k}$$

$$\nabla\phi \times \nabla\psi = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ yz & xz & xy \\ y+z & x+z & x+y \end{vmatrix}$$

$$= \vec{i}\left(xz(x+y) - xy(x+z)\right) - \vec{j}\left(yz(x+y) - xy(y+z)\right) + \vec{k}\left(yz(x+z) - (y+z)xz\right)$$

$$= \vec{i}\left[x^2z + xyz - x^2y - xyz\right]$$
$$- \vec{j}\left[xyz + y^2z - xy^2 - xyz\right]$$

$$+\vec{k}\left[xyz + yz^2 - xyz - xz^2\right]$$

$$= (x^2z - x^2y)\vec{i} - (y^2z - xy^2)\vec{j} + (y^2z - xz)\vec{k}$$

$$\nabla\cdot(\nabla\phi \times \nabla\psi) = \left[\partial/\partial x\,\vec{i} + \partial/\partial y\,\vec{j} + \partial/\partial z\,\vec{k}\right]\cdot$$

$$\left[(x^2z - x^2y)\vec{i} - (y^2z - xy^2)\vec{j} + (yz^2 - xz^2)\vec{k}\right]$$

$$= \partial/\partial x\,(x^2z - x^2y) - \partial/\partial y\,(y^2z - xy^2) + \partial/\partial z\,(yz^2 - xz^2)$$

$$= 2xz - 2xy - 2yz + 2xy + 2zy - 2xz$$

$$\nabla\cdot(\nabla\phi \times \nabla\psi) = 0$$

Unit - II

Vector Integration

line Integral:

let $r = f(t)$ represent a continously differentiable curve denoted by c and $\vec{F}(r)$ be a continuous vector Point function. Then $\frac{dr}{ds}$ is a unit vector function along the tangent at any point P on the curve.

The component of the vector funtion F along this tangent is $\vec{F} \cdot \frac{dr}{ds}$ which is a function of s for points on the curve. then

$$\int_C \vec{F} \cdot \frac{dr}{ds}\, ds = \int \vec{F}\, dv.$$

is called the line integral (or) Tangent line intergral of $\vec{F}(r)$ along c,

let $\vec{F} = F\vec{i} + F_2\vec{j} + F_3\vec{k}$, $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$.

$$\therefore d\vec{r} = dx\vec{i} + dy\vec{j} + dz\vec{k}$$

$$\therefore \int_C \vec{F}\, d\vec{r} = \int_C (F_1\vec{i} + F_2\vec{j} + F_3\vec{k}) \cdot (dx\vec{i} + dy\vec{j} + dz\vec{k})$$

$$\int_C \vec{F}\, d\vec{r} = \int_{t_2}^{t_2} (F_1 dx + F_2 dy + F_3 dz)$$

$$\int_C \vec{F}\, dr = \int_{t_1} \left(F_1 \frac{dx}{dt} + F_2 \frac{dy}{dt} + F_3 \frac{dz}{dt} \right) dt$$

where t_1 and t_2 are the values of

Parameter t for extrementies P and Q of the arc of the curve C. Again if $\vec{r} = x\vec{i} + y\vec{j} + z\vec{K}$, then $\frac{dr}{ds} = \frac{dx}{ds}\vec{i} + \frac{dy}{ds}\vec{j} + \frac{dz}{ds}\vec{k}$ is equal to unit tangent vector t.

$$= \int_C \vec{F}\,\vec{dr} = \int_C F \frac{dr}{ds}\, ds$$

$$= \int_{S_1}^{S_2} \left(F_1 \frac{dx}{ds} + F_2 \frac{dy}{ds} + F_3 \frac{dz}{ds} \right) ds$$

$$= \int_{S_1}^{S_2} \vec{F} \cdot \vec{t}\, ds.$$

where s_1 and s_2 are the values of s for the excdremities P and Q of the arc C.

Vector Integration

- line Integral $\int_C \vec{F} \cdot \vec{dr}$
- Surface integral $\iint_S \vec{F} \cdot \hat{n}\, dr$.
- volume integral $\iiint_V (\nabla \cdot \vec{F})\, d\vec{r}$.

1) Evalute $\int_C \vec{F} \cdot d\vec{r}$ where $\vec{F} = x^2y^2\vec{i} + y\vec{j}$ and the curve C is $y^2 = 4x$ is the xy-plane from $(0,0)$ to $(4,4)$

$$\vec{F} = x^2y^2\vec{i} + y\vec{j}$$

W.K.T, $\vec{r} = x\vec{i} + y\vec{j}$

$$d\vec{r} = dx\vec{i} + dy\vec{j}$$

curve $y^2 = 4x$

$$\int_C \vec{F} \cdot \vec{dr} = \int_{t_1}^{t_2} \vec{F}\, dr.$$

$$= \int_{t_1}^{t_2} (x^2y^2\vec{i} + y\vec{j}).(dx\vec{i} + dy\vec{j})$$

$$= \int_{t_1}^{t_2} (x^2y^2dx + ydy)$$

$$= \int_0^4 x^2y^2dx + \int_0^4 ydy$$

$$= \int_0^4 x^2(4x)dx + \int_0^4 ydy$$

$$= \int_0^4 4x^3dx + \int_0^4 ydy$$

$$= \left[\frac{4x^4}{4}\right]_0^4 + \left[\frac{y^2}{2}\right]_0^4$$

$$= (\cancel{4}^4 - 0^4) + (4^2/2 - 0^2)$$

$$= (256 - 0) + (16/2 - 0)$$

$$= 256 + 8 - 0$$

$$= 264$$

2. compute integral $\int \vec{V}.d\vec{r}$, $\vec{V} = x\vec{i} - y\vec{j} + z\vec{k}$, $x = \cos t$, $y = \sin t$, $z = t^e$ $t = 0$ to $\pi/2$

soln:

Gn, $\vec{V} = x\vec{i} - y\vec{j} + z\vec{k}$

$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$

$d\vec{r} = dx\vec{i} + dy\vec{j} + dz\vec{k}$

$\vec{V}.d\vec{r} = xdx - ydy + zdz.$

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$x = \cos t$ | $y = \sin t$ | $z = t$
$dx = -\sin t\,dt$ | $dy = \cos t\,dt$ | $dz = dt$

$\vec{V}.d\vec{r} = -\cos t \sin t\,dt - \sin t \cos t\,dt + t\,dt$

$= -2 \cos t \sin t\,dt + t\,dt$

$[\because 2\sin\theta\cos\theta = \sin 2\theta]$

$= -\sin 2t\,dt + t\,dt$

$= (-\sin 2t + t)\,dt$

$$\int_C \vec{V}.d\vec{r} = \int_0^{\pi/2} (-\sin 2t + t)\,dt$$

$$= \left[\frac{\cos 2t}{2} + \frac{t^2}{2}\right]_0^{\pi/2}$$

$$= \left[\frac{\cos 2(\pi/2)}{2} - \frac{\cos 2(0)}{2}\right] + \left[\frac{(\pi/2)^2}{2} - \frac{0}{2}\right]$$

$$= \left[\frac{\cos \pi}{2} - \frac{\cos 0}{2}\right] + \left[\frac{\pi^2/4}{2} - 0\right]$$

$$= \left[\frac{-1}{2} - \frac{1}{2}\right] + \left[\frac{\pi^2}{8} - 0\right]$$

$$= \frac{-1}{2} - \frac{1}{2} + \frac{\pi^2}{8} \Rightarrow \frac{-2}{2} + \frac{\pi^2}{8}$$

$$= -1 + \pi^2/8$$

$$= \pi^2/8 - 1$$

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3) $\vec{f} = (y^2 - z^2)\vec{i} + 2yz\vec{j} - x^2\vec{k}$ and C is the curve directed by the vector equation $r(t) = t\vec{i} + t^2\vec{j} + t^3\vec{k}$ $(0 \le t \le 1)$ show that $\int_C \vec{f} \cdot \vec{dr} = 1/35$.

$$\vec{f} = (y^2 - z^2)\vec{i} + 2yz\vec{j} - x^2\vec{k}$$

$$\vec{r} = t\vec{i} + t^2\vec{j} + t^3\vec{k}.$$

$$d\vec{r} = dt\vec{i} + 2t\,dt\vec{j} + 3t^2 dt\vec{k}$$

But $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}.$

$$d\vec{r} = dx\vec{i} + dy\vec{j} + dz\vec{k}.$$

On comparing,

$$x = t, \quad y = t^2, \quad z = t^3.$$

$$dx = dt, \quad dy = 2t\,dt, \quad dz = 3t^2 dt$$

$$\vec{f} \cdot \vec{dr} = (y^2 - z^2)dt + 2yz(2t\,dt) - 3x^2t^2 dt$$

$$= (t^4 - t^6)dt + 4t^6 dt - 3t^4 dt.$$

$$= -2t^4 dt + 3t^6 dt.$$

$$\int_C \vec{f} \cdot \vec{dr} = \int_0^1 (-2t^4 dt + 3t^6 dt)$$

$$= \int_0^1 (-2t^4 + 3t^6)dt.$$

$$= \left[-\frac{2t^5}{5} + 3\frac{t^7}{7}\right]_0^1$$

$$= -\frac{2(1)^5}{5} + \frac{3(1)^7}{7}$$

$$= -2/5 + 3/7$$

$$= \frac{-14+15}{35}$$

$$\int_C \vec{f}\cdot d\vec{r} = 1/35.$$

4) $\vec{f} = xy\vec{i} + z\vec{j} + xyz\vec{k}$. Evaluate $\int_C \vec{f}\cdot d\vec{r}$ from the point $(0,0,0)$ to $(1,1,1)$ there is a curve $x=t,\ y=t^2,\ z=t$

$$\vec{f} = xy\vec{i} + z\vec{j} + xyz\vec{k}$$

$$\vec{r} = t\vec{i} + t^2\vec{j} + t\vec{k}$$

$$d\vec{r} = dt\,\vec{i} + 2t\,dt\,\vec{j} + dt\,\vec{k}$$

$$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$$

$$d\vec{r} = dx\,\vec{i} + dy\,\vec{j} + dz\,\vec{k}$$

On comparing

$$x=t,\ y=t^2,\ z=t$$

$$dx=dt,\ dy=2t\,dt,\ dz=dt$$

$$\vec{f}\cdot d\vec{r} = xy\,dt + z(2t\,dt) + xyz)dt$$

$$= t^3dt + 2t^2dt + t^4dt$$

$$\int_C \vec{f}\cdot d\vec{r} = \int_0^1 (t^3dt + 2t^2dt + t^4dt)$$

$$= \left[\frac{t^4}{4} + \frac{2t^3}{3} + \frac{t^5}{5}\right]_0^1$$

$$= 1/4 + 2/3 + 1/5$$

$$= \frac{15+40+12}{60} = \frac{67}{60}$$

5) Evaluate $\int_c \vec{F}\,\vec{dr}$ if $\vec{F} = (x^2+y^2)\vec{i} - 2xy\vec{j}$ in xy and the curve c is the rectangle bounded by $y=0$, $y=b$, $x=0$, $x=a$.

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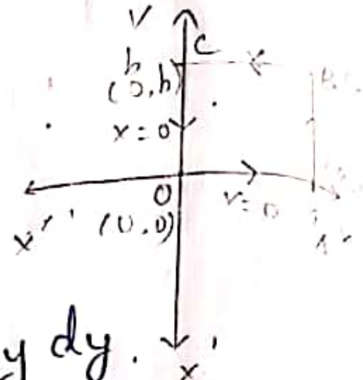


$$\vec{F} = (x^2+y^2)\vec{i} - 2xy\vec{j}$$

$$\vec{r} = x\vec{i} + y\vec{j}$$

$$d\vec{r} = dx\vec{i} + dy\vec{j}$$

$$\vec{F}\cdot d\vec{r} = (x^2+y^2)\,dx - 2xy\,dy.$$

$$\int_C \vec{F}\cdot\vec{dr} = \int_{OA} \vec{f}\cdot\vec{dr} + \int_{AB} \vec{f}\,\vec{dr} + \int_{BC} \vec{f}\,\vec{d} + \int_{OC} \vec{f}\,\vec{dr} \rightarrow (1)$$

On $\underline{OA}$: x varies 0 to a $y=0$

$$\int_{OA} [(x^2+y^2)dx - 2xy\,dy) = \int_0^a x^2\,dx$$

$$= a^3/3 \rightarrow (2)$$

On AB: y varies from 0 to b $x=a \Rightarrow d$

$$\int_{AB} ((x^2+y^2)\,dx - 2xy\,dy) = -\int_0^b 2ay\,dy.$$

$$= -ab^2. \rightarrow (3)$$

On BC x varies from a to 0 $y=b$, $dy=$

$$\int_{BC} ((x^2+y^2)dx - 2xy\,dy) = \int_a^0 (x^2+b^2)dx$$

$$= \left[x^3/3 + b^2x\right]_a^0$$

$$= -a^3/3 - ab^2. \rightarrow (4)$$

On OC y varies from b to 0 $x=0$ dx

b.

$$\int_{OC} ((x^2+y^2)dx - 2xy\,dy) = \int_b^0 (x^2+y^2)dx - 2xy\,dy$$

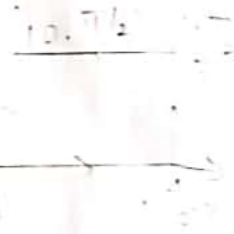
$$= 0 \rightarrow (5)$$

Sub (2), (3), (4), (5) in (1)

$$\int_C \vec{F}\cdot\vec{dr} = a^3/3 - a^3/3 - ab^2 - ab^2 + 0$$

$$= -2ab^2$$

6) Find the circulation around the curve c where $\vec{F} = e^x \sin y\,\vec{i} + e^x \cos y\,\vec{j}$ and c is the rectangle whose vertices are $(0,0), (1,0), (1,\pi/2), (0,\pi/2)$



$$\vec{F} = e^x \sin y\,\vec{i} + e^x \cos y\,\vec{j}$$

$$\vec{r} = x\vec{i} + y\vec{j}$$

$$d\vec{r} = dx\vec{i} + dy\vec{j}$$

$$\vec{F}\cdot\vec{dr} = e^x \sin y\,dx + e^x \cos y\,dy$$

$$\int_C \vec{F}\cdot\vec{dr} = \int_{OA} \vec{F}\cdot\vec{dr} + \int_{AB} \vec{F}\cdot\vec{dr} + \int_{BC} \vec{F}\cdot\vec{dr} + \int_{OC} \vec{F}\cdot\vec{dr} \rightarrow (1)$$

On OA

$y=0$

$dy=0$, x varies from 0 to 1

$$\int_{OA} \vec{F}\cdot\vec{dr} = \int_0^1 e^x \sin y\,dx$$

$$= 0 \rightarrow (2)$$

On AB:

$x = 1$ $\quad$ y varies from 0 to $\pi/2$
$dx = 0$

$$\int_{AB} \vec{F} \cdot \vec{dr} = \int_0^{\pi/2} e^1 \cos y \, dy$$

$$= e \left[\sin y\right]_0^{\pi/2}$$

$$= e(1-0)$$

$$= e \longrightarrow ③$$

On BC:

$y = \pi/2$ $\quad$ x varies from 1 to 0
$dy = 0$

$$\int_{BC} \vec{F} \cdot \vec{dr} = \int_1^0 e^x \, dx$$

$$= \left[e^x\right]_1^0$$

$$= 1 - e \longrightarrow ④$$

On OC:

$x = 0$ $\quad$ y varies from $\pi/2$ to 0
$dx = 0$

$$\int_{OC} \vec{F} \cdot \vec{dr} = \int_{\pi/2}^{0} e^0 \cos y \, dy = \left[\sin y\right]_{\pi/2}^{0}$$

$$= -1 \longrightarrow ⑤$$

Substitute ②, ③, ④, ⑤ in ①

$$\int_C \vec{F} \cdot \vec{dr} = 0 + e + 1 - e - 1 = 0$$

7) Find $\int_C \vec{F} \cdot \vec{dr}$ where $\vec{f} = xy\vec{i} + yz\vec{j} + zx\vec{k}$ and the curve C in $\vec{r} = t\vec{i} + t^2\vec{j} + t^3\vec{k}$ varies from -1 to 1.

$$\vec{f} = xy\vec{i} + yz\vec{j} + zx\vec{k}$$

$$\vec{r} = t\vec{i} + t^2\vec{j} + t^3\vec{k}$$

$$\vec{dr} = dt\vec{i} + 2t\,dt\,\vec{j} + 3t^2dt\,\vec{k}$$

$$\vec{r} = x\vec{i} + y\vec{j} + \vec{k}z$$

$$\vec{dr} = dx\vec{i} + dy\vec{j} + dz\vec{k}$$

On comparing.

$$x = t, \; y = t^2, \; z = t^3.$$

$$dx = dt, \; dy = 2t\,dt, \; dz = 3t^2dt.$$

$$\vec{f} \cdot \vec{dr} = xy\,dt + yz\,(2t\,dt) + zx\,(3t^2dt)$$

$$= t^3dt + 2t^6dt + 3t^6dt.$$

$$\int_C \vec{f} \cdot \vec{dr} = \int_{-1}^{1} \left[t^3dt + 2t^6dt + 3t^6dt\right]$$

$$= \left[\frac{t^4}{4} + 2\,t^7/7 + 3\,t^7/7\right]_{-1}^{1}$$

$$= \left[1/4 + 2/7 + 3/7\right] - \left[1/4 - 2/7 - 3/7\right]$$

$$= \left[\frac{49 + 56 + 84}{196}\right] - \left[\frac{49 - 56 - 84}{196}\right]$$

$$= \left[\frac{189}{196}\right] - \left[\frac{-91}{196}\right]$$

$$= \frac{\cancel{280}^{\cancel{140}\,70}}{\cancel{196}_{\cancel{98}\,49}} = 10/7$$

8) Evaluate $\int_c \vec{F}\cdot\vec{dr}$, $\vec{F} = c\left[(-3a\sin^2\theta\cos\theta)\vec{i}\right.$

$+a[2\sin\theta - 3\sin^3\theta]\vec{j} + b\sin 2\theta\vec{k}$

and the curve 'c' given by

$\vec{r} = a\cos\theta\vec{i} + a\sin\theta\vec{j} + b\theta\vec{k}$ for $\theta = \pi/4$ to π

$$\vec{F} = c\left[(-3a\sin^2\theta\cos\theta)\vec{i} + a(2\sin\theta - 3\sin^3\theta)\vec{j} + b\sin 2\theta\vec{k}\right]$$

$$\vec{r} = a\cos\theta\vec{i} + a\sin\theta\vec{j} + b\theta\vec{k}$$

$$\vec{dr} = -a\sin\theta\, d\theta\vec{i} + a\cos\theta\, d\theta\vec{j} + b\,d\theta\vec{k}$$

$$\vec{F}\cdot\vec{dr} = c(-3a\sin^2\theta\cos\theta)(-a\sin\theta\, d\theta) + ca(2\sin\theta - 3\sin^3\theta)(a\cos\theta\, d\theta) + cb\sin 2\theta \cdot b\,d\theta$$

$$= 3a^2c\sin^3\theta\cos\theta\, d\theta + 2a^2c\sin\theta\cos\theta\, d\theta - 3a^2c\sin^3\theta\cos\theta\, d\theta + b^2c\sin 2\theta\, d\theta$$

$$= a^2c\sin 2\theta\, d\theta + b^2c\sin 2\theta\, d\theta$$

$$= c(a^2+b^2)\sin 2\theta\, d\theta$$

$$\int_c \vec{F}\cdot\vec{dr} = \int_{\pi/4}^{\pi/2} c(a^2+b^2)\sin 2\theta\, d\theta$$

$$= c(a^2+b^2)\int_{\pi/4}^{\pi/2}\sin 2\theta\, d\theta$$

$$= c(a^2+b^2)\left[\frac{-\cos 2\theta}{2}\right]_{\pi/2}^{\pi/2}$$

$$= c(a^2+b^2)\left[\tfrac{1}{2}-0\right]$$

$$= \tfrac{1}{2}\, c(a^2+b^2)$$

11/01/2020

9) If $\vec{A} = (5xy-6x^2)\vec{i} + (2y-4x)\vec{j}$

Evaluate $\int_C \vec{A}\cdot\vec{dr}$ where c is the curve $y=x^3$ in the xy-plane from the point (1,1) to (2,8)

$$\vec{A} = (5xy-6x^2)\vec{i} + (2y-4x)\vec{j}$$

$$\vec{r} = x\vec{i} + y\vec{j}$$

$$d\vec{r} = dx\vec{i} + dy\vec{j}$$

$$y = x^3$$

$$dy = 3x^2\,dx$$

$$\vec{A}\cdot\vec{dr} = \left((5xy-6x^2)\vec{i} + (2y-4x)\vec{j}\right).\left(dx\vec{i}+dy\vec{j}\right)$$

$$= (5xy-6x^2)dx + (2y-4x)dy.$$

$$= (5xy-6x^2)dx + (2y-4x)dy.$$

$$= 5x^4dx - 6x^2dx + (2y-4x)3x^2dx$$

$$= 5x^4dx - 6x^2dx + 6x^5dx - 12x^3dx$$

$$\int_C \vec{A}\,\vec{dr} = \int_1^2 5x^4dx - 6\int_1^2 x^2dx + 6\int_1^2 x^5dx - 12\int_1^2 x^3dx$$

$$= \left[x^5\right]_1^2 - 6\left[\frac{x^3}{3}\right]_1^2 + 6\left[\frac{x^6}{6}\right]_1^2$$

$$+12\left[\frac{x^4}{4}\right]_1$$

$$= 2^5 - 1^5 - 2(2^3 - 1^3) + 8(2^6 - 1^6)$$
$$3\left[2^4 - 1^4\right]$$

$$= (32-1) - 2(8-1) + (64-1) - 3(16$$

$$= 31 - 2(7) + 63 - 3(15)$$

$$= 31 - 14 + 63 - 45$$

$$= 35$$

18) If $\vec{F} = 3xy\vec{i} - y^3\vec{j}$ evaluate $\int_C \vec{F}.\vec{dr}$ where c is the curve in the xy-plane $y =$ from $(0,0)$ to $(1,2)$

$$\vec{F} = 3xy\vec{i} - y^3\vec{j}$$
$$\vec{r} = x\vec{i} + y\vec{j}$$
$$d\vec{r} = dx\vec{i} + dy\vec{j}$$
$$\vec{F}.\vec{dr} = (3xy\vec{i} - y^3\vec{j}).(dx\vec{i} + dy\vec{j})$$
$$= 3xy\,dx - y^3\,dy \rightarrow (1)$$

$$y = 2x^2$$
$$dy = 4x\,dx$$

$$(1) \Rightarrow \vec{F}.\vec{dr} = 3x(2x^2)\,dx - (2x^2)^3(4x\,dx)$$
$$= 6x^3\,dx - 32x^7\,dx$$

$$\int_C \vec{F}.\vec{dr} = \int_0^1 (6x^3\,dx - 32x^7\,dx)$$

$$= \int_0^1 6x^3\,dx - \int_0^1 32x^7\,dx$$

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$$= 6\left[\frac{x^4}{4}\right]_0^1 - 32\left[\frac{x^8}{8}\right]_0^1$$

$$= 3\left[x^4/2\right]_0^1 - 4\left[\frac{x^8}{1}\right]_0^1$$

$$= 3/2 - 4$$

$$= \frac{3-8}{2}$$

$$= -5/2$$

If $\vec{F} = (3x^2+6y)\vec{i} - 14yz\vec{j} + 20xz^2\vec{k}$

Evaluate $\int_C \vec{F}\,\vec{dr}$ from $(0,0,0)$ to $(1,1,1)$

along the part c given by $x=t$, $y=t^2$, $z=t^3$.

$$\vec{F} = (3x^2+6y)\vec{i} - 14yz\vec{j} + 20xz^2\vec{k}$$

$$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$$

$$\vec{dr} = dx\vec{i} + dy\vec{j} + dz\vec{k}$$

$$\vec{F}\cdot\vec{dr} = \left((3x^2+6y)\vec{i} - 14yz\vec{j} + 20xz^2\vec{k}\right).$$
$$\left(dx\vec{i} + dy\vec{j} + dz\vec{k}\right)$$

$$\vec{F}\cdot\vec{dr} = (3x^2+6y)dx - 14yz\,dy + 20xz^2\,dz \rightarrow ①$$

$x=t$ $\quad$ $y=t^2$ $\quad$ $z=t^3$

$dx=dt$ $\quad$ $dy=2t\,dt$ $\quad$ $dz=3t^2dt$

① $\Rightarrow$

$$\vec{F}\cdot\vec{dr} = (3t^2+6t^2)dt - 14(t^2)(t^3)(2t\,dt) + 20(t)(t^3)^2(3t^2dt)$$

$$= 9t^2 - 28t^6dt + 60t^9dt$$

$$\int_C \vec{F}\cdot\vec{dr} = \int_0^1 [9t^2\,dt - 28t^6\,dt + 60t^9\,dt]$$

$$= \int_0^1 9t^2\,dt - \int_0^1 28t^6\,dt + \int_0^1 60t^9\,dt$$

$$= 9\left[\frac{t^3}{3}\right]_0^1 - 28\left[\frac{t^7}{7}\right]_0^1 + 60\left[\frac{t^{10}}{10}\right]_0^1$$

$$= 9(1/3) - 28(1/7) + 60(1/10)$$

$$= 3 - 4 + 6$$

$$= 5$$

12) Evaluate $\int_C \vec{F}\,\vec{dr}$ where $\vec{F} = 3x^2\vec{i} + (2xz-y)\vec{j} + z\vec{k}$ along the straight line c from (0,0,0) to (2,1,3).

$$\vec{F} = 3x^2\vec{i} + (2xz-y)\vec{j} + z\vec{k}$$

$$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$$

$$\vec{dr} = dx\vec{i} + dy\vec{j} + dz\vec{k}$$

$$\vec{F}\cdot\vec{dr} = 3x^2\,dx + (2xz-y)\,dy + z\,dz$$

from (0,0,0) to (2,1,3)

$y=1$

$x = 2 \Rightarrow 2\times 1 = 2y$

$z = 3 \Rightarrow 3\times 1 = 3y$

$$\vec{F}\cdot\vec{dr} = 3x^2\,dx + [2(2y)(3y) - y]\,dy + z\,dz$$

$$= 3x^2\,dx + (12y^2 - y)\,dy + z\,dz$$

$$\int_C \vec{F}\,d\vec{r} = 3\int_0^2 x^2\,dx + \int_0^1 (12y^2 - y)\,dy + \int_0^3 z\,dz$$

$$= 3\left[\frac{x^3}{3}\right]_0^2 + \left[\frac{12y^3}{3} - \frac{y^2}{2}\right]_0^1 + \left[\frac{z^2}{2}\right]_0^3$$

$$= 3\left[\frac{2^3}{3}\right] + \left[\frac{12}{3} - 1/2\right] + \frac{3^2}{2}$$

$$= (8/3)3 + [4 - 1/2] + 9/2$$

$$= 8 + \cancel{7/2} + 9/2$$

$$\cancel{= 8+4+4} \quad = 8 + 16/2 \Rightarrow 8+8$$

$$= 16.$$

23/1/2020 13) Find the workdone in a moving particle once around a circle C in the xy-plane when the circle have its centre at origin & radius 2 and the force field $\vec{F}$ is given by

$$\vec{F} = (2x - y + 2z)\vec{i} + (x + y - z)\vec{j} + (3x - 2y - 5z)\vec{k}$$

Since it is xy-plane, $z = 0$

In a circle radius.

$x = r\cos\theta$	$y = r\sin\theta$
$x = 2\cos\theta$	$y = 2\sin\theta$
$dx = -2\sin\theta\,d\theta$	$dy = 2\cos\theta\,d\theta$

Since circle

$\lim \rightarrow 0$ to 2π

$\vec{r} = x\vec{i} + y\vec{j}$

$d\vec{r} = dx\,\vec{i} + dy\,\vec{j}$

$d\vec{r} = -2\sin\theta\, d\theta\, \vec{i} + 2\cos\theta\, d\theta\, \vec{j}$

$\vec{F} = (4\cos\theta - 2\sin\theta)\vec{i} + (2\cos\theta + 2\sin\theta)\vec{j} + (6\cos\theta - 4\sin\theta)\vec{k}$

$\vec{F}\cdot d\vec{r} = -(4\cos\theta - 2\sin\theta)\, 2\sin\theta\, d\theta + (2\cos\theta + 2\sin\theta)\, 2\cos\theta\, d\theta$

$= (-8\cos\theta\sin\theta + 4\sin^2\theta + 4\cos^2\theta + 4\sin\theta\cos\theta)\, d\theta$

$= (4 - 4\sin\theta\cos\theta)\, d\theta$

$\vec{F}\cdot d\vec{r} = (4 - 2\sin 2\theta)\, d\theta.$

$\int_C \vec{F}\cdot d\vec{r} = \int_0^{2\pi} (4 - 2\sin 2\theta)\, d\theta$

$= \left[4\theta - 2\left(\frac{\cos 2\theta}{2}\right)\right]_0^{2\pi}$

$= 8\pi + 1 - 0 - 1$

$= 8\pi$

14) Find the circulation of $\vec{F}$ around the curve c where $\vec{F} = y\vec{i} + z\vec{j} + x\vec{k}$ and c is the circle $x^2 + y^2 = 1$, $z = 0$

15) If $\vec{F} = xz\vec{i} + yz\vec{j} + z^2\vec{k}$ Evaluate $\int_C \vec{F}\cdot d\vec{r}$ from the pt $(0,0,0)$ to $(1,1,1)$ wher c is the curve

i) $x = t,\ y = t^2,\ z = t^3$

ii) Reitilinear path form (0, 0, 0) to (1, 0, 0) then to (1, 1, 0) to (1, 1, 1)

iii) St line path from (0, 0, 0) to (1, 1, 1)

i) $\vec{F} = xz\vec{i} + yz\vec{j} + z^2\vec{k}$

$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$

$d\vec{r} = dx\vec{i} + dy\vec{j} + dz\vec{k}$.

$\vec{F} \cdot dr = \cancel{xy}\ xz\,dx + yz\,dy + z^2 dz$

i) $x = t, \quad y = t^2, \quad z = t^3$

$dx = dt, dy = 2t\,dt, \quad dz = 3t^2 dt$

$x, y, z \to 0$ to 1.

$\vec{F} \cdot d\vec{r} = t^4 dx + 2t^6 dy + 3t^8 dz$

$\int_C \vec{F} \cdot d\vec{r} = \int_0^1 t^4 dx + 2t^6 dy + 3t^8 dz$.

$= \left[\frac{t^5}{5} + \frac{2t^7}{7} + \frac{3t^9}{9}\right]_0^1$

$= 1/5 + 2/7 + 1/3 - 0$

$= \frac{86}{105}$

ii) $\int_C \vec{F} \cdot d\vec{r} = \int_{AB} + \int_{BC} + \int_{CD} \to (1)$

<u>On AB</u>

x varies 0 to 1

$y = 0 \quad z = 0$.

$dy = 0 \quad dz = 0$

$\int_{AB} \vec{F} \cdot d\vec{r} = \int_0^1 0 + 0 + 0 = 0$

On BC

y varies from 0 to 1

$x = 1$ $\qquad$ ~~$dz \to 0$~~ $\qquad z = 0$

$dx = 0$ $\qquad\qquad dz = 0$

$$\int_{BC} \vec{F} \cdot \vec{dr} = \int_0^1 0+0+0 = 0 \longrightarrow (3)$$

On CD

z varies from 0 to 1

$x = 1$ $\qquad\qquad y = 1$

$dx = 0$ $\qquad\qquad dy = 0$

$$\int_{CD} \vec{F} \cdot \vec{dr} = \int_0^1 0+0+z^2 dz$$

$$= \left[z^3/3 \right]_0^1 = 1/3 \longrightarrow (4)$$

$$\oint_C \int_C \vec{F} \cdot \vec{dr} = 0+0+1/3.$$

$$= 1/3.$$

iii) straight line path from (0,0,0) to (1,1,1)

$$\frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1} = \frac{z-z_1}{z_2-z_1} = t \text{ (say)}$$

$$\frac{x-0}{1-0} = \frac{y-0}{1-0} = \frac{z-0}{1-0} = t.$$

$x = t$, $y = t$, $z = t$

t varies from 0 to 1.

$dx = dt$; $dy = dt$, $dz = dt$

$$\int_C \vec{F}\,\vec{dr} = \int_0^1 t^2 dt + t^2 dt + t^2 dt.$$

$$= 3\int_0^1 t^2 dt.$$

$$= 3\left[\frac{t^3}{3}\right]_0^1$$

$$= (t^3)_0^1$$

$$= 1$$

Find the circulation of $\vec{F}$ around the curve c where $\vec{F} = y\vec{i} + z\vec{j} + x\vec{k}$ and c is the circle $x^2+y^2=1$, $z=0$

Since it is xy-plane, $z=0$

In a circle radius

$x = r\cos\theta$ | $y = r\sin\theta$

$x = 1\cos\theta$ | $y = 1\sin\theta$

$dx = -\sin\theta d\theta$ | $dy = \cos\theta d\theta$

$\lim \to 0 \text{ to } 2\pi$

$\vec{r} = x\vec{i} + y\vec{j}$

$\vec{dr} = dx\vec{i} + dy\vec{j}.$

$\vec{dr} = -\sin\theta d\theta\,\vec{i} + \cos\theta d\theta\,\vec{j}$

$\vec{F} = \sin\theta\vec{i} + \cos\theta\vec{K}$

$$\vec{F}\cdot\vec{dr} = \sin\theta(-\sin\theta)\,d\theta$$

$$= -\sin^2\theta\, d\theta$$

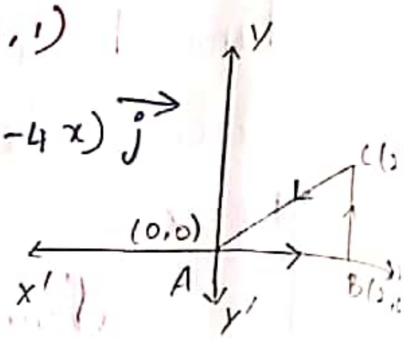
$$\int_C \vec{F}\,\vec{dr} = -\int_0^{2\pi}\left(\frac{1-\cos 2\theta}{2}\right)d\theta$$

$$= -\int_0^{2\pi} \tfrac{1}{2}\,d\theta + \int_0^{2\pi}\left(\frac{\cos 2\theta}{2}\right)d\theta$$

$$= -\left[\theta - \frac{\sin 2\theta}{4}\right]_0$$

$$= -\frac{2\pi}{2} = -\pi$$

16) If $\vec{F} = (2x^2+y^2)\vec{i} + (3y-4x)\vec{j}$ Evaluate $\int_C \vec{F}\cdot\vec{dr}$ around the Δ^{le} ABC whose vertices are $A(0,0)$, $B(2,0)$ & $C(2,1)$



$$\vec{F} = (2x^2+y^2)\vec{i} + (3y-4x)\vec{j}$$

$$\vec{r} = x\vec{i} + y\vec{j}$$

$$\vec{dr} = dx\vec{i} + dy\vec{j}$$

$$\vec{F}\cdot\vec{dr} = (2x^2+y^2)\,dx + (3y-4x)\,dy$$

$$\int_C \vec{F}\cdot\vec{dr} = \int_{AB} + \int_{BC} + \int_{CA}.$$

On AB :

$x \to 0$ to 2

$y = 0 \Rightarrow dy = 0$

$$\int_{AB} \vec{F}\,\vec{dr} = \int_0^2 2x^2\,dx$$

$$= 2\left[\frac{x^3}{3}\right]_0^2$$

$$= 2\left(8/3\right)$$

$$= 16/3.$$

On BC :

$y \to 0$ to 1

$x = 2 \Rightarrow dx = 0$

$$\int_{BC} \vec{F}\cdot\vec{dr} = \int_0^1 (3y - 4(2))\,dy$$

$$= \int_0^1 (3y-8)\,dy$$

$$= \left[\frac{3y^2}{2} - 8y\right]_0^1$$

$$= \left[3/2 - 8\right]$$

$$= \frac{3-16}{2}$$

$$= -13/2$$

On AC:

Eqn of straight line (0,0) & (2,1)

$$\frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1}$$

$$\frac{x-0}{2-0} = \frac{y-0}{1-0}$$

$$x/2 = y$$

$$x = 2y.$$

$$dx = 2dy.$$

$$\vec{F}\cdot\vec{dr} = (2(2y)^2 + y^2)\,2dy + (3y - 4(2y))\,dy$$

$$= (16y^2 + 2y^2 + 3y - 8y)\,dy.$$

$$= (18y^2 - 5y)\,dy$$

$$\int_C \vec{F}\cdot\vec{dr} = \int_0^1 (18y^2 - 5y)\,dy.$$

$$= \left[18\,y^3/3 - 5\frac{y^2}{2}\right]_0^1$$

$$= 18/3 - 5/2$$

$$= \frac{36-15}{6}$$

$$= \frac{21}{6}$$

$$= 7/2.$$

$$\int_C \vec{F} \, \vec{dr} = 16/3 - 13/2 + 7/2 \Rightarrow \frac{14}{6} = 7/3$$

$$\frac{36 \cdot 39+}{6}$$

$$= \frac{28}{6}$$

$= 126 + 45 - 124 + 186 + 10$

If $\vec{F} = (4xy - 3x^2z^2)\vec{i} + 2x^2\vec{j} - 2x^3z\vec{k}$ show that $\int_C \vec{F}\cdot d\vec{r}$ is independent of path c and find the scalar potential ϕ.

$$\vec{F} = (4xy - 3x^2z^2)\vec{i} + 2x^2\vec{j} - 2x^3z\vec{k}$$

To prove $\int_C \vec{F}\, d\vec{r}$ is independent of path c

It is enough to show that $\vec{F}$ is irrotational

$$\text{curl}\,\vec{F} = \vec{\nabla} \times \vec{F}$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 4xy - 3x^2z^2 & 2x^2 & -2x^3z \end{vmatrix}$$

$$= \vec{i}\left[\partial/\partial y\,(-2x^3z) - \partial/\partial z\,(2x^2)\right] - \vec{j}\left[\partial/\partial x\,(-2x^3z) - \partial/\partial z\,(4xy - 3x^2z^2)\right] + \vec{k}\left[\partial/\partial x\,(2x^2) - \partial/\partial y\,(4xy - 3x^2z^2)\right]$$

$= \vec{i}[0-0] - \vec{j}[-6x^2z+6x^2z] + \vec{k}[4x-4x$

$= 0$

$\therefore \vec{F}$ is irrotational

Hence $\int_c \vec{F}\, dr$ is independent of

Path c.

<u>To find ϕ</u>

$$\vec{F} = \nabla \phi$$

$$(4xy-3x^2z^2)\vec{i} + 2x^2\vec{j} - 2x^2z\vec{k} =$$

$$\frac{\partial\phi}{\partial x}\vec{i} + \frac{\partial\phi}{\partial y}\vec{j} + \frac{\partial\phi}{\partial z}\vec{k}$$

Equating the components of $\vec{i}, \vec{j}, \vec{k}$

$$\frac{\partial\phi}{\partial x} = 4xy-3x^2z^2 \longrightarrow (1)$$

$$\frac{\partial\phi}{\partial y} = 2x^2 \longrightarrow (2)$$

$$\frac{\partial\phi}{\partial z} = -2x^2z \longrightarrow (3)$$

On $\int$ing (1) partially w.r to 'x'.

$$\int \partial\phi = \int (4xy-3x^2z^2)\,\partial x$$

$$= \frac{4x^2y}{2} - \frac{3x^3z^2}{3}.$$

$$= 2x^2y - x^3z^2 + f(y,z)$$

On $\int$ing (2) partially w.r to 'y'.

$$\int \partial\phi = \int 2x^2 dy.$$

$$= 2x^2y + f(x,z) \rightarrow (5)$$

On $\int$ing ③ p.w.r to 'z'

$$\int \partial\phi = -\int 2x^3 z \, dz.$$

$$= -2x^3 \frac{z^2}{2} + f(x,y)$$

$$= -x^3 z^2 + f(x,y) \rightarrow ⑥$$

on combining ④, ⑤, ⑥

$$\phi = 2x^2 y - x^3 z^2 + c$$

18) Surface Integral:

consider the surface S, let $\hat{n}$ denote the unit outward normal to the surface S. let R be the projection of the surface S on the xy-plane. Let $\vec{f}$ be the vector valuted function, defined in some region containing the surface S. Then the surface integral of $\vec{f}$ over S is defined by

$$\iint_S \vec{f} \cdot \hat{n} \, ds = \iint_R \frac{\vec{f} \cdot \hat{n}}{|\hat{n} \cdot \vec{k}|} \, dx \, dy.$$

Note:-

In yoz plane,

$$\iint_S \vec{f} \cdot \hat{n} \, ds = \iint_R \frac{\vec{f} \cdot \hat{n}}{|\hat{n} \cdot \vec{i}|} \, dy \, dz.$$

In xoz

$$\iint_S \vec{f} \cdot \hat{n} \, ds = \iint_R \frac{\vec{f} \cdot \hat{n}}{|\hat{n} \cdot \vec{j}|} \, dx \, dz.$$

19) Evaluate $\iint_S \vec{F}\cdot\hat{n}\,ds$ where $\vec{F} = (x+y^2)\vec{i} - 2x\vec{j} + 2yz\vec{k}$ and S is the surface of the plane $2x+y+2z=6$ in the first octant

In xy plane

$$\iint_S \vec{F}\cdot\hat{n}\,ds = \iint_R \frac{\vec{F}\cdot\hat{n}}{|\hat{n}\cdot\vec{k}|}\,dx\,dy$$

given, $\phi = 2x+y+2z=6$

$$\nabla\phi = 2\vec{i} + \vec{j} + 2\vec{k}$$

$$|\nabla\phi| = \sqrt{4+1+4}$$

$$= 3$$

$$\hat{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{2\vec{i}+\vec{j}+2\vec{k}}{3}$$

$$\vec{F}\cdot\hat{n} = \left[(x+y^2)\vec{i} - 2x\vec{j} + 2yz\vec{k}\right]\cdot\left(\frac{2\vec{i}+\vec{j}+2\vec{k}}{3}\right)$$

$$= 1/3\left[2(x+y^2) - 2x + 4yz\right]$$

$$= 1/3\left[2x+2y^2-2x+4yz\right]$$

$$= 2/3\,(y^2+2yz)$$

since $2x+y+2z=6$

$$2z = 6-2x-y$$

$$z = \frac{6-2x-y}{2}$$

$$\vec{F}\cdot\hat{n} = 2/3\left(y^2+\cancel{2}y\left(\frac{6-2x-y}{\cancel{2}}\right)\right)$$

$$= 2/3\left[\frac{y^2+12y-4xy-2y^2}{2}\right]$$

$$= 2/3\left[\cancel{y^2}+6y-2xy-\cancel{y^2}\right]$$

$$= 4/3\,(3y-xy)$$

$$\hat{n}\cdot\vec{k} = \left(\frac{2\vec{i}+\vec{j}+2\vec{k}}{3}\right)\cdot\vec{k}$$

$$= 2/3.$$

$$\frac{\vec{F}\cdot\hat{n}}{|\hat{n}\cdot\vec{k}|} = \frac{\cancel{2}/\cancel{3}\,(3y-xy)}{2/3}$$

$$= 2(3y-xy)$$

To find the limit.

In xy plane, $z=0$

$$2x+y+2z=6$$

$$2x+y=6$$

$$y=6-2x.$$

y varies 0 to $6-2x$

Put $y=0$ in ①,

$$2x=6$$

$$x=3.$$

x varies 0 to 3.

The projection of the surface S on the xy-plane is the region R bounded by the axis $z=0$ & $2x+y=6$

$$\iint_S \vec{F}\cdot\hat{n}\,ds = \iint_R \frac{\vec{F}\cdot\hat{n}}{|\hat{n}\cdot\vec{k}|}\,dx\,dy$$

$$= \int_0^3\int_0^{6-2x} 2(3y-xy)\,dy\,dx$$

$$= \int_0^3 2\left[\frac{3y^2}{2}-\frac{xy^2}{2}\right]_0^{6-2x} dx$$

$$= \int_0^3 (3y^2-xy^2)_0^{6-2x}\,dx$$

$$= \int_0^3 3(6-2x)^2 - x(6-2x)^2 dx$$

$$= \int_0^3 3(36+4x^2-24x) - x(36+4x^2-24x)$$

$$= \int_0^3 108+12x^2-72x-36x-4x^3+24x^2 \, d$$

$$= \int_0^3 -4x^3+36x^2-108x+108 \, dx$$

$$= \left[\frac{-4x^4}{4} + \frac{36x^3}{3} - \frac{108x^2}{2} + 108x\right]_0^3$$

$$= \left[-x^4+12x^3-54x^2+108x\right]_0^3$$

$$= -(3)^4 + 12(3)^3 - 54(3)^2 + 108(3)$$

$$= -81+324-486+324$$

$$= 81$$

20) If $\vec{A} = 2x\vec{i} - xz\vec{k}$ and S is the surface area of the plane (1st octant) $2x+2y+z-4=0$. Prove the following

i) $\iint_S \vec{A}\cdot\hat{n}\,ds = 4$

ii) $\iint_S \vec{r}\cdot\hat{n}\,ds = 8$.

iii) $\iint_S \vec{A}\,ds = 4(2\vec{j}-\vec{k})$

given $\vec{A} = 2x\vec{j} - xz\vec{k}$

$\phi = 2x+2y+z-4$.

$\nabla\phi = 2\vec{i}+2\vec{j}+\vec{k}$

$|\nabla\phi| = \sqrt{4+4+1}$

$= 3$.

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$$\hat{n} = \frac{\nabla\phi}{|\nabla\phi|}$$

$$= \frac{2\vec{i}+2\vec{j}+\vec{k}}{3}$$

$$= \cancel{2x\vec{i} - xz}$$

$$\vec{A}\cdot\hat{n} = 2x\vec{i} - xz\vec{k}\left(\frac{2\vec{i}+2\vec{j}+\vec{k}}{3}\right)$$

$$= \frac{4x - xz}{3}.$$

$$z = -(2x+2y-4)$$

$$\vec{A}\cdot\hat{n} = \frac{4x + x(2x+2y-4)}{3}.$$

$$= \frac{4x+2x^2+2yx-4x}{3}.$$

$$= \frac{2x^2+2yx}{3}$$

$$\hat{n}\cdot\vec{k} = \left(\frac{2\vec{i}+2\vec{j}+\vec{k}}{3}\right)\cdot\vec{k}.$$

$$= 1/3.$$

$$\frac{\vec{A}\cdot\hat{n}}{|\hat{n}\cdot\vec{k}|} = \frac{1/3\,(2x^2+2xy)}{1/3}.$$

$$= 2(x^2+2xy)$$

To find limit:

In xy plane, $z=0$

$$2x+2y=4$$

$$2y = 4-2x$$

$$y = 2-x$$

Put $y=0$ in ①

$$2x = 4$$

$$x = 2$$

y varies $\cancel{2-x}$ 0 to $2-x$ x varies 0 to 2

Scanned with CamScanner

$$\iint_S \vec{A}\cdot\hat{n}\,ds = \iint_R \frac{\vec{A}\cdot\hat{n}}{|\hat{n}\cdot\vec{k}|}\,dy\,dx$$

$$= \int_0^2 \int_0^{2-x} 2(x^2+xy)\,dy\,dx$$

$$= 2\int_0^2 \left[x^2y + \frac{xy^2}{2}\right]_0^{2-x} dx$$

$$= 2\int_0^2 \left[x^2(2-x) + \frac{x}{2}(x^2+4-4x)\right]dx$$

$$= 2\int_0^2 \left(2x^2 - x^3 + \frac{x^3}{2} + \frac{4x}{2} - \frac{4x^2}{2}\right)dx$$

$$= 2\int_0^2 \left[(\cancel{4x} - 2x^2) + x^3/2 + 2x - 2x^2\right]dx$$

$$= 2\left[\frac{-x^4}{4} + \frac{2x^2}{2} + \frac{x^4}{8}\right]_0^2$$

$$= 2\left[\frac{-16}{4} + 8/2 + 16/8\right].$$

$$= 2(-4+4+2)$$

$$= 2(2)$$

$$= 4$$

ii)

$$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$$

$$\vec{r}\cdot\hat{n} = (x\vec{i} + y\vec{j} + z\vec{k})\cdot\left(\frac{2\vec{i} + 2\vec{j} + \vec{k}}{3}\right)$$

$$= \frac{2x+2y+z}{3}.$$

$$\hat{n}\cdot\vec{K} = (x\vec{i} + y\vec{j} + z\vec{k})$$

$$\iint_S \vec{r}\cdot\hat{n}\,ds = \iint_R \frac{\vec{r}\cdot\hat{n}}{|\hat{n}\cdot\vec{k}|}\,dy\,dx$$

$$= \int_0^2\int_0^{2-x} 4\,dy\,dx$$

$$= 4\int_0^2 (y)_0^{2-x}\,dx$$

$$= 4\int_0^2 (2-x)\,dx$$

$$= 4\left[2x - x^2/2\right]_0^2$$

$$= 4\left[4-2\right]$$

$$= 8.$$

iii) $$\iint_S \vec{A}\,ds = \iint_R \frac{\vec{A}}{|\hat{n}\cdot\vec{k}|}\,dy\,dx$$

$$= \int_0^2\int_0^{2-x} \frac{2x\vec{j} - xz\vec{k}}{3}\,dy\,dx$$

$$= 3\int_0^2\int_0^{2-x} \left[2x\vec{j} - x\left[4-2x-2y\right]\vec{k}\right] dy\,dx$$

$$= 3\int_0^2\int_0^{2-x} \left[2x\vec{j} - (4x-2x^2-2xy)\vec{k}\right] dy\,dx$$

$$= 3\int_0^2 \left[2xy\vec{j} - \left[4xy - 2x^2y - xy^2\right]\vec{k}\right]_0^{2-x} dx$$

$$= 3\int_0^2 \left[2x(2-x)\vec{j} - \left[4x(2-x) - 2x^2(2-x) - x(2-x)^2\right]\vec{k}\right] dx$$

$$= 3\int_0^2 \left[(4x-2x^2)\vec{j} - \left[8x - 4x^2 - 4x^2 + 2x^3 - 4x - x^3 + 4x^2\right]\vec{k}\right] dx$$

$$= 3\int_0^2 \left[(4x - 2x^2)\vec{j} - (x^3 - 4x^2 + 4x)\vec{k}\right.$$

$$= 3\left[\left(\frac{4x^2}{2} - \frac{2x^3}{3}\right)\vec{j} - \left(\frac{x^4}{4} - \frac{4x^3}{3} + \frac{4x^2}{2}\right)\vec{k}\right.$$

$$= 3\left[\left(8 - \frac{16}{3}\right)\vec{j} - \left(4 - \frac{32}{3} + 8\right)\vec{k}\right.$$

$$= 3\left[\frac{8}{3}\vec{j} - \frac{4}{3}\vec{k}\right]$$

$$= 8\vec{j} - 4\vec{k}$$

$$= 4(2\vec{j} - \vec{k}).$$

21) Evaluate $\iint_S \vec{F}\cdot\hat{n}\,ds$ where $\vec{F} = z\vec{i} + x\vec{j} - y\vec{k}$ and S is the surface of the curve $x^2+y^2=1$ inclined in the 1st octant between $z=0$ & $z=2$

$$\vec{F} = z\vec{i} + x\vec{j} - y\vec{k}$$

$$\phi = x^2+y^2-1$$

$$\nabla\phi = 2x\vec{i} + 2y\vec{j}$$

$$|\nabla\phi| = \sqrt{4x^2+4y^2}.$$

$$= \sqrt{4(x^2+y^2)}.$$

$$= \sqrt{4}$$

$$= 2$$

$$\hat{n} = \frac{\nabla\phi}{|\nabla\phi|}$$

$$= \frac{2x\vec{i} + 2y\vec{j}}{2}$$

$$= x\vec{i} + y\vec{j}$$

$$\vec{F} \cdot \hat{n} = (z\vec{i} + x\vec{j} - y\vec{k}) \cdot (x\vec{i} + y\vec{j})$$

$$= xz + xy.$$

$$|\hat{n} \cdot \vec{i}| = (x\vec{i} + y\vec{j}) \cdot \vec{i} = x.$$

$$\frac{\vec{f} \cdot \hat{n}}{|\hat{n} \cdot \vec{i}|} = \frac{xz + xy}{x} = z + y$$

To find limit:

In yz plane

$$z \to 0 \text{ to } 2$$

$$x^2 + y^2 = 1$$

$$x = 0 \quad y^2 = 1$$

$$y = \pm 1$$

$$y \to 0 \text{ to } 1$$

$$\iint_S \vec{F} \cdot \hat{n}\, ds = \iint_R \frac{\vec{f} \cdot \hat{n}}{|\hat{n} \cdot \vec{i}|}\, dz\, dy.$$

$$= \int_0^1 \int_0^2 (z+y)\, dz\, dy$$

$$= \int_0^1 \left(\frac{z^2}{2} + yz\right)_0^2 dy$$

$$= \int_0^1 \left(4/2 + 2y\right) dy.$$

$$= \int_0^1 (2 + 2y)\, dy$$

$$= \left[2y + \frac{2y^2}{2}\right]_0^1$$

$$= (2+1) = 3.$$

16) Find the work done in a moving particle
17) a force field $\vec{F}$ given by $\vec{F} = 3xy\vec{i} - 5z\vec{j} + 10x\vec{k}$
along the curve $x = t^2+1,\ y = 2t^2,\ z = t^3$.

$$\vec{F} = 3xy\vec{i} - 5z\vec{j} + 10x\vec{k}$$
$$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$$
$$d\vec{r} = dx\vec{i} + dy\vec{j} + dz\vec{k}$$
$$\vec{f}\cdot d\vec{r} = 3xy\,dx - 5z\,dy + 10x\,dz$$

$$x = t^2+1, \quad y = 2t^2, \quad z = t^3$$
$$dx = 2t\,dt; \quad dy = 4t\,dt; \quad dz = 3t^2 dt$$

$$\vec{f}\cdot d\vec{r} = 3(t^2+1)(2t^2)(2t\,dt) - 5t^3(4t\,dt) + 10(t^2+1)(3t^2 dt)$$
$$= 12t^5 dt + 12t^3 dt - 20t^4 dt + 30t^4 dt + 30t^2 dt$$

$$\int_C \vec{f}\cdot d\vec{r} = \int_1^2 [12t^5 + 12t^3 - 20t^4 + 30t^4 + 30t^2]dt$$
$$= \left[12\frac{t^6}{6} + 12\frac{t^4}{4} - 20\frac{t^5}{5} + 30\frac{t^5}{5} + 30\frac{t^3}{3}\right]$$
$$= \left[2t^6 + 3t^4 - 4t^5 + 6t^5 + 10t^3\right]_1^2$$
$$= \left[2(2)^6 + 3(2)^4 - 4(2)^5 + 6(2)^5 + 10(2)^3\right]$$
$$- \left[2(1)^6 + 3(1)^4 - 4(1)^5 + 6(1)^5 + 10(1)^3\right]$$
$$= \left[2(64) + 3(16) - 4(32) + 6(32) + 10(8)\right]$$
$$- \left[2+3-4+6+10\right]$$

$= [128+48-128+192+80] - [7+10]$

$= [320-17]$

$= 303.$

22) If $\vec{F} = yz\vec{i} + xz\vec{j} - xy\vec{k}$ find $\int_C \vec{F}\cdot\vec{dr}$ where c is the curve

i) $x=t$, $y=t^2$, $z=t^3$ from $(0,0,0)$ to $(2,4,8)$

ii) The rectilinear path from $(0,0,0)$ to $(2,0,0)$ then to $(2,4,0)$ and then to $(2,4,8)$

iii) The straight line path from $(0,0,0)$ to $(2,4,8)$.

$$\vec{F} = yz\vec{i} + xz\vec{j} - xy\vec{k}$$

$$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$$

$$\vec{dr} = dx\vec{i} + dy\vec{j} + dz\vec{k}$$

$$\vec{F}\cdot\vec{dr} = yz\,dx + xz\,dy - xy\,dz$$

$x=t$; $y=t^2$; $z=t^3$

$dx=dt$; $dy=2t\,dt$; $dz=3t^2dt$

$$\vec{F}\cdot\vec{dr} = t^3(t)^2dt + t(t^3)2t\,dt - t\,t^2(3t^2)dt$$

$$\vec{F}\cdot\vec{dr} = t^5dt + 2t^5dt - 3t^5dt$$

$$\vec{F}\cdot\vec{dr} = (t^5+2t^5-3t^5)dt$$

$$\int_C \vec{F}\cdot\vec{dr} = \int_0^2 (t^5+2t^5-3t^5)dt$$

$$\int_C \vec{F}\cdot\vec{dr} = 0$$

ii) $$\int_C \vec{f}\,\vec{dr} = \int_{AB} + \int_{BC} + \int_{CD}.$$

On AB

x varies 0 to 2.

$y=0$, $z=0$

$dy=0$, $dz=0$

$$\int_{AB} \vec{f}\cdot\vec{dr} = \int_0^2 0+0+0$$

$$= 0$$

On BC

y varies from 0 to 4.

$x=2$ $z=0$

$dx=0$ $dz=0$

$$\int_{BC} \vec{f}\cdot\vec{dr} = \int_0^4 0+0+2y\,dz$$

$$= 0$$

On CD

z varies from 0 to 8

$x=2$ $y=4$

$dx=0$ $dy=0$

$$\int_{CD} \vec{f}\cdot\vec{dr} = \int_0^8 0+0-8\,dz.$$

$$= -\int_0^8 8\,dz$$

$$= -8z$$

$$= -64.$$

iii) $\frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1} = \frac{z-z_1}{z_2-z_1} = t$ (say)

$(x_1, y_1, z_1) = (0,0,0)$

$(x_2, y_2, z_2) = (2,4,8)$

$$\frac{x-0}{2-0} = \frac{y-0}{4-0} = \frac{z-0}{8-0} = t$$

$$\frac{x}{2} = \frac{y}{4} = \frac{z}{8} = t$$

$x = 2t$, $y = 4t$, $z = 8t$.

$dx = 2\,dt$, $dy = 4\,dt$, $dz = 8\,dt$.

$t \to 0$ to 1

$$\vec{F}\cdot\vec{dr} = yz\,dx + xz\,dy - xy\,dz$$

$$= 4t(8t)\,dx + (2t)(8t)\,dy - (2t)(4t)\,dz$$

$$= 32t^2(2dt) + 16t^2(4dt) - 8t^2(8dt)$$

$$= 64t^2dt + 64t^2dt - 64t^2dt.$$

$$= (64t^2 + 64t^2 - 64t^2)\,dt.$$

$x \to 0$ to 2

$x = 0 \quad t = x/2 \quad t = 0$

$x = 2 \quad t = 2/2 \quad t = 1$

$$\int_C \vec{F}\cdot\vec{dr} = 64\int_0^1 (t^2)\,dt$$

$$= 64\left(\frac{t^3}{3}\right)_0^1$$

$$= 64\left(\frac{1}{3}\right)$$

$$= \frac{64}{3}.$$

22) If $\vec{A} = (2y+3)\vec{i} + xz\vec{j} + (yz-x)\vec{k}$

P.T $\int_C \vec{A}\cdot\vec{dr} = 8$ where C is the St line Segment joining $(0,0,0)$ to $(2,1,1)$.

$$\vec{A} = (2y+3)\vec{i} + xz\vec{j} + (yz-x)\vec{k}$$

$$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$$

$$\vec{dr} = dx\vec{i} + dy\vec{j} + dz\vec{k}$$

$$\vec{A}\cdot\vec{dr} = (2y+3)\,dx + xz\,dy + (yz-x)\,dz$$

St line Segment joining

$$\frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1} = \frac{z-z_1}{z_2-z_1} = t \text{ (say)}$$

$$(x_1, y_1, z_1) = (0,0,0)$$

$$(x_2, y_2, z_2) = (2,1,1)$$

$$\frac{x-0}{2-0} = \frac{y-0}{1-0} = \frac{z-0}{1-0} = t$$

$$x/2 = y/1 = z/1 = t$$

$$x/2 = y, \quad x/2 = z$$

$$\frac{dx}{2} = dy, \quad \frac{dx}{2} = dz$$

$$\vec{A}\cdot\vec{dr} = \left(2(x/2)+3\right)dx + (x/2)(x)\,dy + \left[(x/2)(x/2)-x\right]dz$$

$$= (x+3)\,dx + \frac{x^2}{2}\left(\frac{dx}{2}\right) + \left(\frac{x^2}{2}-x\right)\frac{dx}{2}$$

$$= (x+3)\,dx + \frac{x^2}{4}dx + \left(\frac{x^2}{4}\right)dx$$

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$\vec{A}\cdot d\vec{r} = (x+3)\,dx$

$$\int_c \vec{A}\cdot d\vec{r} = \int_0^2 (x+3)\,dx$$

$$= \left[\frac{x^2}{2} + 3x\right]_0^2$$

$$= \frac{2^2}{2} + 3(2)$$

$$= \frac{4}{2} + 6$$

$$= \frac{4+12}{2}$$

$$= \frac{16}{2}$$

$$\int_c \vec{A}\cdot d\vec{r} = 8$$

24) Evaluate $\iiint (\nabla\cdot\vec{F})\,dv$, $\vec{F} = x^2\vec{i} + y^2\vec{j} + z^2\vec{k}$ and V is the volume enclosed by the cube $0 \le x, y, z \le 1$

given $\vec{F} = x^2\vec{i} + y^2\vec{j} + z^2\vec{k}$

$\nabla\cdot\vec{F} = 2x + 2y + 2z$

$$\iiint_V (\nabla\cdot\vec{F})\,dv = \int_0^1\int_0^1\int_0^1 (2x+2y+2z)\,dx\,dy\,dz$$

$$= 2\int_0^1\int_0^1\int_0^1 (x+y+z)\,dx\,dy\,dz$$

$$= 2\int_0^1\int_0^1 \left[x^2/2 + xy + xz\right]_0^1 dy\,dz$$

$$= 2\int_0^1\int_0^1 (1/2 + y + z)\,dy\,dz$$

$$= 2\int_0^1 \left(y/2 + y^2/2 + zy\right)_0^1 dz$$

$$= 2\int_0^1 (1/2 + 1/2 + z)\,dz$$

$$= 2\int_0^1 (1+z)\,dz$$

$$= 2\left[z+\frac{z^2}{2}\right]_0^1$$

$$= 2\left[1+1/2\right]$$

$$= 2(3/2)$$

$$= 3$$

25) Evaluate $\iiint_V (\nabla\cdot\vec{F})\,dv$ where $\vec{F} = 2xz\vec{i}+yz\vec{j}+z^2\vec{k}$ and V is the volume obtained by the upper half of the sphere $x^2+y^2+z^2=a^2$.

given $\vec{F} = 2xz\vec{i}+yz\vec{j}+z^2\vec{k}$

$$\nabla\cdot\vec{F} = 2z+z+2z = 5z$$

To find limits.

$$x^2+y^2+z^2=a^2$$

$$z^2 = a^2-x^2-y^2.$$

$$z = \pm\sqrt{a^2-x^2-y^2}$$

$$z \rightarrow 0 \text{ to } \sqrt{a^2-x^2-y^2}.$$

Put $z=0$ in ①

$$x^2+y^2=a^2$$

$$y^2 = a^2-x^2$$

$$= \pm\sqrt{a^2-x^2}$$

$$y \rightarrow 0 \text{ to } \sqrt{a^2-x^2}$$

Put $y=0$, $z=0$ in ①

$$x^2=a^2$$

$$x = \pm a$$

$$x \rightarrow 0 \text{ to } a$$

$$\iiint_V (\nabla\cdot\vec{F})\,dv = \int_0^a \int_0^{\sqrt{a^2-x^2}} \int_0^{\sqrt{a^2-x^2-y^2}} 5z\,dz\,dy\,dx$$

$$= 5\int_0^a \int_0^{\sqrt{a^2-x^2}} \left[\frac{z^2}{2}\right]_0^{\sqrt{a^2-x^2-y^2}} dy\,dx$$

$$= 5/2 \int_0^a \int_0^{\sqrt{a^2-x^2}} \frac{a^2-x^2-y^2}{2}\, dy\,dx$$

$$= 5/2 \int_0^a \left[a^2y - x^2y - y^3/3\right]_0^{\sqrt{a^2-x^2}} dx$$

$$= 5/2 \int_0^a \left[a^2(\sqrt{a^2-x^2}) - x^2(\sqrt{a^2-x^2}) - \frac{(\sqrt{a^2-x^2})^3}{3}\right] dx$$

$$= 5/2 \int_0^a \sqrt{a^2-x^2}\left[a^2-x^2-\frac{a^2-x^2}{3}\right] dx$$

$$= 5/2 \int_0^a \sqrt{a^2-x^2}\,(a^2-x^2)\,(1-1/3)\, dx$$

$$= 5/2 \int_0^a (a^2-x^2)^{3/2}(2/3)\, dx$$

$$= 5/2 \times 2/3 \int_0^a (a^2-x^2)^{3/2} dx$$

$$= 5/3 \int_0^a (a^2-x^2)^{3/2} dx$$

Put $x = a\sin\theta$.

$dx = a\cos\theta\, d\theta$

$x = a \Rightarrow a = a\sin\theta$ | $x = 0$

$1 = \sin\theta$ | $0 = \sin\theta$

$\theta = \pi/2.$ | $0 = \sin\theta$

| $\theta = 0$

As $x \to 0$ to a then $\theta \Rightarrow 0$ to $\pi/2$

$\theta \Rightarrow$

$$\iiint_V (\nabla\cdot\vec{F})\,dv = 5/3 \int_0^{\pi/2} (a^2 - a^2\sin^2\theta)^{3/2} a\cos\theta\, d\theta$$

$$= 5/3 \int_0^{\pi/2} (a^2)^{3/2}(1-\sin^2\theta)^{3/2} a\cos\theta\, d\theta$$

$$= \frac{5a^4}{3} \int_0^{\pi/2} (\cos^2\theta)\cos\theta\, d\theta$$

$$= \frac{5a^4}{3} \int_0^{\pi/2} \cos^3\theta\, d\theta.$$

$$= \frac{5a^4}{3}\left[\frac{4-1}{4}\cdot\frac{4-3}{4-2}\cdot\pi/2\right]$$

$$= \frac{5a^4}{3}\left[3/4 \cdot 1/2 \cdot \pi/2\right]$$

$$= \frac{5\pi a^4}{16}.$$

3/2/2020 conservative Field:

A vector function $\vec{F}$ is called conservative if $\int_{P_1}^{P_2} \vec{F} \cdot d\vec{r}$ is independent of the path joining P_1 and P_2 and conseque $\vec{F} = \nabla\phi$ is irrotational

i.e) $curl\,\vec{F} = 0$

26) If $\vec{P} = (2xy^2+yz)\vec{i} + (2x^2y+xz+2yz^2)\vec{j} + (2y^2z+xy)\vec{k}$ then S.T. P is conservative field

$$curl\,\vec{P} = 0$$

$$\nabla \times \vec{P} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 2xy^2+yz & 2x^2y+xz+2yz^2 & 2y^2z+xy \end{vmatrix}$$

$$= \vec{i}(4yz+x-(x+4yz)) - \vec{j}(y-y) + \vec{k}(4yx+z-(4yx+z))$$

$$= \vec{i}(0) - \vec{j}(0) + \vec{k}(0)$$

$$= 0$$

$$curl\,\vec{P} = 0$$

It is conservative field.

24) Evaluate $\iint_S \vec{F}\cdot\hat{n}\,ds$ if $\vec{F} = 3\vec{i} + x\vec{j} + y^2\vec{k}$ and s is the surface cylinder $x^2+y^2=1$ divides in the first octant between the plane $z=0$ and $z=2$.

given,

$$\vec{F} = 3\vec{i} + x\vec{j} + y^2\vec{k}$$

and $\phi = x^2+y^2-1$

$$\nabla\phi = 2x\vec{i} + 2y\vec{j}$$

$$|\nabla\phi| = \sqrt{4x^2+4y^2}$$

$$= \sqrt{4(x^2+y^2)}$$

$$|\nabla\phi| = 2$$

The unit normal vector

$$\hat{n} = \frac{\nabla\phi}{|\nabla\phi|}$$

$$= \frac{2x\vec{i}+2y\vec{j}}{2}$$

$$= x\vec{i} + y\vec{j}$$

$$\vec{F}\cdot\hat{n} = 3x + xy.$$

$$|\hat{n}\cdot\vec{i}| = x$$

$$\frac{\vec{F}\cdot\hat{n}}{|\hat{n}\cdot\vec{i}|} = 3+y$$

To find the limit.

z varies from 0 to 2.

given Surface $x^2+y^2=1$

$x=0 \Rightarrow y^2=1$

$y = \pm 1$

$\therefore$ y varies from 0 to 1

The projection of the surface

on the xy plane is the region R bounded by the axis and the surface $x^2+y^2=1$

$$\iint_S \vec{F}\cdot\hat{n}\,ds = \int_0^2\int_0^1 (3+y)\,dy\,dz$$

$$= \int_0^2 \left[3y+\frac{y^2}{2}\right]_0^1 dz$$

$$= \int_0^2 (3+1/2)\,dz$$

$$= 7/2 \int_0^2 dz$$

$$= 7/2\,[z]_0^2$$

$$= 7/2\,(2)$$

$$= 7.$$

28) Evaluate $\iint_S \vec{F}\cdot\hat{n}\,ds$ where $\vec{F} = z\vec{i}+x\vec{j}-3y^2\vec{k}$ and S is the surface of the cylinder $x^2+y^2=16$ includes in the first octant between $z=0$ and $z=5$.

$$\vec{F} = z\vec{i}+x\vec{j}-3y^2\vec{k}$$

$$\phi = x^2+y^2-16$$

$$\nabla\phi = \left(\vec{i}\,\partial/\partial x+\vec{j}\,\partial/\partial y+\vec{k}\,\partial/\partial z\right)(x^2+y^2-16)$$

$$\nabla\phi = 2x\vec{i}+2y\vec{j}$$

$$|\nabla\phi| = \sqrt{4x^2+4y^2} = 2\sqrt{x^2+y^2}$$

$$= 2\times 4$$

$$|\nabla\phi| = 8$$

The unit normal vector

$$\hat{n} = \frac{\nabla\phi}{|\nabla\phi|}$$

$$= \frac{2x\vec{i} + 2y\vec{j}}{8}$$

$$= \frac{x\vec{i} + y\vec{j}}{4}$$

$$\vec{F}\cdot\hat{n} = (z\vec{i} + x\vec{j} - 3y^2\vec{k})\cdot\left(\frac{x\vec{i} + y\vec{j}}{4}\right)$$

$$= \frac{zx + xy}{4}$$

$$\hat{n}\cdot\vec{i} = x/4$$

$$|\hat{n}\cdot\vec{i}| = x/4$$

$$\frac{\vec{F}\cdot\hat{n}}{|\hat{n}\cdot\vec{i}|} = \frac{\frac{zx+xy}{4}}{\frac{x}{4}}$$

$$= z+y.$$

$$\iint \frac{\vec{F}\cdot\hat{n}}{\hat{n}\cdot\vec{i}}\,dz\,dy$$

To find the limit.

z varies from 0 to 5.

given plane $x^2+y^2=16$

$$x=0 \Rightarrow y^2 = 16$$
$$y = \pm 4$$

y varies from 0 to 4.

The projection of the surface on the xy plane is the region R bounded by the axis and the surface $x^2+y^2=16$.

29) If $\vec{F} = 2y\vec{i} - z\vec{j} + x^2\vec{k}$ and S is the surface of the parabolic cylinder $y^2 = 8x$ in the first octant bounded by the plane $y = 4$ and $z = 6$ then calculate $\iint_S \frac{\vec{F}\cdot\hat{n}}{|\hat{n}\cdot\vec{i}|}\, ds$ and

$$\hat{n} = \frac{\nabla\phi}{|\nabla\phi|}.$$

$$\vec{F} = 2y\vec{i} - z\vec{j} + x^2\vec{k}$$

$$\phi = y^2 - 8x$$

$$\nabla\phi = -8\vec{i} + 2y\vec{j}$$

$$|\nabla\phi| = \sqrt{64 + 4y^2}$$

$$\hat{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{-8\vec{i} + 2y\vec{j}}{\sqrt{64+4y^2}}.$$

$$\vec{F}\cdot\hat{n} = \frac{-16y - 2yz}{\sqrt{64+4y^2}}.$$

$$|\hat{n}\cdot\vec{i}| = \frac{-8}{\sqrt{64+4y^2}}$$

To find the limit.

$z \to 0$ to 6.

$y \to 0$ to 4.

$$\iint_S \frac{\vec{F}\cdot\hat{n}}{|\hat{n}\cdot\vec{i}|}\, ds = \int_0^6\int_0^4 \frac{-16y - 2yz}{-8}\, dy\, dz$$

$$= 1/8 \int_0^6\int_0^4 (16y + 2yz)\, dy\, dz$$

$$= \frac{1}{4}\int_0^6\int_0^4 (8y + yz)\, dy\, dz.$$

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$$= \frac{1}{4}\int_0^6 \left[\frac{8y^2}{2} + \frac{y^2 z}{2}\right]_0^4 dz.$$

$$= \frac{1}{4}\int_0^6 [64+8z]\, dz$$

$$= \frac{1}{4}\left[64z + \frac{8z^2}{2}\right]_0^6$$

$$= \frac{1}{4}\left[64\times 6 + 4\times 6^2\right]$$

$$= 16\times 6 + 6^2$$

$$= 132.$$

Evaluate $\iint_S \vec{F}\cdot\hat{n}\, ds$ where $\vec{F} = y\vec{i} + zx\vec{j} - z\vec{k}$

30) and S is the surface of the plane $2x+y=6$ in the first octant cut-off by the plane $z=4$.

$$\vec{F} = y\vec{i} + zx\vec{j} - z\vec{k}$$

$$\phi = 2x+y-6$$

$$\nabla\phi = 2\vec{i} + \vec{j}$$

$$|\nabla\phi| = \sqrt{4+1} = \sqrt{5}$$

$$\hat{n} = \frac{2\vec{i}+\vec{j}}{\sqrt{5}}$$

$$\hat{n}\cdot\vec{j} = 1/\sqrt{5}$$

$$\frac{\vec{F}\cdot\hat{n}}{|\hat{n}\cdot\vec{j}|} = \frac{2y+zx}{\frac{1}{\sqrt{5}}}$$

To find limit.

z varies from 0 to 4.

In xz plane

$y=0 \Rightarrow 2x=6 \Rightarrow x=3$

x varies from 0 to 3.

given plane,

$2x+y=6$.

$y=6-2x \rightarrow (2)$

$$\iint_S \vec{F}\cdot \vec{n}\,ds = \int_0^4 \int_0^3 [2(6-2x)+zx]\,dx\,dz$$

$$= \int_0^4 \left[2\left(6x-\frac{2x^2}{2}\right)+\frac{zx^2}{2}\right]_0^3 dz$$

$$= \int_0^4 \left[12x-2x^2+\frac{zx^2}{2}\right]_0^3 dz$$

$$= \int_0^4 \left[12(3)-2(9)+\frac{9z}{2}\right] dz$$

$$= \int_0^4 \left[18+\frac{9z}{2}\right] dz.$$

$$= \left[18z+\frac{9z^2}{4}\right]_0^4$$

$$= 18(4)+\frac{9(4)^2}{4}$$

$$= 108.$$

31) If $\vec{F}=(2x^2-3z)\vec{i}-2xy\vec{j}-4x\vec{k}$ then evaluate $\iiint_V \nabla\cdot\vec{F}\,dv$ and $\iiint_V \nabla\times\vec{F}\,dv$ Where V is the closed region bounded by the planes $x=0$, $y=0$, $z=0$ & $2x+2y+z=4$.

$$\vec{F}=(2x^2-3z)\vec{i}-2xy\vec{j}-4x\vec{k}$$

$$\nabla\cdot\vec{F} = 4x-2x-0 = 2x.$$

$$\nabla\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 2x^2-3z & -2xy & -4x \end{vmatrix}$$

$= \vec{i}(0-0) - \vec{j}(-4+3) + \vec{k}(-2y)$

$= \vec{j} - 2y\vec{k}$

To find ~~the~~ the limit

$2x+2y+z=4$

$z = 4-2y-2x$

$z=0$

$2x+2y=4$

$x+y=2$

$y=2-x$

$y=0,\ z=0 \Rightarrow x=2.$

x varies from 0 to 2.

y varies from 0 to $2-x$.

z varies from 0 to $4-2y-2x$

$$\iiint_V (\nabla\cdot\vec{F})\,dv = \int_0^2\int_0^{2-x}\int_0^{4-2y-2x} 2x\,dz\,dy\,dx$$

$$= \int_0^2\int_0^{2-x}\left[2xz\right]_0^{4-2y-2x} dy\,dx$$

$$= 2\int_0^2\int_0^{2-x} x(4-2y-2x)\,dy\,dx$$

$$= 4\int_0^2\int_0^{2-x}(2x-xy-x^2)\,dy\,dx$$

$$= 4\int_0^2\left[2xy-\frac{xy^2}{2}-x^2y\right]_0^{2-x}dx$$

$$= 4\int_0^2\left[2x(2-x)-\frac{x(2-x)^2}{2} - x^2(2-x)\right]dx$$

$$= 4\int_0^2\left[4x-2x^2-x\left(\frac{4+x^2-4x}{2}\right) - 2x^2+x^3\right]dx \ldots$$

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$$= 4\int_0^2 \left[4x - 4x^2 + x^3 - \frac{4x - x^3 + 4x^2}{2}\right] dx$$

$$= 4\int_0^2 \left(4x - 4x^2 + x^3 - 2x - \frac{x^3}{2} + 2x\right) dx$$

$$= 4\int_0^2 \left(2x - 2x^2 + x^3/2\right) dx$$

$$= 4\left[\frac{2x^2}{2} - \frac{2x^3}{3} + \frac{x^4}{8}\right]_0^2$$

$$= 4\left[4 - \frac{2\times 8}{3} + \frac{4\times 4}{2\times 4}\right]$$

$$= 4\left[6 - \frac{16}{3}\right]$$

$$= 4\left[\frac{18-16}{3}\right] \Rightarrow 4\,(2/3)$$

$$= 8/3$$

$$\iiint_V (\nabla \times \vec{F})\, dv = \int_0^2 \int_0^{2-x} \int_0^{4-2y-2x} (\vec{j} - 2y\vec{k})\, dz\, dy\, dx$$

$$= \int_0^2 \int_0^{2-x} \left[\vec{j}z - 2yz\vec{k}\right]_0^{4-2y-2x} dy\, dx$$

$$= \int_0^2 \int_0^{2-x} \left[\vec{j}(4-2y-2x) - 2y(4-2y-2x)\vec{k}\right] dy\, dx$$

$$= \int_0^2 \int_0^{2-x} \left[(4-2y-2x)\vec{j} - (8y - 4y^2 - 4xy)\vec{k}\right] dy\, dx$$

$$= \int_0^2 \left\{\left[4y - \frac{2y^2}{2} - 2xy\right]_0^{2-x} \vec{j} - 4\left[\frac{2y^2}{2} - \frac{y^3}{3} -\right.\right.$$

$$\left.\frac{xy^2}{2}\right]_0^{2-x}\vec{k}\Big\}\,dx$$

$$= \int_0^2 2\Big\{\Big[4(2-x)-(2-x)^2-2x(2-x)\Big]\vec{j}$$

$$-4\Big[(2-x)^2-\frac{(2-x)^3}{3}-\frac{x(2-x)^2}{2}\Big]\vec{k}\Big\}\,dx$$

$$= \int_0^2 \Big\{(8-4x-4-x^2+4x-4x+2x^2)\vec{j}-$$

$$4\Big[4+x^2-4x-\frac{8-x^3-3(4)x+3(2)(x^2)}{3}-$$

$$\frac{x(4+x^2-4x)}{2}\Big]\vec{k}\Big]\,dx$$

$$= \int_0^2 \Big[(x^2-8x+4)\vec{j}-4\Big[x^2-4x+4-\frac{8-12x+6x^2-x^3}{3}$$

$$-\frac{4x+x^3-4x^2}{2}\Big]\vec{k}\Big]$$

$$= \int_0^2 \Big\{(x^2-8x+4)\vec{j}-4\Big[x^2-4x+4-8/3+4x$$

$$-2x^2+x^3-2x-\frac{x^3}{2}+2x^2\Big]\vec{k}\,dx$$

$$= \int_0^2 \Big[(x^2-8x+4)\vec{j}-4\Big[\frac{x^3}{2}+x^2-2x+4/3\Big]\vec{k}\Big]\,dx$$

$$= \left[\frac{x^3}{3}-\frac{8x^2}{2}+4x\right]_0^2\vec{j}-4\left[\frac{x^4}{8\times 2}+\frac{x^3}{3}-\frac{2x^2}{2}+\frac{4x}{3}\right]_0^2\vec{k}\,dx$$

$$= \Big[8/3-\frac{8(4)}{2}+4(2)\Big]\vec{j}-4\Big[2^4/2\times 8+2^3/3-\frac{2(2)^2}{2}+\frac{4(2)}{3}\Big]\vec{k}$$

$$= \Big[8/3-16+8\Big]\vec{j}-4\Big[1+8/3-4+8/3\Big]\vec{k}$$

$$= \left(8/3 - 8\right)\vec{j} - 4\left(\frac{16}{3} - 3\right)\vec{k}$$

$$= \frac{-16}{3}\vec{j} - 4\left(\frac{16-9}{3}\right)\vec{k}$$

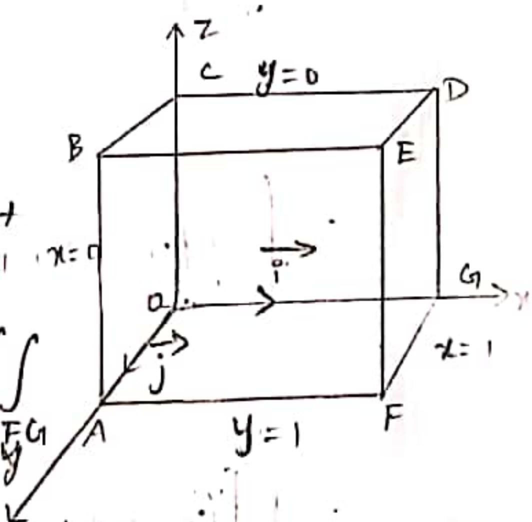
$$= -\frac{16}{3}\vec{j} - 4\left(7/3\right)\vec{k}$$

$$= \frac{-16}{3}\vec{j} - \frac{28}{3}\vec{k}$$

$$= -4/3\,(2\vec{j} - 7\vec{k})$$

30) Evaluate $\iint_S \vec{F}\cdot\hat{n}\,ds$ where $\vec{F} = (x+y)\vec{i} + x\vec{j} + z\vec{k}$ and S is the surface of the cube bounded by $x=0, y=0, z=0; x=1, y=1, z=1$

$\vec{F} = (x+y)\vec{i} + x\vec{j} + z\vec{k}$



$$\iint_S \vec{F}\cdot\hat{n}\,ds = \iint_{OABC} + \iint_{DEFG} + \iint_{ABEF} + \iint_{OCDG} + \iint_{BCDE} + \iint_{OAFG}$$

On the face OABC

$\hat{n} = -\vec{i}$, $x=0$ $dx=0$

$$\iint_{OABC} \frac{\vec{F}\cdot\hat{n}}{|\hat{n}\cdot\vec{i}|}\,dy\,dz = \int_0^1\int_0^1 -(x+y)\,dy\,dz$$

$$= -\int_0^1\int_0^1 y\,dy\,dz$$

$$= -\int_0^1 \left[\frac{y^2}{2}\right]_0^1 dz$$

$$= -1/2\int_0^1 dz$$

$$= -\tfrac{1}{2}[z]_0^1$$

$$= -\tfrac{1}{2} \longrightarrow ②$$

On the face DEFG

$$\hat{n} = \vec{i}, \quad x = 1$$
$$dx = 0$$

$$\iint_{DEFG} \frac{\vec{F}\cdot\hat{n}}{|\hat{n}\cdot\vec{i}|}\, dy\,dz = \int_0^1 \int_0^1 (1+y)\,dy\,dz$$

$$= \int_0^1 \left[y + \frac{y^2}{2}\right]_0^1 dz$$

$$= \int_0^1 (3/2)\,dz$$

$$= 3/2\,(z)_0^1$$

$$= 3/2 \longrightarrow ③$$

On the face ABEF.

$$\hat{n} = \vec{j}, \quad y = 1$$
$$dy = 0$$

$$\iint_{ABEF} \frac{\vec{F}\cdot\hat{n}}{|\hat{n}\cdot\vec{j}|} = \int_0^1 \int_0^1 x\,dx\,dz$$

$$= \int_0^1 \left[\frac{x^2}{2}\right]_0^1 dz.$$

$$= \tfrac{1}{2}\int_0^1 dz$$

$$= \tfrac{1}{2}[z]_0^1$$

$$= \tfrac{1}{2} \longrightarrow ④$$

On the face OCDG:

$$\hat{n} = -\vec{j} \quad y = 0 \quad dy = 0$$

$$\iint_{OCDG} \frac{\vec{F}\cdot\hat{n}}{|\hat{n}\cdot\vec{j}|}\,dx\,dz = \int_0^1\int_0^1 -x\,dx\,dz$$

$$= -\int_0^1 \left[\frac{x^2}{2}\right]_0^1 dz$$

$$= -\int_0^1 \tfrac{1}{2}\,dz.$$

$$= -\tfrac{1}{2}(z)_0^1$$

$$= -\tfrac{1}{2} \rightarrow (5)$$

On the face BCDE

$\therefore \hat{n} = \vec{k}$, $z=1$, $dz=0$

$$\iint_{BCDE} \frac{\vec{F}\cdot\hat{n}}{|\hat{n}\cdot\vec{k}|}\,dx\,dy = \int_0^1\int_0^1 z\,dx\,dy$$

$$= \int_0^1\int_0^1 dy\,dx$$

$$= \int_0^1 (y)_0^1\,dx$$

$$= \int_0^1 (1)\,dx.$$

$$= (x)_0^1$$

On the face OAFG:

$$= 1 \rightarrow (6)$$

$\hat{n} = -\vec{k}$, $z=0$, $dz=0$

$$\iint_{OAFG} \frac{\vec{F}\cdot\hat{n}}{|\hat{n}\cdot\vec{k}|}\,dx\,dy = \int_0^1\int_0^1 -z\,dx\,dy.$$

$$= 0$$

$$\iint_S \vec{F}\cdot\hat{n}\,ds = -\tfrac{1}{2} + \tfrac{3}{2} + \tfrac{1}{2} - \tfrac{1}{2} + 1 + 0$$

$$= \frac{-1+3+2}{2}$$

$$= 4/2$$

$$= 2$$

Volume Integral

consider a closed surface in space enclosing a volume V Then

$\iiint_V \vec{F} \cdot dv$ is defined as the volume integral.

Let $\vec{F} = f_1\vec{i} + f_2\vec{j} + f_3\vec{k}$ Then

$$\iiint_V \vec{F} \cdot dv = \vec{i} \iiint_V f_1 \, dv + \vec{j} \iiint_V f_2 \, dv + \vec{k} \iiint_V f_3 \, dv$$

33) Evaluate $\iint_S \vec{F} \cdot \hat{n} \, ds$ where $\vec{F} = 18z\vec{i} - 12\vec{j} + 3y\vec{k}$ and S is the part of the plane $2x+3y+6z=12$ located in the first octant.

Let $\phi = 2x+3y+6z-12$

The unit surface normal $\hat{n}$,

$$\hat{n} = \frac{\nabla\phi}{|\nabla\phi|}$$

$$\nabla\phi = \left(\vec{i}\, \partial/\partial x + \vec{j}\, \partial/\partial y + \vec{k}\, \partial/\partial z\right).(2x+3y+6z-12)$$

$$= 2\vec{i} + 3\vec{j} + 6\vec{k}$$

$$|\nabla\phi| = \sqrt{4+9+36} = 7$$

$$\hat{n} = \frac{2\vec{i} + 3\vec{j} + 6\vec{k}}{7}$$

$$\vec{f}\cdot\hat{n} = (18z\hat{i} - 12\hat{j} + 3y\hat{k})\cdot\left(\frac{2\hat{i}+3\hat{j}+6\hat{k}}{7}\right)$$

$$= \frac{2(18z) - 12(3) + (3y)(6)}{7}$$

$$= \frac{36z - 36 + 18y}{7}$$

$$\vec{f}\cdot\hat{n} = \frac{36z + 18y - 36}{7} \qquad z = \frac{12-2x-3y}{6}$$

$$\vec{f}\cdot\hat{n} = \frac{1}{7}\left[36\left(\frac{12-2x-3y}{6}\right) + 18y - 36\right]$$

$$= \frac{1}{7}\left[72 - 12x - 18y + 18y - 36\right]$$

$$= \frac{1}{7}\left[72 - 36 - 12x\right]$$

$$\vec{f}\cdot\hat{n} = \frac{1}{7}\left[36 - 12x\right]$$

$$\frac{\vec{f}\cdot\hat{n}}{|\hat{n}\cdot\vec{k}|} = \frac{\frac{36-12x}{7}}{6/7}$$

$$= 6 - 2x$$

To find limit

$$2x + 3y + 6z - 12 = 0$$

In the xy plane, $z = 0$

$$2x + 3y = 12$$

$$y = \frac{12-2x}{3}$$

Put $y = 0 \Rightarrow 2x = 12$

$$x = 6$$

x varies from 0 to 6

y varies from 0 to $\frac{12-2x}{3}$

The projection of the surface on the xy plane is the region R bounded by the axis and the straight line $2x+3y=12$

$$\iint_S \vec{F}\cdot\hat{n}\,ds = \int_0^6 \int_0^{\frac{12-2x}{3}} (6-2x)\,dy\,dx$$

$$= \int_0^6 \left[6y-2xy\right]_0^{\frac{12-2x}{3}} dx$$

$$= \int_0^6 \left[6\left(\frac{12-2x}{3}\right)-2x\left(\frac{12-2x}{3}\right)\right] dx$$

$$= \int_0^6 \left(24-4x-\frac{24x}{3}+\frac{4x^2}{3}\right)dx$$

$$= \int_0^6 \left(24-12x+\frac{4x^2}{3}\right)dx$$

$$= \left[24x-\frac{12x^2}{2}+\frac{4x^3}{9}\right]_0^6$$

$$= 24(6)-6(6^2)+\frac{4(6^3)}{9}$$

$$= 24.$$

2020

7) Evaluate the surface integral $\iint_S (yz\vec{i}+zx\vec{j}+xy\vec{k})$ where S is the surface of the sphere $x^2+y^2+z^2=1$ in the 1^{st} ostant.

$$\vec{F} = yz\vec{i}+zx\vec{j}+xy\vec{k}$$

$$\phi = x^2+y^2+z^2-1$$

$$\nabla\phi = 2x\vec{i}+2y\vec{j}+2z\vec{k}$$

$$|\nabla\phi| = \sqrt{4(x^2+y^2+z^2)}$$

$$= \sqrt{4(1)} = 2.$$

$$\hat{n} = \frac{\nabla \phi}{|\nabla . \phi|}$$

$$= \frac{2x\vec{i} + 2y\vec{j} + \vec{zk}}{2}$$

$$= \vec{xi} + \vec{yj} + \vec{zk}$$

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3/2/2020

Unit - III

Gauss Divergence Theorem:- or ... divergence theorem

The normal surface integral of a vector point function $\vec{F}$ which is continously differentiable over the boundary of a closed region is equal to the volume integral of divergent $\vec{F}$ take through out the region.

i.e) $$\iint_S \vec{F} \cdot \hat{n}\, ds = \iiint_V (\nabla \cdot \vec{F})\, dv$$

1) P.T $\iint \hat{n} \cdot (\nabla \times \vec{F})\, ds = 0$ where $\vec{F}$ is a vector point function f and S is a called surface.

Let $f = (\nabla \times \vec{F})$

By Gauss divergence Thm,

$$\iint_S \vec{F} \cdot \hat{n}\, ds = \iiint_V (\nabla \cdot \vec{F})\, dv.$$

Since $f = \nabla \times \vec{F}$

Now, $\nabla \cdot \vec{F} = \nabla \cdot (\nabla \times \vec{F})$

let $\vec{F} = F_1 \vec{i} + F_2 \vec{j} + F_3 \vec{k}$

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ F_1 & F_2 & F_3 \end{vmatrix}$$

$$= \vec{i}\left[\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right] - \vec{j}\left[\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z}\right] + \vec{k}\left[\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right]$$

$$\nabla\cdot(\nabla\times\vec{F}) = \left(\frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}\right)\cdot$$

$$\left[\vec{i}\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) - \vec{j}\left(\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z}\right) + \vec{k}\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\right]$$

$$= \frac{\partial^2 F_3}{\partial x\partial y} - \frac{\partial^2 F_2}{\partial x\partial z} - \frac{\partial^2 F_3}{\partial x\partial y} + \frac{\partial^2 F_1}{\partial y\partial z} + \frac{\partial^2 F_2}{\partial x\partial z} - \frac{\partial^2 F_1}{\partial z\partial y}$$

$$\nabla\cdot(\nabla\times F) = 0$$

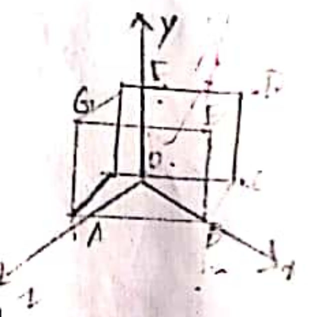
$$\iiint_V \nabla\cdot(\nabla\times\vec{F})\,dv = 0$$

By ① $\Rightarrow$

$$\iint_S \hat{n}\cdot(\nabla\times\vec{F})\,ds = 0$$

Hence Proved.

2) Verify Gauss divergence Theorem, $\vec{F} = x\vec{i} + y\vec{j} + z\vec{k}$ taken over the region bounded by the plane $x=0$, $x=a$, $y=0$, $y=a$, $z=0$, $z=a$.



$$\iint_S \vec{F}\cdot\hat{n}\,ds = \iint_{OAGF} + \iint_{BCDE} + \iint_{ABEG} +$$

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$$\iint_{OCDF} + \iint_{FGED} + \iint_{OABC}$$

On OAGF:

$x = 0 \Rightarrow dx = 0$

$y \rightarrow 0$ to a

$z \rightarrow 0$ to a

$\hat{n} = -\hat{i}$

$$\iint_{OAGF} \vec{F} \cdot \hat{n}\, ds = \int_0^a \int_0^a \frac{-x}{-1}\, dy\, dz$$

$$= 0.$$

On BCDE:

$x = a \Rightarrow dx = 0$

$y \rightarrow 0$ to a

$z \rightarrow 0$ to a

$\hat{n} = \vec{i}$

$$\iint_{BCDF} \vec{F} \cdot \hat{n}\, ds = \int_0^a \int_0^a \frac{x}{1}\, dy\, dz$$

$$= \int_0^a \int_0^a a\, dy\, dz$$

$$= a^3$$

On ABEG

$z = a \Rightarrow dz = 0$

$x \rightarrow 0$ to a

$y \rightarrow 0$ to a

$\hat{n} = \vec{k}$

$$\iint_{ABEG} \vec{F} \cdot \hat{n}\, ds = \int_0^a \int_0^a z\, dx\, dz.$$

$$= \int_0^a \int_0^a a\, dx\, dz$$

$$= a^3$$

On OCDF :

$$z=0 \Rightarrow dz=0$$
$$x \rightarrow 0 \text{ to } a$$
$$y \rightarrow 0 \text{ to } a$$
$$\hat{n} = -\vec{k}$$

$$\iint_{OCDF} \vec{F}\cdot\hat{n}\,ds = \int_0^a\int_0^a \frac{-z}{-1}\,dx\,dy.$$
$$=0$$

On OABC :

$$y=0 \Rightarrow dy=0$$
$$x \rightarrow 0 \text{ to } a$$
$$z \rightarrow 0 \text{ to } a$$
$$\hat{n} = -\vec{j}$$

$$\iint_{OABC} \vec{F}\cdot\hat{n}\,ds = \int_0^a\int_0^a \frac{-y}{-1}\,dx\,dz$$
$$=0$$

$\iint_S \vec{F}\cdot\hat{n}\,ds = \iiint (\nabla\cdot\vec{F})\,ds$

On GEDF :

$$y=a \Rightarrow dy=0$$
$$x \rightarrow 0 \text{ to } a$$
$$z \rightarrow 0 \text{ to } a$$
$$\hat{n} = \vec{j}$$

$$\iint_{GEDF} \vec{F}\cdot\hat{n}\,ds = \int_0^a\int_0^a \frac{y}{1}\,dx\,dz$$
$$= a^3.$$

$$\iint_S \vec{F}\cdot\hat{n}\,ds = 0+a^3+0+a^3+0+a^3.$$
$$=3a^3. \longrightarrow (1)$$

Verification:-

$$\nabla\cdot\vec{F} = \left(\partial/\partial x\,\vec{i} + \partial/\partial y\,\vec{j} + \partial/\partial z\,\vec{k}\right)\cdot\left(x\vec{i}+y\vec{j}+k\vec{z}\right)$$

$= 1+1+1$

$\nabla\cdot\vec{F} = 3.$

$$\iiint_V (\nabla\cdot\vec{F})\,dv = \int_0^a\int_0^a\int_0^a 3\,dx\,dy\,dz$$

$$= 3\int_0^a\int_0^a [x]_0^a\,dy\,dz$$

$$= 3\int_0^a\int_0^a a\,dy\,dz$$

$$= 3\int_0^a [ay]_0^a\,dz.$$

$$= 3\int_0^a [a^2]\,dz$$

$$= 3\,[a^2 z]_0^a$$

$$= 3a^3 \longrightarrow (2)$$

from (1), (2)

$$\iint_S \vec{F}\cdot\hat{n}\,ds = \iiint_V (\nabla\cdot\vec{F})\,dv.$$

Hence Gauss div thm is verified.

4/2/2020 Green's Theorem.

Let R be a closed curve. Let

M and N are continuous function of x and y having continuous partial derivatives in R.

$$\int_C (M\,dx + N\,dy) = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)dx\,dy.$$

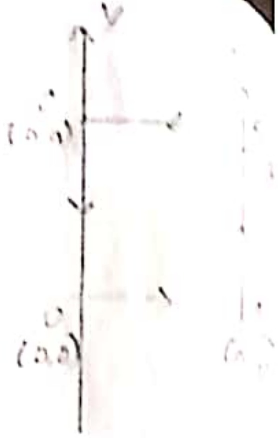
3) Verify Green's Thm for $\int_C x^2\,dx + xy\,dy$ whose c is the curve in the xy-plane given by $x=0,\ x=a,\ y=0,\ y=a.$

$$\int_C (M\,dx + N\,dy) = \int_{OA} + \int_{AB} + \int_{BC} + \int_{CD}$$

On OA

x varies from 0 to a

$y = 0 \quad dy = 0$



$$\int_C M\,dx + N\,dy = \int_0^a x^2\,dx + 0$$

$$= \left[\frac{x^3}{3}\right]_0^a$$

$$= \frac{a^3}{3}$$

On AB

y varies from 0 to a

$x = a. \quad dx = 0$

$$\int_C M\,dx + N\,dy = \int_0^a x^2(0) + ay\,dy.$$

$$= a\left[\frac{y^2}{2}\right]_0^a$$

$$= \frac{a^3}{2}.$$

On BC

x varies from a to 0

$y = a \Rightarrow dy = 0$

$$\int_C M\,dx + N\,dy = -\int_a^0 x^2\,dx + 0$$

$$= -\left[\frac{x^3}{3}\right]_a^0$$

$$= -\frac{a^3}{3}.$$

On CO

y varies from a to 0

$x = 0 \quad dx = 0$

$$\int_C M\,dx + N\,dy = \int_a^0 0 + 0$$

$$= 0$$

$$\int_C (M\,dx + \int N\,dy) = \frac{a^3}{3} + \frac{a^3}{2} - \frac{a^3}{3} + 0$$

$$= a^3/2. \longrightarrow (1)$$

Scanned with CamScanner

Verification.

$$M = x^2 \qquad N = xy.$$

$$\frac{\partial M}{\partial y} = 0 \qquad \frac{\partial N}{\partial x} = y.$$

$$\cancel{\iint_R \left(\frac{\partial M}{\partial y}\right. -}$$

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dy\,dx =$$

$$\int_0^a \int_0^a (y-0)\,dy\,dx$$

$$= \int_0^a \int_0^a (y)\,dx\,dy.$$

$$= \int_0^a \frac{a^2}{2}\,dx$$

$$= \left[\frac{a^3}{2}\right] \longrightarrow ②$$

from ① and ②

$$\int_C (M\,dx + N\,dy) = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dy\,dx$$

Hence Green's Thm is verified.

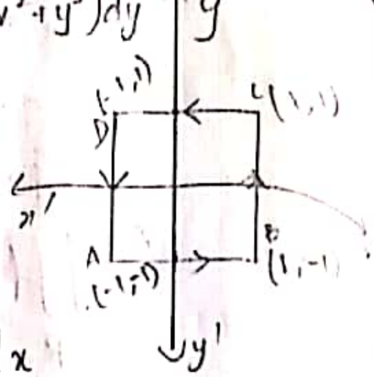
4) Evaluate $\int (xy + x^2)\,dx + (x^2 + y^2)\,dy$ where c is the square formed by the lines. $x=-1, x=1, y=-1, y=1$ using Green's theorem.

$$\int (M\,dx + N\,dy) = \int_{OA} + \int_{AB} + \int_{BC} + \int_{CD}$$

On OA $\quad x$ varies from

On AB:

$(xy+x^2)dx + (x^2+y^2)dy$



x varies from -1 to 1

$y=-1 \quad dy=0$

$$\int_C Mdx + Ndy = \int_{-1}^{1} (-x+x^2)\,dx$$

$$= \left[-\frac{x^2}{2} + \frac{x^3}{3}\right]_{-1}^{1}$$

$$= \left[-1/2 + 1/3\right] - \left[-1/2 + 1/3\right]$$

$$= 2/3$$

On BC

y varies from -1 to 1

$x=1 \quad dx=0$

$$\int_C Mdx + Ndy = \int_{-1}^{1} (y+1)dx + (1+y^2)dy$$

$$= \cancel{(xy+x)_{-1}^{1}} + \left(y+y^3/3\right)_{-1}^{1}$$

$$= \int_{-1}^{1} (1+y^2)\,dy$$

$$= \left[y + y^3/3\right]_{-1}^{1}$$

$$= (1+1/3) - (-1-1/3)$$

$$= 8/3$$

On CD

x varies from 1 to -1

$y=1, \; dy=0$

$$\int_C Mdx + Ndy = \int_{1}^{-1} (x+x^2)\,dx$$

Scanned with CamScanner

$$= \left[\frac{x^2}{2} + \frac{x^3}{3}\right]_1^{-1}$$

$$= [1/2 - 1/3] - [1/2 + 1/3]$$

$$= -2/3$$

On DA

y varies from 1 to −1

$$x = -1 \qquad dx = 0$$

$$= \int_1^{-1} (1+y^2)\, dy$$

$$= \left[y + y^3/3\right]_1^{-1}$$

$$= [-1 - 1/3] - [1 + 1/3]$$

$$= -4/3 - 4/3.$$

$$= -8/3$$

$$\int_C (M\,dx + N\,dy) = 2/3 + 8/3 - 2/3 - 8/3$$

$$= 0 \longrightarrow (1)$$

verification

$$M = xy + x^2, \quad N = x^2 + y^2.$$

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dy\,dx = \int_{-1}^{1}\int_{-1}^{1} (2x - x)\, dy\, dx$$

$$= \int_{-1}^{1}\int_{-1}^{1} x\, dy\, dx$$

$$= \int_{-1}^{1} \left(\frac{x^2}{2}\right)_{-1}^{1} dy.$$

$$= \int_{-1}^{1} [1/2 - 1/2]\, dy$$

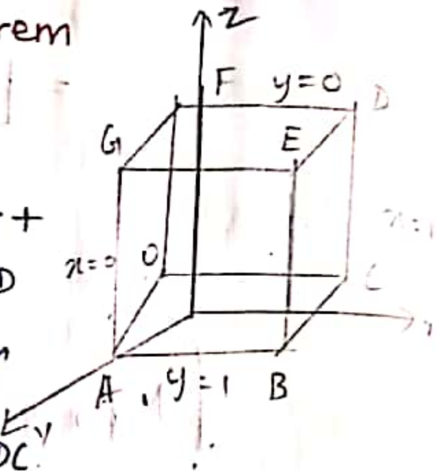
$$= \int_{-1}^{1} 0 \, dy$$

$$= 0 \longrightarrow ②$$

from ① and ②

$$\int_C (M dx + N dy) = \iint_R \left(\frac{\partial N}{\partial x} + \frac{\partial M}{\partial y} \right) dy\, dx$$

5) Evaluate $\int_S \vec{F} \cdot \hat{n} \, ds$ $\vec{F} = 4xz\vec{i} - y^2\vec{j} + yz\vec{k}$ and s is the surface of the cube bounded by $x=0$, $x=1$, $y=0$, $y=1$, $z=0$, $z=1$ divergence theorem



$$\iint_S \vec{F} \cdot \hat{n} \, ds = \iint_{OABC} + \iint_{BCED} + \iint_{GFED} + \iint_{OAGF} + \iint_{ABGE} + \iint_{OFDC}$$

On OABC.

$y = 0 \Rightarrow dy = 0$

$x \rightarrow 0$ to 1 $\quad \hat{n} = -\vec{j}$

$z \rightarrow 0$ to 1

$$\iint_{OABC} \vec{F} \cdot \hat{n} \, ds = \int_0^1 \int_0^1 +y^2 \, dx \, dz.$$

$$= \int_0^1 0 \, dz$$

$$= 0$$

On BCED

$x = 1 \Rightarrow dx = 0$

$y \rightarrow 0$ to 1 $\quad \hat{n} = \vec{i}$

$$\iint_{BCED} \vec{F}\cdot\hat{n}\,ds = \int_0^1\int_0^1 4xz\,dy\,dz$$

$$= \int_0^1 \left[4zy\right]_0^1 dz$$

$$= 4\left(\frac{z^2}{2}\right)_0^1$$

$$= 4/2$$

$$= 2$$

On GIFED

$y = 1 \Rightarrow dy = 0$

$x \to 0$ to 1

$z \to 0$ to 1

$\hat{n} = \vec{j}$

$$\iint_{GIFED} \vec{F}\cdot\hat{n}\,ds = \int_0^1\int_0^1 -y^2\,dx\,dz$$

$$= -\int_0^1 (x)_0^1\,dz$$

$$= -z$$

$$= -1$$

On OAGIF:

$x = 0 \Rightarrow dx = 0$

$y \to 0$ to 1

$z \to 0$ to 1

$\hat{n} = -\vec{i}$

$$\iint_{OAGIF} \vec{F}\cdot\hat{n}\,ds = -\int_0^1\int_0^1 4(0)z\,dx\,dy$$

$$= 0$$

On ABGE.

$z = 1 \Rightarrow dz = 0$

$x \to 0$ to 1

$y \to 0$ to 1

$$\hat{n} = \vec{k}$$

$$\iint_{ABGF} \vec{F}\cdot\hat{n}\,ds = \int_0^1\int_0^1 yz\,dx\,dy$$

$$= \int_0^1 \left(y^2/2\right)_0^1 dx$$

$$= 1/2 \int_0^1 dx$$

$$= 1/2.$$

On OFDC

$z = 0 \quad dz = 0$

$x \to 0 \text{ to } 1$

$y \to 0 \text{ to } 1$

$$\hat{n} = -\vec{k}$$

$$\iint_{OFDC} \vec{F}\cdot\hat{n}\,ds = -\int_0^1\int_0^1 yz\,dx\,dy.$$

$$= 0$$

$$\iint_S \vec{F}\cdot\hat{n}\,ds = 0+2-1+1/2.$$

$$= 3/2.$$

verification

$$\nabla\cdot\vec{F} = \left(\partial/\partial x\,\vec{i} + \partial/\partial y\,\vec{j} + \partial/\partial z\,\vec{k}\right)\left(4xz\vec{i} - y^2\vec{j} + yz\vec{k}\right)$$

$$= 4xz\,\partial/\partial x - y^2\,\partial/\partial y + yz\,\partial/\partial z$$

$$= 4z - 2y + y.$$

$$= 4z - y$$

$$\iiint_V (\nabla\cdot\vec{F})\,dv = \int_0^1\int_0^1\int_0^1 (4z-y)\,dx\,dy\,dz$$

$$= \int_0^1\int_0^1 \left(\frac{4z^2}{2} - yz\right)_0^1 dx\,dy$$

$$= \int_0^1 \int_0^1 (2-y)\, dx\, dy.$$

$$= \int_0^1 (2y - y^2/2)\, dx$$

$$= \int_0^1 (2 - 1/2)\, dx$$

$$= 3/2 \int_0^1 dx.$$

$$= 3/2.$$

6) Using Green's Thm evaluate $\int_c (2x-y)\,dx + (x+y)\,dy$ where c is the boundary of circle $x^2+y^2=a^2$ in xoy plane.

Proof

Green's Thm

$$\int_c M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dy\, dx$$

given that $\int_c (2x-y)\,dx + (x+y)\,dy$

$M = 2x - y \quad N = x+y.$

$\frac{\partial M}{\partial y} = -1 \quad \frac{\partial N}{\partial x} = 1$

To find limit.

$x^2+y^2=a^2$

$y^2 = a^2 - x^2$

$y = \sqrt{a^2-x^2}.$

$y \to 0$ to $\sqrt{a^2-x^2}.$

Put $y=0$ in ①.

$x^2 = a^2$

$x = \pm a.$

$x \to 0$ to a

By Green's Thm

$$\int_C (2x-y)dx+(x+y)dy = 4\int_0^a \int_0^{\sqrt{a^2-x^2}} (1-(-1))\, dy\, dx$$

$$= 4\int_0^a \int_0^{\sqrt{a^2-x^2}} 2\, dy\, dx.$$

$$= 4\times 2\int_0^a (y)_0^{\sqrt{a^2-x^2}}\, dx$$

$$= 8\int_0^a \sqrt{a^2-x^2}\, dx$$

$$= 8\left[\frac{a^2}{2} \sin^{-1}(x/a)\right]_0^a$$

$$= 8\left[\frac{a^2}{2} \sin^{-1}(1) - \frac{a^2}{2}\sin^{-1}(0)\right]$$

$$= 8\left[\frac{a^2}{2} \cdot \frac{\pi}{2}\right]$$

$$= 2\pi a^2.$$

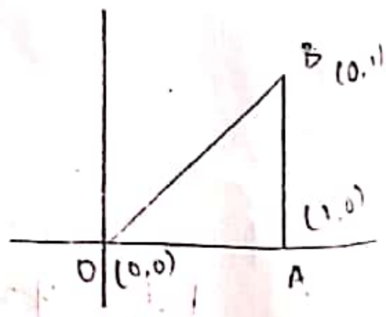
7) Verify Green's thm for

$\int_C (3x^2-8y^2)\,dx + (4y-6xy)\,dy$ bounded by

$x=0,\ y=0,\ x+y=1$

$$\int_C (3x^2-8y^2)\,dx+(4y-6xy)\,dy = \int_{OA} + \int_{AB} + \int_{OB}.$$

On OA.



x varies from 0 to 1

$y=0 \quad dy=0$

$$= \int_0^1 3x^2\, dx$$

$$= 3\left(x^3/3\right)_0^1$$

$$= 1$$

On AB

y varies from 0 to 1

$x = 1 \quad dx = 0$

$$= \int_0^1 (4y - 6y)\,dy$$

$$= \int_0^1 -2y\,dy$$

$$= -2\left[\frac{y^2}{2}\right]_0^1$$

$$= -1$$

~~On OB~~ On AB

$x \rightarrow 1$ to 0

$x + y = 1$

$y = 1 - x$

$dy = -dx$

$$\int_{AB} (3x^2 - 8y^2)\,dx + (4y - 6xy)\,dy = \int_1^0 (3x^2 - 8(1-x)^2\,dx + [4(1-x) - 6x(1-x)] - dx$$

$$= \int_1^0 (3x^2 - 8 + 16x - 8x^2)\,dx - (4 - 4x - 6x + 6x^2)\,dx$$

$$= \int_1^0 (-11x^2 + 26x - 12)\,dx$$

$$= \left[-\frac{11x^3}{3} + \frac{26x^2}{2} - 12x\right]_1^0$$

$$= -\left[\frac{-11}{3} + 13 - 12\right]$$

$$= 11/3 - 1 = 18/3.$$

On OB:-

y varies from 1 to 0

$x=0 \Rightarrow dx=0$

$$\int_{OB} (3x^2-8y^2)dx+(4y-6xy)dy$$

$$= \int_1^0 4y\,dy$$

$$= 4\left[\frac{y^2}{2}\right]_1^0$$

$$= 2(y^2)_1^0$$

$$= 2(-1)$$

$$= -2.$$

$$\int_C (3x^2-8y^2)dx+(4y-6xy)dy = 1+\frac{8}{3}-2$$

$$= 5/3$$

Verification:

$M=3x^2-8y^2 \qquad N=4y-6xy.$

$\frac{\partial M}{\partial y}=-16y \qquad \frac{\partial N}{\partial x}=-6y.$

$$\iint\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)dx\,dy$$

find limit

$x+y=1$

$y=1-x$

$y \to 0$ to $1-x$

$y=0 \Rightarrow x=1$

$x \to 0$ to 1

$$= \int_0^1\int_0^{1-x}(-6y+16y)dx\,dy$$

$$= \int_0^1\int_0^{1-x}(-6y+16y)\,dx\,dy \Rightarrow \int_0^1\left(-\frac{6y^2}{2}+\frac{16y^2}{2}\right)_0^{1-x}dx$$

$$= \int_0^1\left(-3(1-x)^2+8(1-x)^2\right)dx \Rightarrow \int_0^1 -3(1+x^2-2x)+8(1+x^2-2x)\,dx$$

$$= \int_0^1(-3-\cancel{x^2}+\cancel{6x}+8+\cancel{x^2}-\cancel{6x})dx \Rightarrow \int_0^1 5\,dx$$

$$= 5/3.$$

Hence Green's Thm verifyed.

8) If $\vec{F} = 2yx\vec{i} - zy\vec{j} + x^2\vec{k}$ Evaluate $\int_S \vec{F}\cdot\hat{n}\,ds$ where S denote the entire surface of the cube bounded by the co-ordinates planes and the planes $x=a$, $y=a$, $z=a$ by the application of Gauss's Thm and verify it by direct evaluation of surface Integral.

$$\iint_S \vec{F}\cdot\hat{n}\,ds = \iint_{OAGF} + \iint_{BCDE} + \iint_{ABEG} + \iint_{OCDE} + \iint_{FGBE} + \iint_{OABC}.$$

On OAGF

$x=0 \Rightarrow dx=0$

$y \to 0$ to a

$z \to 0$ to a

$\hat{n} = -\vec{i}$

$$\iint_{OAGF} \vec{F}\cdot\hat{n}\,ds = \int_0^a\int_0^a 2yx\,dy\,dz.$$

$$= \int_0^a\int_0^a 2y(0)\,dy\,dz.$$

$$=0.$$

On BCDE

$x=a \quad dx=0$

$y \to 0$ to a

$z \to 0$ to a

$\hat{n} = \vec{i}$

$$\iint_{BCDE} \vec{F}\cdot\hat{n}\,ds = \int_0^a\int_0^a 2yx\,dy\,dz$$

$$= \int_0^a\int_0^a 2ya\,dy\,dz$$

$$= \int_0^a \left[\frac{2y^2}{2} a\right]_0^a dz$$

$$= \int_0^a a^3 \, dz$$

$$= (a^3 z)_0^a .$$

$$= a^4 .$$

On ABEG:-

$$z = a \Rightarrow dz = 0$$

$$x \to 0 \text{ to } a$$

$$y \to 0 \text{ to } a$$

$$\hat{n} = \vec{k}$$

$$\iint_{ABEG} \vec{F} \cdot \hat{n} \, ds = \int_0^a \int_0^a x^2 \, dx \, dy$$

$$= \int_0^a \left[x^3/3\right]_0^a dy .$$

$$= \int_0^a \frac{a^3}{3} \, dy .$$

$$= \left[a^3 \, y/3\right]_0^a .$$

$$= \frac{a^4}{3} .$$

On OCDE

$$z = 0 \Rightarrow dz = 0$$

$$x \to 0 \text{ to } a$$

$$y \to 0 \text{ to } a$$

$$\hat{n} = -\vec{k}$$

$$\iint_{OCDE} \vec{F} \cdot \hat{n} \, ds = \int_0^a \int_0^a -x^2 \, dx \, dy .$$

$$= -\int_0^a \left[x^3/3\right]_0^a dy .$$

$$= -\left[a^3/3 \; y\right]_0^a$$

$$= -a^4/3$$

On FGDE.

$y = a \Rightarrow dy = 0$

$x \to 0$ to a

$z \to 0$ to a.

$\hat{n} = -\vec{j}$

$$\iint_{FGDE} \vec{F} \cdot \hat{n}\, ds = \int_0^a \int_0^a -zy\, dx\, dz.$$

$$= -a \int_0^a \int_0^a z\, dx\, dz.$$

$$= \int_0^a [xza]_0^a\, dz$$

$$= \int_0^a (-za^2)\, dz.$$

$$= -\left(\frac{z^2 a^2}{2}\right)_0^a$$

$$= \frac{-a^4}{2}$$

On OABC

$y = 0 \Rightarrow dy = 0$

$x \to 0$ to a

$z \to 0$ to a

$\hat{n} = -\vec{j}$

$$\iint_{OABC} \vec{F} \cdot \hat{n}\, ds = \int_0^a \int_0^a zy\, dx\, dz.$$

$$= 0$$

$$\iint_S \vec{F} \cdot \hat{n}\, ds = 0 + a^4 + \frac{a^4}{3} - \frac{a^4}{3} - \frac{a^4}{2} + 0$$

$$= \tfrac{1}{2} a^4 \to (1)$$

Verification

$$\nabla \cdot \vec{F} = \partial/\partial x\,(2yx) + \partial/\partial y\,(-zy) + \partial/\partial z\,(x^2).$$

$$= 2y - z$$

$$\iiint_V (\nabla\cdot\vec{F})\,dv = \int_0^a\int_0^a\int_0^a 2y - z\,dx\,dy\,dz$$

$$= \int_0^a\int_0^a \left[2xy - zx\right]_0^a dy\,dz$$

$$= \int_0^a\int_0^a (2ay - za)\,dy\,dz$$

$$= \int_0^a \left[\frac{2ay^2}{2} - zay\right]_0^a dz$$

$$= \int_0^a (a^3 - za^2)\,dz.$$

$$= \left[a^3 z - \frac{z^2}{2}a^2\right]_0^a$$

$$= a^4 - \frac{a^4}{2}$$

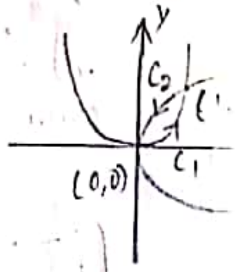
$$= \tfrac{1}{2}a^4 \longrightarrow ②$$

from ① & ②

Hence Gauss's thm is verified.

9) verify Green's thm for $\int_C (3x^2 - 8y^2)dx + (4y - 6xy)dy$

$y = \sqrt{x}$, $y = x^2$.



$y^2 = x$ $\quad$ $y = x^2$

$y = (y^2)^2$

$y = y^4$

$y^4 - y = 0$

$y(y^3 - 1) = 0$

$y = 0, y = 1$

$y = 0 \Rightarrow y = x^2$

$x = 1$

$x = 1 \Rightarrow y = 1$

On C_1. $\quad x^2 = y \Rightarrow dy = 2x\,dx$

$x \to 0$ to 1

$= \int_0^1 (3x^2 - 8x^4)dx + (4x^2 - 6x^3)\, 2x\, dx.$

$= \left[\left[3x^3/3 - \frac{8x^5}{5}\right] + \left[8\frac{x^4}{4} - 12\frac{x^5}{5}\right]\right]_0^1$

$= \left[x^3 - \frac{8x^5}{5} + 2x^4 - \frac{12}{5}x^5\right]_0^1.$

$= 1 - 8/5 + 2 - 12/5.$

$= 3 - \frac{20}{5}.$

$= \frac{15-20}{5} = \frac{-5}{5} = -1$

On C_2. $y^2 = x \Rightarrow dx = 2y\,dy.$

$y \to 1$ to 0.

$= \int_1^0 (3y^4 - 8y^2)2y\,dy + (4y - 6y^3)dy.$

$= \int_1^0 (6y^5 - 16y^3)\,dy + (4y - 6y^3)\,dy.$

$= -\left[\frac{6y^6}{6} - \frac{16y^4}{4} + \frac{4y^2}{2} - \frac{6y^4}{4}\right]_1^0$

$= -[1 - 4 + 2 - 3/2] = -[-1 - 3/2]$

$= 1 + 3/2 \Rightarrow 5/2.$

Verification

$\int_C = \int_{C_1} + \int_{C_2}.$

$= -1 + 5/2$

$\boxed{= 3/2}$

$\longrightarrow$ $x \to 0$ to 1

$y \to x^2$ to $\sqrt{x}$.

$M = 3x^2 - 8y^2$ $\quad N = 4y - 6xy$

$\frac{\partial M}{\partial y} = -16y$ $\quad \frac{\partial N}{\partial x} = -6y$

$$= \int_0^1 \int_{x^2}^{\sqrt{x}} (10y)\, dx\, dy$$

$$= \int_0^1 \left[\cancel{10}^{5}\frac{y^2}{2}\right]_{x^2}^{\sqrt{x}} dx$$

$$= 5\int_0^1 (y^2)_{x^2}^{\sqrt{x}}\, dx$$

$$= 5\int_0^1 x - x^4\, dx$$

$$= 5\left[\frac{x^2}{2} - \frac{x^5}{5}\right]_0^1$$

$$= 5\left[1/2 - 1/5\right]$$

$$= 5\left[\frac{5-2}{10}\right]$$

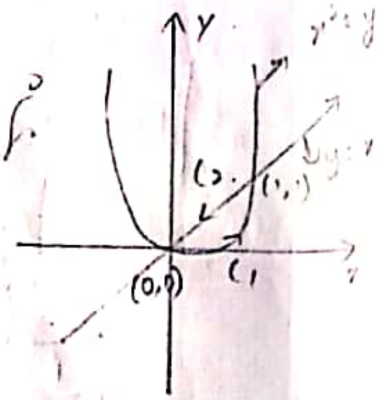
$$= 5\left[3/10\right]$$

$$\boxed{= 3/2}$$

Hence Green's is verified.

10) Verify Green's thm in the plane for $\int_c (xy+y^2)\,dx + x^2 dy$ where c is the closed curve of the region bounded by $y=x$ & $y=x^2$

$y=x^2$ is a parabola & $y=x$ is a st line thro' the origin both intersecting at $(0,0)$ & $(1,1)$



On C_1

$y = x^2$

$dy = 2x\,dx$

$x \to 0$ to 1

$$\int_{C_1} (xy+y^2)\,dx + x^2 dy =$$

$$= \int_0^1 (x^3 + y^1 x^4)\,dx + x^2 (2x\,dx)$$

$$= \left[\frac{x^4}{4} + \frac{x^5}{5} + 2\frac{x^4}{4}\right]_0^1$$

$$= 1/4 + 1/5 + 2/4$$

$$= 3/4 + 1/5$$

$$= \frac{15+4}{20} = \frac{19}{20}.$$

On C_2.

$x = y$

$y \to 1$ to 0 $\quad dx = dy$.

$$\int_c (xy+y^2)dx + x^2 dy = \int_1^0 (y^2 + y^2)dy + y^2 dy$$

$$= \left[y^3/3 + y^3/3 + y^3/3\right]_1^0$$

$$= -\left[1/3 + 1/3 + 1/3\right]$$

$$= -[3/3]$$

$$= -1.$$

$$\int_c = \int_{c1} + \int_{c_2}.$$

$$= \frac{19}{20} - 1$$

$$= \frac{19-20}{20}$$

$$= -\frac{1}{20}$$

Verification

$x \to 0$ to 1

$y \to x^2$ to x.

$M = xy + y^2 \qquad N = x^2$

$$\frac{\partial M}{\partial y} = x + 2y \qquad \frac{\partial N}{\partial x} = 2x$$

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx\,dy = \int_0^1 \int_{x^2}^{x} (-x - 2y + 2x)\, dx\,dy$$

$$= \int_0^1 \int_{x^2}^{x} (2y - x)\, dx\,dy$$

$$= \int_0^1 \left(2y^2/2 - xy \right)_{x^2}^{x} dx$$

$$= \int_0^1 (y^2 - xy)_{x^2}^{x} dx$$

$$= \int_0^1 \left[(x^2 - x^2) - (x^4 - x^3) \right] dx$$

$$= \int_0^1 -x\, dx$$

$$= \int_0^1 \int_{x^2}^{x} (x - 2y)\, dx\,dy$$

$$= \int_0^1 \left[xy - 2y^2/2 \right]_{x^2}^{x} dx$$

$$= \int_0^1 \left[[x^2 - x^2] - [x^3 - x^4] \right] dx$$

$$= \int_0^1 x^4 - x^3\, dx$$

$$= \left[\frac{x^5}{5} - \frac{x^4}{4} \right]_0^1$$

$= 1/5 - 1/4.$

$= \frac{4-5}{20}$

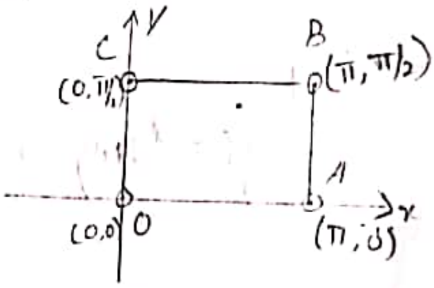
$= -1/20$

Hence Green's theorem is Verified.

Evaluate by Green's theorem $\int_C (e^{-x} \sin y\, dx + e^{-x} \cos y\, dy)$ where c is rectance with vertices $(0,0)$, $(\pi, 0)$, $(\pi, \pi/2)$, $(0, \pi/2)$ + hence verify Green's thm.

$$\int_C = \int_{OA} + \int_{AB} + \int_{BC} + \int_{OC}$$

$$\int_C (e^{-x} \sin y\, dx + e^{-x} \cos y\, dy)$$



On OA

x varies from 0 to π

$y = 0 \quad dy = 0$

$$= \int_0^{\pi} e^{-x} \cos y\, dy + e^{-x} \sin y\, dx$$

$$= \int_0^{\pi} \cos y\, dy$$

$$= \int_0^{\pi} (e^{-x} \sin y\, dx)$$

$$= \left[-e^{-x} \sin y \right]_0^{\pi}$$

On OA = 0

On AB

y varies from 0 to $\pi/2$

$x = \pi \quad dx = 0$

$$\int_C (e^{-x}\sin y\, dx + e^{-x}\cos y\, dy) = \int_0^{\pi/2} (e^{-\pi}\cos y)\, dy$$

$$= e^{-\pi}\int_0^{\pi/2}\cos y\, dy$$

$$= e^{-\pi}(\sin y)_0^{\pi/2}$$

$$= e^{-\pi}\sin(\pi/2)$$

$$= e^{-\pi}$$

On BC

x varies from π to 0

$y = \pi/2 \quad dy = 0$

$$\int_C (e^{-x}\sin y\, dx + e^{-x}\cos y\, dy) = \int_{\pi}^{0} (e^{-x}\sin \pi/2)\, dx$$

$$= [-e^{-x}]_{\pi}^{0} \Rightarrow -[e^{-0} - e^{-\pi}]$$

$$= -1 + e^{-\pi}$$

On OC

y varies from $\pi/2$ to 0

$x = 0 \quad dx = 0$

$$\int_C (e^{-x}\sin y\, dx + e^{-x}\cos y\, dy) = \int_{\pi/2}^{0} e^{-0}\cos y\, dy$$

$$= [\sin y]_{\pi/2}^{0}$$

$$= -\sin \pi/2$$

$$= -1$$

$$\text{On } \int_C = e^{-\pi} + e^{-\pi} - 1 - 1 + 0$$

$$= 2e^{-\pi} - 2$$

Verification

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\, dy.$$

$$M = e^{-x} \sin y.$$

$$\frac{\partial M}{\partial y} = e^{-x} \cos y$$

$$N = e^{-x} \cos y.$$

$$\frac{\partial N}{\partial x} = -e^{-x} \cos y.$$

$$= \int_0^{\pi} \int_0^{\pi/2} (-e^{-x} \cos y - e^{-x} \cos y)\, dx\, dy.$$

$$= -2 \int_0^{\pi} \int_0^{\pi/2} (e^{-x} \cos y)\, dx\, dy$$

$$= -2 \int_0^{\pi} \left[e^{-x} \sin y\right]_0^{\pi/2} dx.$$

$$= -2 \int_0^{\pi} (e^{-x})\, dx$$

$$= -2 \left[-e^{-x}\right]_0^{\pi}$$

$$= -2\left[-e^{-\pi} + e^{-0}\right]$$

$$= 2e^{-\pi} - 2.$$

Hence Green's Thm is verified

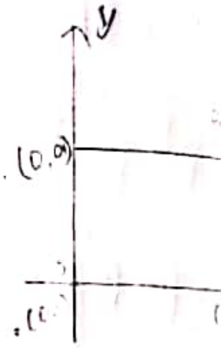
stoke's Theorem:-

Let S be a open surface bounded by a closed curve. let $\vec{F}$ be the vector point. function defined on S $\hat{n}$ be the unit outward drawn normal at any

Point P on it, then.

$$\int_C \vec{F}\cdot\vec{dr} = \iint_S (\nabla\times\vec{F})\cdot\vec{n}$$

12) Verify Stoke's thm for $\vec{F} = x^2 i$
taken around the square in xy-p
whose sides are $x=0$, $x=a$, $y=0$, $y=a$



On OA
$$\vec{F} = x^2\vec{i} + xy\vec{j}$$
$$\vec{r} = x\vec{i} + y\vec{j}$$
$$\vec{dr} = dx\vec{i} + dy\vec{j}$$
$$\vec{F}\cdot\vec{dr} = x^2dx + xy\,dy$$

$$\int_C \vec{F}\cdot\vec{dr} = \int_{OA} + \int_{AB} + \int_{BC} + \int_{OC}$$

On OA.

x varies 0 to a

$y=0 \quad dy=0$

$$\int_{OA} \vec{F}\cdot dr = \int_0^a x^2\,dx$$
$$= \left[x^3/3\right]_0^a$$
$$= a^3/3$$

On AB y varies 0 to a

$x=a \quad dx=0$

$$\int_{AB} = a\int_0^a y\,dy.$$
$$= a\left[y^2/2\right]_0^a$$
$$= a^3/2$$

On BC

x varies a to 0

$y = a \quad dy = 0$

$$\int_{BC} = \int_a^0 x^2\, dx$$

$$= \left[x^3/3 \right]_a^0$$

$$= -a^3/3.$$

On CO

y varies a to 0

$x = 0 \quad dx = 0$

$$\int_{CO} = \int_a^0 0\,dx + 0y\,dy.$$

$$= 0$$

$$\int_C \vec{F} \cdot \vec{dr} = \int_{OA} + \int_{AB} + \int_{BC} + \int_{OC}$$

$$= a^3/3 + a^3/2 - a^3/3 + 0$$

$$= a^3/2. \longrightarrow (1)$$

Verification

$$\vec{F} = x^2 \vec{i} + xy \vec{j}$$

$$\text{curl} \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z. \\ x^2 & xy & 0 \end{vmatrix}$$

$$= \vec{k} \left[\partial/\partial x (xy) - \partial/\partial y (x^2) \right]$$

$$\text{curl} \vec{F} = y\vec{k}.$$

n = outward normal is $\vec{k}$

W.K.T $x \to 0$ to a, $y = 0$ to a

$\hat{n}\cdot\hat{k} = 1$

$(\text{curl}\,\vec{F})\cdot\hat{n} = y$

$$\iint_S (\text{curl}\,\vec{F})\cdot\hat{n}\,ds = \int_0^a\int_0^a \frac{(\text{curl}\,\vec{F})\cdot\hat{n}}{|\hat{n}\cdot\vec{k}|}\,dx\,dy$$

$$= \int_0^a\int_0^a \frac{y}{1}\,dx\,dy$$

$$= \int_0^a [xy]_0^a\,dy$$

$$\iint_c \vec{F}\cdot\vec{dr} = \iint (\text{curl}\,\vec{F})\cdot\hat{n}\,ds$$

$$= \int_0^a ay\,dy$$

$$= a\int_0^a y\,dy$$

$$= a\left[\frac{y^2}{2}\right]_0^a$$

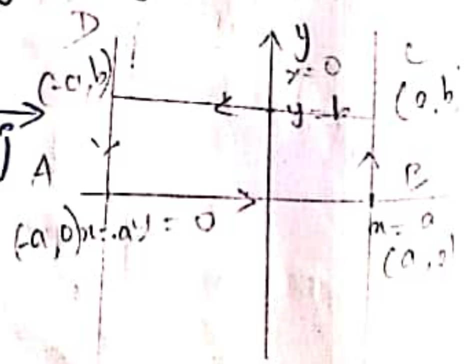
$$= a\left(\frac{a^2}{2}\right)$$

$$= \frac{a^3}{2} \rightarrow ②$$

from ① & ②

$$\int_c \vec{F}\cdot d\vec{r} = \iint_S (\text{curl}\,\vec{F})\cdot\hat{n}\,ds.$$

13) Verify stoke's thm $\vec{F} = (x^2+y^2)\vec{i} - 2xy\vec{j}$ taken around the rectangle bounded by the line $x = \pm a$, $y = 0$, $y = b$.



$\vec{F} = (x^2+y^2)\vec{i} - 2xy\vec{j}$

$\vec{r} = x\vec{i} + y\vec{j}$

$\vec{dr} = dx\vec{i} + dy\vec{j}$

$\vec{F}\cdot\vec{dr} = (x^2+y^2)\,dx - 2xy\,dy.$

$$\int_C \vec{F}\cdot d\vec{r} = \int_{AB} + \int_{BC} + \int_{CD} + \int_{DA}$$

On AB

x varies from $-a$ to a

$y = 0 \quad dy = 0$

$$\int_{AB} \vec{F}\cdot\vec{dr} = \int_{-a}^{a} x^2\,dx$$

$$= \left[x^3/3\right]_{-a}^{a}$$

$$= a^3/3 - (-a)^3/3$$

$$= 2a^3/3$$

On BC

y varies from 0 to b

$x = a \quad dx = 0$

$$\int_{BC} \vec{F}\cdot\vec{dr} = (a^2+y^2)(0) - \int_0^b 2ay\,dy$$

$$= -2a\left[\frac{y^2}{2}\right]_0^b$$

$$= -ab^2$$

On CD

x varies from a to $-a$

$y = b \quad dy = 0$

$$\int_{CD} \vec{F}\cdot\vec{dr} = \int_{+a}^{-a} (x^2+b^2)\,dx$$

$$= \int_a^{-a} x^2\,dx + b^2\int_a^{-a} dx$$

$$= \left(x^3/3\right)_a^{-a} + b^2(x)_a^{-a}$$

$$= \left(-a^3/3 - a^3/3\right) + b^2(-a-a)$$

$= -2a^3/3 - 2ab^2.$

On DA

y varies from b to 0

$x = a \quad dx = 0$

$$\int_{DA} \vec{F} \cdot \vec{dr} = 2ay\,dy$$

$$= 2\left[a\, y^2/2\right]_b^0$$

$$= a(y^2)_b^0$$

$$= -ab^2.$$

$$\int_C \vec{F} \cdot \vec{dr} = \int_{AB} + \int_{BC} + \int_{CD} + \int_{DA}$$

$$= 2a^3/3 - ab^2 - 2a^3/3 - 2ab^2 - ab^2$$

$$= -4ab^2 \longrightarrow (1)$$

Verification

$$\vec{F} = (x^2+y^2)\vec{i} - 2xy\,\vec{j}$$

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^2+y^2 & -2xy & 0 \end{vmatrix}$$

$$= 0 + 0 + \vec{k}\left[\partial/\partial x(-2xy) - \partial/\partial y(x^2+y^2)\right]$$

$$= (-2y - 2y)\vec{k}$$

$$= (-4y)\vec{k}$$

$x \to -a$ to a, $y = 0$ to b

$\hat{n} \cdot \vec{k} = 1$

$(\text{curl}\,\vec{F}) \cdot \hat{n} = -4y$

$$= \int_{-a}^{a} \int_{0}^{b} \frac{(\text{curl}\,\vec{F}) \cdot \hat{n}}{|\hat{n} \cdot \vec{k}|} \, dx\,dy$$

$$= \int_{-a}^{a} \int_{0}^{b} \frac{-4y}{1} \, dx\,dy$$

$$= -4 \int_{-a}^{a} \left(y^2/2 \right)_0^b \, dx$$

$$= -4 \int_{-a}^{a} \left[b^2/2 \right] dx$$

$$= -\frac{4b^2}{2} \left[x \right]_{-a}^{a}$$

$$= -\frac{4b^2}{2} \left[a + a \right]$$

$$= \frac{-4b^2}{2} (2a)$$

$$= -4ab^2 \longrightarrow ②$$

from ① & ②

$$\int_C \vec{F} \cdot dr = \iint_S (\text{curl} \cdot \vec{F}) \cdot \hat{n} \, ds$$

Verity stoke's Thm for $\vec{F} = (2x-y)\vec{i} - yz^2\vec{j} - y^2z\vec{k}$ where S is the upper half surface of the sphere $x^2+y^2+z^2 = 1$ and c is its boundary.

Stoke's Theorem

$$\int_C \vec{F} \cdot \vec{dr} = \iint_S (\text{curl}\,\vec{F}) \cdot \hat{n}\, ds$$

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Clearly c is the upper half of the ...

is circle $x^2+y^2=1$

$\therefore \quad \vec{F} = (2x-y)\vec{i} - yz^2\vec{j} - y^2z\vec{k}$

$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$

$\vec{dr} = dx\vec{i} + dy\vec{j} + dz\vec{k}$

$\vec{F}\cdot\vec{dr} = (2x-y)\,dx - yz^2\,dy - y^2z\,dz$

Since it in xy plane $z=0 \Rightarrow dz=0$

The parametric eqn of circle.

$x = r\cos t \qquad y = \sin t$

$dx = -\sin t\,dt$

$\vec{F}\cdot\vec{dr} = (2x-y)\,dx$

$t \rightarrow 0$ to 2π

$$\int_c \vec{F}\cdot\vec{dr} = \int_0^{2\pi} (2\cos t - \sin t)(-\sin t\,dt)$$

$$= \int_0^{2\pi} (-2\sin t\cos t + \sin^2 t)\,dt$$

$$= -\int_0^{2\pi} \sin 2t\,dt + \int_0^{2\pi} \frac{1-\cos 2t}{2}\,dt$$

$$= \left[\frac{\cos 2t}{2}\right]_0^{2\pi} + \left[t/_2 - 1/_2\,\frac{\sin 2t}{2}\right]$$

$$= \frac{\cos 2(2\pi)}{2} - \frac{\cos 2(0)}{2} + \frac{2\pi}{2} - 1/_4 \sin 2(2\pi)$$

$$= \frac{\cos 4\pi}{2} - \frac{\cos 0}{2} + \pi - 1/_4 \sin 4\pi$$

$$= 1/_2 - 1/_2 + \pi - 0$$

$$= \pi \longrightarrow (1)$$

Verification

$$\text{curl}\,\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 2x-y & -yz^2 & -y^2 z \end{vmatrix}$$

$$= \partial/\partial y\,[-2yz+2yz] - \vec{j}\,(0-0)$$
$$+\vec{k}\,[0+1]$$

$$\text{curl}\,\vec{F} = \vec{k}.$$

Given

$$\phi = x^2+y^2+z^2-1$$
$$\nabla\phi = 2x\vec{i} + 2y\vec{j} + 2z\vec{k}$$
$$|\nabla\phi| = \sqrt{2x^2+2y^2+2z^2}$$
$$= \cancel{\sqrt{4+4+4}} \quad = \sqrt{4(x^2+y^2+z^2)}$$
$$\cancel{= \sqrt{12}} \quad = \sqrt{4(1)}$$
$$= \sqrt{4} = 2.$$

$$\hat{n} = \frac{2x\vec{i}+2y\vec{j}+2z\vec{k}}{2}$$
$$\hat{n} = x\vec{i}+y\vec{j}+z\vec{k}$$
$$\text{curl}\,\vec{F}\cdot\hat{n} = \vec{k}\cdot(x\vec{i}+y\vec{j}+z\vec{k})$$
$$= z$$
$$\hat{n}\cdot\vec{k} = z$$

Find the limit

$$x^2+y^2=1$$
$$y=\pm\sqrt{1-x^2}$$
$$y \rightarrow -\sqrt{1-x^2} \text{ to } \sqrt{1-x^2}.$$
$$y=0 \Rightarrow x^2=1$$
$$x=\pm 1$$
$$x \rightarrow -1 \text{ to } 1$$

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$$\int_C \vec{F}\cdot\hat{n}\, ds = \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{z}{z}\, dy\, dx.$$

$$= \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dy\, dx$$

$$= 2\times \int_{0}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dy\, dx.$$

$$= 4 \int_{0}^{1}\int_{0}^{\sqrt{1-x^2}} dy\, dx.$$

$$= 4 \int_{0}^{1} (y)_{0}^{\sqrt{1-x^2}}\, dx$$

$$= 4 \int_{0}^{1} \sqrt{1-x^2}\, dx.$$

$$= 4 \left[x/2 \sqrt{a^2-x^2} + \frac{a^2}{2} \sin^{-1}(x/a) dx\right]_0^1$$

$$= 4 \left[x/2 \sqrt{1-x^2} + 1/2 \sin^{-1} x\, dx\right]_0^1$$

$$= 2 \left[x\sqrt{1-x^2} + \sin^{-1} x\, dx\right]_0^1$$

$$= 2\left[\left[1\sqrt{1-1} + \sin^{-1}(1)\right] dx - \left[0\sqrt{1-0} + \sin^{-1}(0)\right]\right]$$

$$= 2\left[0 + \pi/2 - 0\right]$$

$$= 2\left[\pi/2\right]$$

$$= \pi \longrightarrow ②$$

Verified ① and ②

Unit - IV

Fourier Series.

Defn:.

Consider the series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos nx + b_n \sin nx]$$

where

$$a_0 = \frac{1}{\pi} \int_{\lambda}^{\lambda+2\pi} f(x)\, dx$$

$$a_n = \frac{1}{\pi} \int_{\lambda}^{\lambda+2\pi} f(x) \cos nx\, dx.$$

$$b_n = \frac{1}{\pi} \int_{\lambda}^{\lambda+2\pi} f(x) \sin nx\, dx.$$

Then the series $f(x)$ is called a Fourier Series.

Periodic function.

A function $f: R \to R$ is said to be periodic if there exists a positive number ω such that $f(x+\omega) = f(x)$ for all real number x. Then ω is called periodic of f.

Ex.

$\sin 2x$ and $\cos 2x$ are periodic functions with period 2π.

identities:-

$\int_{\lambda}^{\lambda+2\pi} \cos nx\, dx = 0$ where n is an integer

2) $\int_{\lambda}^{\lambda+2\pi} \sin nx \, dx = 0$ where n is an integer

3) $\int_{\lambda}^{\lambda+2\pi} \cos mx \cos nx \, dx = 0$ if $m \neq n$ & m, n are integer

4) $\int_{\lambda}^{\lambda+2\pi} \sin mx \sin nx \, dx = 0$ if $m \neq n$ & m, n are integer

5) If $m = n$ and m, n are integers, then

$$\int_{\lambda}^{\lambda+2\pi} \cos mx \cos nx \, dx = \int_{\lambda}^{\lambda+2\pi} \cos^2 mx \, dx = \pi$$

$$\int_{\lambda}^{\lambda+2\pi} \sin mx \sin nx \, dx = \int_{\lambda}^{\lambda+2\pi} \sin^2 mx \, dx = \pi$$

$$\int_{\lambda}^{\lambda+2\pi} \sin mx \cos nx \, dx = \frac{1}{2} \int_{\lambda}^{\lambda+2\pi} \sin 2mx \, dx = 0$$

Thm

Let $f(x)$ be the periodic function with period 2π. suppose $f(x)$ can be represented as a trignometry series.

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos nx + b_n \sin nx]$$

$$= \frac{a_0}{2} + a_1 \cos x + a_2 \cos 2x + \cdots$$

$$+ b_1 \sin x + b_2 \sin 2x + \cdots \rightarrow (1)$$

To find a_0:

Integrating (1) on both side, we've

$$\int_{\lambda}^{\lambda+2\pi} f(x) dx = \int_{\lambda}^{\lambda+2\pi} \frac{a_0}{2} dx + \int_{\lambda}^{\lambda+2\pi} a_1 \cos x \, dx + \int_{\lambda}^{\lambda+2\pi} a_2 \cos 2x + \cdots$$

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$$+\int_{\lambda}^{\lambda+2\pi} b_1 \sin x\, dx + \int_{\lambda}^{\lambda+2\pi} b_2 \sin 2x\, dx + \cdots$$

$$\int_{\lambda}^{\lambda+2\pi} f(x)\,dx = \frac{a_0}{2}\int_{\lambda}^{\lambda+2\pi} dx.$$

$$= \frac{a_0}{2}\left[x\right]_{\lambda}^{\lambda+2\pi}$$

$$= \frac{a_0}{2}\left[\lambda+2\pi-\lambda\right]$$

$$= \frac{a_0}{2}\left[2\pi\right] \Rightarrow a_0\pi$$

$$\frac{1}{\pi} = a_0$$

$$\int_{\lambda}^{\lambda+2\pi} f(x)\,dx = \frac{1}{\pi}$$

To find a_n:

Multiply ① by $\cos nx$ on both sides & Integrate.

$$\int_{\lambda}^{\lambda+2\pi} f(x)\cos nx\,dx = \int_{\lambda}^{\lambda+2\pi} \frac{a_0}{2}\cos nx\,dx + \int_{\lambda}^{\lambda+2\pi} a_1 \cos x \cos nx\,dx +$$

$$\int_{\lambda}^{\lambda+2\pi} a_2 \cos 2x \cos nx\,dx + \cdots + \int_{\lambda}^{\lambda+2\pi} b_1 \sin x \cos nx\,dx +$$

$$+\int_{\lambda}^{\lambda+2\pi} b_2 \sin 2x \cos nx\,dx + \cdots$$

$$\int_{\lambda}^{\lambda+2\pi} f(x)\cos nx\,dx = \int_{\lambda}^{\lambda+2\pi} a_n \cos^2 nx\,dx$$

$$= a_n \int_{\lambda}^{\lambda+2\pi} \cos^2 nx\, dx.$$

$$= a_n \pi$$

$$a_n = \frac{1}{\pi} \int_{\lambda}^{\lambda+2\pi} f(x) \cos nx\, dx$$

To find b_n

Multiply ① by $\sin nx$ on both sides & Integrate

$$\int_{\lambda}^{\lambda+2\pi} f(x) \sin nx\, dx = \int_{\lambda}^{\lambda+2\pi} \frac{a_0}{2} \sin nx\, dx + \int_{\lambda}^{\lambda+2\pi} a_1 \sin nx \cos x\, dx$$

$$+ \int_{\lambda}^{\lambda+2\pi} a_2 \cos 2x \sin nx + \cdots\cdots + \int_{\lambda}^{\lambda+2\pi} b_1 \sin x \sin nx\, dx$$

$$+ \int_{\lambda}^{\lambda+2\pi} b_2 \sin 2x \sin nx\, dx + \cdots\cdots + \int_{\lambda}^{\lambda+2\pi} b_n \sin^2 nx\, dx$$

$$\int_{\lambda}^{\lambda+2\pi} f(x) \sin nx\, dx = \int_{\lambda}^{\lambda+2\pi} b_n \sin^2 nx\, dx$$

$$= b_n \int_{\lambda}^{\lambda+2\pi} \sin^2 nx\, dx.$$

$$= b_n \pi$$

$$b_n = \frac{1}{\pi} \int_{\lambda}^{\lambda+2\pi} f(x) \sin nx\, dx.$$

Show that $x^2 = \pi^2/3 + 4\sum_{n=1}^{\infty} \frac{(-1)^n \cos nx}{n^2}$ in the interval $-\pi \le x \le \pi$. Deduce that

i) $\frac{1}{1^2} - \frac{1}{2^2} + \ldots \ldots \pm \frac{\pi^2}{12}$

ii) $1/1^2 + 1/2^2 + \ldots\ldots = \frac{\pi^2}{6}$.

iii) $1/1^2 + 1/3^2 + \ldots\ldots = \frac{\pi^2}{8}$.

W.K.T, Fourier Series is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

Given $-\pi \le x \le \pi$ $f(x) = x^2$.

$$a_0 = 1/\pi \int_{\lambda}^{\lambda+2\pi} f(x)\,dx.$$

$$= 1/\pi \int_{-\pi}^{\pi} x^2\,dx.$$

$$= 1/\pi \left[x^3/3\right]_{-\pi}^{\pi}$$

$$= 1/\pi \left[\frac{\pi^3}{3} + \frac{\pi^3}{3}\right]$$

$$= \frac{2\pi^2}{3}$$

$$a_n = 1/\pi \int_{\lambda}^{\lambda+2\pi} f(x)\,dx$$

$$= 1/\pi \int_{-\pi}^{\pi} x^2 \cos nx\,dx$$

$u = x^2$ $dv = \cos nx\,dx$

$du = 2x\,dx$ $v = \frac{\sin nx}{n}$

$$= 1/\pi \left\{ \left[\frac{x^2 \sin nx}{n}\right]_{-\pi}^{\pi} - 2/n \int_{-\pi}^{\pi} x \sin nx\,dx \right\}$$

$$= \frac{1}{\pi}\left[0 - \frac{2}{n}\int_{-\pi}^{\pi} x \sin nx \, dx\right]$$

$$a_n = \frac{-2}{n\pi}\int_{-\pi}^{\pi} x \sin nx \, dx$$

$u = x \quad dv = \sin nx \, dx$

$du = dx \quad v = \frac{-\cos nx}{n}$

$$a_n = \frac{-2}{n\pi}\left\{\left[-\frac{x\cos nx}{n}\right]_{-\pi}^{\pi} - \int_{-\pi}^{\pi} \frac{\cos nx}{n} dx\right\}$$

$$= \frac{2}{n^2\pi}\left\{\pi(-1)^n + \pi(-1)^n) + \left[\frac{\sin nx}{n}\right]_{-\pi}^{\pi}\right\}$$

$$= \frac{2}{n^2\pi}\left\{2\pi(-1)^n + (0-0)\right\}$$

$$= \frac{4\pi(-1)^n}{n^2\pi} = \frac{4(-1)^n}{n^2}.$$

$$b_n = \frac{1}{\pi}\int_{\lambda}^{\lambda+2\pi} f(x)\sin nx \, dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 \sin nx \, dx.$$

$u = x^2 \quad dv = \sin nx \, dx.$

$du = 2x\,dx \quad v = -\frac{\cos nx}{n}.$

$$b_n = \frac{1}{\pi}\left\{\left[\frac{-x^2\cos nx}{n}\right]_{-\pi}^{\pi} + \int_{-\pi}^{\pi} \frac{\cos nx}{n} 2x\,dx\right\}$$

$$= \frac{1}{n\pi}\left\{-\pi^2(-1)^n + \pi^2(-1)^n + 2\int_{-\pi}^{\pi} x\cos nx \, dx\right\}$$

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$$= \frac{\pm 2}{n\pi} \int_{-\pi}^{\pi} x \cos nx\, dx.$$

$u = x \qquad dv = \cos nx\, dx$

$du = dx \qquad v = \frac{\sin nx}{n}$

$$b_n = \frac{2}{n\pi} \left\{ \left[\frac{x \sin nx}{n} \right]_{-\pi}^{\pi} \mp \int_{-\pi}^{\pi} \frac{\sin nx}{n}\, dx \right\}$$

$$= \frac{2}{n^2\pi} \left\{ 0 \mp \int_{-\pi}^{\pi} \sin nx\, dx \right\}$$

$$= \frac{2}{n^2\pi} \left[\frac{\pm \cos nx}{n} \right]_{-\pi}^{\pi}$$

$$= \frac{-2}{n^3\pi} \left[(-1)^n - (-1)^n \right]$$

$$= 0$$

$$f(x) = \frac{2\pi^2}{3 \times 2} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2} \cos nx.$$

$$\boxed{x^2 = \frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{(-1)^n \cos nx}{n^2}} \rightarrow ①$$

i) $\frac{1}{1^2} - \frac{1}{2^2} + \cdots\cdots = \frac{\pi^2}{12}.$

Put $x = 0$ in ①

$$0 = \frac{\pi^2}{3} + 4 \left[-\frac{1}{1^2} + \frac{1}{2^2} - \frac{1}{3^2} + \cdots \right]$$

$$-\frac{\pi^2}{3} = -4 \left[\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} + \cdots \right]$$

$$\frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} + \cdots\cdots \rightarrow ②$$

ii) $\frac{1}{1^2} + \frac{1}{2^2} + \cdots\cdots = \frac{\pi^2}{6}.$

Put $x = \pi$ in ①.

$$\pi^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n \cos n\pi}{n^2}$$

$$\pi^2 - \frac{\pi^2}{3} = 4\sum_{n=1}^{\infty} \frac{(-1)^n(-1)^n}{n^2}$$

$$\frac{2\pi^2}{3} = 2\sum_{n=1}^{\infty} \frac{(-1)^{2n}}{n^2}$$

$$\frac{\pi^2}{6} = 1/1^2 + 1/2^2 + 1/3^2 + \cdots$$

iii) $1/1^2 + 1/3^2 + \cdots\cdots = \frac{\pi^2}{8}$

② + ③

$$\left(1/1^2 + 1/2^2 + 1/3^2 + \cdots\cdots\right) + \left(1/1^2 - 1/2^2 + 1/3^2 - \cdots\right)$$

$$= \frac{\pi^2}{12} + \frac{\pi^2}{6}$$

$$\frac{2}{1^2} + \frac{2}{3^2} + \cdots\cdots = \frac{3\pi^2}{12}$$

$$2\left[1/1^2 + 1/3^2 + 1/5^2 + \cdots\right] = \frac{\cancel{3}\pi^2}{\cancel{12}\,4}$$

$$1/1^2 + 1/3^2 + 1/5^2 + \cdots\cdots = \frac{\pi^2}{8}.$$

27/2/2022) A function is defined as follows

$$f(x) = \begin{cases} -x, & \text{when } -\pi < x \leq 0 \\ x, & \text{when } 0 < x \leq \pi \end{cases}$$

S.T $f(x) = \pi/2 - \frac{4}{\pi}\left[\frac{1}{1^2}\cos x + 1/3^2 \cos 3x + \cdots\right]$

Deduce that $\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{8}$

(or)

Express $f(x) = |x|$, $-\pi < x < \pi$ as Fourier series and hence deduce $\frac{\pi^2}{8} = \sum_{n=1} \frac{1}{(2n-1)^2}$

W.K.T, Fourier Series is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

$$a_0 = \frac{1}{\pi} \int_{\lambda}^{\lambda+2\pi} f(x)dx$$

$$= \frac{1}{\pi} \left\{ \int_{-\pi}^{0} -x dx + \int_{0}^{\pi} x dx \right\}$$

$$= \frac{1}{\pi} \left\{ \left[-\frac{x^2}{2} \right]_{-\pi}^{0} + \left[\frac{x^2}{2} \right]_{0}^{\pi} \right\}$$

$\int v du$

$$= \frac{1}{\pi} \left\{ \frac{\pi^2}{2} + \frac{\pi^2}{2} \right\}$$

$$= \frac{\pi^2}{\pi}$$

$\sin n\pi = 0$

$\cos n\pi = (-1)^n$

$$a_0 = \pi$$

$$a_n = \frac{1}{\pi} \int_{\lambda}^{\lambda+2\pi} f(x) \cos nx dx$$

$= -\sin x$

$= \cos x$

$$= \frac{1}{\pi} \left\{ -\int_{-\pi}^{0} x \cos nx dx + \int_{0}^{\pi} x \cos nx dx \right.$$

$u = x$ $\quad dv = \cos nx dx$

$du = dx$ $\quad v = \frac{\sin nx}{n}$

$$= \frac{1}{\pi} \left\{ -\left[\frac{x \sin nx}{n} \right]_{-\pi}^{0} + \int_{-\pi}^{0} \frac{\sin nx}{n} dx \right.$$

$$\left. + \left[x \frac{\sin nx}{n} \right]_{0}^{\pi} - \int_{0}^{\pi} \frac{\sin nx}{n} dx \right\}$$

$$= \frac{1}{\pi} \left\{ 0 - \frac{\pi \sin(-n\pi)}{n} + \int_{-\pi}^{0} \frac{\sin nx}{n} dx + \frac{\pi \sin n\pi}{n} \right.$$

$$-0-\int_0^{\pi}\frac{\sin nx}{n}\,dx$$

$$a_n = \frac{1}{\pi}\left\{\pi(0) + \frac{1}{n}\left[-\frac{\cos nx}{n}\right]_{-\pi}^{0} + \pi(\ldots) - \frac{1}{n}\left[\frac{-\cos nx}{n}\right]_0^{\pi}\right\}$$

$$= \frac{1}{\pi}\left\{\frac{1}{n^2}\left[-1+(-1)^n\right] + \frac{1}{n^2}\left[(-1)^n \ldots\right]\right\}$$

$$= \frac{1}{n^2\pi}\left[-1+(-1)^n+(-1)^n-1\right]$$

$$a_n = \frac{2(-1)^n-2}{n^2\pi} = \frac{2((-1)^n-1)}{n^2\pi}$$

$a_n = 0$, when n is even

$a_n = \frac{-4}{n^2\pi}$ when n is odd.

$$b_n = \frac{1}{\pi}\int_{\lambda}^{\lambda+2\pi} f(x)\sin nx\,dx$$

$$= \frac{1}{\pi}\left\{-\int_{-\pi}^{0} x\sin nx\,dx + \int_0^{\pi} x\sin nx\,dx\right\}$$

$u = x$ $\quad dv = \sin nx\,dx$

$du = dx$ $\quad v = -\frac{\cos nx}{n}$

$$b_n = \frac{1}{\pi}\left\{\left[\frac{x\cos nx}{n}\right]_{-\pi}^{0} - \int_{-\pi}^{0}\frac{\cos nx}{n}\,dx - \left[\frac{x\cos nx}{n}\right]_0^{\pi} + \int_0^{\pi}\frac{\cos nx}{n}\,dx\right.$$

$$= \frac{1}{\pi}\left\{0+\pi(-1)^n - \frac{1}{n}\left[\frac{\sin nx}{n}\right]_{-\pi}^{0} - \pi(-\ldots) + 0 + \frac{1}{n}\left[\frac{\sin nx}{n}\right]_0^{\pi}\right.$$

$$= \frac{1}{\pi} \left\{ -1/n^2 (0-0) + 1/n^2 (0-0) \right\}$$

$$= \frac{1}{\pi} (0)$$

$$b_n = 0$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos nx + b_n \sin nx]$$

$$f(x) = \frac{\pi}{2} + \sum_{n=\text{odd}} \frac{-4}{n^2 \pi} \cos nx$$

$$f(x) = \frac{\pi}{2} - \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{\cos(2n-1)x}{(2n-1)^2} \rightarrow (2)$$

Put $x = 0$ in (2),

$$0 = \pi/2 - \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}$$

$$-\frac{\pi}{2} = -\frac{4}{\pi} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}$$

$$\frac{\pi^2}{8} = \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}$$

$$\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\cdots$$

Even and odd function

If $f(x) = \begin{cases} -\frac{\pi}{4}, & \text{if } -\pi < x < 0 \\ \frac{\pi}{4}, & \text{if } 0 < x < \pi \end{cases}$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

$$a_0 = \frac{1}{\pi} \int_{\alpha}^{\alpha+2\pi} f(x)\,dx$$

$$= \frac{1}{\pi} \left\{ \int_{-\pi}^{0} -\frac{\pi}{4}\,dx + \int_{0}^{\pi} \frac{\pi}{4}\,dx \right\}$$

$$= \frac{1}{\pi} \left\{ \left[-\frac{\pi}{4} (x)_{-\pi}^{0} + \frac{\pi}{4} (x)_{0}^{\pi} \right] \right\}$$

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$$= \frac{1}{\pi}\left[-\frac{\pi^2}{4} + \frac{\pi^2}{4}\right]$$

$$= \frac{1}{\pi}(0)$$

$$a_0 = 0$$

$$a_n = \frac{1}{\pi}\int_{\lambda}^{\lambda+2\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} -\frac{\pi}{4}\cos nx\,dx + \int_{0}^{\pi}\frac{\pi}{4}\cos nx\,dx\right]$$

$$= \frac{\pi}{\pi(4)}\left[-\int_{-\pi}^{0}\cos nx\,dx + \int_{0}^{\pi}\cos nx\,dx\right]$$

$$= \frac{1}{4}\left[-(\sin nx)_{-\pi}^{0} + (\sin nx)_{0}^{\pi}\right]$$

$$= \frac{1}{4}\left[-(0-0) + (0-0)\right]$$

$$a_n = 0$$

$$b_n = \frac{1}{\pi}\int_{\lambda}^{\lambda+2\pi} f(x)\sin nx\,dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} -\frac{\pi}{4}\sin nx\,dx + \int_{0}^{\pi}\frac{\pi}{4}\sin nx\,dx\right]$$

$$= \frac{1}{4}\left[-\int_{-\pi}^{0}\sin nx\,dx + \int_{0}^{\pi}\sin nx\,dx\right]$$

$$= \frac{1}{4}\left[+\left(\frac{\cos nx}{n}\right)_{-\pi}^{0} - \left(\frac{\cos nx}{n}\right)_{0}^{\pi}\right]$$

$$= \frac{1}{4n}\left[\cos n(-\pi) - \cos n(0)\right.$$

$$= \frac{1}{4n}\left[\cos n(-\pi) - \cos n(0)\right.$$

$$\left. - \cos n(\pi) + \cos n(0)\right]$$

$a_n = 0$

$f(x) = -\pi/4 \sin nx$

$f(-x) = \pi/4 \sin nx$

$f(x) = \frac{\pi}{4} \sin nx$

$f(-x) = -\pi/4 \sin nx$

$$f(x) = \frac{2}{\pi}\int_0^{\pi} f(x) \sin nx \, dx .$$

$$= \frac{2}{\pi}\int_0^{\pi} \pi/4 \sin nx \, dx .$$

$$= \frac{1}{2}\left[\frac{-\cos nx}{n}\right]_0^{\pi}$$

$$f(x) = \sum_{n=1}^{\infty} \frac{\sin(2n-1)x}{(2n-1)} = \frac{1}{2n}\left[-(-1)^n + 1\right] = \frac{1}{2n}\left[1 - (-1)^n\right]$$

0 if n is even

$1/n$ if n is odd.

$$f(x) = \sum_{n=\text{odd}}^{\infty} -1/n \sin nx$$

Even and odd function.

$f(x) = f(-x)$ then $f(x)$ is said to be an even function

$f(x) = -f(-x)$ then $f(x)$ is said to be an odd function.

Ex, x^2, $\cos x$ = even function

x^3, $\sin x$ = odd function.

Properties of odd & even function.

i) $\int_{-a}^{a} f(x) = 0$ $f(x)$ is odd.

ii) $\int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx$ if ...

Proof:-

$$\int_{-a}^{a} f(x)dx = \int_{-a}^{0} f(x)dx + \int_{0}^{a} f(x)dx$$

In the first integral on the R.H.S.

Put $x = -y$ As $x \to -a$ to a

$dx = -dy$ $y \to a$ to 0.

$$\int_{-a}^{a} f(x)dx = \int_{+a}^{0} f(-y)(-dy) + \int_{0}^{a} f(x)dx.$$

$$= +\int_{0}^{a} f(-y)dy + \int_{0}^{a} f(x)dx$$

$$= \int_{0}^{a} f(-x)dx + \int_{0}^{a} f(x)dx$$

$$= \int_{0}^{a} [f(-x) + f(x)]dx$$

If $f(x)$ is odd function, then

$f(x) = -f(-x)$

$f(-x) = -f(x)$

$$\int_{-a}^{a} f(x)dx = \int_{0}^{a} [-f(x) + f(x)]dx$$

$$= 0$$

If $f(x)$ is even function then,

$f(x) = f(-x)$

$$\int_{-a}^{a} f(x)dx = \int_{0}^{a} (f(x) + f(x))dx$$

$$= 2\int_{0}^{a} f(x)dx$$

Derivation:-

$f(x)$ can be expanded as a Fourier Series in the form,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos nx + b_n \sin nx]$$

where.

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\,dx.$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx\,dx.$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx\,dx.$$

case (i)

If $f(x)$ is odd function.

$f(x) \cos nx$ is an odd function

$f(x) \sin nx$ is an even function.

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx\,dx = 0$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx\,dx$$

$$= \frac{2}{\pi} \int_{0}^{\pi} f(x) \sin nx\,dx.$$

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\,dx = 0$$

Then Fourier series becomes,

$$\boxed{f(x) = \sum_{n=1}^{\infty} b_n \sin nx}$$

$$b_n = \frac{2}{\pi} \int_{0}^{\pi} f(x) \sin nx\,dx.$$

when $f(x)$ is odd function.

case (ii),

If $f(x)$ is even function then

$f(x)\cos nx$ is an even function

$f(x)\sin nx$ is an odd function

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

$$= \frac{2}{\pi}\int_{0}^{\pi} f(x)\cos nx\,dx.$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx. = 0$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

$$= \frac{2}{\pi}\int_{-\pi}^{\pi} f(x)\,dx.$$

Then Fourier series becomes,

$$\boxed{f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx.}$$

where $a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx.$

$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx.$

Express $f(x)=x$ in the interval $-\pi$ to π as a Fourier Series with period π.

$$f(x) = x$$

$$f(-x) = -x$$

$\therefore$ $f(x)$ is an odd function

The Fourier Series,

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx \longrightarrow (1)$$

$$\therefore b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} x\sin nx\,dx$$

$u = x \qquad dv = \sin nx$

$du = dx \qquad v = -\frac{\cos nx}{n}$

$$b_n = \frac{2}{\pi}\left\{\left[-\frac{x\cos nx}{n}\right]_0^{\pi} - \int_0^{\pi} -\frac{\cos nx}{n}\,dx\right\}$$

$$= \frac{2}{\pi}\left\{-\frac{\pi\cos n\pi}{n} + 0 + \frac{1}{n}\left[\frac{\sin nx}{n}\right]_0^{\pi}\right\}$$

$$= \frac{2}{\pi}\left\{\frac{-\pi(-1)^n}{n} + \frac{1}{n}(0-0)\right\}$$

$$= \frac{-2\pi(-1)^n}{\pi n}$$

$$b_n = \frac{2(-1)^{n+1}}{n}$$

$\therefore$ (1) becomes

$$f(x) = \sum_{n=1}^{\infty}\frac{2(-1)^{n+1}}{n}\sin nx$$

$$= 2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin nx$$

Express $f(x) = |\sin x|$ in the interval $-\pi$ to π

$f(x) = |x|$

$x, 0 \le x \le \pi$

$-x, -\pi \le x \le 0$

$\sin x \rightarrow 0 \le x \le \pi$

$-\sin x \rightarrow -\pi \le x \le 0$

$f(x) = |\sin x|$

$f(-x) = |\sin(-x)| = |-\sin x|$

$= |\sin x|$ $f(x)$ is an even function

$\therefore$ The Fourier Series is.

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx.$$

$$a_0 = \frac{2}{\pi} \int_0^{\pi} f(x)\, dx$$

$$= \frac{2}{\pi} \int_0^{\pi} \sin x\, dx$$

$$= \frac{2}{\pi} \left[-\cos x\right]_0^{\pi}$$

$$= \frac{2}{\pi} \left[-\cos\pi + \cos 0\right]$$

$$= \frac{2}{\pi} [1+1]$$

$$= \frac{4}{\pi}$$

$$a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cos nx\, dx.$$

$\sin B \cos A$

$$= \frac{2}{\pi} \int_0^{\pi} \sin x \cos nx\, dx$$

$\frac{1}{2} \sin(A+B) - \sin(A-B)$

$$= \frac{2}{\pi} \int_0^{\pi} \frac{1}{2} \left[\sin(n+1)x - \sin(n-1)x\right]$$

$$= \frac{1}{\pi} \left[\frac{\cos(n+1)x}{n+1} + \frac{\cos(n-1)x}{n-1}\right]_0^{\pi}$$

$$= \frac{1}{\pi} \left[\frac{-(-1)^{n+1}}{n+1} + \frac{(-1)^{n-1}}{n-1} + \frac{1}{n+1} - \frac{1}{n-1}\right.$$

$$= \frac{1}{\pi} \left[\frac{(-1)^n(-1)^2}{n+1} + \frac{(-1)^n(-1)^{-1}}{n-1} + \frac{n-1-n}{n^2-1}\right.$$

$$= \frac{1}{\pi}\left[(-1)^n\left[\frac{1}{n+1} - \frac{1}{n-1}\right] - \frac{2}{n^2-1}\right]$$

$$= \frac{1}{\pi}\left\{(-1)^n\left[\frac{n-1-n-1}{n^2-1}\right] - \frac{2}{n^2-1}\right\}$$

$$= \frac{1}{\pi}\left[\frac{2(-1)^n}{n^2-1} - \frac{2}{n^2-1}\right]$$

$$= \frac{-2}{(n^2-1)\pi}\left[(-1)^n+1\right]$$

$$a_n = \begin{cases} \frac{-4}{(n^2-1)\pi}, & n \text{ is even} \\ 0, & n \text{ is odd.} \end{cases}$$

$$a_1 = \frac{2}{\pi}\int_0^{\pi} f(x)\cos x \, dx$$

$$= \frac{2}{\pi}\int_0^{\pi} f(x)\cos x \, dx$$

$2\sin x \cos x = \sin 2x$

$$= \frac{2}{\pi}\int_0^{\pi} \sin x \cos x \, dx$$

$$= \frac{1}{\pi}\int_0^{\pi} \sin 2x \, dx$$

$$= \frac{1}{\pi}\left[-\frac{\cos 2x}{2}\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[-\frac{1}{2} + \frac{1}{2}\right]$$

$$= 0.$$

$n = 2\pi$

$n = 1, n = 2, n = 3$

$2, 4, 6, \ldots\ldots$

$\therefore$ ① becomes

$$f(x) = \frac{4\pi}{\pi 2} + 0 + \sum_{n=\text{even}}^{\infty} \frac{4}{(n^2-1)\pi}$$

$$= \frac{2}{\pi} - \frac{4}{\pi} \cdot \sum_{n=1}^{\infty} \frac{\cos 2nx}{4n^2-1}.$$

Show that in the range of 0 to 2π

The Fourier Series of e^x is

$$\frac{e^{2\pi}-1}{\pi}\left[\frac{1}{2}+\sum_{n=1}^{\infty}\left(\frac{\cos nx}{n^2+1}-\sum_{n=1}^{\infty}\frac{n(\sin nx)}{n^2+1}\right)\right.$$

$f(x)=e^x \quad 0 \leq x \leq 2\pi$

The Fourier Series is

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx)$$

$$a_0=\frac{1}{\pi}\int_0^{2\pi} f(x)\,dx$$

$$=\frac{1}{\pi}\int_0^{2\pi} e^x\,dx$$

$$=\frac{1}{\pi}\left[\frac{e^x}{1}\right]_0^{2\pi}$$

$$=\frac{1}{\pi}\left[e^{2\pi}-1\right]$$

$\int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}[a\cos bx+b\sin bx]$

$$a_0=\frac{e^{2\pi}-1}{\pi}$$

$\int e^{ax}\sin bx\,dx=\frac{e^{ax}}{a^2+b^2}[a\sin bx-b\cos bx]$

$$a_n=\frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx$$

$$=\frac{1}{\pi}\int_0^{2\pi} e^x\cos nx\,dx$$

$$=\frac{1}{\pi}\left[\frac{e^x}{1+n^2}(\cos nx+n\sin nx)\right]_0^{2\pi}$$

$$=\frac{1}{\pi}\left[\frac{e^{2\pi}}{1+n^2}(\cos 2\pi n+n\sin n2\pi)-\frac{e^0}{1+n^2}(1-0)\right]$$

$$=\frac{1}{\pi}\left[\frac{e^{2\pi}}{1+n^2}(1+n(0))-\frac{1}{1+n^2}\right]$$

$$=\frac{1}{\pi}\left[\frac{e^{2\pi}}{1+n^2}-\frac{1}{1+n^2}\right]$$

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$$a_n = \frac{e^{2\pi} - 1}{\pi(1+n^2)}$$

$$b_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin nx \, dx$$

$$= \frac{1}{\pi} \int_0^{2\pi} e^x \sin nx \, dx$$

$$= \frac{1}{\pi} \left[\frac{e^x}{1+n^2} (\sin nx - n \cos nx) \right]_0^{2\pi}$$

$$= \frac{1}{\pi} \left[\frac{e^{2\pi}}{1+n^2} (\sin 2\pi n - n \cos 2\pi n) - \frac{e^0}{1+n^2} (\sin 0 - n \cos 0) \right]$$

$$b_n = \frac{1}{\pi} \left[\frac{e^{2\pi}}{1+n^2} (0 - n) - \frac{1}{1+n^2} (0 - n) \right]$$

$$= \frac{1}{\pi} \left[\frac{-n e^{2\pi}}{1+n^2} + \frac{n}{1+n^2} \right]$$

$$= \frac{-n(e^{2\pi} - 1)}{\pi(n^2+1)}$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

$$= \frac{e^{2\pi} - 1}{2\pi} + \sum_{n=1}^{\infty} \left[\frac{e^{2\pi} - 1}{\pi(n^2+1)} \cos nx - \frac{n(e^{2\pi} - 1)}{\pi(n^2+1)} \sin nx \right]$$

$$= \frac{e^{2\pi} - 1}{2\pi} + \frac{e^{2\pi} - 1}{\pi} \sum_{n=1}^{\infty} \left(\frac{\cos nx}{n^2+1} - \frac{n \sin nx}{n^2+1} \right)$$

$$= \frac{e^{2\pi} - 1}{2\pi} \left[1 + \sum_{n=1}^{\infty} \left(\frac{\cos nx}{n^2+1} - \frac{n \sin nx}{n^2+1} \right) \right]$$

Unit - V

1) $f(x) = e^x \quad -\pi < x < \pi$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} e^x\,dx$$

$$= \frac{1}{\pi}\left[e^x\right]_{-\pi}^{\pi}$$

$$a_0 = \frac{1}{\pi}\left[e^{\pi} - e^{-\pi}\right]$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} e^{x}\cos nx\,dx$$

$$= \frac{1}{\pi}\left[\frac{e^x}{1+n^2}(\cos nx + n\sin nx)\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[\left(\frac{e^{\pi}}{1+n^2}(\cos n\pi + n\sin n\pi)\right) - \left(\frac{e^{-\pi}}{1+n^2}(\cos n(-\pi) - \pi \sin n(-\pi)\right)\right]$$

$\cos n\pi = \cos n(-\pi)$

$\cos n\pi$

$\sin n(-\pi) = -\sin n\pi$

$$= \frac{1}{\pi}\left[\frac{e^{\pi}}{1+n^2}((-1)^n + \pi(0)) - \frac{e^{-\pi}}{1+n^2}((-1)^n + \pi(0))\right]$$

$$a_n = \frac{1}{\pi(1+n^2)}\left[e^{\pi}(-1)^n - e^{-\pi}(-1)^n\right]$$

$$a_n = \frac{(-1)^n}{\pi(1+n^2)}\left[e^{\pi} - e^{-\pi}\right]$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} e^{x}\sin nx\,dx$$

$$= \frac{1}{\pi}\left[\frac{e^{x}}{1+n^2}(\sin nx - n\cos nx)\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{e^{\pi}}{1+n^2}(\sin n\pi - n\cos n\pi) - \frac{e^{-\pi}}{1+n^2}(\sin n(-\pi) - n\cos n(-\pi))\right]$$

$$= \frac{1}{\pi}\left[\frac{e^{\pi}}{1+n^2}(0 - n(-1)^n) - \frac{e^{-\pi}}{1+n^2}(0 - n(-1)^n)\right]$$

$$b_n = \frac{-n(-1)^n}{\pi(1+n^2)}\left[e^{\pi} - e^{-\pi}\right]$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n\cos nx + b_n\sin nx)$$

$$= \frac{e^{\pi}-e^{-\pi}}{2\pi} + \sum_{n=1}^{\infty}\left[\frac{(-1)^n}{\pi(1+n^2)}(e^{\pi}-e^{-\pi})\cos nx + \frac{-n(-1)^n}{\pi(1+n^2)}(e^{\pi}-e^{-\pi})\sin nx\right]$$

$$= \frac{e^{\pi}-e^{-\pi}}{\pi}\left[\frac{1}{2} + \sum_{n=1}^{\infty}\left(\frac{(-1)^n}{1+n^2}\cos nx - \frac{n(-1)^n}{1+n^2}\sin nx\right)\right]$$

Express $f(x) = \frac{1}{2}(\pi - x)$ as a Fourier Series with period 2π to be valid in the interval 0 to 2π.

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n\cos nx + b_n\sin nx)$$

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} \tfrac{1}{2}(\pi - x)\,dx$$

$$= \tfrac{1}{2\pi}\left[\pi x - x^2/2\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\left[\pi(2\pi) - (2\pi)^2/2 - \pi(0) + 0/2\right]$$

$$= \tfrac{1}{2\pi}\left[2\pi^2 - \frac{4\pi^2}{2}\right]$$

$$= \frac{1}{2\pi}\left[\frac{4\pi^2 - 4\pi^2}{2}\right]$$

$$a_0 = 0$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} (\pi - x)\cos nx\,dx$$

$u = \pi - x \quad dv = \cos nx\,dx$

$du = -dx \quad v = \frac{\sin nx}{n}$

$$= \frac{1}{2\pi}\int_0^{2\pi} (\pi - x)\cos nx\,dx$$

$$\int u\,dv = uv - \int v\,du.$$

$$= \left[\frac{(\pi - x)\sin nx}{n}\right]_0^{2\pi} + \int_0^{2\pi} \frac{\sin nx}{n}\,dx$$

$$= \left[\frac{(\pi - 2\pi)\sin 2n\pi}{n} - \frac{(\pi - 0)\sin 0}{n}\right]$$

$$= \frac{1}{2\pi}\left[- \frac{1}{n}\left[\frac{\cos nx}{n}\right]_0^{2\pi}\right]$$

$$= \frac{1}{2\pi}\left[0 - \frac{1}{n^2}(\cos 2n\pi - \cos 0)\right)$$

$$= -\tfrac{1}{n^2}(1-1)$$

$$= 0$$

$$a_n = \tfrac{1}{2\pi}(0)$$

$$a_n = 0$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} (\pi - x)\sin nx\,dx$$

$u = \pi - x \quad dv = \sin nx\,dx$

$du = -dx \quad v = -\frac{\cos nx}{n}$

$$b_n = \frac{1}{2\pi}\left[-\frac{(\pi-x)\cos nx}{n}\right]_0^{2\pi} - \int_0^{2\pi}\frac{\cos nx}{n}\,dx$$

$$= \frac{1}{2\pi}\left[\left(\frac{-(\pi-2\pi)}{n}\cos 2n\pi\right) - \left(\frac{-(\pi-0)\cos 0}{n}\right)\right] - \frac{1}{n}\left[\frac{\sin nx}{n}\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\left[\frac{\pi}{n}(1) + \frac{\pi}{n}(1)\right] - \frac{1}{n^2}\left[\sin 2n\pi - \sin 0\right]$$

$$= \frac{2\pi}{n}\cdot\frac{1}{2\pi}$$

$$b_n = \frac{1}{n}$$

$$f(x) = 0 + \sum_{n=1}^{\infty}(1/n \sin nx)$$

$$= \sum_{n=1}^{\infty}\frac{1}{n}\sin nx$$

$$= \sin x + \frac{\sin 2x}{2} + \cdots$$

$$f(x) = \frac{1}{2}(\pi - x)$$

$$\frac{1}{2}(\pi - x) = \sin x + \frac{\sin 2x}{2} + \cdots$$

let $x = \pi/2$.

$$\frac{1}{2}(\pi - \pi/2) = \sin \pi/2 - \frac{\sin 2(\pi/2)}{2} + \cdots$$

$$\frac{1}{2}\left(\frac{2\pi - \pi}{2}\right) = 1 + 0 - 1/3 + \cdots$$

$$\pi/4 = 1 - 1/3 + \cdots$$

3) obtain the Fourier Series function for that $f(x) = \pi^2 - x^2$, $-\pi < x < \pi$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi}(\pi^2 - x^2)\,dx$$

$$= \frac{1}{\pi}\left[\pi^2 x - x^3/3\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[\pi^2 \pi - \pi^3/3 - \pi^2(-\pi) - -\pi^3/3\right]$$

$$= \frac{1}{\pi}\left[\pi^3 - \pi^3/3 + \pi^3 - \pi^3/3\right]$$

$$= \frac{1}{\pi}\left[2\pi^3 - 2\pi^3/3\right]$$

$$= \frac{1}{\pi}\left[\frac{6\pi^3 - 2\pi^3}{3}\right]$$

$$= \frac{1}{\pi}\,\frac{4\pi^3}{3}$$

$$a_0 = \frac{4\pi^2}{3}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi}(\pi^2 - x^2)\cdot\cos nx\,dx$$

$u = \pi^2 - x^2$ $\quad$ $dv = \cos nx\,dx$

$du = -2x\,dx$ $\quad$ $v = \frac{\sin nx}{n}$

$$\int u\,dv = \left[\frac{(\pi^2 - x^2)\sin nx}{n}\right]_{-\pi}^{\pi} + \int_{-\pi}^{\pi}\frac{\sin nx}{n}\,2x\,dx$$

$$= \left[\frac{(\pi^2 - \pi^2)\sin n\pi}{n} - \frac{(\pi^2 - \pi^2)\sin n(-\pi)}{n}\right] + 2/\pi\int_{-\pi}^{\pi} x\sin nx\,dx$$

$$= 0 + \frac{2}{n}\int_{-\pi}^{\pi} x\sin nx\,dx$$

$$= 2/n\int_{-\pi}^{\pi} x\sin nx\,dx.$$

$u = x$ $\quad$ $dv = \sin nx\,dx$

$du = dx$ $\quad$ $v = -\frac{\cos nx}{n}$

$$\int u\,dv = \left[\frac{-x\cdot\cos nx}{n}\right]_{-\pi}^{\pi} + \int_{-\pi}^{\pi}\frac{\cos nx}{n}\,dx$$

$$= \left[\frac{-\pi\cos n\pi}{n} - \frac{\pi\cos n(-\pi)}{n}\right] + 1/n\left[\frac{\sin nx}{n}\right]_{-\pi}^{\pi}$$

$$= \left[\frac{-2\pi\cos n\pi}{n}\right] + 1/n^2\left[\sin n\pi - \sin n(-\pi)\right]$$

$$= \frac{-2n(-1)^n}{n} + 1/n^2(0)$$

$$= -\frac{2\pi}{n}(-1)^n$$

$$a_n = \frac{1}{\pi}\left[2/n \cdot -\frac{2\pi(-1)^n}{n}\right]$$

$$a_n = \frac{4(-1)^{n+1}}{n^2}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi}(\pi^2-x^2)\sin nx\,dx$$

$u = \pi^2 - x^2$

$du = -2x\,dx$ | $dv = \sin nx\,dx$

$v = -\frac{\cos nx}{n}$

$$\int_{-\pi}^{\pi}(\pi^2-x^2)\sin nx\,dx = \left[-\frac{(\pi^2-x^2)\cos nx}{n}\right]_{-\pi}^{\pi} - \int_{-\pi}^{\pi}\frac{\cos nx}{n}\,2x\,dx$$

$$= \left[-\frac{(\pi^2-\pi^2)\cos n\pi}{n}\right] - \left[-\frac{(\pi^2-x^2)\cos n(-\pi)}{n}\right]$$

$$-\frac{2}{n}\left[\frac{\sin nx}{n}\right]_{-\pi}^{\pi}$$

$$= 0 - 2/n^2\left[\sin n\pi - \sin n(-\pi)\right]$$

$b_n = 0$

$$f(x) = \frac{4\pi^2}{3\times 2} + \sum_{n=1}^{\infty}\left[\frac{4(-1)^{n+1}}{n^2}\cos nx\right]$$

$$= \frac{2\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}\cos nx$$

Unit - IV

obtain $f(x) = a. \; 0 < x < \pi$ $f(x) = -a$ $\pi < x < 2\pi$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

$$a_0 = \frac{1}{\pi} \int_{\lambda}^{\lambda+2\pi} f(x)\,dx$$

$$= \frac{1}{\pi}\left[\int_0^{\pi} a\,dx + \int_{\pi}^{2\pi} -a\,dx\right]$$

$$= \frac{1}{\pi}\left[a(x)_0^{\pi} - a(x)_{\pi}^{2\pi}\right]$$

$$= \frac{1}{\pi}(0)$$

$a_0 = 0$

$$a_n = \frac{1}{\pi}\int_{\lambda}^{\lambda+2\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\left[\int_0^{\pi} a \cos nx\,dx + \int_{\pi}^{2\pi} -a \cos nx\,dx\right]$$

$$= \frac{a}{\pi}\left[\int_0^{\pi} \cos nx\,dx - \int_{\pi}^{2\pi} \cos nx\,dx\right]$$

$$= \frac{a}{\pi}\left[\left(\frac{\sin nx}{n}\right)_0^{\pi} - \left(\frac{\sin nx}{n}\right)_{\pi}^{2\pi}\right]$$

$$= \frac{a}{\pi}\left[\sin\pi - \sin 0 - \sin 2\pi + \sin\pi\right]$$

$$= \frac{a}{\pi}[0]$$

$a_n = 0$

$$b_n = \frac{1}{\pi}\int_{\lambda}^{\lambda+2\pi} f(x)\sin nx\,dx$$

$$= \frac{1}{\pi}\left[\int_0^{\pi} a \sin nx\,dx + \int_{\pi}^{2\pi} -a \sin nx\,dx\right]$$

$$= \frac{a}{\pi}\left[\left(\frac{-\cos nx}{n}\right)_0^{\pi} + \left(\frac{\cos nx}{n}\right)_{\pi}^{2\pi}\right]$$

$$= \frac{a}{n\pi}\left[-\cos n(\pi) + \cos n(0) + \cos n(2\pi) - \cos n(\pi)\right]$$

$$= \frac{a}{n\pi}\left[-(-1)^n + 1 + 1 - (-1)^n\right]$$

$$= \frac{a}{n\pi}\left[2 - 2(-1)^n\right]$$

$$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos nx + b_n \sin nx\right]$$

$$f(x) = \frac{0}{2} + \sum_{n=1}^{\infty}\left[(0)\cos nx + \left[\frac{a}{n\pi}\left(2-2(-1)^n\right)\sin nx\right]\right]$$

$$f(x) = \sum_{n=1}^{\infty} \frac{a}{n\pi}\left[2 - 2(-1)^n \sin nx\right]$$

Express $f(x)$ interm of fourier series

$$f(x) = \frac{(\pi - x)^2}{4}, \quad 0 \le x \le 2\pi$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos nx + b_n \sin nx\right]$$

$$a_0 = \frac{1}{\pi}\int_{\lambda}^{\lambda+2\pi} f(x)\,dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} \frac{(\pi - x)^2}{4}\,dx$$

$$= \frac{1}{4\pi}\int_0^{2\pi} (\pi^2 + x^2 - 2x\pi)\,dx$$

$$= \frac{1}{4\pi}\left[\pi^2 x + x^3/3 - \frac{2x^2\pi}{2}\right]_0^{2\pi}$$

$$= \frac{1}{4\pi}\, \pi^2(2\pi) + \frac{8\pi^3}{3} - \frac{4\pi^3}{2}$$

$$= 2\pi \cdot \frac{1}{4\pi}\left[2\pi^3 + \frac{8\pi^3}{3} - 4\pi^3\right]$$

A function $f(x)$ is defined within the range $(0, 2\pi)$ by the relation

$$f(x) = \begin{cases} x & \text{in the range} \\ 2\pi - x & \text{in the range } (\pi, 2\pi) \end{cases}$$

express $f(x)$ as a F.S in the range $(0, 2\pi)$

Ans ÷

$a_0 = \pi$

$$a_n = \frac{2}{\pi n^2}\left[(-1)^n - 1\right]$$

$$b_n = 0$$

$$f(x) = \pi/2 + \sum_{n=1}^{\infty} \frac{2}{n^2 \pi}\left[(-1)^n - 1\right] \cos nx$$

$$= \frac{1}{4\pi}\left(\frac{8\pi^3}{3} - 2\pi^3\right)$$

$$= \frac{1}{4\pi}\left(\frac{8\pi^3 - 6\pi^3}{3}\right)$$

$$= \frac{1}{4\pi}\left[\frac{2\pi^3}{3}\right]$$

$$= \frac{\pi^2}{6}$$

$$a_n = \frac{1}{\pi} \int_{\lambda}^{\lambda+2\pi} f(x) \cos nx \, dx$$

$$= \frac{1}{\pi} \int_0^{2\pi} \frac{(\pi - x)^2}{4} \cos nx \, dx.$$

$$= \frac{1}{4\pi} \int_0^{2\pi} (\pi - x)^2 \cos nx \, dx$$

$u = (\pi - x)^2$ $\quad dv = \int \cos nx \, dx$

$du = -2(\pi - x)dx$ $\quad v = \frac{\sin nx}{n}$

$$\int_0^{2\pi} (\pi - x)^2 \cos nx \, dx = \left[\frac{(\pi - x)^2 \sin nx}{n}\right]_0^{2\pi} - \int_0^{2\pi} \frac{\sin nx}{n} \cdot -2(\pi - x)dx$$

$$= \left[\frac{(\pi - 2\pi)^2 \sin 2n\pi}{n}\right] - \frac{(\pi - 0)^2 \sin 0}{n} + 2/n \int_0^{2\pi} (\pi - x) \sin nx \, dx$$

$$= 0 + \frac{2}{n} \int_0^{2\pi} (\pi - x) \sin nx \, dx$$

$$= 2/n \int_0^{2\pi} (\pi - x) \sin nx \, dx$$

$u = \pi - x$ $\quad dv = \sin nx \, dx$

$du = -dx$ $\quad v = -\frac{\cos nx}{n}$

$$= \left[-\frac{(\pi - x)\cos nx}{n}\right]_0^{2\pi} - \int_0^{2\pi} \frac{\cos nx}{n} \, dx$$

$$= \left[-\frac{(\pi - 2\pi)\cos 2n\pi}{n} - \left(-\frac{(\pi-0)\cos 0}{n}\right)\right]$$

$$- 1/n \left[\frac{\sin nx}{n}\right]_0^{2\pi}$$

$$= \left[\frac{\pi}{n} + \pi/n\right] - 1/n^2 \left[\sin n 2\pi - \sin 0\right]$$

$$= 2\pi/n - 1/n^2 (0)$$

$$= \frac{2\pi}{n}$$

$$a_n = \frac{1}{4\pi}\left[\frac{2}{n}\cdot\frac{2\pi}{n}\right]$$

$$a_n = 1/n^2$$

$$b_n = \frac{1}{\pi}\int_{\lambda}^{\lambda+2\pi} f(x)\sin nx\,dx$$

$$= \frac{1}{4\pi}\int_0^{2\pi} (\pi - x)^2 \sin nx\,dx$$

$u = (\pi - x)^2$ $\quad$ $dv = \sin nx\,dx$

$du = -2(\pi - x)dx$ $\quad$ $v = -\frac{\cos nx}{n}$

$$= \left[\frac{-(\pi - x)^2\cos nx}{n}\right]_0^{2\pi} - \int_0^{2\pi}\frac{\cos nx}{n}\,2(\pi - x)dx$$

$$= \left[\frac{-(\pi - 2\pi)^2 \cos n 2\pi}{n} - \left(-\frac{(\pi - 0)^2\cos 0}{n}\right)\right]$$

$$-\frac{2}{n}\int_0^{2\pi}(\pi - x)\cos nx\,dx$$

$$= \left(\frac{\pi^2}{n} + \frac{\pi^2}{n}\right) - 2/n\int_0^{2\pi}(\pi - x)\cos nx\,dx$$

$$= -2/n\int_0^{2\pi}(\pi - x)\cos nx\,dx$$

$u = \pi - x$ $\quad$ $dv = \cos nx\,dx$

$du = -dx$ $\quad$ $v = \frac{\sin nx}{n}$

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$$= \left[\frac{(\pi - x)\sin nx}{n}\right]_0^{2\pi} + \int_0^{2\pi} \frac{\sin nx}{n}\,dx$$

$$= \left[\frac{(\pi - 2\pi)\sin 2n\pi}{n} - \frac{(\pi - 0)\sin 0}{n}\right] + \frac{1}{n}\left[\frac{\cos nx}{n}\right]_0^{2\pi}$$

$$= 0 + \frac{1}{n^2}(\cos 2n\pi - \cos 0)$$

$$= \frac{1}{n^2}(0 - 0)$$

$$= 0$$

$$b_n = \left[\frac{1}{4\pi}\left(-2n(0)\right)\right]$$

$$b_n = 0$$

$$f(x) = \frac{\pi^2}{6\times 2} + \sum_{n=1}^{\infty}\left(\frac{1}{n^2}\cos nx + 0\right)$$

$$= \frac{\pi^2}{12} + \sum_{n=1}^{\infty} \frac{1}{n^2}\cos nx.$$

A function $f(x)$ is defined within the range $(0, \pi)$ by the relation $f(x) = \begin{cases} x & \text{in the range } (0, \pi) \\ 2\pi - x & \text{in the range } (\pi, 2\pi) \end{cases}$ express $f(x)$ as a Fourier Series in the range $(0, 2\pi)$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n\cos nx + b_n \sin nx\right]$$

$$a_0 = \frac{1}{\pi}\int_{\lambda}^{\lambda+2\pi} f(x)\,dx$$

$$= \frac{1}{\pi}\left[\int_0^{\pi} x\,dx + \int_{\pi}^{2\pi} 2\pi - x\,dx\right]$$

$$= \frac{1}{\pi}\left[\left(\frac{x^2}{2}\right)_0^{\pi} + \left(2\pi x - \frac{x^2}{2}\right)_{\pi}^{2\pi}\right]$$

$$= \frac{1}{\pi}\left[\frac{\pi^2}{2} + 2\pi(2\pi) - \frac{(2\pi)^2}{2} - 2\pi(\pi) + \frac{\pi^2}{2}\right]$$

$$= \frac{1}{\pi}\left[\frac{\pi^2}{2} + 4\pi^2 - \frac{4\pi^2}{2} - 2\pi^2 + \frac{\pi^2}{2}\right]$$

$$= \frac{1}{\pi}\left[\frac{2\pi^2}{2} + 4\pi^2 - 2\pi^2 - \frac{4\pi^2}{2}\right]$$

$$= \frac{1}{\pi}\left[3\pi^2 - \frac{4\pi^2}{2}\right]$$

$$= \frac{1}{\pi}\left[\frac{6\pi^2 - 4\pi^2}{2}\right]$$

$$= \frac{1}{\pi}\left[\frac{2\pi^2}{2}\right]$$

$$= \frac{1}{\pi}\left[\pi^2\right]$$

$$a_0 = \pi$$

$$a_n = \frac{1}{\pi}\int_{\lambda}^{\lambda+2\pi} f(x)\cos nx\, dx$$

$$= \frac{1}{\pi}\left[\int_0^{\pi} x\, dx + \int_{\pi}^{2\pi} 2\pi - x\, \cancel{dx}\right]\cos nx$$

$$= \frac{1}{\pi}\int_0^{\pi} x\cos nx\, dx + \int_{\pi}^{2\pi}(2\pi - x)\cos nx\, d$$

$$\int x\cos nx\, dx \Rightarrow u = x \quad dv = \cos nx$$
$$du = dx \quad v = \frac{\sin nx}{n}$$

$$\int 2\pi - x \cos nx\, dx \Rightarrow u = 2\pi - x \quad dv = \cos nx$$
$$du = dx \quad u = \frac{\sin nx}{n}$$

$$= \frac{1}{\pi}\left[\left(x\cdot\frac{\sin nx}{n}\right)_0^{\pi} - \int_0^{\pi}\frac{\sin nx}{n}\, dx\right]$$

$$+ \left[\left((2\pi - x)\frac{\sin nx}{n}\right)_0^{\pi} - \int_{\pi}^{2\pi}\frac{\sin nx}{n} - dx\right]$$

$$= \frac{1}{n\pi}\left[-\int_0^{\pi}\sin nx\, dx + \int_{\pi}^{2\pi}\sin nx\, dx\right]$$

$$= \frac{1}{n\pi}\left[\left(\frac{\cos nx}{n}\right)_0^{\pi} - \left(\frac{\cos nx}{n}\right)_{\pi}^{2\pi}\right]$$

$$= \frac{1}{n^2\pi}\left[\cos n\pi - \cos n(0) - \cos n(2\pi) + \cos n\pi\right]$$

$$= \frac{1}{n^2\pi}\left[(-1)^n - 1 - 1 + (-1)^n\right]$$

$$= \frac{1}{n^2\pi}\left[(-2)^n - 2\right]$$

$$= \frac{2}{n^2\pi}\left[(-1)^n - 1\right]$$

$$b_n = \frac{1}{\pi}\int_{\lambda}^{\lambda+2\pi} f(x)\, \sin nx\, dx$$

$$= \frac{1}{\pi}\left[\int_0^{\pi} x \sin nx\, dx + \int_{\pi}^{2\pi} (2\pi - x)\sin nx\, dx\right]$$

$\int x \sin nx\, dx \Rightarrow u = x \quad dv = \sin nx$

$du = dx \quad v = -\frac{\cos nx}{n}$

$\int (2\pi - x)\sin nx\, dx \Rightarrow u = 2\pi - x \quad dv = \sin nx$

$du = -dx \quad v = -\frac{\cos nx}{n}$

$$= \frac{1}{\pi}\left[-\left(x \frac{\cos nx}{n}\right)_0^{\pi} + \int_0^{\pi} \frac{\cos nx}{n}\, dx\right]$$

$$+ \left[\left(-(2\pi - x)\frac{\cos nx}{n}\right)_{\pi}^{2\pi} + \int_{\pi}^{2\pi} \frac{\cos nx}{n} - dx\right.$$

$$= \frac{1}{n\pi}\left[(\pi \cos n\pi) + \int_0^{\pi} \cos nx\, dx\right.$$

$$\left. - (\pi \cos n\pi) + \int_{\pi}^{2\pi} \cos nx\, dx\right]$$

$$= \frac{1}{n\pi}\left[-\left(\frac{\sin nx}{n}\right)_0^{\pi} + \left(\frac{\sin nx}{n}\right)_{\pi}^{2\pi}\right]$$

$$= \frac{1}{n\pi}\left(-(0-0) + (0-0)\right)$$

$$b_n = 0$$

$$f(x) = \frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{2}{n^2\pi}\left[(-1)^n - 1\right] \cos nx.$$

Unit-V

13/3/2020 Half range fourier series:

Defn:-

It is required to obtain the fourier series expansion of a function in an interval [0, l] where l is half the period such a expansion is called half range fourier series.

It is half and conuinent to obtain a Fourier series of a function to holds for a range which is half the Period of the Fourier series.

i.e) To expand f(x) in the range $(0, \pi)$ in a Fourier Series of period 2π

In the half range f(x) can be expressed as a series obtain of cosines alone or sine alone.

... identities are used,

① $\int_0^{\pi} \cos mx\,dx = 0$, if m is integer

② $\int_0^{\pi} \cos mx \cos nx\,dx = 0$, if $m \neq n$ & m & n are integers

③ $\int_0^{\pi} \sin mx \sin nx\,dx = 0$ if $m \neq n$ & m & n are integers

④ $\int_0^{\pi} \cos mx \cos nx\,dx = \int_0^{\pi} \cos^2 x\,dx$, if $m = n$ $= \pi/2$

⑤ $\int_0^{\pi} \sin mx \sin nx\,dx = \int_0^{\pi} \sin^2 x\,dx$, if $m = n = \frac{\pi}{2}$

Development of cosine series:-

let $f(x)$ be expressed as a series contains cosines only.

let $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \rightarrow ①$

If we integrate ① on both sides between 0 to π.

$$\int_0^{\pi} f(x)\,dx = \int_0^{\pi} \frac{a_0}{2}\,dx + \int_0^{\pi} \sum_{n=1}^{\infty} a_n \cos nx\,dx$$

$$\int_0^{\pi} f(x)\,dx = \frac{a_0}{2}\int_0^{\pi} dx + \sum_{n=1}^{\infty} a_n \int_0^{\pi} \cos nx\,dx.$$

$$= \frac{a_0}{2}[x]_0^{\pi} + \sum_{n=1}^{\infty} a_n \left[\frac{\sin nx}{n}\right]_0^{\pi}$$

$$= \frac{a_0}{2}(\pi) + \sum_{n=1}^{\infty} a_n(0-0)$$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx$$

Multiplying both side of ① by $\cos nx$ & $\int$ing between to 0 to π.

$$\int_0^{\pi} f(x)\cos nx\,dx = \int_0^{\pi} \frac{a_0}{2}\cos nx\,dx + \int_0^{\pi} \sum_{n=1}^{\infty} a_n \cos nx \cos nx\,dx$$

$$= \frac{a_0}{2}\left[\frac{\sin nx}{n}\right]_0^{\pi} + \sum_{n=1}^{\infty} a_n \int_0^{\pi} \cos^2 nx\, d$$

$$= \frac{a_0}{2}(0-0) + \sum_{n=1}^{\infty} a_n \pi/2 .$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\, dx.$$

Development of Sine Series:-

let $f(x)$ be expanded as a Series containing Sine only.

$$\text{let } f(x) = \sum_{n=1}^{\infty} b_n \sin nx \rightarrow (1)$$

Multiplying $\sin nx$ on both side & $\int$ between 0 to π

$$\int_0^{\pi} f(x)\sin nx\, dx = \int_0^{\pi} \sum_{n=1}^{\infty} b_n \sin nx \sin$$

$$= \sum_{n=1}^{\infty} b_n \int_0^{\pi} \sin^2 nx\, dx$$

$$= \sum_{n=1}^{\infty} b_n (\pi/2)$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\, dx$$