9/12/19

Linear Algebra

Unit-I

Vector Spaces

Vector spaces - Definition and examples - Subspaces - linear transformation - Span of a set

Unit-II

Linear Independence - Basis and dimension - Rank and Nullity.

Unit-III

Matrix and Inner Product Space

Matrix of a linear transformation - Inner Product space - Definition and examples - Orthogonality - Gram Schmidt orthogonalisation Process - orthogonal complement.

Unit-IV

Theory of Matrices

Algebra of Matrices - The inverse of a Matrix - Elementary transformation - Rank of a matrix

Unit-V

characteristic equation and bilinear forms

characteristic equation and

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cayley - Hamilton theorem - Eigen Values and Eigen vectors

Text Book

1) Arumugan.S and Thangapandi Isaac A modern Algebra. Scitech publications Ltd

Unit	Chapter	Section
I	5	5.1 to 5.4
II	5	5.5 to 5.7
III	5,6	5.8, 6.1 to 6.3
IV	7	7.1 to 7.5
V	7	7.7, 7.8

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Unit – I

12/12/19 **Defn : Group**

A non-empty set G together with a binary operation $* : G \times G \to G$ is called a group if the following conditions are satisfied

i) $*$ is associative

i.e) $a * (b * c) = (a * b) * c$ for all $a, b, c \in G$

ii) there exist an element $e \in G$ such that $a * e = e * a = a \; \forall \; a \in G$

'e' is called the identity element of G.

iii) For any element a in G there exist an element $a' \in G$ such that

$$a * a' = a' * a = e$$

a' is called the inverse of a.

Defn : Abelian

A Group G is said to be abelian if $ab = ba$ for all $a, b \in G$.

Vector space

5.1 Definition and Example

Defn : vector space

A non-empty set V is said to be a Vector space over a Field F if.

i) V is abelian group under an operation called addition which we

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denote by '+'.

ii) For every $\alpha \in F$ and $v \in V$, there is defined an element αv in V subject to the following conditions.

i) $\alpha(u+v) = \alpha u + \alpha v \quad \forall\, u, v \in V$ and $\alpha \in F$

ii) $(\alpha+\beta)u = \alpha u + \beta u \quad \forall\, u \in V$ & $\alpha, \beta \in F$

iii) $\alpha(\beta u) = (\alpha\beta)u \quad \forall\, u \in V$

iv) $1u = u \quad \forall\, u \in V$

Remark:

1) The elements of F are called Scalars and the elements of V are called vectors.

2) The rule which associates with each scalar $\alpha \in F$ and a vector $v \in V$, a vector αv is called the scalar multiplicatio

Thus a scalar multiplication given rule to a function from

$$F \times V \to V \text{ defined by } (\alpha, v) \to \alpha v$$

Examples:

1) $R \times R$ is a vector space over R under addition and scalar multiplication defined by,

$$(x_1, x_2) + (y_1, y_2) = (x_1 + y_1, x_2 + y_2)$$

$$\alpha(x_1, x_2) = (\alpha x_1, \alpha x_2)$$

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Proof: clearly binary operation '+' is commutative and associative

And $(0,0)$ is the zero element

The inverse of (x_1, x_2) is $(-x_1, -x_2)$

Hence $(R \times R, +)$ is an abelian group

Now let $u = (x_1, x_2)$ and

$v = (y_1, y_2)$

let $\alpha, \beta \in R$

Then $\alpha(u+v) = \alpha[(x_1, x_2) + (y_1, y_2)]$

$= \alpha[(x_1, y_1) + (x_2, y_2)]$

$= \alpha(x_1, y_1) + \alpha(x_2, y_2)$

$= (\alpha x_1, \alpha y_1) + (\alpha x_2, \alpha y_2)$

$= \alpha(x_1, x_2) + \alpha(y_1, y_2)$

$\alpha(u+v) = \alpha u + \alpha v$

Now, $(\alpha+\beta)u = (\alpha+\beta)(x_1, x_2)$

$= (\alpha+\beta)x_1, (\alpha+\beta)x_2$

$= (\alpha x_1 + \beta x_1), (\alpha x_2 + \beta x_2)$

$= (\alpha x_1, \alpha x_2) + (\beta x_1, \beta x_2)$

$(\alpha+\beta)u = \alpha u + \beta u$

$\alpha(\beta u) = \alpha(\beta(x_1, x_2))$

$= \alpha(\beta x_1, \beta x_2)$

$= \alpha\beta x_1, \alpha\beta x_2$

$= \alpha\beta(x_1, x_2)$

$\alpha(\beta u) = (\alpha\beta)u$

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obviously $1 \cdot u = u$

$\therefore$ $R \times R$ is a vector

space over R.

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② $R^n = \{(x_1, x_2, \ldots x_n) / x_i \in R,\ 1 \leq i \leq n\}$. Then R^n is a vector space over R under addition and Multiplication defined by,

$$(x_1, x_2, \ldots, x_n) + (y_1, y_2, \ldots y_n) = (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n)$$

and

$$\alpha(x_1, x_2, \ldots, x_n) = (\alpha x_1, \ldots, \alpha x_n)$$

Proof:-

Clearly the binary operation is commutative and associative

$(0, 0, \ldots 0)$ is the zero element

The inverse of

$(x_1, x_2, \ldots, x_n)$ is $(-x_1, -x_2, \ldots, -x_n)$

$\therefore$ $(R^n, +)$ is abelian group.

Now let $u = (x_1, x_2, \ldots, x_n)$

$v = (y_1, y_2, \ldots, y_n)$

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let $\alpha, \beta \in R$

$$\alpha(u+v) = \alpha\left[(x_1, x_2, \ldots, x_n) + (y_1, y_2, \ldots, y_n)\right]$$

$$= \alpha\left[(x_1+y_1), (x_2+y_2), \ldots, (x_n+y_n)\right]$$

$$= \alpha x_1 + \alpha y_1, \alpha x_2 + \alpha y_2, \ldots, \alpha x_n + \alpha y_n$$

$$= (\alpha x_1, \alpha x_2, \ldots, \alpha x_n) + (\alpha y_1, \alpha y_2, \ldots, \alpha y_n)$$

$$= \alpha(x_1, x_2, \ldots, x_n) + \alpha(y_1, y_2, \ldots, y_n)$$

$\alpha(u+v) = \alpha u + \alpha v \quad \forall \alpha \in R, u, v \in R^n.$

III^{ly} $(\alpha+\beta)u = \alpha u + \beta u$ and

$\alpha(\beta u) = (\alpha\beta)u$

obviously $1 \cdot u = u$

$\therefore R^n$ is a vector space over R.

3. Let v denote the set of all solution of the differential Equation $2\frac{d^2y}{dx^2} - 7\frac{dy}{dx} + 3y = 0$

The v is a vector space over R.

Proof:-

let $f, g \in V$ and $\alpha \in R$:

then, $2\frac{d^2f}{dx^2} - 7\frac{df}{dx} + 3f = 0 \longrightarrow (1)$

$2\frac{d^2g}{dx^2} - 7\frac{dg}{dx} + 3g = 0 \longrightarrow (2)$

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From ① & ②

$$2\left[\frac{d^2f}{dx^2}+\frac{d^2g}{dx^2}\right]-7\left[\frac{df}{dx}+\frac{dg}{dx}\right]+3[f+g]=$$

$$2\left[\frac{d^2}{dx^2}(f+g)\right]-7\left[\frac{d}{dx}(f+g)\right]+3[f+g]=0$$

$\therefore f+g \in V$.

Hence

$$2\frac{d^2(\alpha f)}{dx^2}-7\frac{d(\alpha f)}{dx}+3(\alpha f)=0$$

$\therefore$ Hence $\alpha f \in V$

Since the operations are usual addition and scalar multiplication the axioms of the vector space are true.

Hence V is a vector space over R.

4) let R^+ be the set of all positive real numbers Define addition and scalar multiplication as follows $u+v=uv$ for all $u,v \in$ $\alpha u = u^\alpha$ for all $u, v \in R^+$ and $\alpha \in R$ Then R^+ is a real ~~number~~ vector space

R^+ is positive real number. Positive real number must have commutative and associative

$u+e=e+u=e$

$u+e=u$

clearly $(R^+,+)$ abelian group with identity 1.

Now $\alpha(u+v)=\alpha(uv)$

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$= (uv)^{\alpha}$

$= u^{\alpha} \cdot v^{\alpha}$

$= u^{\alpha} + v^{\alpha}$

$\boxed{\alpha(u+v) = \alpha u + \alpha v}$

$(\alpha+\beta)u = u^{(\alpha+\beta)}$

$= u^{\alpha} \cdot u^{\beta}$

$= \alpha u \cdot \beta u$

$\boxed{(\alpha+\beta)u = \alpha u + \beta u}$

$\alpha(\beta u) = (\beta u)^{\alpha} \Rightarrow (u^{\beta})^{\alpha}$ u வின் reciprocal நான் v என்று வைத்து எழுதவும்

$= u^{\alpha\beta}$

$\alpha(\beta u) = (\alpha\beta)u$.

Obviously $1 \cdot u = u$

Hence R^+ is a vector space over R.

$xe = x$

$\boxed{e = 1}$ identity

$a + (-a) = (-a) + a = e$

$u + (-u) = e$

$u + v = uv$

$uv = e$

$v = 1/u$

$u(1/u) = e$

$\boxed{1 = e}$

Theorem : 5.1

Let V be a vector space over a field F, Then

i) $\alpha \cdot 0 = 0 \; \forall \; \alpha \in F$ (0+0) : α

ii) $0 \cdot v = 0 \; \forall \; v \in V$

iii) $(-\alpha)v = \alpha(-v) = -(\alpha v) \; \forall \; \alpha \in F$ and $v \in V$.

iv) $\alpha v = 0 \Rightarrow \alpha = 0$ or $v = 0$

Proof :-

i) $\alpha \cdot 0 = \alpha \cdot (0+0)$

$= 0\alpha + 0 \cdot \alpha$

$= 0 + 0$

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$= 0$

$\alpha \cdot 0 = 0 \quad \forall \alpha \in F$

ii) $0 \cdot v = (0+0)v$

$= 0 \cdot v + 0 \cdot v$

$= 0 + 0$

$= 0$

$0 \cdot v = 0 \quad \forall \ v \in V$

iii) $0 = [\alpha - \alpha]v$

$= [\alpha + (-\alpha)]v$

$= \alpha v + (-\alpha)v$

$\boxed{-\alpha v = (-\alpha)v} \longrightarrow ①$

And $0 = [v - v]\alpha$

$= [v + (-v)]\alpha$

$= \alpha v + \alpha(-v)$

$-\alpha v = \alpha(-v)$

Equating eqn ① & ② we get,

$-\alpha v = (-\alpha)v = \alpha(-v)$

iv) let $\alpha v = 0$.

If $\alpha = 0$, there is nothing to prove

If $\alpha \neq 0$, Then $\alpha^{-1} \in F$.

Now, $v = 1 \cdot v$

$= (\alpha^{-1} \cdot \alpha)v$

$= \alpha^{-1}(\alpha v)$

$= \alpha^{-1}(0)$

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$V = 0$

Subspaces.

Definition:

let V be a vector space over a Field F. A non-empty subset W of V is called a subspace of V if W itself is a vector space over F under the operations of V.

Theorem: 5.2

let V be a vector space over F. A non-empty subset W of V is a subspace of V iff W is closed with respect to vector addition and scalar multiplication in V.

Proof:-

let W be a subspace of V.

then W itself is a vector space and hence W is closed with respective vector addition and scalar multiplication

conversly let W be a non-empty subset V such that $u, v \in W \Rightarrow u+v \in W$ and $u \in W, \alpha \in F \Rightarrow \alpha u \in W$.

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we prove that W is a subspace of V

Since W is non-empty, there exist a element $u \in W$

$$0u = 0 \in W$$

Also $v \in W \Rightarrow (-1)v = -v \in W$

Thus W contain 0 and additive inverse of each of its elements.

Hence W is an additive subgroup of V

Also $u \in W$ and $\alpha \in F \Rightarrow \alpha u \in W$

Since the elements of W are the elements of V the other axioms of a vector space are true in W

Hence W is a subspace of V.

Theorem : 5-3.

let V be a vector space over a field F. A non-empty subset W of V is a subspace of V iff $u, v \in W$ and $\alpha, \beta \in F \Rightarrow \alpha u + \beta v \in W$

Proof :- let W be a subspace of V

let $u, v \in W$ and $\alpha, \beta \in F$

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. then by thm 5.2 αu and $\beta v \in W$ and hence $\alpha u + \beta v \in W$

conversly,

let $u, v \in W$ and $\alpha, \beta \in F \Rightarrow \alpha u + \beta u \in W$

Taking $\boxed{\alpha = \beta = 1}$, we get $u, v \in W \Rightarrow u + v \in W$

Taking $\beta = 0$ we get,

$\alpha \in F$ and $u \in W \Rightarrow \alpha u \in W$

Hence by thm 5.2, W is a subspace of V.

Examples:

1) $\{0\}$ and V are subspaces of any vector space V. They are called the trivial subspace of V. $(V, +)$, V, $\{0\}$ ஆகியும் குறிந்தால் அது trivial subspace of V

2) $W = \{(a, 0, 0) / a \in R\}$ is a subspace of R^3 for, let $u = (a, 0, 0)$, $V = (b, 0, 0) \in W$ and $\alpha, \beta \in R$

Then $\alpha u + \beta V = \alpha(a, 0, 0) + \beta(b, 0, 0)$

$= (\alpha a + \beta b, 0, 0) \in W$

Hence W is a subspace of R^3.

3) In R^3, $W = \{(Ka, kb, Kc) / K \in R\}$ is a subspace of R^3.

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let $u = (Ka, Kb, Kc)$

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$v = (La, Lb, Lc)$ and $\alpha, \beta \in R$

$\alpha u + \beta v = \alpha(Ka, Kb, Kc) + \beta(La, Lb, Lc)$

$= ((\alpha K + \beta L)a, (\alpha K, \beta L)b, (\alpha K, \beta L)c) \in W$

$\therefore$ W is a subspace of R^3.

Problem

1) Prove that the intersection of two subspaces of a vector space of is a subspaces.

Let A and B be two subspace of a vector space V over a Field F

we claim that $A \cap B$ is a subspace of V.

clearly $0 \in A \cap B$ and hence $A \cap B$ is non-empty

Now, let $u, v \in A \cap B$ and $\alpha, \beta \in F$

Then $u, v \in A$ & $u, v \in B$

$\therefore \alpha u + \beta v \in A$ & $\alpha u + \beta v \in B$

(Since A and B are subspace

Hence $A \cap B$ is a subspace of V.

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2) P.T the union of two subspaces of a vector space need not be a subspace.

let $A = \{(a,0,0) \mid a \in R\}$ and

$B = \{(0,b,0) \mid b \in R\}$

clearly A and B are subspaces of R^3

However $A \cup B$ is not a subspaces of R^3

For $(1,0,0) \in (0,1,0) \in A \cup B$.

இரண்டும் 0 வருவதால் இது vector space ஆகாது

But, $(1,0,0) + (0,1,0) = (1,1,0) \notin A \cup B$

3) P.T the union of two subspace of a vector space is a subspace iff one is contained in the other.

let H and K be two subspaces of V such that one is contained in the other Hence either $H \subseteq K$ (or) $K \subseteq H$

$H \cup K = K$ (or) $H \cup K = H$

Hence $H \cup K$ is a subspace of V.

conversely,

Suppose $H \cup K$ is a subspace of V we claim that $H \subseteq K$ or $K \subseteq H$

Suppose that H is not containing

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K and K is not containing H. Then there exist elements a, b such that

$a \in H$ and $a \notin K \longrightarrow ①$

$b \in K$ and $b \in H \longrightarrow ②$

Clearly $a, b \in H \cup K$ since $H \cup K$ is a subspace of V,

$ab \in H \cup K$

Hence $ab \in H$ (or) $ab \in K$

case(i):- let $ab \in H$

Since $a \in H$, $a^{-1} \in H$

Hence $a^{-1}(ab) = b \in H$ which is contradiction to ②.

case(ii):- let $ab \in K$ Since

$b \in K$, $b^{-1} \in K$

Hence $(ab)b^{-1} = a \in K$ which is contradiction to ①

Hence our assumption that H is not containing in K and K is not containing in H is false.

(or)

$H \subseteq K$ and $K \subseteq H$

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1) If A and B are subspaces of V prove that $A+B = \{v \in V / v = a+b, a \in A, b \in B\}$ is a subspace of V. Futher show that A+B is the smallest subspace containing A and B. (ie) If W is any any subspace of V containing A and B then, W contains A+B.

containing என்றால்
$A \subseteq A+B$

let $v_1, v_2 \in A+B$ and $\alpha \in F$ (எந்த எண்ணாகவும்)

Then $v_1 = a_1+b_1$, $v_2 = a_2+b_2$ where $a_1, a_2 \in A$ and $b_1, b_2 \in B$.

contained என்றால்
ஒன்றுள்ளே
$A+B \subseteq A$

$$v_1+v_2 = (a_1+b_1)+(a_2+b_2)$$
$$= (a_1+a_2)+(b_1+b_2) \in A+B$$

Also $\alpha(a_1+b_1) = \alpha a_1 + \alpha b_1 \in A+B$.

Hence A+B is a subspace of V.

clearly $A \subseteq A+B$ and $B \subseteq A+B$

Now, let W be any subspace of V containing A and B.

We shall prove that $A+B \subseteq W$.

let $v \in A+B$. Then $v = a+b$ where $a \in A$ and $b \in B$

Since $A \subseteq W$, $a \in W$. similarly $b \in W$

$\therefore a+b = v \in W$

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$\therefore A+B \subseteq W$ so that $A+B$ is the smallest subspace of V containing A and B.

5) Let A and B be subspace of a vector space V. Then $A \cap B = \{0\}$ iff every vector $v \in A+B$ can be uniquely expressed in the form $v = a+b$ where $a \in A$ and $b \in B$

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let $A \cap B = \{0\}$, Let $v \in A+B$

let $v = a_1+b_1 = a_2+b_2$ where $a_1, a_2 \in A$ $b_1, b_2 \in B$

Then $a_1 - a_2 = b_2 - b_1$

But $a_1 - a_2 \in A$ and $b_1 - b_2 \in B$

Hence $a_1 - a_2, b_1 - b_2 \in A \cap B$.

Since $A \cap B = \{0\}$, $a_1 - a_2 = 0$ and $b_2 - b_1 = 0$ so that $a_1 = a_2$ and $b_1 = b_2$. Hence the expression of v in the form $a+b$ where $a \in A$ and $b \in B$ is unique.

conversely suppose that any element in $A+B$ can be uniquely expressed in the form $a+b$ where

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$a \in A$ and $b \in B$

We claim that $A \cap B = \{0\}$

If $A \cap B \neq \{0\}$, let $v \in A \cap B$ and $v \neq 0$

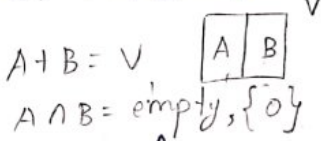
$v + (-v)$ and $0 + 0$ we have two values But we need only one value

Then $0 = v - v = 0 + 0$. Thus 0 has been expressed in the form $a+b$ in two different ways which is a contradiction. Hence so it is wrong.

$A \cap B = \{0\}$.

Definition

Let A and B be subspaces of a vector space V. Then V is called the direct sum of A and B if (i) $A + B = V$



ii) $A \cap B = \{0\}$.

If V is the direct sum of A and B we write $V = A \oplus B$ (direct sum)

Note:- $V = A \oplus B$ iff every element of V can be uniquely expressed in the form $a+b$ where $a \in A$ and $b \in B$.

Examples:-

1) In $V_3(R)$ let $A = \{(a,b,0) \mid a,b \in R\}$ and $B = \{(0,0,c) \mid c \in R\}$, clearly A and B are subspaces of V and $A \cap B = \{0\}$. Also let $v = (a,b,c) \in V_3(R)$.

Then $v = (a,b,0) + (0,0,c)$ so that

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$$A+B = V_3(R)$$

Hence $V_3(R) = A \oplus B$

2) In $M_2(R)$, let A be the set of all matrices of the form $\begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}$ and B be the set of all matrices of the form $\begin{bmatrix} 0 & 0 \\ c & d \end{bmatrix}$. clearly A and B are subspaces of $M_2(R)$ and

$$A \cap B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \text{ and } A+B = M_2(R)$$

A direct sum of

Hence $M_2(R) = A \oplus B$. B

26/12/19 Theorem 5.4

02/01/2020 Let V be a vector space over F and W a subspace of V. Let $V/W = \{W+v / v \in V\}$. Then V/W is a vector space over F under the following operations

i) $(W+v_1)+(W+v_2) = W+v_1+v_2$

ii) $\alpha(W+v_1) = W+\alpha v_1$

Proof:- Since W is a subspace of V it is a subgroup of $(V,+)$

Since $(V,+)$ is abelian, W is a normal subgroup of $(V,+)$ so that (i). is a well

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defined operation

Now we shall prove that (ii) is a well defined operation

(1) $\alpha(W+v_1) = \alpha(W+v_2)$

(2) $W+v_1 = W+v_2 \Rightarrow v_1 - v_2 \in W$ (4)

(3) $W+v_1 - W - v_2 \Rightarrow \alpha(v_1 - v_2) \in W$ (since W is a subspace) (5)

$v_1 - v_2 \in W$

$W+v_1 = W+v_2 \Rightarrow \alpha v_1 - \alpha v_2 \in W$

So $v_1 = \alpha v_1$, $v_2 = \alpha v_2 \Rightarrow \alpha v_1 \in W + \alpha v_2$

$\alpha v_1 - \alpha v_2 \in V \Rightarrow W + \alpha v_1 = W + \alpha v_2$

$W + \alpha v_1 = W + \alpha v_2$

Hence (ii) is a well defined operation

Now, let $W+v_1, W+v_2, W+v_3 \in V/W$

Then $(W+v_1) + [(W+v_2) + (W+v_3)]$ — $A+(B+C) = (A+B)+C \Rightarrow$ associative.

$= (W+v_1) + (W+v_2+v_3)$

$= W+v_1+v_2+v_3$

$= (W+v_1+v_2) + (W+v_3)$

Hence + is associative

add identity $=0$ Multi identity $=1$

$W+0 = W \in V/W$ is the additive identity

W + identity (0)

identity element.

For $(W+v_1) + (W+0) = W+v_1 = (W+0) + (W+v_1)$

add $-v$ Multi v^{-1}

for all $v_1 \in V$

Also $W-v_1$ is the additive inverse of $W+v_1$.

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Hence V/W is a group under $+$.

Futher $(W+v_1)+(W+v_2) = W+v_1+v_2$

$= W+v_2+v_1$

$= (W+v_2)+(W+v_1)$

Hence V/W is an abelian group.

Now, let $\alpha, \beta \in F$.

$\alpha[(W+v_1)+(W+v_2)] = \alpha(W+v_1+v_2)$

$= W+\alpha(v_1+v_2)$

$= W+\alpha v_1+\alpha v_2$

$= (W+\alpha v_1)+(W+\alpha v_2)$

$= \alpha(W+v_1)+\alpha(W+v_2)$

$(\alpha+\beta)(W+v_1) = W+(\alpha+\beta)v_1$

$= W+\alpha v_1+\beta v_1$

$= (W+\alpha v_1)+(W+\beta v_1)$

$= \alpha(W+v_1)+\beta(W+v_1)$

$\alpha[\beta(W+v_1)] = \alpha(W+\beta v_1)$

$= W+\alpha\beta v_1$

$= \alpha\beta(W+v_1)$

$1(W+v_1) = W+1v_1$

$= W+v_1$

Hence V/W is a vector space

The vector space V/W is called the quotien

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space of V by W.

Linear Transformation

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Definition homomor

2M [[Let V and W be vector spaces over a field F. A mapping $T: V \longrightarrow W$ is called a homomorphism if

(a) $T(u+v) = T(u) + T(v)$ and

(b) $T(\alpha u) = \alpha T(u)$ where $\alpha \in F$ and $u, v \in V$]

A homomorphism T of vector spaces homomor is also called a linear transformation]

(i) If T is 1-1 then T is called 2M monomorphism

(ii) If T is onto then T is called an epimorphism

(iii) If T is 1-1 and onto T is called is an isomorphism

(iv) Two vector spaces V and W are said to be isomorphic if there exists an isomorphism T from V to W and we write $V \cong W$ isomorphic the condition

type homomorphism, 1-1, onto...

v) A linear transformation $T: V \to F$

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is called a linear functional

Example

1) $T: V \to W$ defined by $T(v) = 0$ for all $v \in V$ is a trivial linear transformation.

2) $T: V \to V$ defined by $T(v) = v$ for all $v \in V$ is the identity linear transformation.

3) Let V be a vector space over a field F and W a subspace of V. Then $T: V \to V/W$ defined by $T(v) = W + v$ is a linear transformation for,

$$T(v_1 + v_2) = W + (v_1 + v_2)$$
$$= (W + v_1) + (W + v_2)$$
$$= T(v_1) + T(v_2)$$

Also $T(\alpha v_1) = W + \alpha v_1$
$$= \alpha(W + v_1)$$
$$= \alpha T(v_1)$$

This is called the natural homomorphism from V to V/W. clearly T is onto and hence T is an epimorphism

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Theorem 5.5 T is mapping from V into W

Let $T: V \to W$ be a linear transformation

Then $T(V) = \{T(v) / v \in V\}$ is a subspace of W.

Proof:-

let w_1 and $w_2 \in T(V)$ and $\alpha \in F$.

Then there exist $v_1, v_2 \in V$ such that $T(v_1) = w_1$ and $T(v_2) = w_2$

Hence $w_1 + w_2 = T(v_1) + T(v_2)$

$= T(v_1 + v_2) \in T(V)$

Similarly $\alpha w_1 = \alpha T(v_1) = T(\alpha v_1) \in T(V)$

Hence T(V) is subspace of W.

Definition

Let V and W be vector spaces over a field F and $T: V \to W$ be a linear transformation. Then the <u>kernel of T</u> is defined to be $\{v / v \in V \text{ and } T(v) = 0\}$ and is denoted by Ker T.

Thus $\text{Ker } T = \{v / v \in V \text{ and } T(v) = 0\}$

For examples. in example 1, Ker T = V

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Note:-

let $T: V \to W$ be a linear transformation. Then T is a monomorphism iff $\ker T = \{0\}$

Theorem 5.6

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Fundamental theorem of homomorphism

Statement

Let V and W be vector spaces over a field F and $T: V \to W$ be an epimorphism (only onto)

Then

(i) $\ker T = V_1$ is a subspace of V and

(ii) $\frac{V}{V_1} \cong W$

Proof

i) Given $V_1 = \ker T$

$\ker T = \{v / v \in V \text{ and } T(v) = 0\}$

clearly $T(0) = 0$. Hence $0 \in \ker T = V_1$

$\therefore V_1$ is nonempty subset of V.

let $u, v \in \ker T$ and $\alpha, \beta \in F$.

$\therefore T(u) = 0$ and $T(v) = 0$

Now $T(\alpha u + \beta v) = T(\alpha u) + T(\beta v)$

$= \alpha T(u) + \beta T(v)$

$= \alpha(0) + \beta(0) = 0$

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$\therefore \alpha u + \beta v \in \ker T$ $T(v) = 0$ coz $v \in \ker T$

$T(\alpha u + \beta v) = 0$ coz $(\alpha u + \beta v) \in \ker T$

$\therefore$ By theorem 5.3 $\ker T$ is a subspace of V.

Theorem 5.3

$\alpha u + \beta v \in \ker T \Leftarrow \alpha u + \beta v \in W$

$\ker T$ is a subspace of V W is a subspace of V

ii) We define a map $\phi : \frac{V}{V_1} \to W$ by

$\frac{A}{C} \to B$ operation

$$\boxed{\phi(V_1 + v) = T(v)}$$ well defined

ϕ is well defined. Let $V_1 + v = V_1 + w$.

$\therefore \boxed{v \in V_1 + w}$

$\therefore v = v_1 + w$ where $v_1 \in V_1$.

$\therefore T(v) = T(v_1 + w) = T(v_1) + T(w)$

$= 0 + T(w)$

$= T(w)$

$\therefore \phi(V_1 + v) = \phi(V_1 + w)$

well defined

ϕ is 1-1 one-one

$\phi(V_1 + v) = \phi(V_1 + w)$

$\phi(V_1 + v) = T(v) \Rightarrow T(v) = T(w)$

$\phi(V_1 + w) = T(w) \Rightarrow T(v) - T(w) = 0$

$\Rightarrow T(v) + T(-w) = 0$

$\Rightarrow T(v - w) = 0$

$\Rightarrow v - w \in \ker T = V_1$

$\Rightarrow v \in V_1 + w$.

one to one $\Rightarrow V_1 + v = V_1 + w$.

ϕ is onto.

$T : V \to W$

27/12/19 let $w \in W$. Since T is onto, there exists $v \in V$ such that $T(v) = w$.

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$\therefore \phi(V_1+v) = w$

$\therefore \phi(V_1+v) = w$.

homomorphism

ϕ is a homomorphism.

$$\phi[(V_1+v)+(V_1+w)] = \phi[V_1+(v+w)]$$

$$\phi(V_1+v) = T(v) \quad = T(v+w)$$

$$\boxed{\phi(V_1+(v+w)) = T(v+w)} \quad = T(v)+T(w)$$

$$= \phi(V_1+v) + \phi(V_1+w)$$

$T\alpha(u) = \alpha T(u)$

Also $\phi[\alpha(V_1+v)] = \phi(V_1+\alpha v)$

$$= T(\alpha v)$$
$$= \alpha T(v)$$
$$= \alpha\phi(V_1+v)$$

$\therefore \phi$ is an isomorphism from $\frac{V}{V_1}$ onto W.

$$\therefore \frac{V}{V_1} \cong W.$$

Theorem 5.7

Let V be a vector space over a field F. Let A and B be subspaces of V.

Then $\frac{A+B}{A} \cong \frac{B}{A\cap B}$.

V is vector space so A and B is also vector space

Proof:- V

$\frac{A+B}{A} = \{A+a+b \mid a\in A, b\in B\}$

We know that $A+B$ is a subspace of V containing A.

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Hence $\frac{A+B}{A}$ is also a vector space over F.

An element of $\frac{A+B}{A}$ is of the form $A+(a+b)$ where $a \in A$ and $b \in B$. But $A+a=A$.

Hence an element of $\frac{A+B}{A}$ is of the form $A+b$.

Now, consider $f: B \to \frac{A+B}{A}$ defined by

$f(b) = A+b$ element

pre-image

clearly f is onto

Also $f(b_1+b_2) = A+(b_1+b_2)$

$\phi[(V_1+v)+(V_1+w)] = (A+b_1)+(A+b_2)$

$= \phi[V_1+(v+w)] = f(b_1)+f(b_2)$

and $f(\alpha b_1) = A+\alpha b_1$

$= \alpha(A+b_1)$

$= \alpha f(b_1)$

Hence f is a linear transformation.

Let K be the kernel of f.

Then $K = \{b \mid b \in B, A+b = A\}$.

Now, $A+b = A$ iff $b \in A$. Hence $K = A \cap B$

Hence by theorem 5.6. $\frac{B}{A \cap B} \cong \frac{A+B}{A}$.

Theorem: 5.8

Let V and W be vector spaces

over a field F. Let $L(V,W)$ represent

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Set of all linear transformations from V to W. Then L(V, W) itself is a vector space over F under addition and scalar multiplication defined by $(f+g)(v) = f(v) + g(v)$ and $(\alpha f)(v) = \alpha f(v)$.

Proof :-

let $f, g \in L(V, W)$ and $v_1, v_2 \in V$.

Then $(f+g)(v_1+v_2) = f(v_1+v_2) + g(v_1+v_2)$
(vector addition)

$= f(v_1) + f(v_2) + g(v_1) + g(v_2)$

$= f(v_1) + g(v_1) + f(v_2) + g(v_2)$

$= (f+g)(v_1) + (f+g)(v_2)$

Also $(f+g)(\alpha v) = f(\alpha v) + g(\alpha v)$
(scalar multiplication)

$= \alpha f(v) + \alpha g(v)$

$= \alpha [f(v) + g(v)]$

$= \alpha (f+g)(v)$.

Hence $(f+g) \in L(V, W)$.

Now, $(\alpha f)(v_1+v_2) = (\alpha f)(v_1) + (\alpha f)(v_2)$

$= \alpha f(v_1) + \alpha f(v_2)$

$= \alpha [f(v_1) + f(v_2)]$

$= \alpha f(v_1+v_2)$

Also $(\alpha f)(\beta v) = \alpha [f(\beta v)] \Rightarrow \alpha [\beta f(v)]$

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$= \beta[\alpha f(v)] = \beta[(\alpha f)(v)]$.

Hence $\alpha f \in L(V,W)$.

Addition defined on $L(V,W)$ is obviously commutative and associative

The function $f : V \to W$ defined by $f(v) = 0$ for all $v \in V$ is clearly a linear transformation and is the additive identity of $L(V,W)$

Futher $(-f) : V \to W$ defined by $(-f)(v) = -f(v)$ is the additive inverse of f.

Thus $L(V,W)$ is an abelian group under addition.

The remaining axioms for a vector space can be easily verified

Hence $L(V,W)$ is a vector space over F.

Span of a set

Definition :-

Let V be a vector space over a field F.

Let $v_1, v_2, \ldots\ldots v_n \in V$. Then an

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element of the form $\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n$ (Scalar, vector) where $\alpha_i \in F$ is called a linear combination of the vectors $v_1, v_2, \ldots, v_n$.

Definition.

Let S be a non-empty subset of vector space V. Then the set of all linear combinations of finite sets of elements of S is called the linear span of S and denoted by $L(S)$.

Note:-

Any element of $L(S)$ is of the form

$\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n$ where $\alpha_1, \alpha_2, \ldots \alpha_n \in F$

Theorem 5.9

Let V be a vector space over a field F and S be a non empty subset of V. Then

i) $L(S)$ is a subspace of V.

ii) $S \subseteq L(S)$. (Subset)

iii) If W is any subspace of V such that $S \subseteq W$, then $L(S) \subseteq W$ (i.e) $L(S)$ is the smallest subspace of V containing S.

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i) let $v, w \in L(S)$ and $\alpha, \beta \in F$.

Then $v = \alpha_1 v_1 + \alpha_2 v_2 + \ldots\ldots + \alpha_n v_n$ where $v_i \in S$ and $\alpha_i \in F$.

Also, $w = \beta_1 w_1 + \beta_2 w_2 + \ldots\ldots + \beta_m w_m$

where $w_j \in S$ and $\beta_j \in F$.

Now, $\alpha v + \beta w = \alpha(\alpha_1 v_1 + \alpha_2 v_2 + \ldots\ldots + \alpha_n v_n) +$

$\beta(\beta_1 w_1 + \beta_2 w_2 + \ldots\ldots + \beta_m w_n)$.

(scalar, vector)

$= (\alpha\alpha_1) v_1 + \ldots\ldots + (\alpha\alpha_n) v_n + (\beta\beta_1) w_1$

Scalar × vector = linear combination $+ \ldots\ldots + (\beta\beta_m) w_n$.

$\therefore \alpha v + \beta w$ is also a linear combination of a finite number of elements of S.

Hence $\alpha v + \beta w \in L(S)$

$\therefore L(S)$ is a subspace of V.

ii) Let $u \in S$

Then $u = 1u \in L(S)$ (scalar, vector). Hence $S \subseteq L(S)$.

iii) Let W be any subspace of V such that $S \subseteq W$.

We claim that $L(S) \subseteq W$.

Let $u \in L(S)$

Then $u = \alpha_1 u_1 + \alpha_2 u_2 + \ldots\ldots + \alpha_n u_n$

where $u_i \in S$ and $\alpha_i \in F$

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Since $S \subseteq W$ we have $u_1, u_2, \ldots, u_n \in W$

$\therefore a_1 u_1 + a_2 u_2 + \ldots + a_n u_n \in W$.

(since W is a subspace of V)

$\therefore u \in W$. Hence $L(S) \subseteq W$.

Note: $L(S)$ is called the subspace spanned (generated) by the set S.

Examples.

1) In $V_3(R)$ let $e_1 = (1,0,0)$, $e_2 = (0,1,0)$ and $e_3 = (0,0,1)$

a) Let $S = \{e_1, e_2\}$

Then $L(S) = \{\alpha e_1 + \beta e_2 / \alpha, \beta \in R\}$

$= \{(\alpha, \beta, 0) / \alpha, \beta \in R\}$

b) Let $S = \{e_1, e_2, e_3\}$. Then,

$L(S) = \{\alpha e_1 + \beta e_2 + \gamma e_3 / \alpha, \beta, \gamma \in R\}$

$= \{(\alpha, \beta, \gamma) / \alpha, \beta, \gamma \in R\}$

$= V_3(R)$

$\alpha e_1 = \alpha(1,0,0)$

$= (\alpha, 0, 0)$

$\beta e_2 = \beta(0,1,0)$

$= (0, \beta, 0)$

Thus $V_3(R)$ is spanned by $\{e_1, e_2, e_3\}$.

2) In $V_n(R)$, let $e_1 = (1, 0, \ldots 0)$,

$e_2 = (0, 1, 0, \ldots, 0), \ldots, e_n = (0, 0, \ldots 1)$

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Let $S = \{e_1, e_2, \ldots\ldots, e_n\}$. Then.

$$L(S) = \{\alpha_1 e_1 + \alpha_2 e_2 + \ldots\ldots + \alpha_n e_n \mid \alpha_i \in R\}.$$

$$= \{(\alpha_1, \alpha_2, \ldots\ldots, \alpha_n) \mid \alpha_i \in R\}$$

$$= V_n(R)$$

Thus $V_n(R)$ is spanned by $\{e_1, e_2, \ldots\ldots, e_n\}$.

Theorem 5.10

Let V be a vector space over a field F. Let $S, T \subseteq V$. then (Normally set, linear combination)

a) $S \subseteq T \Rightarrow L(S) \subseteq L(T)$

b) $L(S \cup T) = L(S) + L(T)$ $\Rightarrow$ $A = B$, $A \subseteq B$, $B \subseteq A$

c) $L(S) = S$ iff S is a subspace of V.

a) Let $S \subseteq T$, let $s \in L(S)$.

Then $s = \alpha_1 s_1 + \alpha_2 s_2 + \ldots\ldots + \alpha_n s_n$ where $s_i \in S$ and $\alpha_i \in F$.

Now, since $S \subseteq T$, $s_i \in T$.

Hence $\alpha_1 s_1 + \alpha_2 s_2 + \ldots\ldots + \alpha_n s_n \in L(T)$.

Thus $L(S) \subseteq L(T)$.

b) Let $s \in L(S \cup T)$.

Then $s = \alpha_1 s_1 + \alpha_2 s_2 + \ldots\ldots + \alpha_n s_n \in L(T)$.

where $s_i \in S \cup T$ and $\alpha_i \in F$.

Without loss of generality we can assume that

$s_1, s_2, \ldots, s_m \in S$.

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and

$$s_{m+1}, \ldots\ldots, s_n \in T.$$

Hence $\alpha_1 s_1 + \alpha_2 s_2 + \ldots\ldots + \alpha_m s_m \in L(S)$ ar

$$\alpha_{m+1} s_{m+1} + \ldots\ldots\ldots + \alpha_n s_n \in L(T)$$

$$\therefore S = (\alpha_1 s_1 + \ldots\ldots + \alpha_m s_m) + (\alpha_{m+1} s_{m+1} + \ldots\ldots + \alpha_n s_n) \in L(S) + L(T$$

Hence $L(S \cup T) \subseteq L(S) + L(T)$

Also by (a) $L(S) \subseteq L(S \cup T)$ and $L(T) \subseteq L(S \cup T)$.

Hence $L(S) + L(T) \subseteq L(S \cup T)$

Hence $L(S \cup T) = L(S) + L(T)$

c) Let $L(S) = S$. By theorem 5.9 $L(S) = S$ is a subspace of V.

Conversely, let S be a subspace V. Then the smallest subspace containing S is S itself Hence $L(S) = S$

corollary linear combination

$$L[L(S)] = L(S).$$

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Unit - II

Linear Independence.

In $V_3(R)$, let $S = \{e_1, e_2, e_3\}$. we have seen that $L(S) = V_3(R)$.

Thus S is a subset of $V_3(R)$ which spans the whole space $V_3(R)$.

Definition:-

Let V be a vector space over a field F. V is said to be finite dimensional if there exists a finite subset S of V such that $L(S) = V$.

examples:-

1) $V_3(R)$ is a finite - dimensional vector space

2) $V_n(R)$ is a finite - dimensional vector space, since $S = \{e_1, e_2, \ldots, e_n\}$ is a finite subset of $V_n(R)$ such that $L(S) = V_n(R)$

In general if F is any field $V_n(F)$ is a finite dimensional vector space over F.

3) In $M_2(R)$ consider that set S consisting of the matrices

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} ; \quad B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} ;$$

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$C = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$; $D = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$. Then

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = aA + bB + cC + dD.$$

Hence $L(S) = M_2(R)$ so that $M_2(R)$ is finite dimensional.

Sol 4) Let V be the set of all polynomials in $F(x)$ of degree $\leq n$. Let $S = \{1, x, x^2, \ldots, x^n\}$.

$x^2 + y'x' + 3y^2$ this equation is polynomials

Then $L(S) = V$ and hence V is finite dimensional.

5) C is a finite dimensional vector space over R, since $L(\{1, i\}) = C$.

set of all complex number.

Note:

All the vector spaces we have considered above are finite dimensional. However there are vector spaces which cannot be spanned by a finite number of vectors. For example, consider $R[x]$. Let S be any finite subset of $R[x]$. Let f be a polynomial of maximum degree in S. let $\deg f = n$. Then any element of $L(S)$ is a polynomial of degree $\leq n$ and hence $L(S) \neq R[x]$ Thus $R[x]$ is not finite-dimensional.

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Throughout the rest of this chapter all the vector spaces we consider are finite dimensional vector space.

Although we have defined what is meant by a finite dimensional space we have not yet defined what is meant by the dimension of a vector space. We now proceed to introduce the concepts necessary to define the demension of a finite dimensional vector space

consider the vectors $e_1 = (1,0,0)$, $e_2 = (0,1,0)$, $e_3 = (0,0,1)$ in $V_3(\mathbb{R})$

Suppose that $\alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3 = 0$.

Then $(\alpha_1,0,0) + (0,\alpha_2,0) + (0,0,\alpha_3) = (0,0,0)$

$\therefore (\alpha_1, \alpha_2, \alpha_3) = (0,0,0)$

$\therefore \alpha_1 = \alpha_2 = \alpha_3 = 0$

(i.e) $\alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3 = 0$ iff

$\alpha_1 = \alpha_2 = \alpha_3 = 0$

Thus a linear combination of the vectors, e_1, e_2 and e_3 will yield the zero vector iff all the coefficients are zero.

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Definition

Let V be a vector space over a field F. A finite set of vectors $[v_1, v_2, \ldots, v_n]$ in V is said to be **linearly independent** if

$$\alpha_1 v_1 + \alpha_2 v_2 + \ldots \alpha_n v_n = 0$$

$$\Rightarrow \alpha_1 = \alpha_2 = \ldots = \alpha_n = 0$$

If $v_1, v_2, \ldots, v_n$ are not linearly independent, then they are said to be linearly dependent.

Note:-

If $v_1, v_2, \ldots, v_n$ are linearly dependent then there exist scalars $\alpha_1, \alpha_2, \ldots, \alpha_n$ not all zero, such that $\alpha_1 v_1 + \ldots + \alpha_n v_n = 0$

Examples:-

1) In $V_n(F)$, $\{e_1, e_2, \ldots, e_n\}$ is a linearly independent set of vectors, for

$$\alpha_1 e_1 + \alpha_2 e_2 + \ldots + \alpha_n e_n = 0$$

$$\Rightarrow \alpha_1 (1, 0, \ldots, 0) + \alpha_2 (0, 1, \ldots, 0) + \ldots$$
$$\alpha_n (0, 0, \ldots, 1) = (0, 0, \ldots$$

$$\Rightarrow (\alpha_1, \alpha_2, \ldots, \alpha_n) = (0, 0, \ldots, 0)$$

$$\Rightarrow \alpha_1 = \alpha_2 = \ldots = \alpha_n = 0$$

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2. In $V_3(R)$ the vectors $(1,2,1), (2,1,0)$ and $(1,-1,2)$ are linearly independent. For, let

$$\alpha_1(1,2,1) + \alpha_2(2,1,0) + \alpha_3(1,-1,2) = (0,0,0).$$

$$\therefore (\alpha_1 + 2\alpha_2 + \alpha_3, 2\alpha_1 + \alpha_2 - \alpha_3, \alpha_1 + 2\alpha_3) = (0,0,0)$$

$$\therefore \alpha_1 + 2\alpha_2 + \alpha_3 = 0 \longrightarrow (1)$$

$$2\alpha_1 + \alpha_2 - \alpha_3 = 0 \longrightarrow (2)$$

$$\alpha_1 + 2\alpha_3 = 0 \longrightarrow (3)$$

Solving equations (1), (2) and (3) we get

$$\alpha_1 = \alpha_2 = \alpha_3 = 0.$$

$\therefore$ The given vectors are linearly independent.

3. In $V_3(R)$ the vectors $(1,4,-2), (-2,1,3)$ and $(-4,11,5)$ are linearly dependent, For, let

$\alpha_1(1,4,-2) + \alpha_2(-2,1,3)$ and $(-4,11,5)$ are linearly dependent. For, let.

$$\alpha_1(1,4,-2) + \alpha_2(-2,1,3) + \alpha_3(-4,11,5) = (0,0,0)$$

$$\therefore \alpha_1 - 2\alpha_2 - 4\alpha_3 = 0 \longrightarrow (1)$$

$$4\alpha_1 + \alpha_2 + 11\alpha_3 = 0 \longrightarrow (2)$$

$$-2\alpha_1 - 3\alpha_2 + 5\alpha_3 = 0 \longrightarrow (3)$$

From (1) and (2)

$$\frac{\alpha_1}{-18} = \frac{\alpha_2}{-27} = \frac{\alpha_3}{9} = k \text{ (say)}$$

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$\therefore \alpha_1 = -18k, \alpha_2 = -27k, \alpha_3 = 9k$

These values of α_1, α_2 and α_3 for any k satisfy (3) also.

Taking $k = 1$ we get.

$\alpha_1 = -18, \alpha_2 = -27, \alpha_3 = 9$ as a non-trivial solution

Hence the three vectors are linearly dependent.

4. let V be a vector space over a field F. Then any subset S of V containing the zero vector is linearly dependent.

$S \subseteq V$ S subset

Proof:-

let $S = \{0, v_1, \ldots, v_n\}$

clearly $\alpha 0 + 0v_1 + 0v_2 + \ldots + 0v_n = 0$

where α is any element of F. Hence for any $\alpha \neq 0$ we get a non-trivial linear combination of vectors in S giving the zero vector. Hence S is linearly dependent.

Theorem 5.11

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Any subset of a linearly independent set is linearly independent.

Proof:-

Let V be a vector space over a field

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let $S = \{v_1, v_2, \ldots\ldots, v_n\}$ be a linearly independent set.

Let S' be a subset of S. without loss of generality we take $S' = \{v_1, v_2, \ldots\ldots, v_k\}$ where $k \leq n$.

Suppose S' is a linearly dependent set. Then there exist $\alpha_1, \alpha_2, \ldots\ldots, \alpha_k$ in F not all zero, such that

$$\alpha_1 v_1 + \alpha_2 v_2 + \ldots\ldots + \alpha_k v_k = 0$$

Hence $\alpha_1 v_1 + \alpha_2 v_2 + \ldots\ldots + \alpha_k v_k + 0 v_{k+1} + \ldots\ldots + 0 v_n = 0$ is a non-trivial linear combination giving the zero vector.

Here S is a linearly dependent set which is a contradiction.

Hence S' a linearly independent.

21/12/19 **Theorem 5.12.**

Any set containing a linearly dependent set is also linearly dependent. (contained $\subset$, containing $\supset$)

Proof:-

Let V be a vector space. Let S be a linearly dependent set. Let $S' \supset S$.

If S' is linearly independent S

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is also linearly independent (by theorem 5, which is a contradiction. Hence S' is linearly dependent.

Theorem · 5·13.

Let $S = \{v_1, v_2, \ldots\ldots, v_n\}$ be a linearly independent set of vectors in a vector space V over a field F. Then every element of $L(S)$ can be uniquely written in the form

$\alpha_1 v_1 + \alpha_2 v_2 + \ldots\ldots + \alpha_n v_n$, where $\alpha_i \in F$.

Proof :-

By definition every elements of $L(S)$ is of the form

$$\alpha_1 v_1 + \alpha_2 v_2 + \ldots\ldots + \alpha_n v_n$$

Now, let $\alpha_1 v_1 + \alpha_2 v_2 + \ldots\ldots + \alpha_n v_n = \beta_1 v_1 + \beta_2 v_2 + \ldots\ldots + \beta_n v_n$

Hence $(\alpha_1 - \beta_1)v_1 + (\alpha_2 - \beta_2)v_2 + \ldots\ldots + (\alpha_n - \beta_n)v_n = 0$

this is linearly independent means

Since S is a linearly independent set,

$\alpha_i - \beta_i = 0$ for all i.

$(\alpha_i - \beta_i) = 0$

$\alpha_i - \beta_i = 0$

$i = 1, 2, \ldots$

$\therefore \alpha_i = \beta_i$ for all i. Hence the theorem

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Monday Test

Theorem 5.14.

$S = \{v_1, v_2, \ldots\ldots, v_n\}$ is a linearly dependent set of vectors in V iff there exists a vector $v_k \in S$ such that v_k is a linear combination of the preceding vectors $v_1, v_2, \ldots\ldots, v_{k-1}$.

Proof:-

Suppose $v_1, v_2, \ldots\ldots, v_n$ are linearly dependent.

Then there exist $\alpha_1, \alpha_2, \ldots\ldots, \alpha_n \in F$, not all zero, such that

$$\alpha_1 v_1 + \alpha_2 v_2 + \ldots\ldots + \alpha_n v_n = 0.$$

Let k be the largest integer for which $\alpha_k \neq 0$.

Then $\alpha_1 v_1 + \alpha_2 v_2 + \ldots\ldots + \alpha_k v_k = 0$

$$\therefore \alpha_k v_k = -\alpha_1 v_1 - \alpha_2 v_2 \ldots\ldots - \alpha_{k-1} v_{k-1}$$

$$\therefore v_k = (-\alpha_k^{-1}\alpha_1) v_1 + \ldots\ldots + (-\alpha_k^{-1}\alpha_{k-1}) v_{k-1}.$$

$\therefore v_k$ is a linear combination of the Preceding vectors.

Conversely, suppose there exists a

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Vector v_k such that

$$v_k = \alpha_1 v_1 + \cdots\cdots + \alpha_{k-1} v_{k-1}.$$

Hence $-\alpha_1 v_1 - \cdots\cdots - \alpha_{k-1} v_{k-1} + v_k + 0 v_{k+1} + \cdots + 0 v_{n=1}$

Since the coefficient of $v_k = 1$, we have

$S = \{v_1, \ldots\ldots, v_n\}$ is linearly dependent.

Examples:-

In $V_3(R)$, let $S = \{(1,0,0), (0,1,0), (0,0,1), (1,1,1)\}$.

Here $(1,1,1) = (1,0,0) + (0,1,0) + (0,0,1)$.

Thus $(1,1,1)$ is a linear combination of the precending vectors. Hence S is a linearly dependent set.

Theorem 5.15.

28/1/2020

Let V be a vector space over F. Let $S = \{v_1, v_2, \ldots\ldots, v_n\}$ and $L(S) = W$. Then there exists a linearly independent subset S' of S such that $L(S') = W$.

Proof:-

Let $S = \{v_1, v_2, \ldots\ldots, v_n\}$.

If S is linearly independent there

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is nothing to prove.

If not, let v_k be the first vector in S which is a linear combination of the preceding vectors.

v_k இல்லாத v_k elementat பயன்படாத S_1 -ல்

Let $S_1 = \{v_1, v_2, \ldots, v_{k-1}, v_{k+1}, \ldots, v_n\}$.

i.e) S_1 is obtained by deleting the vector v_k from S.

S' பதிலாக S_1, $L(S') = L(S_1)$

We claim that $L(S_1) = L(S) = W$.

$A = B$
$A \subseteq B$
$B \subseteq A$

Since $S_1 \subseteq S$, $L(S_1) \subseteq L(S)$. (refer theorem 5.10).

$L(S_1) = W$
$W \subseteq L(S_1)$
$L(S_1) \subseteq W$

Not, let $v \in L(S)$.

$L(S_1) \subseteq W$
$W = L(S)$
$L(S_1) \subseteq L(S)$

Then $v = \alpha_1 v_1 + \ldots + \alpha_k v_k + \ldots + \alpha_n v_n$.

Now, v_k is a linear combination of the preceding vectors.

$L(S') \subseteq W$
$W \subseteq L(S')$

Let $v_k = \beta_1 v_1 + \ldots + \beta_{k-1} v_{k-1}$.

Hence $v = \alpha_1 v_1 + \ldots + \alpha_{k-1} v_{k-1} + \alpha_k (\beta_1 v_1 + \ldots + \beta_{k-1} v_{k-1}) + \alpha_{k+1} v_{k+1} + \ldots + \alpha_n v_n$

v can be expressed as a linear combination of the vectors of S_1 so that $v \in L(S_1)$

Hence $L(S) \subseteq L(S_1)$

Thus $L(S) = L(S_1) = W$.

Now, if S_1 is linearly independent,

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the proof is complete.

If not, we continue the above Process of removing a vector from S_1, S_2 ... which is a linear combination of the preceeding vectors until we arrive at a linearly independent subset S' of S such that

$$L(S') = W.$$

5.6 Basis and Dimension

28/01/2020 **Defintion.** 2M

[A linearly independent subset S of S is the which spans. a vector space V which spans the whole space V. then S' is called a basis of the vector space.] 2M

$L(S) = V$ condition

Theorem 5.16.

Any finite-dimensional vector space V contains a finite number of linearly independent vectors which span V. i.e) A finite dimensional vector space has a basis consisting of a finite number of vectors.

28/01/2020

Proof:-

Since V is finite dimensional vect

exists a finite subset S of V such that

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$L(S) = V$. By theorem 5.15 this set S contains a linearly independent subset $S' = \{v_1, v_2, \ldots\ldots, v_n\}$ such that $L(S') = L(S) = V$.

$L(S') = L(S_1) = W$

Hence S' is a basic for V.

Theorem 5.17.

Let V be a vector space over a field F. Then $S = \{v_1, v_2, \ldots\ldots, v_n\}$ is a basis for V iff every element of V can be uniquely expressed as a linear combination of elements of S.

Proof:- Let S be a basis for V.

Then by definition S is linearly independent and $L(S) = V$. Hence by theorem 5.13 every element of V can be uniquely expressed as a linear combination of elements of S.

conversely, suppose every elements of V can be uniquely expressed as a linear combination of elements of S.

clearly $L(S) = V$.

Now, let $\alpha_1 v_1 + \alpha_2 v_2 + \ldots\ldots + \alpha_n v_n = 0$.

$\alpha_1 = \alpha_2 = \alpha_3 \ldots\ldots = \alpha_n = 0$

Also, $0v_1 + 0v_2 + \ldots\ldots + 0v_n = 0$.

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Thus we have expressed 0 as a linear combination of vectors of S in two arrays.

$\therefore$ By hypothesis $\alpha_1 = \alpha_2 = \ldots\ldots = \alpha_n = 0$

Hence S is linearly independent. Hence S is a basis.

Examples :-

1) $S = \{(1,0,0), (0,1,0), (0,0,1)\}$ is a basis for $V_3(R)$ for, $(a,b,c) = a(1,0,0) + b(0,1,0) + c(0,0,1)$.

$a=\alpha$, $b=\beta$, $c=\gamma$

$\therefore$ Any vector (a,b,c) of $V_3(R)$ has been expressed uniquely as a linear combination of the elements of S and hence S is a basis for $V_3(R)$.

2) $S = \{e_1, e_2, \ldots\ldots, e_n\}$ is a basis for $V_n(F)$. This is known as the standard basis of $V_n(F)$.

3) $S = \{(1,0,0), (0,1,0), (1,1,1)\}$ is a basis for $V_3(R)$.

Proof :-

We shall show that any element

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(a,b,c) of $V_3(R)$ can be uniquely expressed as a linear combination of the vectors of S.

Let $(a,b,c) = \alpha(1,0,0) + \beta(0,1,0) + \gamma(1,1,1)$

$= (\alpha,0,0) + (0,\beta,0) + (\gamma,\gamma,\gamma)$

$= (\alpha+0+\gamma,\ 0+\beta+\gamma,\ 0+0+\gamma)$

$(\alpha+\gamma,\ \beta+\gamma,\ \gamma)$

Then $\alpha+\gamma = a$, $\beta+\gamma = b$, $\gamma = c$.

Hence $\alpha = a-c$ and $\beta = b-c$.

Thus $(a,b,c) = (a-c)(1,0,0) + (b-c)(0,1,0) + c(1,1,1)$

$\therefore S$ is a basis for $V_3(R)$.

4) $S = \{1\}$ is a basis for the vector Space R over R.

5) $S = \left\{\begin{bmatrix}1&0\\0&0\end{bmatrix}, \begin{bmatrix}0&1\\0&0\end{bmatrix}, \begin{bmatrix}0&0\\1&0\end{bmatrix}, \begin{bmatrix}0&0\\0&1\end{bmatrix}\right\}$ is a basis for $M_2(R)$, since any matrix $\begin{bmatrix}a&b\\c&d\end{bmatrix}$ can be uniquely written as.

$$\begin{bmatrix}a&b\\c&d\end{bmatrix} = a\begin{bmatrix}1&0\\0&0\end{bmatrix} + b\begin{bmatrix}0&1\\0&0\end{bmatrix} + c\begin{bmatrix}0&0\\1&0\end{bmatrix} + d\begin{bmatrix}0&0\\0&1\end{bmatrix}$$

6) $\{1, i\}$ is a basis for the vector Space C over R.

7) let V be the set of all polynomials of degree $\leq n$ in $R[x]$. Then $\{1, x^1, x^2, \ldots, x^n\}$ is a basis for V.

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8) $\{(1,0),(i,0),(0,1),(0,i)\}$ is a basis, for the vector space $C \times C$ over R, for

$(a+ib, c+id) = a(1,0)+b(i,0)+c(0,1)+d(0,i)$

Must see for sem

V.V.I 9) $S=\{(1,0,0),(0,1,0),(1,1,1),(1,1,0)\}$ Spans the vector space $V_3(R)$ but is not a basis

Proof:-

Let $S' = \{(1,0,0),(0,1,0),(1,1,1)\}$

Then $L(S') = V_3(R)$ (refer example 3)

Now, Since $S \supseteq S'$, we get $L(S) = V_3(R)$

Thus S spans $V_3(R)$.

But S is linearly dependent since

$(1,1,0) = (1,0,0)+(0,1,0)$

Hence S is not a basis.

see for sem

V.V.I Must 10) $S=\{(1,0,0),(1,1,0)\}$ is linearly independent but not a basis of $V_3(R)$

Proof:- $(\alpha,0,0)+(\beta,\beta,0)=(0,0,0)$

Let $\alpha(1,0,0)+\beta(1,1,0)=(0,0,0)$,

Then $\alpha+\beta=0$ and $\beta=0$. $\alpha+\beta=0$ $\beta=0 \Rightarrow \alpha=0$

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$\therefore \alpha = \beta = 0$. Hence S is linearly independent.

Also $L(S) = \{(a,b,0) / a,b \in R\} \neq V_3(R)$.

$\therefore$ S is not a basis.

Theorem 5.18. Test for

Let V be a vector space over a field F.

Let $S = \{v_1, v_2, \ldots\ldots, v_n\}$ span V, let. $L(S) = V$

$S = \{w_1, w_2, \ldots\ldots, w_n\}$ be a linearly independent set of vectors in V. then $\boxed{m \leq n.}$

Span $= L(S)$

Proof:-

Since $L(S) = V$, every vector in V and in particular w_1, is a linear combination of $v_1, v_2, \ldots\ldots, v_n$.

Hence $S_1 = \{w_1, v_1, v_2, \ldots\ldots, v_n\}$ is a linearly dependent set of vectors. Hence there exists a vector $v_K \neq w_1$ in S_1 which is a linear combination of the preceding vectors.

Let $S_2 = \{w_1, v_1, \ldots\ldots, v_{K-1}, v_{K+1}, \ldots\ldots, v_n\}$.

clearly, $L(S_2) = V$.

Hence w_2 is a linear combination of the vectors in S_2.

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Hence $S_2 = \{w_2, w_1, v_1, \ldots, v_{k-1}, v_{k_1}, \ldots, v_n\}$

is linearly dependent. Hence there exists a vector in S_2 which is a linear combination of the preceding vectors. Since the w_i's are linearly independent, this vector cannot be w_2 or w_1 and hence must be some v_j where $j \neq k$ (say, with $j > k$). Deletion of v_j from the set S_2 gives the set.

$$S_4 = \{w_2, w_1, v_1, v_2, \ldots, v_{k-1}, v_{k+1}, \ldots v_{j-1}, v_{j+1}, \ldots, v_n\}$$

of n vectors spanning V.

In this process, at each step we insert one vector from $\{w_1, w_2, \ldots, w_n\}$ and delete one vector from $\{v_1, v_2, \ldots, v_n\}$

If $m > n$ after repeating this process n times, we arrive at the set $\{w_n, w_{n-1}, \ldots, w_1\}$ which spans V.

Hence w_{n+1} is a linear combination of $w_1, w_2, \ldots, w_n$. Hence $\{w_1, w_2, \ldots, w_n, w_{n+1}, \ldots, w_m\}$

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is linearly dependent which is a contradiction

Hence $m \leq n$.

Theorem 5:19

S = linearly independent
$L(S) = V$ - dent

4/2/2020

Any two bases of a finite dimensional vector space V have the same number of elements.

Proof :-

Since V is finite dimensional, it has a basis say $S = \{v_1, v_2, \ldots\ldots, v_n\}$.

Let $S' = \{w_1, w_2, \ldots\ldots, w_m\}$ be any other basis for V.

Now, $L(S) = V$ and S' is a set of m linearly independent vectors. Hence by theorem 5.18, $m \leq n$.

$S = \{v_1, v_2, \ldots, v_n\}$ Span V
$S' = \{w_1, w_2, \ldots, w_m\}$ linear indepe - ndond

Also since $L(S') = V$ and S' is set of n linearly independent vectors, $n \leq m$.

$m \leq n$

$n \leq m$ and $n \geq m$ Hence $m = n$.

$m = n$. $S = \{v_1, v_2, \ldots, v_n\}$ linear independent
$S' = \{w_1, w_2, \ldots, w_n\}$ Span V

Definition.

Let V be a finite dimensional vector space over a field F. The number of elements in any basis of V is called the dimension of V and is denoted by $\dim V$.

Basis Number of elements

$n \leq m$

dimensional

$L(S) = V$

$L(S) = V$

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Examples.

1. $\dim V_n(R) = n$, since $\{e_1, e_2, \ldots$ is a basis of $V_n(R)$

2. $M_2(R)$ is a vector space of dimension 4 over R since $\left\{\begin{bmatrix}1&0\\0&0\end{bmatrix}, \begin{bmatrix}0&1\\0&0\end{bmatrix}, \begin{bmatrix}0&0\\1&0\end{bmatrix}, \begin{bmatrix}0&0\\0&1\end{bmatrix}\right\}$ is a basis for $M_2(R)$

3.

Theorem 5.20

Let V be a vector space of dimension n. Then

i) any set of m vectors where $m > n$ is linearly dependent.

V is a vector space
m vectors
any set

ii) any set of m vectors where $m < n$ cannot span V.

Proof :-

i) Let $S = \{v_1, v_2, \ldots, v_n\}$ be a basis for V. Hence $L(S) = V$.

Let $S' = \{w_1, w_2, \ldots\}$ be any set consisting of m vectors where $m > n$. Suppose S' is linearly independent. Since S spans V by Theorem 5.18 $m \leq n$ which is a contradiction.

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Hence S' is linearly dependent.

ii) Let S' be a set consisting of m vectors where $m < n$. Suppose $L(S') = V$

So I take. $\Leftarrow L(S)' \neq V$

Now, $S = \{v_1, v_2, \ldots, v_n\}$ is a basis for V and hence linearly independent.

Hence by theorem 5.18 $n \leq m$ which is a contradiction. $\downarrow m < n$ Span V

Hence S' cannot span V.

$L(S) = V$

02/01/2020

Theorem 5.21.

Let V be a finite dimensional vector space over a field F. Any linearly independent set of vectors in V is part of a basis.

5/2/2020

Proof :-

Let $S = \{v_1, v_2, \ldots, v_r\}$ be a linearly independent set of vectors.

if $L(S) = V$ then S itself is a basis.

If $L(S) \neq V$, choose an element $v_{r+1} \in V - L(S)$

Now, consider $S_1 = \{v_1, v_2, \ldots, v_r, v_{r+1}\}$.

We shall prove that S_1 is linearly independent by showing that no vector in S_1 is a linear combination of the preceding vectors. (refer theorem 5.14)

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Since $\{v_1, v_2, \ldots, v_r\}$ is linearly independent, v_i where $1 \leq i \leq r$ is not a linear combination of the preceeding vectors

Also $v_{r+1} \notin L(S)$ and hence v_{r+1} is not a linear combination of $v_1, v_2, \ldots, v_r$.

v_{r+1} $L(S)$-లో లేదు కాబట్టి $v_{r+1} \notin L$

Hence S_1 is linearly independent

If $L(S_1) = V$, then S_1 is a basis for V. If not we take an element $v_{r+2} \in V - L(S_1)$ and proceed as before. Since the dimension of V is finite, this process must stop at a certain stage giving the required basis containing S.

Theorem 5.22.

Let V be a finite dimensional vector space over a field F. Let A be a subspace of V. Then there exists a subspace B of V such that $V = A \oplus B$.

Proof:-

Let $S = \{v_1, v_2, \ldots, v_r\}$ be a basis of A.

By theorem 5.21 we can find

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$w_1, w_2, \ldots\ldots, w_s \in V$ such that such that

$S' = \{v_1, v_2, \ldots, v_r, w_1, w_2, \ldots\ldots, w_s\}$ is a basis of V.

Now, let $B = L(\{w_1, w_2, \ldots\ldots, w_s\})$

We claim that $A \cap B = \{0\}$ and $V = A + B$.

Now, let $v \in A \cap B$. Then $v \in A$ and $v \in B$

$\hookrightarrow (\alpha_1 v_1 + \alpha_2 v_2 + \ldots\ldots + \alpha_r v_r)$

Hence $v = \alpha_1 v_1 + \ldots\ldots + \alpha_r v_r$.

$= \beta_1 w_1 + \ldots\ldots + \beta_s w_s$ $\quad B = \beta_1 w_1 + \beta_2 w_2 + \ldots\ldots + \beta_s w_s$

$\therefore \alpha_1 v_1 + \ldots\ldots + \alpha_r v_r - \beta_1 w_1 - \ldots\ldots - \beta_s w_s = 0$

Now, since S' is linearly independent, $\alpha_i = 0 = \beta_j$ for all i and j

Hence $v = 0$. Thus $A \cap B = \{0\}$.

Now, let $v \in V$

Then $v = (\alpha_1 v_1 + \ldots\ldots + \alpha_r v_r) + (\beta_1 w_1 + \ldots\ldots + \beta_s w_s) \in A + B$.

Hence $A + B = V$ so that $V = A \oplus B$

Definition :-

10/02/2020

Let V be a vector space and $S = \{v_1, v_2, \ldots\ldots, v_n\}$ be a set of independent vectors in V. Then S is called a maximal linearly independent set if for every $v \in V - S$,

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the set $\{v, v_1, v_2, \ldots, v_n\}$ is linearly dependent.

Definition:-

Let $S = \{v_1, v_2, \ldots, v_n\}$ be a set of vectors in V and let $\boxed{L(S) = V}$. Then S is called a minimal generating set if for any $v_i \in S$,

$$L(S - \{v_i\}) \neq V$$

Theorem 5.23.

Let V be a vector space over a field F. Let $S = \{v_1, v_2, \ldots, v_n\} \subseteq V$. Then the following are equivalent.

i) S is a basis for V.

ii) S is a maximal linearly independent set.

iii) S is a minimal generating set.

$\boxed{\text{i)} \Rightarrow \text{ii)}}$ Let $S = \{v_1, v_2, \ldots, v_n\}$ be a basis for V. Then by theorem 5.20 any $n+1$ vectors in V are linearly dependent and hence S is a maximal linearly independent set.

$\boxed{\text{ii)} \Rightarrow \text{i)}}$ Let $S = \{v_1, v_2, \ldots, v_n\}$ be a

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maximal linearly independent set. Now to Prove that S is a basis for V we shall show that $L(S) = V$.

Obviously $L(S) \subseteq V$

Now, let $v \in V$.

If $v \in S$, then $v \in L(S)$. (since $S \subseteq L(S)$)

If $v \notin S$, $S' = \{v_1, v_2, \ldots, v_n, v\}$ is a linearly dependent set (since S is a maximal linearly independent set).

$\therefore$ There exists a vector in S' which is a linear combination of the Preceeding vectors.

Since $v_1, v_2, \ldots, v_n$ are linearly independent, this vector must be v, Thus v is a linear combination of $v_1, v_2, \ldots, v_n$.

Therefore $v \in L(S)$.

Hence $V \subseteq L(S)$. Thus $\boxed{V = L(S)}$.

$\boxed{(i) \Rightarrow (iii)}$ Let $S = \{v_1, v_2, \ldots, v_n\}$ be a basis. Then $L(S) = V$.

If S is not minimal, there exists $v_i \in S$ such that $L(S - \{v_i\}) = V$.

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Since S is linearly independent, $S - \{v_i\}$ also linearly independent. Thus $S - \{v_i\}$ basis consisting of $n-1$ elements which is contradiction.

Hence S is a minimal generating

iii) $\Rightarrow$ i) Let $S = \{v_1, v_2, \ldots, v_n\}$ be minimal generating set. To prove that S is basis, we have to show that S is linearly independent and $L(S) = V$

If S is linearly dependent, there exist a vector v_k which is a linear combination of the preceeding vectors.

Clearly $L(S - \{v_k\}) = V$ contradicting the minimality of S.

Thus S is linearly independent and Since $L(S) = V$, S is a basis for V.

Theorem 5.24.

11/2/2020

Any vector space of dimension n over a field F is isomorphic to $V_n(F)$.

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Proof:-

Let V be a vector space of dimension n. let $\{v_1, v_2, \ldots, v_n\}$ be a basis for V.

Then we know that if $v \in V$, v can be written uniquely as $v = \alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_n v_n$, where $\alpha_i \in F$.

any vector space
we define an isomorphic to $V_n(F)$
$f : V \to V_n(F)$
$f(\alpha_i v_i)$

Now, consider the map given by: $L(S) = V$ (every element of V is a linear combination)

$f : A \to B$
$f(a) = b$

$$f(\alpha_1 v_1 + \ldots + \alpha_n v_n) = (\alpha_1, \alpha_2, \ldots, \alpha_n)$$

clearly f is 1-1 and onto.

isomorphic: homomorphism, onto, one to one

let $v, w \in V$.

Then $v = \alpha_1 v_1 + \ldots + \alpha_n v_n$ and

$w = \beta_1 v_1 + \ldots + \beta_n v_n$.

$$f(v+w) = f[(\alpha_1+\beta_1)v_1 + \ldots + (\alpha_n+\beta_n)v_n]$$

$$= ((\alpha_1+\beta_1), (\alpha_2+\beta_2), \ldots, (\alpha_n+\beta_n))$$

$$= (\alpha_1, \alpha_2, \ldots, \alpha_n) + (\beta_1, \beta_2, \ldots, \beta_n)$$

$$f(v+w) = f(v) + f(w)$$

Also $f(\alpha v) = f(\alpha\alpha_1 v_1 + \ldots + \alpha\alpha_n v_n)$.

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$= (\alpha\alpha_1, \alpha\alpha_2, \ldots\ldots, \alpha\alpha_n)$

$= \alpha(\alpha_1, \alpha_2, \ldots\ldots, \alpha_n)$

$= \alpha f(v)$.

Hence f is an isomorphism of V to $V_n(F)$ so it is isomorphic

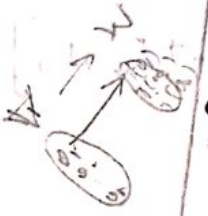
corollary.

Any two vector space of the same dimension over a field F are isomorphic, for if the vector spaces are of dimension n, each is isomorphic to $V_n(F)$ and hence they are isomorphic.

basis = L.I + span

Theorem 5.25

Let V and W be vector space over a field F. Let $V \to W$ be an isomorphism. Then T maps a basis of V onto a basis of W.



V → W, basis V → basis W

Proof :-

Let $\{v_1, v_2, \ldots\ldots, v_n\}$ be a basis for V.

S is spanned by ⇐ S is spanned by $L(S) = V$

We shall prove that $T(v_1), T(v_2), \ldots\ldots, T(v_n)$ are linearly independent and that they span W.

Now, $\alpha_1 T(v_1) + \alpha_2 T(v_2) + \ldots\ldots + \alpha_n T(v_n) = 0$

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$\Rightarrow T(\alpha_1 v_1) + T(\alpha_2 v_2) + \ldots\ldots + T(\alpha_n v_n) = 0$

$\Rightarrow T(\alpha_1 v_1 + \alpha_2 v_2 + \ldots\ldots + \alpha_n v_n) = 0$

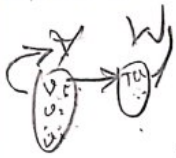
$\Rightarrow \alpha_1 v_1 + \alpha_2 v_2 + \ldots\ldots + \alpha_n v_n = 0$ (since T is linearly independent $S = \{v_1, v_2, \ldots, v_n\}$ 1-1) $\alpha_1 v_1 + \alpha_2 v_2 + \ldots\ldots v_n v_n = 0$

$\Rightarrow \alpha_1 = \alpha_2 = \ldots\ldots = \alpha_n = 0$

(since $v_1, v_2, \ldots\ldots, v_n$ are linearly independent).

$S' = \{T(v_1), T(v_2), \ldots\ldots, T(v_n)\}$ are linearly independent. $L(S') = V$

Now, let $w \in W$. Then since T is onto, there exists a vector $v \in V$ such that $T(v) = w$



Let $v = \alpha_1 v_1 + \ldots\ldots + \alpha_n v_n$.

Then $w = T(v)$

$v_1, v_2 \ldots v_n$ basis of V $= T(\alpha_1 v_1 + \ldots\ldots + \alpha_n v_n)$.

$T(v_1), T(v_2) \ldots T(v_n)$ L.I. of V $= \alpha_1 T(v_1) + \ldots\ldots + \alpha_n T(v_n)$

Thus w is a linearly combination of the vectors

$$T(v_1), \ldots\ldots, T(v_n)$$

$\therefore T(v_1), \ldots\ldots, T(v_n)$ span W and hence is a basis for W.

corollary:-

Two finite dimensional vector spaces V and W over a field F are isomorphic iff they have the same dimension.

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Theorem 5.26. Let V and W be finite dimensional vector spaces over a field F. Let $\{v_1, v_2, \ldots, v_n\}$ be a basis for V and let $w_1, w_2, \ldots, w_n$ be any n vectors in W (not necessarily distinct). Then there exists a unique linear transformation $T : V \to W$ such that $T(v_i) = w_i$, $i = 1, 2, \ldots, n$.

Proof :-

Let $v = \alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_n v_n \in V$

We define $T(v) = \alpha_1 w_1 + \alpha_2 w_2 + \ldots + \alpha_n w_n$

$T : V \to W$ Now, let $x, y \in V$.

Let $x = \alpha_1 v_1 + \ldots + \alpha_n v_n$ and

$y = \beta_1 v_1 + \ldots + \beta_n v_n$.

$x + y = (\alpha_1 + \beta_1) v_1 + \ldots + (\alpha_n + \beta_n) v_n$

$T(x+y) = (\alpha_1 + \beta_1) w_1 + \ldots + (\alpha_n + \beta_n) w_n$

$= (\alpha_1 w_1 + \ldots + \alpha_n w_n) + (\beta_1 w_1 + \ldots + \beta_n w_n)$

$= T(x) + T(y)$.

Similarly $T(\alpha x) = \alpha T(x)$.

Hence T is a linear transformation.

Also $v_1 = 1 v_1 + 0 v_2 + \ldots + 0 v_n$.

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~~Hence $T(v_i) = w_i$ for all $i = 1, 2, \ldots, n$.~~

Hence $T(v_1) = 1w_1 + 0w_2 + \ldots 0w_n = w_1$.

Similarly $T(v_i) = w_i$ for all $i = 1, 2, \ldots, n$.

Now, to prove the uniqueness, let $T' : V \to W$ be any other linear transformation such that $T'(v_i) = w_i$. (So I take T')

let $v = \alpha_1 v_1 + \ldots + \alpha_n v_n \in V$

$T'(v) = \alpha_1 T'(v_1) + \ldots + \alpha_n T'(v_n)$

$= \alpha_1 w_1 + \ldots + \alpha_n w_n = T(v)$

Hence $T = T'$.

Remark:

The above theorem shows that a linear transformation is completely determined by its values on the elements of a basis

Theorem 5.27 13/2/2020

Let V be a finite dimensional vector space over a field F. Let W be a subspace of V. Then

i) $\dim W \leq \dim V$.

ii) $\dim \frac{V}{W} = \dim V - \dim W$.

Proof:-

i) let $S = \{w_1, w_2, \ldots, w_m\}$ be a ($L(S) = W$ and linearly independent)

basis for W. Since W is a subspace of V, S is a

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5.21

part of a basis for V.

Hence $\dim W \leq \dim V$.

ii) Let $\dim V = n$ and $\dim W = m$.

Let $S = \{w_1, w_2, \ldots, w_m\}$ be a basis for W.

Clearly S is a linearly independent set of vectors in V.

Hence S is a part of a basis in V.

Let $\{w_1, w_2, \ldots w_m, v_1, v_2, \ldots, v_r\}$ be a basis for V. Then $\boxed{m+r=n}$.



Now, we claim $S' = \{W+v_1, W+v_2, \ldots, W+v_r\}$ is a basis for $\frac{V}{W}$.

$$\alpha_1(W+v_1) + \alpha_2(W+v_2) + \ldots + \alpha_r(W+v_r) = W+0$$

$$\Rightarrow (W+\alpha_1 v_1) + (W+\alpha_2 v_2) + \ldots + (W+\alpha_r v_r) = W$$

$$\Rightarrow W + (\alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_r v_r) = W.$$

$$\Rightarrow \alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_r v_r \in W.$$

$$\alpha_1 = \alpha_2 = \ldots = \alpha_r = 0.$$

Now, Since $\{w_1, w_2, \ldots, w_n\}$ is a basis for W.

$$\alpha_1 v_1 + \ldots + \alpha_r v_r = \beta_1 w_1 + \ldots + \beta_m w_m$$

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$\therefore \alpha_1 v_1 + \cdots + \alpha_r v_r - \beta_1 w_1 - \cdots - \beta_m w_m = 0$

$\therefore \alpha_1 = \alpha_2 = \alpha_r = \beta_1 = \beta_2 = \cdots = \beta_m = 0$

$\therefore S'$ is a linearly independent set.

Now, let $W + v \in \frac{V}{W}$.

$v = \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_r v_r$

$n = m + r$

Let $v = \alpha_1 v_1 + \cdots + \alpha_r v_r + \beta_1 w_1 + \cdots + \beta_m w_m$

Then $W + v = W + (\alpha_1 v_1 + \cdots + \alpha_r v_r + \beta_1 w_1 + \cdots + \beta_m w_m)$

$= W + (\alpha_1 v_1 + \cdots + \alpha_r v_r)$

(Since $\beta_1 w_1 + \cdots + \beta_m w_n \in W$)

$= (W + \alpha_1 v_1) + \cdots + (W + \alpha_r v_r)$

$= \alpha_1 (W + v_1) + \cdots + \alpha_r (W + v_r)$

Hence S' spans $\frac{V}{W}$ so that S' is a basis for $\frac{V}{W}$.

Span $\alpha_1(W + v_1)$

W'

$v : v$

$\dim \frac{V}{W} = r = n - m$

$\dim \frac{V}{W} = r$

$m + r = n$

$r = n - m$

$= \dim V - \dim W$.

$= \dim V - \dim W$

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Theorem 5.28.

4/1/2020

Let V be a finite dimensional vector space over a field F. Let A and B be subspaces of V. Then $\dim(A+B) = \dim A + \dim B - \dim(A \cap B)$.

Proof:-

A and B are subspaces of V. Hence

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$A \cap B$ is subspace of V.

Let $\dim(A \cap B) = r$

Let $S = \{v_1, v_2, \ldots, v_r\}$ be a basis for $A \cap B$

Since $A \cap B$ is a subspace of A and is a part of a basis for A and B.

Let $\{v_1, v_2, \ldots, v_r, u_1, u_2, \ldots, u_s\}$ be a basis for A and $\{v_1, v_2, v_3, \ldots, w_1, w_2, \ldots, w_t\}$ be a basis for B

we shall prove that $S' = \{v_1, \ldots, v_r, u_1, \ldots, u_s, w_1, \ldots, w_t\}$ is a basis for $A+B$.

Let $\alpha_1 v_1 + \ldots + \alpha_r v_r + \beta_1 u_1 + \ldots + \beta_s u_s + \gamma_1 w_1 + \ldots + \gamma_t v_t = 0$

then $\beta_1 u_1 + \ldots \beta_s u_s = -(\gamma_1 w_1 + \ldots + \gamma_t v_t) - (\alpha_1 v_1 + \ldots + \alpha_r v_r) \in$

Hence $\beta_1 u_1 + \ldots + \beta_s u_s \in B$.

Also $\beta_1 u_1 + \ldots + \beta_s u_s \in A$.

Hence $\beta_1 u_1 + \ldots + \beta_s u_s \in A \cap B$.

$\therefore \beta_1 u_1 + \ldots + \beta_s u_s = \delta_1 v_1 + \ldots + \delta_r v_r$

$\therefore \beta_1 u_1 + \ldots + \beta_s u_s - \delta_1 v_1 - \ldots - \delta_r v_r =$

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$\therefore \beta_1 = \ldots\ldots = \beta_s = \alpha_1 = \ldots\ldots = \alpha_r = 0$

(Since $\{u_1, \ldots\ldots, u_s, v_1, \ldots\ldots, v_r\}$ is linearly independent)

Similarly we can prove $\gamma_1 = \gamma_2 = \ldots\ldots = \gamma_t = 0$

$\therefore \alpha_i = \beta_j = \gamma_k = 0$ for $1 \leq i \leq r$.

$1 \leq j \leq s$; $1 \leq k \leq t$.

Thus S' is a linearly independent set.

Clearly S' spans $A+B$.

$\therefore S'$ is a basis for $A+B$.

Hence $\dim(A+B) = r+s+t$.

Also $\dim A = r+s$; $\dim B = r+t$ and

$\dim(A \cap B) = r$.

$\therefore \dim A + \dim B - \dim A \cap B = (r+s)+(r+t)-r$

$= r+s+t$.

$= \dim(A+B)$

Aliter:-

By theorem 5-7 $\frac{A+B}{A} = \frac{B}{A \cap B}$.

Hence $\dim\left[\frac{A+B}{A}\right] = \dim\left[\frac{B}{A \cap B}\right]$

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$\therefore \dim(A+B) - \dim A = \dim B - \dim(A$

$\therefore \dim(A+B) = \dim A + \dim B - \dim(A$

Cordlary:

If $V = A \oplus B$, $\dim V = \dim A + \dim$

Proof:-

$V = A \oplus B \Rightarrow A+B = V$ and $A \cap B = \{0$

$\therefore \dim(A \cap B) = 0$.

Hence $\dim V = \dim A + \dim B$.

5.7 Rank and Nullity.

Definition:-

Let $T: V \to W$ be a linear transformation. Then the dimension of $T(V)$ is called the rank of T. The dimension of $\ker T$ is called the nullity of T.

$\{v \mid v \in V, T(v)$

$\ker T$

Theorem 5.29.

Let $T: V \to W$ be a linear transformation. Then $\dim V = \text{rank } T + \text{nullity } T$.

Proof:-

$V/V_1 = W$

We know that $V/\ker T = T(V)$.

$\therefore \dim V - \dim(\ker T) = \dim(T(V))$.

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$\therefore \dim V - \text{nullity } T = \text{rank } T$.

$\therefore \dim V = \text{nullity } T + \text{rank } T$.

Note:-

Ker T is also called null space of T.

Example:

Let V denote the set of all polynomials of degree $\leq n$ in $R[x]$. Let $T: V \to V$ be defined by $T(f) = \frac{df}{dx}$. We know that T is a linear transformation. Since $\frac{df}{dx} = 0 \iff f$ is constant, Ker T consists of all constant polynomials. The dimension of this subspace of V is 1. Hence nullity T is 1. Since $\dim V = n+1$, rank $T = n$.

Definition:-

A linear transformation $T: V \to W$ is called non-singular if T is 1-1.



otherwise T is called singular.

$T: V \to W$
$T(v) = 0 \ \forall v \in V$
Rank T = 0
dim Ker T = dim V

$T: V \to V$
$T(v) = v \ \forall v \in V$
Rank T = dim V
Nullity T = 0

$T: V_3(R) \to V_3(R)$
$T(a,b,c) = (a,0,0)$
Rank T = 1
Nullity T = 2

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4.1.2020. 1) Define one-one.

A function $f: A \to B$ is one-one (injective) if distinct element in A have distinct images in B under of

. In other words f is 1-1

if $x, y \in A$ and $x \neq y \Rightarrow f(x) \neq f(y)$ or

equivalently, $x = y \Rightarrow f(x) = f(y)$.

2) Define onto

The mapping f is called onto (surjective) if the range of f is equal to B. Thus if f is onto, every element of B has a pre-image in A

3) Defn Bijection.

If $f: A \to B$ is both 1-1 and onto then f is called a bijection

In this case every element of B has exactly one pre-image in A.

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Unit - III

5.8 Matrix of a Linear Transformation

Let V and W be finite dimensional vector space over a field F. let $\dim V = m$ and $\dim W = n$. Fix an ordered basis $\{v_1, v_2, \ldots, v_m\}$ for V and an ordered basis $\{w_1, w_2, \ldots, w_n\}$ for W.

Let $T: V \to W$ be a linear transformation. We have seen that T is completely specified by the elements $T(v_1), T(v_2), \ldots, T(v_m)$. Now, let

$$T(v_1) = a_{11}w_1 + a_{12}w_2 + \ldots + a_{1n}w_n$$

$$T(v_2) = a_{21}w_1 + a_{22}w_2 + \ldots + a_{2n}w_n \quad \hookrightarrow (1)$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$T(v_m) = a_{m1}w_1 + a_{m2}w_2 + \ldots + a_{mn}w_n$$

Hence $T(v_1), T(v_2), \ldots, T(v_m)$ are completely specified by the mn elements a_{ij} of the field F. These a_{ij} can be conveniently arranged in the form of m rows and n columns as follows:

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$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

Such an array of mn elements of F arranged in m rows and n columns is known as m x n matrix over the field F and is denoted by (a;j). Thus to every linear transformation T there is associated any with it an mxn matrix over F. Conversely any mxn matrix over F defines a linear transformation $T: V \longrightarrow W$ given by the formula ①.

Note:-

The mxn matrix which we have associated with a linear transformation $T: V \longrightarrow W$ depends on the choice of the basis for V and W.

For example, consider the linearly transformation $T: V_2(R) \longrightarrow V_2(R)$ given by $T(a;b) = (a, a+b)$. choose $\{e_1, e_2\}$ as a basis both for the domain and the range.

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Then $T(e_1) = (1,1) = e_1 + e_2$

$e_1 = 1$

$e_2 = 1$

$T(e_2) = (0,1) = e_2$

$e_1 = (1,0), e_2 = (0,1)$

Hence the matrix representing

T is $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.

Now, we choose $\{e_1, e_2\}$ as a basis for the domain and $\{(1,1), (1,$ as a basis for the range.

let $w_1 = (1,1)$ and $w_2 = (1,-1)$

$T: V \to W$

e_1 (1,0) → (1,1)

e_2 (0,1) → (1,−1)

Then $T(e_1) = (1,1) = w_1$.

and $T(e_2) = (0,1) = (1/2)w_1 - (1/2)$

Hence the matrix representing T is

$\begin{bmatrix} 1 & 0 \\ 1/2 & 1/2 \end{bmatrix}$

$T(a,b) = (a, a+b)$

$T(e_1) = (1,1) = w_1$

$1/2\, w_1 - 1/2\, w_2$

$1/2\,(1,1) - 1/2\,(1,-1)$

$(1/2, 1/2) + (-1/2, 1/2)$

$(0, \frac{2}{2}) = (0,1)$

Solved problems

Problem 1.

Obtain the matrix representing the linear transformation $T: V_3(R) \to V_3(R)$ given by $T(a,b,c) = (3a, a-b, 2a+b+c)$ w.r.t the standard basis $\{e_1, e_2, e_3\}$

$T(e_1) = T(1,0,0) = (3,1,2) = 3e_1 + e_2 + 2e_3$

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$$T(e_2) = T(0,1,0) = (0,-1,1) = -e_2 + e_3$$

$$T(e_3) = T(0,0,1) = (0,0,1) = e_3$$

Thus the matrix representing T is

$$\begin{bmatrix} 3 & 1 & 2 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

Problem 2.

Find the linear transformation $T: V_3(R) \to V_3(R)$ determined by the matrix $\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ -1 & 3 & 4 \end{bmatrix}$ w.r.t the standard basis $\{e_1, e_2, e_3\}$.

$$T(e_1) = e_1 + 2e_2 + e_3 = (1,2,1).$$

$$T(e_2) = 0e_1 + e_2 + e_3 = (0,1,1).$$

$$T(e_3) = -e_1 + 3e_2 + 4e_3 = (-1,3,4).$$

Now, $(a,b,c) = a(1,0,0) + b(0,1,0) + c(0,0,1)$

$$= ae_1 + be_2 + ce_3.$$

$$T(a,b,c) = T(ae_1 + be_2 + ce_3)$$

$$= aT(e_1) + bT(e_2) + cT(e_3)$$

$$= a(1,2,1) + b(0,1,1) + c(-1,3,4)$$

$$\therefore T(a,b,c) = (a-c, 2a+b+3c, a+b+4c)$$

This is the required linear transformation

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Definition:- 2 Mark

[Let $A = (a_{ij})$ and $B = (b_{ij})$ be two $m \times n$ matrices. We define the sum of these two matrices by $A + B = (a_{ij} + b_{ij})$] 2 Mark

Note that we have defined addition only for two matrices having the same number of rows and the same number of columns.

Definition.

Let $A = (a_{ij})$ be an arbitrary matrix over a field F. Let $\alpha \in F$. We define $\alpha A = (\alpha a_{ij})$.

Theorem 5.30:

The set $M_{m \times n}(F)$ of all $m \times n$ matrices over the field F is a vector space of dimension mn over F under matrix addition and scalar multiplication defined above

Let $A = (a_{ij})$ and $B = (b_{ij})$ be two $m \times n$ matrices over the field F

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i) The addition of mxn matrices is a binary operation which is both commutative and associative. ii) The mxn matrix whose entries are 0 is the identity matrix and $(-a_{ij})$ is the inverse matrix of (a_{ij}). iii) Thus the set of all mxn matrices over the field F is an abelian group with respect to addition. The verification of the following axioms are straight forward.

$(M_{mxn}(F), +)$ is abelian group

a) $\alpha(A+B) = \alpha A + \alpha B$

b) $(\alpha+\beta)A = \alpha A + \beta A$.

c) $(\alpha\beta)A = \alpha(\beta A)$

d) $1A = A$.

$e_{ij} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Hence the set of all mxn over F is a vector space over F.

Now, we shall prove that the dimension of this vector space is mn. Let E_{ij} be the matrix with entry 1 in the $(i,j)^{th}$ place and 0 in the other places. We have mn matrices of this form. Also any matrix $A = (a_{ij})$ can be written as $A = \sum a_{ij} E_{ij}$.

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Hence A is a linear combination of the matrices E_{ij}. Further these mn matrices E_{ij} are linearly independent (↑ scalar). Hence these mn matrices form a basis for the space of all mxn matrices. Therefore the dimension of the vector space is mn.

Theorem 5-31.

Let V and W be two finite dimensional vector space over a field. Let dim V = m and dim W = n. Then L(V, W) is a vector space of dimension mn over F.

Let V and W be vector spaces over a field F. let L(V, W) represent the set of all linear transformations form V to W. Then L(V, W) itself is a vector space over F under

Proof:- By Theorem 5.8, L(V, W) is a vector space over F. Now, we shall Prove that the vector space L(V, W) is isomorphic to the vector space $M_{mxn}(F)$. Since $M_{mxn}(F)$ is of dimension mn; it follows that L(V, W) is also of dimension mn.

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Fix a basis $\{v_1, v_2, \ldots, v_m\}$ for V and a basis $\{w_1, w_2, \ldots, w_n\}$ for W.

We know that any linear transformation.

$T \in L(V, W)$ can be represented by an $m \times n$ matrix over F.

$M : L(V, W) \to$

Let T be represented by $M(T)$.

This function $M : L(V, W) \to M_{m \times n}(F)$ is clearly 1-1 and onto.

Let $T_1, T_2 \in L(V, W)$ and $M(T_1) = (a_{ij})$ and $M(T_2) = (b_{ij})$

$\sum_{j=1}^{n} a_{ij} w_{ij}$.

addition and scalar multiplication defined by
$T(u+v) = T(u) + T(v)$
$T(\alpha u) = \alpha T(u)$

$$M(T_1) = (a_{ij}) \Rightarrow T_1(v_i) = \sum_{j=1}^{n} a_{ij} w_j$$

$$M(T_2) = (b_{ij}) \Rightarrow T_2(v_i) = \sum_{j=1}^{n} b_{ij} w_j$$

$$\therefore (T_1 + T_2)(v_i) = \sum_{j=1}^{n} (a_{ij} + b_{ij}) w_j.$$

$$M(T_1 + T_2) = (a_{ij} + b_{ij})$$

$$= (a_{ij}) + (b_{ij})$$

$$= M(T_1) + M(T_2)$$

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Similarly $M(\alpha T_1) = \alpha M(T_1)$

Hence M is the required isomorphism from $L(V, W)$ to $M_{m \times n}(F)$.

6. Inner Product Spaces

6.0 Introduction.

Upto this point we have dealt with the algebraic properties of a vector space and these properties are consequences of the basic operation, namely, vector addition and scalar multiplication defined in the vector space. We know that in the usual three dimensional vector space $V_3(R)$ it is possible to talk about the length of a vector and angle between to vectors. these concepts of length and angle can be defined in terms of the usual "dot product" or "scalar product" of two vectors. The dot product of $u = (a_1, b_1, c_1)$ and $v(a_2, b_2, c_2)$ is defined by

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$u \cdot v = a_1a_2 + b_1b_2 + c_1c_2$.

We note that the length of u is given by $\sqrt{u \cdot u}$ and the angle θ between u and v is determined by $\cos\theta = \frac{u \cdot v}{\sqrt{u \cdot u}\sqrt{v \cdot v}}$. Hence u and v are Prependicular or orthogonal iff $u \cdot v = 0$

An inner product on a vector space is a generalisation of the dot Product and in terms of such an inner product we can define the length of a vector and angle between two vectors. Our study about angle will be restricted to the concept of Perpendicularity of two vectors.

Throughout this section we shall deal only with vector space over the field F of real or complex numbers.

6.1 Definition and Examples.

Definition

Let V be a vector space over F. An Linner Product on V is a functior

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which assings to each ordered pa
of vectors u, v in V a scalar in F
denoted by (u,v) satisfying the follo
conditions.

i) $\langle u+v, w\rangle = \langle u, w\rangle + \langle v, w\rangle$

ii) $\langle \alpha u, v\rangle = \alpha \langle u, v\rangle$

iii) $\langle u, v\rangle = \overline{\langle v, u\rangle}$, where $\overline{\langle v, u\rangle}$ is the complex conjugate of $\langle u, v\rangle$

iv) $\langle u, u\rangle \geq 0$ and $\langle u, u\rangle = 0$ iff $u = 0$.

A vector space with an inner product defined on it is call an inner product space. An inner Product space is called an Euclidean space or unitary space according as F is the field of real numbers or complex numbers.

Define Euclidean space unitary space

Note 1:

If F is the field of real numbers the condition iii) takes the form $(u,v) = (v,u)$. Further (iii) asserts that (u,u) is always real and hence

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(iv) is meaningful whether F is the field of real or complex numbers.

Note 2:-

$\langle u, \alpha v \rangle = \bar{\alpha} \langle u, v \rangle$.

Let V be a vector space over F.

For, $\langle u, \alpha v \rangle = \overline{\langle \alpha v, u \rangle}$

$= \overline{\alpha \langle v, u \rangle}$

$= \bar{\alpha} \overline{\langle v, u \rangle}$

$= \bar{\alpha} \langle u, v \rangle$

Note 3:-

$\langle u, v+w \rangle = \langle u, v \rangle + \langle u, w \rangle$

For, $\langle u, v+w \rangle = \overline{\langle v+w, u \rangle}$

$= \overline{\langle v, u \rangle + \langle w, u \rangle}$

$= \overline{\langle v, u \rangle} + \overline{\langle w, u \rangle}$

$= \langle u, v \rangle + \langle u, w \rangle$

Note 4:-

$\langle u, 0 \rangle = \langle 0, v \rangle = 0$

For, $\langle u, 0 \rangle = \langle u, 00 \rangle \Rightarrow 0 \langle u, 0 \rangle$

$= 0$

Similarly $\langle 0, v \rangle = 0$.

Examples:-

1) $V_n(R)$ is a real inner product space with inner product defined by.

$\langle x, y \rangle = x_1 y_1 + x_2 y_2 + \ldots\ldots + x_n y_n$ where

$x = (x_1, x_2, \ldots\ldots, x_n)$ and

$y = (y_1, y_2, \ldots\ldots, y_n)$

This is called the standard inner Product on

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Proof:-

Let $x, y, z \in V_n(R)$ and $\alpha \in R$.

i) $\langle x+y, z \rangle = (x_1+y_1)z_1 + (x_2+y_2)z_2 + \ldots + (x_n+y_n)z_n$

$= (x_1z_1 + x_2z_2 + \ldots + x_nz_n) + (y_1z_1 + y_2z_2 + \ldots + y_nz_n)$

$= (x, z) + (y, z)$

ii) $\langle \alpha x, y \rangle = \alpha x_1y_1 + \alpha x_2y_2 + \ldots + \alpha x_ny_n$

$= \alpha(x_1y_1 + x_2y_2 + \ldots + x_ny_n)$

$= \alpha(x, y)$

iii) $\langle x, y \rangle = x_1y_1 + x_2y_2 + \ldots + x_ny_n$

$= y_1x_1 + y_2x_2 + \ldots + y_nx_n$

$= (y, x)$

iv) $\langle x, x \rangle = x_1^2 + x_2^2 + \ldots + x_n^2 \geq 0$ and

$\langle x, x \rangle = 0$ iff $x_1 = x_2 = \ldots = x_n = 0$

$\therefore (x, x) = 0$ iff $x = 0$.

2) $V_n(C)$ is a complex inner product space with inner product defined by

$\langle x, y \rangle = x_1\overline{y_1} + x_2\overline{y_2} + \ldots + x_n\overline{y_n}$ where

$x = (x_1, x_2, \ldots, x_n)$ and

$y = (y_1, y_2, \ldots, y_n)$

Let $x, y, z \in V_n(C)$ and $\alpha \in C$.

i) $\langle x+y, z \rangle = (x_1+y_1)\overline{z_1} + (x_2+y_2)\overline{z_2} + \ldots + (x_n+y_n)\overline{z_n}$

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$$= (x_1\bar{z}_1 + x_2\bar{z}_2 + \cdots + x_n\bar{z}_n) + (y_1\bar{z}_1 + y_2\bar{z}_2 + \cdots + y_n\bar{z}_n)$$

$$= (x,z) + (y,z)$$

ii) $\langle \alpha x, y \rangle = \alpha x_1\bar{y}_1 + \alpha x_2\bar{y}_2 + \cdots + \alpha x_n\bar{y}_n$

$$= \alpha(x_1\bar{y}_1 + x_2\bar{y}_2 + \cdots + x_n\bar{y}_n)$$

$$= \alpha(x,y).$$

iii) $\overline{\langle y,x \rangle} = y_1\bar{x}_1 + y_2\bar{x}_2 + \cdots + y_n\bar{x}_n$

$$= \bar{y}_1 x_1 + \bar{y}_2 x_2 + \cdots + \bar{y}_n x_n$$

$$= x_1\bar{y}_1 + x_2\bar{y}_2 + \cdots + x_n\bar{y}_n$$

$$= (x,y)$$

iv) $\langle x,x \rangle = x_1\bar{x}_1 + \cdots + x_n\bar{x}_n$

$$= |x_1|^2 + |x_2|^2 + \cdots + |x_n|^2 \geq 0$$

and $\langle x,x \rangle = 0$ iff $x = 0$

3) Let V be the set of all continuous real valued functions defined on the closed interval $[0,1]$ V is a real inner product space with inner product defined by.

$$\langle f,g \rangle = \int_0^1 f(t)g(t)dt$$

Proof:-

Let $f,g,h \in V$ and $\alpha \in \mathbb{R}$.

i) $\langle f+g,h \rangle = \int_0^1 [f(t)+g(t)]\,h(t)dt$

$$= \int_0^1 f(t)h(t)dt + \int_0^1 g(t)h(t)dt$$

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$= (f+h) + (g+h)$

ii) $\langle \alpha f, g \rangle = \int_0^1 \alpha f(t)\, g(t)\, dt$.

$= \alpha \int_0^1 f(t)\, g(t)\, dt$.

$= \alpha (f, g)$.

iii) $\langle f, g \rangle = \int_0^1 f(t)\, g(t)\, dt$.

$= \int_0^1 g(t)\, f(t)\, dt$.

$= (g, f)$

iv) $\langle f, f \rangle = \int_0^1 [f(t)]^2 dt \geq 0$ and

$\langle f, f \rangle = 0$ iff $f = 0$

Definition.

Let V be an inner product space and let $x \in V$. The norm or length of x, denoted by $\|x\|$ is defined by $\|x\| = \sqrt{\langle x, x \rangle}$ $\| \ \| \rightarrow$ norm (or) length

x is called a unit vector if

$\|x\| = 1$

Problem :1.

Let V be the vector space of polynomials with inner product given by $(f+g) = \int_0^1 f(t) g(t)$

Let $f(t) = t+2$ and $g(t) = t^2 - 2t - 3$.

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Find i) $\langle f, g \rangle$ ii) $\|F\|$

i) $\langle f, g \rangle = \int_0^1 f(t)\, g(t)\, dt$

$= \int_0^1 (t+2)(t^2 - 2t - 3)\, dt$

$= \int_0^1 (t^3 - 7t - 6)\, dt$

$= \left[\frac{t^4}{4} - \frac{7t^2}{2} - 6t\right]_0^1$

$= \left[1/4 - 7/2 - 6\right]$

$= -\frac{37}{4}$

ii) $\|F\|^2 = \langle f, f \rangle$

$= \int_0^1 [f(t)]^2\, dt$

$= \int_0^1 [t+2]^2\, dt$

$= \int_0^1 (t^2 + 4t + 4)\, dt$

$= \left[t^3/3 + \frac{4t^2}{2} + 4t\right]_0^1$

$= 1/3 + 2 + 4$

$\|F\|^2 = 19/3$

$\|F\| = \frac{\sqrt{19}}{\sqrt{3}}$

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Theorem : 6.1

The norm defined in an inner Product space V has the following

Properties

i) $\|x\| \geq 0$ and $\|x\| = 0$ iff $x = 0$

ii) $\|\alpha x\| = |\alpha| \|x\|$

iii) $|\langle x, y \rangle| \leq \|x\| \|y\|$ (Schwarz inequality)

iv) $\|x+y\| \leq \|x\| + \|y\|$ (Triangle inequality)

i) $\|x\| = \sqrt{\langle x, x \rangle} \geq 0$ and $\|x\| = 0$ iff $x = 0$

ii) $\|\alpha x\|^2 = \langle \alpha x, \alpha x \rangle$

$= \alpha \langle x, \alpha x \rangle$

$= \alpha \bar{\alpha} \langle x, x \rangle$

$\langle x, x \rangle = \|x\|^2$ $= |\alpha|^2 \|x\|^2$

Hence $\|\alpha x\| = |\alpha| \|x\|$

iii) The inequality is trivially true when $x = 0$ or $y = 0$. Hence let $x \neq 0$ and $y \neq 0$ consider $z = y - \frac{\langle y, x \rangle}{\|x\|^2} x$.

Then $0 \leq \langle z, z \rangle$

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$$= \left\langle y - \frac{\langle y, x \rangle}{\|x\|^2} x,\ y - \frac{\langle y, x \rangle}{\|x\|^2} x \right\rangle$$

$$= \langle y, y \rangle - \frac{\langle \overline{y, x} \rangle}{\|x\|^2} \langle y, x \rangle - \frac{\langle y, x \rangle}{\|x\|^2} \langle x, y \rangle + \frac{(y, x)(\overline{y, x})}{\|x\|^2 \|x\|^2} \langle x, x \rangle$$

$z = y - \frac{\langle y, x \rangle}{\|x\|}$

$$= \|y^2\| - \frac{\langle \overline{y, x} \rangle \langle y, x \rangle}{\|x\|^2} - \frac{\langle y, x \rangle \langle x, y \rangle}{\|x\|^2} + \frac{\langle y, x \rangle \langle \overline{y, x} \rangle}{\|x\|^2}$$

$$= \|y\|^2 - \frac{\langle \overline{x, y} \rangle \langle x, y \rangle}{\|x\|^2}$$

$$\therefore 0 \leq \|x\|^2 \|y\|^2 - |\langle x, y \rangle|^2$$

$$\therefore |\langle x, y \rangle| \leq \|x\| \|y\|$$

iv) $$\|x+y\|^2 = \langle x+y, x+y \rangle$$

$$= \langle x, x \rangle + \langle x, y \rangle + \langle y, x \rangle + \langle y, y \rangle$$

$$= \|x\|^2 + \langle x, y \rangle + \langle \overline{x, y} \rangle + \|y\|^2$$

$$= \|x\|^2 + 2 \operatorname{Re} \langle x, y \rangle + \|y\|^2$$

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$$\leq \|x\|^2 + 2|\langle x, y\rangle| + \|y\|^2$$

$$\leq \|x\|^2 + 2\|x\|\|y\| + \|y\|^2 \quad \text{by (1)}$$

$$\leq (\|x\| + \|y\|)^2$$

$$\therefore \|x+y\| \leq \|x\| + \|y\|$$

6.2 Orthogonality

Definition.

Let V be an inner product space and let $x, y \in V$. x is said to be orthogonal to y if $\langle x, y\rangle = 0$

Note 1 :-

x is orthogonal to $y \Rightarrow \langle x, y\rangle = 0$

$\Rightarrow \overline{\langle x, y\rangle} = \overline{0}$

$\Rightarrow \langle y, x\rangle = 0$

$\Rightarrow y$ is orthogonal to x.

Thus x and y are orthogonal iff $\langle x, y\rangle = 0$

Note 2 :-

x is orthogonal to $y \Rightarrow \alpha x$ is orthogonal to y.

Note 3 :-

x_1 and x_2 are orthogonal to $y \Rightarrow x_1 + x_2$ is orthogonal to y.

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Note 4:-

0 is orthogonal to every vector in V and is the only vector with this property.

Definition

Let V be an inner product space. A set S of vectors in V is said to be an orthogonal set if any two distinct vectors in S are orthogonal.

Definition

S is said to be an orthonormal set if S is orthogonal and $\|x\| = 1$ for all $x \in S$

Examples:-

The standard basis $\{e_1, e_2, \ldots, e_n\}$ in R^n or C^n is an orthogonal set with respect to the standard inner product.

Theorem 6.2

Let $S = \{v_1, v_2, \ldots, v_n\}$ be an orthogonal set of non-zero vectors in an inner product space V. Then S is linearly independent.

Proof :-

Let $\alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_n v_n = 0$

Then $\langle \alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_n v_n, v_1 \rangle = \langle 0, v_1 \rangle = 0$

$\therefore \alpha_1 \langle v_1, v_1 \rangle + \alpha_2 \langle v_2, v_1 \rangle + \ldots + \alpha_n \langle v_n, v_1 \rangle = 0$

$\therefore \alpha_1 \langle v_1, v_1 \rangle = 0$ (since S is orthogonal)

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$\therefore \alpha_1 = 0$ (since $v_1 \neq 0$)

Similarly $\alpha_2 = \alpha_3 = \cdots\cdots = \alpha_n = 0$

Hence S is linearly independent.

Theorem 6.3

Let $S = \{v_1, v_2, \ldots, v_n\}$ be an orthogonal set of non-zero vectors in V. Let $v \in V$ and $v = \alpha_1 v_1 + \alpha_2 v_2 + \cdots\cdots + \alpha_n v_n$.

Then $\alpha_k = \dfrac{\langle v, v_k \rangle}{\|v_k\|^2}$

Proof:-

$$\langle v, v_k \rangle = \langle \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n, v_k \rangle$$

$$= \alpha_1 \langle v_1, v_k \rangle + \alpha_2 \langle v_2, v_k \rangle + \cdots$$
$$\cdots + \alpha_k \langle v_k, v_k \rangle + \cdots + \alpha_k \langle v_n, v_k \rangle$$

$$= \alpha_k \langle v_k, v_k \rangle \quad \text{(since } S \text{ is orthogonal)}$$

$$= \alpha_k \|v_k\|^2$$

$$\therefore \alpha_k = \frac{\langle v, v_k \rangle}{\|v_k\|^2}$$

Theorem 6.4 Gram-Schmidt orthogonalisation Process

Every finite dimensional inner Product space has an orthonormal basis.

Proof:-

Let V be a finite dimensional

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inner product space. Let $\{v_1, v_2, \ldots, v_n\}$ be a basis for V. From this basis we shall construct an orthonormal basis $\{w_1, w_2, \ldots, w_n\}$ by means of a construction known as Gram-Schmidt orthogonalisation process.

Frist we take $\boxed{w_1 = v_1}$

Let $w_2 = v_2 - \frac{\langle v_2, w_1 \rangle}{\|w_1\|^2} w_1$.

We claim that $w_2 \neq 0$. For, if $w_2 = 0$ then v_2 is a scalar multiple of w_1 and hence of v_1 which is a contradiction since v_1, v_2 are linearly independent.

Also $\langle w_2, w_1 \rangle = \left\langle v_2 - \frac{\langle v_2, w_1 \rangle}{\|w_1\|^2} w_1, w_1 \right\rangle$

$= \left\langle v_2 - \frac{\langle v_2, v_1 \rangle}{\|v_1\|^2} v_1, v_1 \right\rangle \quad (\because w_1 = v_1)$

$= \langle v_2, v_1 \rangle - \frac{\langle v_2, v_1 \rangle}{\|v_1\|^2} \langle v_1, v_1 \rangle$

$= \langle v_2, v_1 \rangle - \langle v_2, v_1 \rangle$

$= 0$

Now, suppose that we have constructed non-zero orthogonal vectors $w_1, w_2, \ldots, w_k$

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Then put

$$w_{k+1} = v_{k+1} - \sum_{j=1}^{k} \frac{\langle v_{k+1}, w_j \rangle}{\|w_j\|^2}$$

we claim that $w_{k+1} \neq 0$. For, if $w_{k+1} = 0$ then v_{k+1} is a linear combination of $w_1, w_2, \ldots, w_k$ and hence is a linear combination of $v_1, v_2, \ldots, v_k$ which is a contradiction since $v_1, v_2, \ldots, v_{k+1}$ are linearly independent.

Also

$$\langle w_{k+1}, w_i \rangle = \langle v_{k+1}, w_i \rangle - \sum_{j=1}^{k} \frac{\langle v_{k+1}, w_j \rangle}{\|w_j\|^2} \langle w_j, w_i \rangle$$

$$= \langle v_{k+1}, w_i \rangle - \frac{\langle v_{k+1}, w_i \rangle}{\|w_i\|^2} \langle w_i, w_i \rangle$$

$$= \langle v_{k+1}, w_i \rangle - \langle v_{k+1}, w_i \rangle$$

$$= 0$$

Thus continuing in this way we ultimately obtain a non-zero orthogonal set $\{w_1, w_2, \ldots, w_n\}$.

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By theorem 6.2 this set is linearly independent and hence a basis.

To obtain an orthonormal basis we replace each w_i by $\frac{w_i}{\|w_i\|}$

Problem 1.

Apply Gram-Schmidt process to construct an orthonormal basis for $V_3(R)$ with the standard inner product for the basis $\{v_1, v_2, v_3\}$ where $v_1 = (1, 0, 1)$; $v_2 = (1, 3, 1)$ and $v_3 = (3, 2, 1)$

Solution:-

Take $w_1 = v_1 = (1, 0, 1)$

Then $\|w_1\|^2 = \langle w_1, w_1 \rangle = 1^2 + 0^2 + 1^2 = 2$

and $\langle w_1, v_2 \rangle = 1 + 0 + 1 = 2$

Put $w_2 = v_2 - \frac{\langle v_2, w_1 \rangle}{\|w_1\|^2} w_1$

$= (1, 3, 1) - (1, 0, 1)$

$= (0, 3, 0)$

$\therefore \|w_2\|^2 = 9$

Also, $\langle w_2, v_3 \rangle = 0 + 6 + 0 = 6$ and

$\langle w_1, v_3 \rangle = 3 + 0 + 1 = 4.$

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Now, $w_3 = v_3 - \frac{\langle v_3, w_1 \rangle}{\|w_1\|^2} w_1 - \frac{\langle v_3, w_2 \rangle}{\|w_2\|^2} w_2$

$= (3,2,1) - \frac{4}{2}(1,0,1) - 0$

$= (3,2,1) - 2(1,0,1) - 0$

$= (1,0,-1)$ orthonormal

$\therefore \|w_3\|^2 = 2 \quad \frac{w_3}{\|w_3\|} = \frac{w_1}{\|w_1\|} = \frac{(1,0,1)}{\sqrt{2}}$

$= \frac{1}{\sqrt{2}}, \frac{0}{\sqrt{2}}$

$\therefore$ the orthogonal basis is $\{(1,0,1), (0,3,0), (1,0,-1)\}$

Hence the orthonormal basis is

$\left\{\left[\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right], (0,1,0), \left[\frac{1}{\sqrt{2}}, 0, \frac{-1}{\sqrt{2}}\right]\right\}$

Problem 2 :-

Let V be the set of all polynomials of degree ≤ 2 together with the zero Polynomial. V is a real inner product space with inner product defined by $\langle f, g \rangle = \int_{-1}^{1} f(x) g(x) dx$. Starting with the basis $\{1, x, x^2\}$, obtain an orthonormal basis for V.

Solution:-

Let $v_1 = 1$; $v_2 = x$ and $v_3 = x^2$

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Let $w_1 = v_1$.

Then $\|w_1\|^2 = \langle w_1, w_1 \rangle$

$$= \int_{-1}^{1} 1 \, dx = 2.$$

Hence $\|w_1\| = \sqrt{2}$

$$w_2 = v_2 - \frac{\langle v_2, w_1 \rangle}{\|w_1\|^2} w_1$$

$$= x - \frac{1}{2} \int_{-1}^{1} x \, dx$$

$$\boxed{w_2 = x}$$

$$\|w_2\|^2 = \langle w_2, w_2 \rangle = \int_{-1}^{1} x^2 dx.$$

$$= 2/3.$$

Now $w_3 = v_3 - \frac{\langle v_3, w_1 \rangle}{\|w_1\|^2} w_1 - \frac{\langle v_3, w_2 \rangle}{\|w_2\|^2} w_2$

$$= x^2 - \frac{1}{2} \int_{-1}^{1} x^2 dx - \left[\frac{3x}{2}\right] \int_{-1}^{1} x^3 dx$$

$$= x^2 - 1/3.$$

$$\|w_3\|^2 = \langle w_3, w_3 \rangle = \int_{-1}^{1} \left(x^2 - \frac{1}{3}\right)^2 dx$$

$$= \frac{8}{45}$$

Hence the orthogonal basis is $\{1, x, x^2 - 1/3\}$.

∴ The required orthonormal basis is

$$\left\{ \frac{1}{\sqrt{2}}, \frac{\sqrt{3}}{\sqrt{2}} x, \frac{\sqrt{10}}{4} (3x^2 - 1) \right\}$$

$\frac{x^2 - 1/3}{\sqrt{8/45}} \Rightarrow \frac{3x^2 - 1}{\sqrt{8}/\sqrt{45}}$

$\frac{8}{45}$

$\langle v_2, w_1 \rangle = \int_{-1}^{1} x \cdot 1 \, dx$

$= \left[\frac{x^2}{2}\right]_{-1}^{1}$

$= [1/2 - 1/2]$

$= 0$

$3^2 + 1^2 = 10$

$x^4 + 1/9 - \frac{2x^2}{3}$

$\left[\frac{x^5}{5} + x/9 - \frac{2x^3}{9}\right]_{-1}^{1}$

$[1/5 + 1/9 - 2/9] - [-1/5 - 1/9 + 2/9]$

$= [1/5 + 1/9] - [-1/5 + 1/9]$

$1/5 - 1/9 + 1/5 - 1/9 = 2/5 - 2/9 = \frac{18 - 10}{45} = \frac{8}{45}$

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Problem 3:-

Find a vector of unit length which is orthogonal to $(1,3,4)$ in $V_3(R)$ with standard inner product.

Solution:-

Let $x=(x_1,x_2,x_3)$ be any vector orthogonal to $(1,3,4)$. Then $x_1+3x_2+4x_3=0$

Any solution of this equation gives a vector orthogonal to $(1,3,4)$. For example $x=(1,1,-1)$ is orthogonal to $(1,3,4)$

$x_1+3x_2+4x_3=0$

$x_1=1$

$x_2=1$

$1+3+4x_3=0$

$4+4x_3=0$

$\boxed{x_3=-1}$

$(1,1,-1)$

Also $\|x\|=\sqrt{3}$. Hence a unit vector orthogonal to $(1,2,3)$ is given by $\left(\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right)$.

Note:-

The set of all vectors orthogonal to $(1,3,4)$ are the points lying on the plane $x+3y+4z=0$, which is a two dimensional subspace of $V_3(R)$

Problem 4:-

Find an orthogonal basis containing the vector $(1,3,4)$ for $V_3(R)$ with

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the standard inner product.

Solution:

$(4,1,-1)$ is a vector orthogonal to $(1,3,4)$ (refer problem 3 above).

Now, let $y=(y_1,y_2,y_3)$ be a vector orthogonal to both $(1,3,4)$ and $(1,1,-1)$.

Then $y_1+3y_2+4y_3=0$

$y_1+y_2-y_3=0$

Any solution of this system of equations gives a vector orthogonal to $(1,3,4)$ and $(1,1,-1)$

For example $(7,-5,2)$ is one such vector (by cross multiplication method).

Hence $\{(1,3,4),(1,1,-1),(7,-5,2)\}$ is an orthogonal basis containing $(1,3,4)$

6.3 Orthogonal Complement.

Definition

Let V be an inner product space. Let S be a subset of V. The orthogonal complement of S, denoted by $S^{\perp}$, is the set of all vectors in V.V.I which are orthogonal to every vector of S

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i.e) $S^{\perp} = \{x / x \in V \text{ and } \langle x, u \rangle = 0 \text{ for all } u \in S\}$

S^{-} - complement of S

$S^{\perp}$ - orthogonal complement of S

Examples

1. $V^{\perp} = \{0\}$ and $\{0\}^{\perp} = V$ since 0 is the only vector which is orthogonal to every vector.

2. Let $S = \{(x, 0, 0) / x \in R\} \subseteq V_3(R$ with standard inner product. Then

$$S^{\perp} = \{(0, y, z) / y, z \in R\}.$$

i.e) The orthogonal complement of the x-axis is the yz-plane.

Theorem 6.5

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If S is any subset of V then $S^{\perp}$ is a subspace of V. In the end must read 5.3 Theorem

Proof:- clearly $0 \in S^{\perp}$ and hence $S^{\perp} \neq \phi$

Now, let $x, y \in S^{\perp}$ and $\alpha, \beta \in F$.

Then $\langle x, u \rangle = \langle y, u \rangle = 0$ for all $u \in S$.

$\alpha(0) + \beta(0) = 0$

$\langle \alpha x + \beta y, u \rangle = \alpha \langle x, u \rangle + \beta \langle y, u \rangle =$

$\langle \alpha x, u \rangle + \langle \beta y, u \rangle$

$\alpha \langle x, u \rangle + \beta \langle y, u \rangle$ for all $u \in S$

$\alpha x + \beta y \in S^{\perp}$

$\alpha(0) + \beta(0) = 0$

$\therefore \alpha x + \beta y \in S^{\perp}$. Hence $S^{\perp}$ is a subspace of

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Theorem 6.6.

Let V be a finite dimensional inner product space. Let W be a subspace of V. Then V is the direct sum of W and $W^{\perp}$ (i.e) $V = W \oplus W^{\perp}$

direct sum $V = A + B$

Proof :-

We shall prove that $A \cap B = \{0\}$

We Want to Prove that

i) $W \cap W^{\perp} = \{0\}$ and
ii) $W + W^{\perp} = V$

i) Let $v \in W \cap W^{\perp}$. Then $v \in W$ and $v \in W^{\perp}$.

Now, $v \in W^{\perp} \Rightarrow v$ is orthogonal to every vector in W.

In particular, v is orthogonal to itself.

$\therefore \langle v, v \rangle = 0$ and hence $v = 0$.

Hence $W \cap W^{\perp} = \{0\}$

ii) Let $\{v_1, v_2, \ldots, v_r\}$ be an orthonormal basis for W. Let $v \in V$

consider..

$$v_0 = v - \langle v, v_1 \rangle v_1 - \langle v, v_2 \rangle v_2, \ldots, -(v, v_r) v_r$$

$$\therefore \langle v_0 v_i \rangle = \langle v, v_i \rangle - \langle v, v_i \rangle \langle v_i, v_i \rangle$$

(since $\langle v_i, v_j \rangle = 0$ if $i \neq j$)

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$= \langle v, v_i \rangle - \langle v, v_i \rangle$

(since $\langle v_i, v_i \rangle = 1$)

$= 0$

$\therefore v_0$ is orthogonal to each of $v_1, v_2, \ldots, v_r$ and hence is orthogonal to every vector in W. Hence $v_0 \in W^\perp$ and

$$v = [\langle v, v_1 \rangle v_1 + \langle v, v_2 \rangle v_2 + \ldots + \langle v, v_r \rangle v_r] + v_0 \in W + W^\perp$$

$\therefore V = W \oplus W^\perp$

Hence the theorem.

corollary.

$$\dim V = \dim W + \dim W^\perp$$

Proof.

$$\dim V = \dim (W \oplus W^\perp)$$
$$= \dim W + \dim W^\perp$$

Theorem 6.7

Let V be a finite dimensional inner product space. Let W be a subspace of V. Then $(W^\perp)^\perp = W$.

Proof

Let $w \in W$. Then for any $u \in W^\perp$, $\langle w, u \rangle = 0$

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Hence $w \in (W^{\perp})^{\perp}$. Thus

$W \subseteq (W^{\perp})^{\perp} \longrightarrow ①$

Now by Theorem 6.6, $V = W \oplus W^{\perp}$.

Also $V = W^{\perp} \oplus (W^{\perp})^{\perp}$.

Hence $\dim W = \dim (W^{\perp})^{\perp} \longrightarrow ②$

From ① and ② we get $W = (W^{\perp})^{\perp}$

Problem 1.

Let V be an inner product space and let S_1 and S_2 be subsets of V. Then $S_1 \subseteq S_2 \Rightarrow S_2^{\perp} \subseteq S_1^{\perp}$

$S^{\perp} = \{x / x \in V \text{ and } \langle x, u \rangle = 0 \; \forall u \in S\}$

$S_1 \subseteq S_2$

S_2 ഒരവ്യൂ set

S_1 ചെറിയ set

Solution

Let $u \in S_2^{\perp}$

Then $\langle u, v \rangle = 0$ for all $v \in S_2$.

But $S_1 \subseteq S_2$. Hence $\langle u, v \rangle = 0$ for all $v \in S_1$

Hence $u \in S_1^{\perp}$. Thus $S_2^{\perp} \subseteq S_1^{\perp}$.

Problem 2.

Let W_1 and W_2 be subspaces of a finite dimensional inner product space. Then

i) $(W_1 + W_2)^{\perp} = W_1^{\perp} \cap W_2^{\perp}$

ii) $(W_1 \cap W_2)^{\perp} = W_1^{\perp} + W_2^{\perp}$.

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i) We know that $W_1 \subseteq W_1 + W_2$

$\therefore (W_1 + W_2)^{\perp} \subseteq W_1^{\perp}$ (by the Problem)

Similarly, $(W_1 + W_2)^{\perp} \subseteq W_2^{\perp}$

Hence $(W_1 + W_2)^{\perp} \subseteq W_1^{\perp} \cap W_2^{\perp}$ ⟶ ①

Now, let $w \in W_1^{\perp} \cap W_2^{\perp}$

Then $w \in W_1^{\perp}$ and $w \in W_2^{\perp}$

$\therefore \langle w, u \rangle = 0$ for all $u \in W_1$ and W_2

Now, let $v \in W_1 + W_2$

Then $v = v_1 + v_2$ where $v_1 \in W_1$ and $v_2 \in W_2$

$$\therefore \langle w, v \rangle = \langle w, v_1 + v_2 \rangle$$
$$= \langle w, v_1 \rangle + \langle w, v_2 \rangle$$
$$= 0 + 0 \quad (\text{since } v_1 \in W_1 \text{ and } v_2 \in W_2)$$
$$= 0$$

Hence $w \in (W_1 + W_2)^{\perp}$

$\therefore W_1^{\perp} \cap W_2^{\perp} \subseteq (W_1 + W_2)^{\perp}$ ⟶ ②

From ① and ② we get

$$(W_1 + W_2)^{\perp} = W_1^{\perp} \cap W_2^{\perp}$$

ii) Proof is similar to that of (i)

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Examples:-

1) Let V be an inner product space Let S be any subspace of V. Then show that $S^{\perp} = [L(S)]^{\perp}$.

2) Find a basis for the orthogonal complement of the subspace spanned by $(2, 1, -2)$ in $V_3(R)$.

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26/2/2020

Unit - IV

Theory of Matrices.

Algebra of Matrices.

We have already seen that a matrix A is an array of mn numbers a_{ij} where $1 \leq i \leq m$, $1 \leq j \leq n$ arranged in m rows and n columns as follows:

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

We shall denote this matrix by the symbol (a_{ij}). If $m = n$, A is called a square matrix of order n.

Definition:-

Two matrices $A = (a_{ij})$ and $B = (b_{ij})$ are said to be equal if A and B have the same number of rows and columns and the corresponding entries in the two matrices are same.

Addition of Matrices.

We have already defined the addition of two $m \times n$ matrices $A = (a_{ij})$ and $B = (b_{ij})$ by $A + B = (a_{ij} + b_{ij})$

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We note that we can add two matrices iff they have the same number of rows and columns.

Example.

If $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 9 & 5 \end{bmatrix}$ and $\begin{bmatrix} 0 & 4 \\ 2 & 1 \\ -1 & 0 \end{bmatrix}$ then

$$A+B = \begin{bmatrix} 1 & 6 \\ 5 & 5 \\ 8 & 5 \end{bmatrix}$$

Remark.

The set of all mxn matrices is an abelian group under matrix addition. The mxn matrix with each entry 0 is the zero matrix and is denoted by 0 and the additive inverse of matrix $A = (a_{ij})$ is $(-a_{ij})$ and is denoted by $-A$.

If $A = (a_{ij})$ is any matrix and α is any number (real or complex) we have defined the matrix αA by

$$\alpha A = (\alpha a_{ij})$$

The set of all mxn matrices over the field R under matrix addition and scalar multiplication define above is a vector space. This result is true if R is

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replaced by $\mathbb{C}$ or by any field F

We now proceed to define multiplication of matrices. We have already defined the multiplication of 2×2 matrices. which we generalise in the following definition

Definition.

Let $A = (a_{ij})$ be an $m \times n$ matrix and $B = (b_{ij})$ be an $n \times p$ matrix. We define he product AB as the $m \times p$ matrix (a_{ij}) where the ij^{th} entry c_{ij} is given by

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}$$

$$= \sum_{k=1}^{n} a_{ik}b_{kj}$$

Note 1:

The product AB of two matrices is defined only when the number of columns of A is equal to the number of rows of B

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Note 2:

The entry a_{ij} of the product AB is found by multiplying i^{th} row of A and the j^{th} column of B. To multiply a row and a column, we multiply the corresponding entries and add.

Examples.

1) Let $A = \begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & 2 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix}$ and

$B = \begin{bmatrix} 1 & 1 \\ 1 & 5 \\ 3 & 2 \\ 1 & 0 \end{bmatrix}$. A is a 3×4 matrix and B is a 4×2 matrix. Hence the product AB is a 3×2 matrix and

$$AB = \begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & 2 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 5 \\ 3 & 2 \\ 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 10 & 5 \\ 6 & 12 \\ 2 & 1 \end{bmatrix}$$

Note that in this example the product BA is not defined. Even if the product BA is defined, AB need not be equal to BA.

2) Let $A = \begin{bmatrix} 2 & 4 & 0 \\ 9 & 3 & 1 \\ 4 & 7 & 2 \end{bmatrix}$ and

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$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Then

$AI = IA = A$. (verify)

3) Consider the square matrix of order n given by

$$I_n = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$$

Let A be any mxn matrix. Then $I_n A = A$. Also if A is an mxn matrix, $AI_n = A$. If A is any mxn matrix, $AI_n = I_n A = A$.

I_n is called the identity matrix of order n. We shall denote the identity matrix of any order by the symbol I.

Solved problems:-

Problem 1:

Show that the matrix $A = \begin{bmatrix} 2 & -3 & 1 \\ 3 & 1 & 3 \\ -5 & 2 & -4 \end{bmatrix}$ satisfies the equation $A(A-I)(A+2I) = 0$.

$$A - I = \begin{bmatrix} 2 & -3 & 1 \\ 3 & 1 & 3 \\ -5 & 2 & -4 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

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$$= \begin{bmatrix} 1 & -3 & 1 \\ 3 & 0 & 3 \\ -5 & 2 & -5 \end{bmatrix}$$

$$A+2I = \begin{bmatrix} 4 & -3 & 1 \\ 3 & 3 & 3 \\ -5 & 2 & -2 \end{bmatrix}$$

Now,

$$A(A-I)(A+2I) = \begin{bmatrix} 2 & -3 & 1 \\ 3 & 1 & 3 \\ -5 & 2 & -4 \end{bmatrix} \begin{bmatrix} 1 & -3 & 1 \\ 3 & 0 & 3 \\ -5 & 2 & -5 \end{bmatrix} \begin{bmatrix} 4 & -3 & 1 \\ 3 & 3 & 3 \\ -5 & 2 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} -12 & -4 & -12 \\ -9 & -3 & -9 \\ 21 & 7 & 21 \end{bmatrix} \begin{bmatrix} 4 & -3 & 1 \\ 3 & 3 & 3 \\ -5 & 2 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= 0.$$

Hence $A(A-I)(A+2I) = 0$

Problem 2:

Prove that $\begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}^n = \begin{bmatrix} \lambda^n & n\lambda^{n-1} \\ 0 & \lambda^n \end{bmatrix}$

we prove this result by induction on n.

When $n=1$ result is obviously true

Let us assume that the result is true for $n=k$.

$$\therefore \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}^k = \begin{bmatrix} \lambda^k & k\lambda^{k-1} \\ 0 & \lambda^k \end{bmatrix}$$

$$\therefore \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}^k \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} \lambda^k & k\lambda^{k-1} \\ 0 & \lambda^k \end{bmatrix} \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$$

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$$\begin{bmatrix} \lambda^{k+1} & \lambda^k + k\lambda^k \\ 0 & \lambda^{k+1} \end{bmatrix} = \begin{bmatrix} \lambda^{k+1} & (k+1)\lambda^k \\ 0 & \lambda^{k+1} \end{bmatrix}$$

k இருக்க வேண்டிய இடத்தில் எல்லாம் k+1 இருக்கும் அப்போ இது true ஆகும்.

$\therefore$ The result is true for $n = k+1$

Hence the result is true for all Positive integers n.

Theorem 7.1

10/3/2020

Let A be an $m \times n$ matrix, B an $n \times p$ matrix and C a $p \times q$ matrix. Then $A(BC) = (AB)C$. Associative

Proof:-

Let $A = (a_{ij})$, $B = (b_{ij})$ and $C = (c_{ij})$, Let us find the rs^{th} entry in $A(BC)$

The r^{th} row in A is $a_{r1}, a_{r2}, \ldots, a_{rn}$

The s^{th} column in BC consists of the elements $\sum b_{1j} c_{js}, \ldots, \sum b_{nj} c_{js}$.

Hence the rs^{th} entry in $A(BC)$ is

$$a_{r1} \sum b_{1j} c_{js} + \ldots + a_{rn} \sum b_{nj} c_{js}$$

$$= \sum_{i=1}^{n} a_{ri} \sum_{j=1}^{p} b_{ij} c_{js}$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{p} a_{ri} b_{ij} c_{js}$$

$A \rightarrow m \times n$ matrix
$B \rightarrow n \times p$ matrix
$C \rightarrow p \times q$ matrix
$AB \Rightarrow m \times p$ matrix
$BC \rightarrow n \times q$ matrix

Let us now find the rs^{th} entry in $(AB)C$

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The r^{th} row in AB is.

$$\sum a_{ri}b_{i1}, \sum a_{ri}b_{i2}, \ldots\ldots, \sum a_{ri}b_{ip}$$

The s^{th} column in C is $c_{1s}, c_{2s}, \ldots, c_{ps}$.

Hence the rs^{th} entry in $(AB)C$ is

$$(\sum a_{ri}b_{i1})c_{1s} + (\sum a_{ri}b_{i2})c_{2s} + \ldots\ldots +$$

$$(\sum a_{ri}b_{ip})c_{ps} = \sum_{i=1}^{n}\sum_{j=1}^{p} a_{ri}b_{ij}c_{js}.$$

Thus $A(BC) = (AB)C$

Theorem 7.2.

Let U, V, W be vector space of dimension m, n and p respectively over a field F with respective bases $\{u_1, u_2, \ldots, u_m\}, \{v_1, v_2, \ldots, v_n\}$ and $\{w_1, w_2, \ldots, w_p\}$. Let $T_1 : V \to V$ and $T_2 : V \to W$ be linear transformations and $M(T_1)$ and $M(T_2)$ their corresponding matrices with respect to these bases.

Then $M(T_2 \circ T_1) = M(T_1)M(T_2)$

($\circ$: composite)

Proof:- $M(T_1)$ and is an $m \times n$ matrix and $M(T_2)$ is an $n \times p$ matrix. Hence the Product $M(T_1)M(T_2)$ is defined and is an $m \times p$ matrix.

Let $M(T_1) = (a_{ij})$ $M(T_2) = (b_{ij})$

Then. $T_1(u_i) = \sum_{j=1}^{n} a_{ij}v_j$ and

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$$T_2(v_j) = \sum_{k=1}^{p} b_{jk} w_k$$

$$\therefore (T_2 \circ T_1)(u_i) = T_2\left(\sum_{j=1}^{n} a_{ij} v_j\right)$$

$$= \sum_{j=1}^{n} a_{ij} T_2(v_j)$$

$$= \sum_{j=1}^{n} a_{ij} \sum_{k=1}^{p} b_{jk} w_k$$

Thus $M(T_2 \circ T_1) = M(T_1) M(T_2)$

Note1:-

Thus multiplication of two matrices is equivalent to the composition of their corresponding linear transformations in the reverse order. Since composition of linear transformation is associative we get matrix multiplication is associative

Note 2:-

Let $M_n(F)$ denote the set of all square matrices of order n over the field F. Then matrix multiplication is an associative binary operation on $M_n(F)$ If $A, B, C \in M_n(F)$ the two distributive laws

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$A(B+C) = AB+AC$ and $(A+B)C = AC+BC$ can be verified.

Since $M_n(F)$ is already an abelian group under matrix addition we see that $M_n(F)$ is a ring.

Definition:-

Let $A = (a_{ij})$ be an $m \times n$ matrix. Then the $n \times m$ matrix $B = (b_{ij})$ where $b_{ij} = a_{ji}$ is called the transpose of the matrix A and it is denoted by A^T. Thus A^T is obtained from the matrix A by interchanging its rows and columns and the

$(i,j)^{th}$ entry of A^T = $(j,i)^{th}$ entry of A.

For example, if $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 0 & 1 \\ 0 & 3 & 1 & 5 \end{bmatrix}$ then

$$A^T = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 3 \\ 3 & 0 & 1 \\ 4 & 1 & 5 \end{bmatrix}$$

clearly if A is an $m \times n$ matrix, then A^T is an $n \times m$ matrix.

Theorem 7.3:-

Let A and B be two $m \times n$ matrices

Then i) $(A^T)^T = A$.

ii) $(A+B)^T = A^T + B^T$.

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i) The $(i,j)^{th}$ entry of $(A^T)^T$.

$= (j,i)^{th}$ entry of A^T

$= (i,j)^{th}$ entry of A.

$\therefore (A^T)^T = A$.

ii) The $(i,j)^{th}$ entry of $(A+B)^T$.

$= (j,i)^{th}$ entry of $A+B$

$= (j,i)^{th}$ entry of $A+(j,i)^{th}$ entry of B.

$= (i,j)^{th}$ entry of $A^T +$ $(i,j)^{th}$ entry of B^T.

$= (i,j)^{th}$ entry of $(A^T + B^T)$

$(A+B)^T = A^T + B^T$.

Theorem 7.4.

Let A be an $m \times n$ matrix and B be an $n \times p$ matrix. Then $(AB)^T = B^T A^T$.

Proof:-

By hypothesis AB is defined and it is an $m \times p$ matrix. Hence $(AB)^T$ is a $p \times m$ matrix.

Futher B^T is a $p \times n$ matrix and A^T is an $n \times m$ matrix..

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Hence, the product $B^T A^T$ is defined and it is a $p \times m$ matrix.

Now, let $A = (a_{ij})$, $B = (b_{ij})$ and

$$(AB) = (c_{ij})$$

The $(i,j)^{th}$ entry of $(AB)^T$ =

$$AB = c_{ij} = \sum_{K=1}^{n} a_{iK} b_{Kj}.$$

$\therefore$ The $(i,j)^{th}$ entry of $(AB)^T = c_{ij}$

$$= \sum_{K=1}^{n} a_{jK} b_{Ki}.$$

Now the i^{th} row of B^T is the i^{th} column of B and it consists of the elements $b_{1i}, b_{2i}, \ldots\ldots, b_{ni}$. Also the j^{th} column of A^T is the j^{th} row of A and it consists of the elements $a_{j1}, a_{j2}, \ldots\ldots, a_{jn}$.

Hence the $(i,j)^{th}$ entry of $B^T A^T = b_{1i} a_{j1} + b_{2i} a_{j2} + \ldots\ldots + b_{ni} a_{jn}$

$A = m \times n \Rightarrow A^T = n \times m$

$B = n \times p \Rightarrow B^T = p \times n$

$$= \sum_{K=1}^{n} b_{Ki} a_{jK}$$

$B^T A^T = p \times m$

$= (i,j)^{th}$ entry of $(AB)^T$.

Hence $(AB)^T = B^T A^T$.

Definition:-

Let $A = (a_{ij})$ be a matrix with entries from the field of complex numbers. The conjugate of A, denoted by $\overline{A}$, is defined by $\overline{A} = (\overline{a_{ij}})$

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$\overline{A}^T$ is called the conjugate transpose of the matrix A.

For example.

if $A = \begin{bmatrix} 2 & 2+i & -i \\ 1+i & -3 & 4+3i \end{bmatrix}$ then

$$\overline{A} = \begin{bmatrix} 2 & 2-i & i \\ 1-i & -3 & 4-3i \end{bmatrix}$$

$$\overline{A}^T = \begin{bmatrix} 2 & 1-i \\ 2-i & -3 \\ i & 4-3i \end{bmatrix}$$

Theorem 7.5:

Let A and B be matrices with entries from C. Then.

(i) $\overline{(\overline{A})} = A$.

(ii) $\overline{A+B} = \overline{A} + \overline{B}$

(iii) $\overline{kA} = \overline{k}\,\overline{A}$, where $k \in C$.

(iv) $A = \overline{A} \iff$ all entries of A are real.

(v) $\overline{AB} = \overline{A}\,\overline{B}$ provided AB is defined.

(vi) $(\overline{A})^T = \overline{A^T}$

The proof of the above results are immediate consequences of the corresponding properties of complex numbers.

7.2 Types of Matrices. 12/2/2020

Definition:-

An 1xn matrix is called a row matrix.

Thus a row matrix consists of 1 row and n columns.

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It is of the form $(a_{11}, a_{12}, a_{13}, \ldots, a_{1n})$.

Definition:-

An $m \times 1$ matrix is called a column matrix. Thus a column matrix consists of m rows and 1 columns and it is of the form $\begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}$

Definition:-

Let $A = (a_{ij})$ be a square matrix. Then the elements $a_{11}, a_{22}, \ldots, a_{nn}$ are called the diagonal elements of A and the diagonal elements constitute what is known as the principal diagonal of the matrix A. A square matrix is called a diagonal matrix if all the entries which do not belong to the principal are zero. Hence in a diagonal matrix $a_{ij} = 0$ if $i \neq j$.

For example $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ is a diagonal matrix.

Definition:-

A diagonal matrix in which all the entries of the principal diagonal are equal is called a scalar matrix.

For example $\begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$ is a scalar matrix

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Definition:-

A Square matrix (a_{ij}) is called upper triangular matrix if all the entri above the principal diagonal are zero.

Hence $a_{ij}=0$ whenever $i<j$ in an upper triangular matrix. $\begin{bmatrix} 0 & 0 & 3 \\ 0 & 4 & 5 \\ 3 & 2 & 1 \end{bmatrix}$

Definition:-

A Square matrix (a_{ij}) is called a lo triangular matrix if all the entries below the principal diagonal are zero.

Hence $a_{ij}=0$ whenever $i>j$ in an lower triangular matrix.

For example, $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{bmatrix}$ is lower triangular. $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 2 & 3 & 0 \\ 2 & 3 & 2 & 4 \end{bmatrix}$ is upper triangular

clearly a Square matrix is a diagonal matrix iff it is both lower triangular and upper triangular.

Definition:-

A Square matrix $A=(a_{ij})$ is said to be Symmetric if $a_{ij}=a_{ji}$ forall i,j.

Transposes

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Example:-

$\begin{bmatrix} a & b \\ b & a \end{bmatrix}, \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}, \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 0 & 0 & 5 \\ 3 & 0 & 6 & 7 \\ 4 & 5 & 7 & 8 \end{bmatrix}$ are

symmetric matrices.

Theorem : 7.6

A square matrix A is symmetric

iff $A = A^T$.

Proof:- Let A be a symmetric matrix.

Then the $(i,j)^{th}$ entry of A

= $(j,i)^{th}$ entry of A. (symmetric)

$(i,j)^{th}$ entry of A = $(i,j)^{th}$ entry of A^T.

Hence $A = A^T$.

conversely let $A = A^T$.

Then $(i,j)^{th}$ entry of A.

Symmetric means = $(i,j)^{th}$ entry of A^T.

$(j,i)^{th}$ entry of A. = $(j,i)^{th}$ entry of A.

$a_{ij} = a_{ji}$

Hence A is symmetric.

Theorem : 7.7

12/3/2020

Let A be any square matrix

Then $A + A^T$ is symmetric

Proof:-

$(A + A^T)^T = A^T + (A^T)^T$

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$= A^T + A$.

$= A + A^T$.

Hence $A + A^T$ is symmetric

$A = A^T$

$(A+A^T)^T = A^T + (A^T)^T$

$= A^T + A$

$A + A^T$ is symmetric

Theorem: 7.8

Let A and B be symmetric matrices of order n. Then

(i) $A+B$ is symmetric.

(ii) AB is symmetric iff $AB = BA$.

(iii) $AB+BA$ is symmetric

(iv) If A is symmetric, then KA is symmetric where $K \in F$.

Proof:-

(i) $(A+B)^T = A^T + B^T$ — transpose எடுக்க வேண்டும்

= $A+B$ (since A and B are symmetric) — இதில் Transpose-ஐ கணக்கில் எடுத்துக் கூடாது

$\therefore A+B$ is symmetric.

(ii) AB is symmetric

$\iff (AB)^T = AB$.

$\iff B^T A^T = AB$ (by theorem 7.4) ↓ condition

$\iff BA = AB$

$(AB)^T = B^T A^T$

(iii) $(AB+BA)^T = (AB)^T + (BA)^T$

$= B^T A^T + A^T B^T$.

$= BA + AB$ (since A and B are symmetric)

$= AB + BA$.

$\therefore AB+BA$ is symmetric.

(iv) $(KA)^T = KA^T = KA$ (since A is symmet

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$\therefore$ kA is symmetric.

Definition:-

A square matrix $A = (a_{ij})$ is said to be skew symmetric if $a_{ij} = -a_{ij}$, for all i, j.

$$A = -A^T$$

Note: Let A be a skew symmetric matrix. Then $a_{ii} = -a_{ii}$. Hence $2a_{ii} = 0$ (ie) $a_{ii} = 0$ for all i. Thus in a skew symmetric matrix all the diagonal entries are zero

$\begin{bmatrix} 0 & -a \\ a & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & -2 & 1 \\ 2 & 0 & -3 \\ -1 & 3 & 0 \end{bmatrix}$ are examples of skew symmetric matrices.

$(i,j)^{th}$ entry of A $= -(j,i)^{th}$ entry of A.
$= -(i,j)^{th}$ entry of A^T
$A = -A^T$

Theorem : 7.9.

A square matrix A is skew symmetric matrix iff $A = -A^T$.

Proof. is similar to that of Theorem 7.6.

Theorem : 7.10

Let A be any square matrix. Then $A - A^T$ is skew symmetric.

$A = A^T \rightarrow$ Symmetric
$A^T = A$
$\hookrightarrow$ Skew symmetric

Proof:-

$$(A - A^T)^T = A^T - (A^T)^T$$
$$= A^T - A.$$
$$= -(A - A^T)$$

Hence $A - A^T$ is skew symmetric.

Theorem : 7.11.

Any square matrix A can be expressed uniquely as the (sum / addition) of a symmetric matrix and a skew symmetric

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matrix.

Proof:-

Let A be any square matrix.

Then $A+A^T$ is a symmetric matrix (by Theorem 7.7).

let A be any square matrix
Then $A+A^T$ is symmetric

$\therefore \frac{1}{2}(A+A^T)$ is also a symmetric matrix

Also, $\frac{1}{2}(A-A^T)$ is a skew symmetric matrix (by Theorem 7.10)

$= \frac{1}{2}A + \frac{1}{2}A^T + \frac{1}{2}A - \frac{1}{2}A^T$

$= \frac{1}{2}A + \frac{1}{2}A$

$= A$

Now, $A = \frac{1}{2}(A+A^T) + \frac{1}{2}(A-A^T)$

$\therefore$ A is the sum of a symmetric matrix and a skew symmetric matrix.

Now, to prove the uniqueness,

let $A = S+R$ where S is a symmetric matrix and R is a skew symmetric matrix

we claim that $S = \frac{1}{2}(A+A^T)$ and $R = \frac{1}{2}(A-A^T)$.

$A - A^T = S + R - S + R$

$= 2R$

$R = \frac{1}{2}(A-A^T)$

$A = S+R \rightarrow (1)$

$A^T = S-R \rightarrow (2)$

$A^T = (S+R)^T$

$= S^T + R^T$

$= S - R$ (Since S is symmetric and R is skew symmetric)

$\rightarrow S = \frac{1}{2}(A+A^T)$

$A+A^T = S+R+S-R$

$A+A^T = 2S$

S addition (1) and (2)

R subtract (1) and (2)

Condition

Symmetric = S.
Skew Symmetric = −R.

$\therefore A^T = S-R$

From (1) and (2) we get

$S = \frac{1}{2}(A+A^T)$ and $R = \frac{1}{2}(A-A^T)$.

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Theorem 7:12.

Let A and B be skew symmetric matrices of order n. Then.

i) A+B is skew symmetric

ii) KA is skew symmetric, where $K \in F$.

iii) A^{2n} is a symmetric matrix and A^{2n+1} is a skew symmetric matrix where n is any positive integer. skew symmetric $A = -A^T$

Proof:-

let A, B be skew symmetric.

i) $(A+B)^T = A^T + B^T$.

$= -A - B$ (by Theorem 7.9) $A^T = -A$, $B^T = -B$

$= -(A+B)$

$\therefore A+B$ is skew symmetric.

ii) proof is similar to that of (i) $(kA)^T = kA^T$

iii) Let m be any positive integer $(kA)^T = k(-A)$, $(kA)^T = -kA$.

Then $(A^m)^T = (AA \ldots \text{m times})^T$.

$= A^T A^T \ldots A^T$ (m times)

$= (-A)(-A) \ldots (-A)$ (m times)

(since $A^T = -A$)

$= (-1)^m A^m$.

$\therefore (A^m)^T = \begin{cases} A^m & \text{if } m \text{ is even} \\ -A^m & \text{if } m \text{ is odd.} \end{cases}$

A^m is symmetric when m is even and skew symmetric when m is odd.

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Definition:-

A square matrix $A=(a_{ij})$ is said to be a Hermitian matrix if $a_{ij}=\overline{a_{ji}}$ for all i,j. A is said to be a skew Hermitian matrix iff $a_{ij}=-\overline{a_{ji}}$, for all i,j

$(\overline{A})^T=(\overline{A^T})$

$A=-(\overline{A})^T$

Examples:-

$\begin{bmatrix} 1 & -1+2i & 3+4i \\ -1-2i & -2 & 3 \\ 3-4i & 3 & 2 \end{bmatrix}$ is a Hermitian matrix

$\begin{bmatrix} 0 & -a+ib \\ a+ib & 0 \end{bmatrix}$, $\begin{bmatrix} ib & c+id \\ -c+id & ib \end{bmatrix}$ are Skew Hermitian matrices

$A=\begin{bmatrix} 0 & -a+ib \\ a+ib & 0 \end{bmatrix}$

$\overline{A}=\begin{bmatrix} 0 & -a-ib \\ a-ib & 0 \end{bmatrix}$ $\Leftarrow$ $\overline{A}^T=\begin{bmatrix} 0 & a-ib \\ -a-ib & 0 \end{bmatrix}$ $\Leftarrow$ $\overline{A}^T=\begin{bmatrix} 0 & -a-ib \\ a-ib & 0 \end{bmatrix}$

Note:

1) Any Hermitian matrix over R is a symmetric matrix and any skew Hermitian matrix over R is a skew symmetric matrix

2) Let $A=(a_{ij})$ be a Hermitian matrix. Then $a_{ii}=\overline{a_{ii}}$ and hence a_{ii} is real for all i.

3) Let $A=(a_{ij})$ be a skew Hermitian matrix. Then $a_{ii}=-\overline{a_{ii}}$, and hence $a_{ii}=0$ or purely imaginary for all i.

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Theorem : 7.13.

Let A be a square matrix.

i) A is Hermitian iff $A = \overline{A}^T$

ii) A is skew Hermitian iff $A = -\overline{A}^T$

Proof :-

The result is an immediate consequence of the definition

Theorem : 7.14

Let A and B be square matrices of the same order. Then.

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i) A, B are Hermitian $\Rightarrow A+B$ is Hermitian

ii) A, B are skew Hermitian $\Rightarrow A+B$ is skew Hermitian

iii) A is Hermitian $\Rightarrow iA$ is skew Hermitian

iv) A is skew Hermitian $\Rightarrow iA$ is Hermitian

v) A is Hermitian and K is real $\Rightarrow KA$ is Hermitian.

vi) A is skew Hermitian and K is real $\Rightarrow KA$ is skew Hermitian

vii) A, B are Hermitian $\Rightarrow AB+BA$ is Hermitian

viii) A, B are Hermitian $\Rightarrow AB-BA$ is skew Hermitian.

Proof :- We shall prove (i), (iii) and (vii)

i) $(\overline{A+B})^T = (\overline{A}+\overline{B})^T$

$= \overline{A}^T + \overline{B}^T$

$= A+B$ (since A and B are Hermiti

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$\therefore A+B$ is Hermitian.

iv) $(\overline{iA})^T = (-i\overline{A})^T$

$= -i\overline{A}^T$

$= -i(-A)$

$= iA$.

iii) $\overline{-(iA)}^T = (-i\overline{A})^T$

$A = -\overline{A}^T = i\overline{A}^T$

$= iA$ (since A is Hermitian)

$\therefore iA$ is skew Hermitian

vii) $(\overline{AB+BA})^T = (\overline{AB} + \overline{BA})^T$

$= (\overline{A}\,\overline{B} + \overline{B}\,\overline{A})^T$

$= (\overline{A}\,\overline{B})^T + (\overline{B}\,\overline{A})^T$

$(\overline{AB})^T = \overline{B}^T\overline{A}^T$

$= \overline{B}^T\overline{A}^T + \overline{A}^T\overline{B}^T$

$= BA + AB$

$= AB + BA$.

$\therefore AB+BA$ is Hermitian

Theorem 7:15

Let A be any square matrix

Then

i) $A + \overline{A}^T$ is Hermitian

ii) $A - \overline{A}^T$ is skew Hermitian.

Proof:-

i) Let $A + \overline{A}^T = B$

Then $\overline{B} = \overline{A} + A^T$

$\therefore \overline{B}^T = (\overline{A} + A^T)^T$

$= \overline{A}^T + A$.

$= B$

Hence $A + \overline{A}^T$ is Hermitian

$\overline{B} = \overline{A} - A^T$

$\overline{B}^T = (\overline{A} - A^T)^T$

$= \overline{A}^T - A$

$= -\overline{A}^T - A$

$\overline{B} = -(\overline{A}^T + A)$

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ii) Proof is similar to that of (i).

Theorem : 7.16

Any square matrix A can be uniquely expressed as the sum of a Hermitian matrix and a skew Hermitian matrix

Proof :-

The proof is similar to that of Theorem : 7.11.

Definition

A real square matrix A is said to be orthogonal if $\boxed{AA^T = A^TA = I}$.

Example.

$A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ is an orthogonal matrix (verify).

Theorem : 7.17.

Let A and B be orthogonal matrices of the same order. Then.

i) A^T is orthogonal. $\Rightarrow AA^T = A^TA = I$

ii) AB is orthogonal.

Proof :-

i) $A^T(A^T)^T = A^TA = I$

(since A is orthogonal).

Similarly we can prove $(A^T)^TA^T = I$.

$\therefore A^T$ is orthogonal

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ii) $(AB)(AB)^T = (AB)(B^T A^T)$

$= A(BB^T)A^T$

$= AIA^T$ (since B is orthogonal)

$= AA^T$

$= I$

Similarly $(AB)^T(AB) = I$

Hence AB is orthogonal.

Definition:-

A square matrix A is said to be an unitary matrix if $A\bar{A}^T = \bar{A}^T A = I$

For example $\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$ is unitary

Note:-

Any unitary matrix over R is an orthogonal matrix.

$AA^T = A^TA = I$ orthogonal

$A\bar{A}^T = \bar{A}^TA = I$ unitary matrix

Theorem: 7.18

If A and B are unitary matrices of the same order, then AB is also an unitary matrix.

Proof:-

Similar to the proof of ii) of Theorem: 7.17. $A\bar{A}^T$

1) Definition. Nilpotent.

A square matrix A is said to be nilpotent if $A^n = 0$ for some positive interger n.

A any matrix

Definition involutory.

A square matrix A is said to be involutory if $A^2 = I$

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7.3 The Inverse of a Matrix.

A 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ has an inverse iff $|A| = ad - bc \neq 0$ and the inverse of A is given by $\frac{1}{|A|}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$. such matrices are called non-singular. In this section we shall describe the method of finding the inverse of any non-singular matrix of order n.

$A^{-1} = \frac{1}{|A|}\,\text{adj}\,A$

modulus. தரம்பம் ஏன்று நாராண்

Determinants: We can associate with any nxn matrix $A = (a_{ij})$ over a field F an element of F given by the

determinant $\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$

Its value can be determined in the usual way and it is denoted by $|A|$. For example

i) If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then $|A| = ad - bc$.

ii) If $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 1 & 2 & 1 \end{bmatrix}$ then

$$|A| = \begin{vmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 1 & 2 & 1 \end{vmatrix} = 1$$

Definition:-

2 Mark A square matrix A is said to be singular if $|A| = 0$

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2 Mark A is called a non-singular matrix if $|A| \neq 0$

Remark:-

The rule for multiplying two matrices is same as the rule for multiplying two determinants.

Hence if A and B are two $n \times n$ matrices

$$|AB| = |A||B|$$

Theorem: 7:19

The product of any two non-singular matrices is non-singular.

Proof:-

Let A and B be two non-singular matrices of the same order.

Then $|A| \neq 0$ and $|B| \neq 0$

$$\therefore |AB| = |A||B| \neq 0$$

Hence AB is non-singular.

Note:- Addition-ல் ஆனால் Multiplying-ல் need not non-singular

Sum of two non-singular matrices need not be non-singular.

For, if A is any non-singular matrix then $-A$ is also a non-singular matrix and $A + (-A)$ is the zero matrix which

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is obviously a singular matrix.

Definition :-

Let $A = (a_{ij})$ be an $n \times n$ matrix. If we delete the row and the column containing the element a_{ij} we obtain a square matrix of order $n-1$ and the determinant of this square matrix is called the minor of the element a_{ij} and is denoted by M_{ij}

The minor M_{ij} multiplied by $(-1)^{i+j}$ is called the cofactor of the element a_{ij} and is denoted by A_{ij}

$$\therefore \boxed{A_{ij} = (-1)^{i+j} M_{ij}}$$

Examples

$$\text{let } A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

corresponding to the 9 elements a_{ij}, we get 9 minors of A. For example, the minor of a_{11} is.

$$M_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} \text{ and the minor of } a_{23} \text{ is}$$

$$M_{23} = \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix}$$

The cofactor of a_{11} is $A_{11} = (-1)^2 M_{11} = M_{11}$.

The cofactor of a_{23} is $A_{23} = (-1)^{2+3} M_{23}$

$$= -M_{23}$$

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Definition :-

let $A = (a_{ij})$ be a square matrix. Let A_{ij} denote the cofactor of a_{ij}. The transpose of the matrix (A_{ij}) is called the adjoint or adjugate of the matrix A and is denoted by adj A.

Thus the $(i,j)^{th}$ entry of adj A is A_{ji}.

$A = (a_{ij})$

A_{ij} = Co-factor of a_{ij}

Note :-

If A is a square matrix of order n then adj A is also a square matrix of order n.

Example :

let $A = \begin{bmatrix} 1 & 0 & 2 \\ 3 & 1 & -1 \\ -2 & 1 & 3 \end{bmatrix}$

Then $A_{11} = \begin{vmatrix} 1 & -1 \\ 1 & 3 \end{vmatrix} = 4$

$A_{12} = \begin{vmatrix} 3 & -1 \\ -2 & 3 \end{vmatrix} = -7$

Similarly other co-factors can be calculated and we get

$$\text{adj } A = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix} = \begin{bmatrix} 4 & 2 & -2 \\ -7 & 7 & 7 \\ 5 & -1 & 1 \end{bmatrix}$$

we notice that

$$A(\text{adj } A) = \begin{bmatrix} 1 & 0 & 2 \\ 3 & 1 & -1 \\ -2 & 1 & 3 \end{bmatrix} \begin{bmatrix} 4 & 2 & -2 \\ -7 & 7 & 7 \\ 5 & -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 14 & 0 & 0 \\ 0 & 14 & 0 \\ 0 & 0 & 14 \end{bmatrix}$$

$$= (\text{adj } A)\, A. \text{ (verify)}$$

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Theorem: 7:20

Let A be any square matrix of order n. Then $(adj A)A = A(adj A) = |A|I$ where I is the identity matrix of order n.

Proof:-

The $(i,j)^{th}$ element of $(A(adj A))$

$$= \sum_{k=1}^{n} a_{ik} A_{jk}$$

$$= \begin{cases} 0 & \text{if } i \neq j \\ |A| & \text{if } i = j \end{cases}$$

$$\therefore A(adj A) = \begin{bmatrix} |A| & 0 & \cdots & 0 \\ 0 & |A| & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & |A| \end{bmatrix}$$

$$= |A|I.$$

Similarly, $(adj A)A = |A|I$.

Hence $(adj A)A = A(adj A)$

$$= |A|I.$$

Note:-

Suppose $|A| \neq 0$. Now, consider the matrix $B = \frac{1}{|A|} adj A$.

Then $AB = A\left[\frac{1}{|A|}(adj A)\right]$

$$= \frac{1}{|A|}(A\, adj A)$$

$$= \frac{1}{|A|}|A|I$$

$$= I.$$

Similarly $BA = I$. Thus $AB = BA = I$.

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Definition:-

Let A be a square matrix of order n. A is said to be invertible if there exists a square matrix B of order n such that $AB = BA = I$ and B is called the inverse of A and is denoted by A^{-1}

Note:-

The invertible matrices are precisely the units of the ring $M_n(F)$.

Theorem: 7.21

A square matrix A of order n is non-singular iff A is invertible

Proof:-

Suppose A is invertible

Then there exists a matrix B such that

$$AB = BA = I.$$

Hence $|AB| = |I| = 1$

$\therefore |A||B| = 1.$

Hence $|A| \neq 0$ so that A is non-singular

conversely, let A be non-singular.

Hence $|A| \neq 0$

Now, consider the matrix $B = \frac{1}{|A|} \text{adj}$

Then $AB = BA = I$. (refer the note above)

$\therefore$ A is invertible and B is the inverse of A.

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Solved problems:-

Problem 1.

compute the inverse of the matrix

$$A = \begin{bmatrix} 2 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & 2 & 2 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 2 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & 2 & 2 \end{vmatrix} = -1$$

Since $|A| \neq 0$, A is non-singular.

Hence A^{-1} exists and is given by

$$A^{-1} = \frac{\text{adj } A}{|A|}.$$

Now, we find $\text{adj } A = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}$

where A_{ij} $(i, j = 1, 2, 3)$ are cofactors of a_{ij}.

$$A_{11} = \begin{vmatrix} 6 & -5 \\ -2 & 2 \end{vmatrix} = 2;$$

$$A_{12} = -\begin{vmatrix} -15 & -5 \\ 5 & 2 \end{vmatrix} = 5$$

$$A_{13} = \begin{vmatrix} -15 & 6 \\ 5 & -2 \end{vmatrix} = 0$$

$$A_{21} = -\begin{vmatrix} -1 & 1 \\ -2 & 2 \end{vmatrix} = 0$$

$$A_{22} = \begin{vmatrix} 2 & 1 \\ 5 & 2 \end{vmatrix} = -1$$

$$A_{23} = -\begin{vmatrix} 2 & -1 \\ 5 & -2 \end{vmatrix} = -1$$

$$A_{31} = \begin{vmatrix} -1 & 1 \\ 6 & -5 \end{vmatrix} = -1$$

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$$A_{32} = -\begin{vmatrix} 2 & 1 \\ -15 & -5 \end{vmatrix} = -5$$

$$A_{33} = \begin{vmatrix} 2 & -1 \\ -15 & 6 \end{vmatrix} = -3$$

Hence $\text{adj } A = \begin{bmatrix} 2 & 0 & -1 \\ 5 & -1 & -5 \\ 0 & -1 & -3 \end{bmatrix}$

$$\therefore A^{-1} = \frac{1}{-1}\begin{bmatrix} 2 & 0 & -1 \\ 5 & -1 & -5 \\ 0 & -1 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} -2 & 0 & 1 \\ -5 & 1 & 5 \\ 0 & 1 & 3 \end{bmatrix}$$

Problem 2:

If $\omega = e^{\frac{2\pi i}{3}}$ find the inverse of the matrix $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{bmatrix}$

We note that $\omega^3 = 1$

$\therefore |A| = 3(\omega^2 - \omega)$. (verify)

Since $|A| \neq 0$, A is non-singular. Hence A^{-1} exists and is given by $A^{-1} = \frac{\text{adj } A}{|A|}$

Now, $\text{adj } A = \begin{bmatrix} \omega^2-\omega & \omega^2-\omega & \omega^2-\omega \\ \omega^2-\omega & \omega-1 & 1-\omega^2 \\ \omega^2-\omega & 1-\omega^2 & \omega-1 \end{bmatrix}$ (verify)

$$A^{-1} = \frac{1}{3(\omega^2-\omega)}\begin{bmatrix} \omega^2-\omega & \omega^2-\omega & \omega^2-\omega \\ \omega^2-\omega & \omega-1 & 1-\omega^2 \\ \omega^2-\omega & 1-\omega^2 & \omega-1 \end{bmatrix}$$

$$= \frac{1}{3\omega}\begin{bmatrix} \omega & \omega & \omega \\ \omega & 1 & -\omega \\ \omega & -\omega & 1 \end{bmatrix} -1/\omega$$

$-1/\omega$

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Problem 3.

Show that a square matrix A is orthogonal iff $\boxed{A^{-1} = A^T}$.

Suppose A is orthogonal. Then $AA^T = I$.

$\therefore |AA^T| = |I| = 1$

$\therefore |A||A^T| = 1$. $\quad |A^T| = |A|$

$\therefore |A||A| = 1$

$\therefore \boxed{|A| \neq 0 \text{ and hence } A \text{ is non-singular}}$

$\therefore A^{-1}$ exists.

Now, $A^{-1}(AA^T) = A^{-1}I$

$\therefore (A^{-1}A)A^T = A^{-1}$.

$IA^T = A^{-1}$

$A^T = A^{-1}$

conversely, let $A^T = A^{-1}$.

Then $AA^T = AA^{-1} = I$. Similarly $A^TA = I$.

Hence A is orthogonal.

Problem 4 :-

Show that a square matrix A is involutory iff $A = A^{-1}$.

Suppose A is involutory. Then $A^2 = I$.

Hence $|A^2| = 1$

$\therefore |A^2| = |A||A| = 1$.

$|A| \neq 0$ and hence A is non-singular.

$\therefore A^{-1}$ exists.

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Now, $A^{-1}(AA) = A^{-1}I$.

$$(A^{-1}A)A = A^{-1}.$$

$$IA = A^{-1}$$

$$A = A^{-1}$$

Conversely, let $A = A^{-1}$

Then $A^2 = AA = AA^{-1} = I$, $A^2 = I$.

$\therefore$ A is involutory.

Theorem: 7.22

Let V and W be vector spaces of dimension n over a field F with bases $v_1, v_2, \ldots\ldots, v_n$ and $w_1, w_2, \ldots\ldots, w_n$ respectively. Then a linear transformation $T: V \to W$ is non-singular iff the associated matrix is non-singular.

Proof :-

$\Longleftarrow$ Let $T: V \to W$ be a non-singular linear transformation ie 1-1 and onto

non singular ie inverse exist ஆகும்

Then T is 1-1 and onto

Hence $T^{-1}: W \to V$ is also a linear transformation.

Let A and B be the matrices representing the linear transformations T and T^{-1} with respect to the chooses bases.

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By theorem 7.2, multiplication of the matrices A and B is equivalent to the composition of the corresponding linear transformation T and T^{-1}

Also TOT^{-1} and $T^{-1}OT$ are identity transformations.

Hence $AB = BA = I$. Thus A has an inverse B.

Hence A is non-Singular.

Conversely, let A be a non-Singular matrix. Then A^{-1} exists.

let $S : W \longrightarrow V$ be the linear transformation determined by the matrix A^{-1}.

It is easily verified that

$TOS = S \circ T = I$

Hence T has an inverse linear transformation S.

Hence T is a non-singular linear transformation.

7.4. Elementary Transformations.

Definition

Let A be an mxn matrix over a field F. An elementary row-operation on A is any one of the following three types

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1) The interchange of any two rows

2) Multiplication of a row by a non-zero element c in F.

3) Addition of any multiple of one row with any other row.

Similarly we define an elementary column operation on A as any one of the following three types.

1) The interchange of any two columns.

2) Multiplication of a column by a non-zero element c in F.

3) Addition of any multiple of one column with any other column.

Example:-

let $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \\ 3 & -1 \end{bmatrix}$, $A_1 = \begin{bmatrix} 3 & -1 \\ 2 & 1 \\ 1 & 2 \end{bmatrix}$

$A_2 = \begin{bmatrix} 2 & 2 \\ 4 & 1 \\ 6 & -1 \end{bmatrix}$, $A_3 = \begin{bmatrix} 1 & 2 \\ 5 & 7 \\ 3 & -1 \end{bmatrix}$. A_1 is obtained from A by interchanging the first and third rows.

A_2 is obtained from A by multiplying the first column of A by 2.

A_3 is obtained from A by adding to the second row the multiple by 2 of the first row

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Notation:

We shall employ the following notations for elementary transformations:

i) Interchange of i^{th} and j^{th} rows will be denoted by $R_i \longleftrightarrow R_j$.

ii) Multiplying of i^{th} row by a non-zero element $c \in F$ will be denoted by

$$R_i \longrightarrow cR_i.$$

iii) Addition of k times the j^{th} row to the i^{th} row will be denoted by.

$$R_i \longrightarrow R_i + kR_j.$$

The corresponding column operations will be denoted by writing C in the place of R.

Definition:

An $m \times n$ matrix B is said to be row equivalent (column equivalent) to an $m \times n$ matrix A if B can be obtained from A by a finite succession of elementary row operations (columns operations)

A and B are said to be equivalent if B can be obtained from A by a finite succession of elementary row or column operation.

If A and B are equivalent: We write $A \sim B$.

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Exercise:-

Prove the row equivalence, column equivalence and equivalence are equivalence relations in the set of all $m \times n$ matrices.

Definition:-

A matrix obtained from the identity matrix by applying a single elementary row or column operation is called an elementary matrix.

For example. $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $\begin{bmatrix} 4 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}$ are elementary matrices obtained from the identity matrix $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ by applying the elementary operations $R_1 \longleftrightarrow R_2$

$R_1 \to 4R_1$, $R_3 \to R_3 + 2R_2$ respectively.

Exercise:-

Give examples of elementary matrices of order 4.

Theorem 7:2.3

Any elementary matrix is non-singular.

Proof:-

The determinant of the identity matrices of any order is 1. Hence the determinant of an elementary matrix

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obtained by interchanging any two rows is -1. The determinant of an elementary matrix obtained by multiplying any row by $k \neq 0$ is k. The determinant of an elementary matrix obtained by adding a multiple of one row with another row is 1. Hence any elementary matrix is non-singular.

Theorem : 1.24 :

Let A be an $m \times n$ matrix and B be an $n \times p$ matrix. Then every elementary row (column) operation of the product AB can be obtained by subjecting the matrix A (matrix B) to the same elementary row (column) operation.

Proof :-

Let $R_1, R_2, \ldots, R_m$ denote the rows of the matrix A and $C_1, C_2, \ldots, C_p$ denote the columns of B. By the definition of matrix multiplication.

$$AB = \begin{bmatrix} R_1C_1 & R_1C_2 & \cdots & R_1C_p \\ R_2C_1 & R_2C_2 & \cdots & R_2C_p \\ \vdots & \vdots & \cdots & \vdots \\ R_mC_1 & R_mC_2 & \cdots & R_mC_p \end{bmatrix}$$

It is obvious from the above representation of AB that if we apply any elementary row operation on A the matrix AB is also

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subjected to the same elementary row operation. Also if we apply any elementary columns operation on B the matrix AB is also subjected to the same elementary column operation.

Theorem 7.25.

Each elementary row operation on an mxn matrix A is equivalent to pre-multiplying the matrix A by the corresponding elementary mxm matrix

Proof:-

Since A is an mxn matrix we can write $A = IA$ where I is the identity matrix of order m. By theorem 7.24 an elementary row operation IA is equivalent to the same row operation on I. But an elementary row operation on I gives an elementary matrix. Hence by pre-multiplying A by the corresponding elementary matrix we get the required row operation on A.

Note:-

Similarly each elementary column operation of an mxn matrix A is equivalent to post-multiplying the matrix A by the

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corresponding elementary $n \times n$ matrix

corollary 1.

If two $m \times n$ matrices A and B are row equivalent then $A = PB$ where P is a non-singular $m \times m$ matrix.

Proof:-

Since A is row equivalent to B, A can be obtained from B by applying successive elementary row operations. Hence $A = E_1 E_2 \ldots\ldots E_n B$: where each E_i is an elementary matrix. Since each E_i is non-singular. $A = PB$ where $P = E_1 E_2 \ldots E_n$ and P is non-singular.

corollary 2:

If row matrices A and B are column equivalent then $A = BQ$ where Q is a non-singular matrix.

corollary 3:

If two $m \times n$ matrices A and B are equivalent then $A = PBQ$ where P is a non-singular $m \times m$ matrix and Q is a non-singular $n \times n$ matrix

corollary 4

The inverse of an elementary matrix is again an elementary matrix.

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Proof :-

Let E be an elementary matrix obtained from I by applying some elementary operations. If we apply the reverse operation on E, then E is carried back to I. Let E^* be the elementary matrix corresponding to the reverse operation.

Then $E^*E = EE^* = I$. Hence $E^* = E^{-1}$

Hence E^{-1} is also an elementary matrix.

Canonical From of a matrix.

We now use elementary row and columns operations to reduce any matrices to a simple form called the canonical form of a matrix.

Theorem 7.26

By successive applications of elementary row and column operations, any non-zero mxn matrix A can be reduced to a diagonal matrix D in which the diagonal entries are either 0 or 1 and all the 1's Proceeding all the zeros on the diagonal In other words, any non-zero mxn matrix is equivalent to a matrix of the form

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$\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$ where I_r is the $r \times r$ identity matrix and 0 is the zero matrix.

Proof:-

We shall prove the theorem by induction on the number of rows of A. Suppose A has just one row. Let $A = (a_{11}, a_{12} \ldots a_{1n})$.

Since $A \neq 0$, by interchanging columns if necessary, we can bring a non-zero entry c to the position a_{11}.

Multiplying A by c^{-1} we get 1 as the first entry.

Other entries in A can be made zero by adding suitable multiples of 1. Thus the result is true when $m=1$.

Now, suppose that the result is true for any non-zero matrix with $m-1$ rows.

Let A be a non-zero $m \times n$ matrix. By permuting rows and columns we can bring some non-zero entry c to the position a_{11}.

Multiplying the first row by c^{-1} we get 1 as the first entry.

All other entries in the first column can be made zero by adding suitable

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multiples of the first row to each other row.

Similarly all the other entries in first row can be made zero.

This reduces A to a matrix of the form $B = \begin{bmatrix} I_1 & 0 \\ 0 & C \end{bmatrix}$ where C is an $(m-1)\times(n-1)$ matrix.

Now by induction hypothesis C can be reduced to the desired from by elementary row and column operations.

Hence A is equivalent to a matrix of the required form.

corollary 1:

If A is an mxn matrix there exist non-singular square matrices P and Q of orders m and n respectively such that $PAQ = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$

The result follows from corollary 3 of theorem 7.25.

corollary 2.

Any non-singular square matrix A of order n is equivalent to the identity

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matrix

Proof:-

By corollary 1. $PAQ = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$

Since P, A, Q are all non-singular $\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$ is non-singular. This is possible iff $\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} = I_n$.

corollary 3:-

Any non-singular matrix A can be expressed as a product of elementary matrices.

Proof:-

By corollary 2, $PAQ = I_n$. Hence $A = P^{-1}Q^{-1}$. Further by corollary 4 of theorem 7.25. P^{-1} and Q^{-1} are products of elementary matrices.

Hence A is a product of elementary matrices.

Note:-

The inverse of a non-singular matrix A can be computed by using elementary transformations. let A be a non-singular matrix of order n. Then $AA^{-1} = A^{-1}A = I$. Now the non-singular matrix A^{-1} can be expressed as the product of elementary matrices.

Let $A^{-1} = E_1, E_2, \ldots, E_n$.

Then. $I = A^{-1}A = E_1, E_2, \ldots, E_n A$.

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Thus every non-singular matrix A can be reduced to I by pre-multiplying A by elementary matrices.

Hence A can be reduced to the identity matrix by applying successive elementary row operations.

Now, $A = IA$. Reduce the matrix A in the left hand side to I by applying successive elementary row operations and apply the same elementary row operations to the factor I in the right hand side.

Then we get $I = BA$ so that $B = A^{-1}$.

Solved problems:

Problem 1:-

Reduce the matrix $A = \begin{bmatrix} 1 & 2 & -1 \\ 1 & 1 & 2 \\ 2 & 4 & -2 \end{bmatrix}$ to the canonical From.

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 1 & 1 & 2 \\ 2 & 4 & -2 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & -1 \\ 0 & -1 & 3 \\ 0 & 0 & 0 \end{bmatrix} \begin{matrix} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - 2R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 3 \\ 0 & 0 & 0 \end{bmatrix} \begin{matrix} C_2 \to C_2 - 2C_1 \\ C_3 \to C_3 + C_1 \end{matrix}$$

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$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad C_3 \to C_3 + 3C_2.$$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad R_2 \to -R_2.$$

Problem : 2.

Find the inverse of the matrix

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 3 & 1 & -1 \\ -2 & 1 & 3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 2 \\ 3 & 1 & -1 \\ -2 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A.$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -7 \\ 0 & 1 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} A, \quad R_2 \to R_2 - 3R_1, \quad R_3 \to R_3 + 2R_1$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -7 \\ 0 & 0 & 14 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 5 & -1 & 1 \end{bmatrix} A, \quad R_3 \to R_3 - R_2$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{2}{7} & \frac{1}{7} & -\frac{1}{7} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{5}{14} & -\frac{1}{14} & \frac{1}{14} \end{bmatrix} \begin{matrix} R_1 \to R_1 - \frac{1}{7}R_3 \\ R_2 \to R_2 + \frac{1}{2}R_3 \\ R_3 \to \frac{1}{4}R_3 \end{matrix}$$

$$\Rightarrow A^{-1} = \begin{bmatrix} \frac{2}{7} & \frac{1}{7} & -\frac{1}{7} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{5}{14} & -\frac{1}{14} & \frac{1}{14} \end{bmatrix}$$

Definition

Let A and B be two square matrices of order n. B is said to be similar to A if there exists a nxn non-singular matrix P such that $B = P^{-1}AP$.

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Solved problems.

Problem 1:

Similarity of matrices is an equivalence relation in the set of all $n \times n$ matrices.

Proof:-

Let S be the set of all $n \times n$ matrices

Let $A \in S$

Since $A = I^{-1} A I$ and I is non-singular, A is similar to A.

Hence similarity of matrices is reflexive

Now, let $A, B \in S$ and let A be similar to B:

$\therefore A = P^{-1} B P$ where $P \in S$ is a non-singular matrix

Now, $P^{-1} B P = A \Rightarrow P P^{-1} B P P^{-1} = P A P^{-1}$

$\Rightarrow B = P A P^{-1}$

$\Rightarrow B = (P^{-1})^{-1} A (P^{-1})$

Since P is non-singular $P^{-1} \in S$ is also non-singular

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$\therefore$ B is similar to A.

Hence similarity of matrices is Symmetric

Now, let $A, B, C \in S$

let A be similar to B to B be similar to C. Hence there exist non-singular matrices. $P, Q \in S$ such that

$$A = P^{-1}BP \text{ and } B = Q^{-1}CQ$$

Now, $A = P^{-1}BP$

$= P^{-1}(Q^{-1}CQ)P$

$= (P^{-1}Q^{-1})CQP$

$= (QP)^{-1}CQP$

Since $P, Q \in S$ are non-singular, $QP \in S$ is also non-singular

Hence A is similar to C.

$\therefore$ similarity of matrices is transitive.

Hence similarity of matrices is an equivalence relation.

Problem 2:-

If A and B are similar matrices show that their determinants are same.

Let A and B be two similar matrices

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$\therefore$ There exists a non-singular matrix P such that $B = P^{-1}AP$.

$$\text{Now, } |B| = |P^{-1}AP|$$
$$= |P^{-1}||A||P|$$
$$= |A| \quad \left(\text{Since } |P^{-1}| = \frac{1}{|P|}\right)$$

Hence the result.

7.5 Rank of a Matrix

We now proceed to introduce the concept of the rank of a matrix.

Definition

Let $A = (a_{ij})$ be an $m \times n$ matrix. The rows $R_i = (a_{i1}, a_{i2}, \ldots, a_{in})$ of A can be thought of as elements of F^n. The subspace of F^n generated by the m rows of A is called the row space of A.

Similarly, the subspace of F^m generated by the n columns of A is called the column space of A.

The dimension of the row space (column space) of A is called the row rank (column rank) of A.

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Theorem: 7.27.

Any two row equivalent matrices have the same row space and have the same row rank.

Proof:- Let A be an mxn matrix.

It is enough if we prove that the row space of A is not altered by any elementary row operation.

Obviously the row space of A is not altered by an elementary row operation of the type $R_i \longleftrightarrow R_j$.

Now, consider the elementary row operation

$R_i \longrightarrow cR_i$ where $c \in F - \{0\}$.

Since $L(\{R_1, R_2, \ldots, R_i, \ldots, R_n\}) = L(\{R_1, R_2, \ldots, cR_i, \ldots, R_n\})$ the row space of A is not altered by this type of elementary row operation.

Similarly we can easily prove that the row space of A is not altered by an elementary row operation of the type $R_i \longrightarrow R_i + cR_j$.

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Hence row equivalent matrices have the same row space and hence the same row rank.

Similarly we can prove the following theorem.

Theorem 7.28.

Any two column equivalent matrices have the same column rank.

Theorem :7.29 :-

The row rank and the column rank of any matrix are equal.

Proof :-

let $A = (a_{ij})$ be an $m \times n$ matrix.

let $R_1, R_2, \ldots, R_m$ denote the rows of A.

Hence $R_i = (a_{i1}, a_{i2}, \ldots, a_{in})$.

Suppose the row rank of A is r.

Then the dimension of the row space is r.

Let $v_1 = (b_{11}, \ldots, b_{1n})$, $v_2 = (b_{21}, \ldots$

$\ldots v_r = (b_{r1}, \ldots, b_{rn})$ be a basis for the row space of A.

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Then each row is a linear combination of the vectors $v_1, v_2, \ldots, v_r$.

let, $R_1 = K_{11}v_1 + K_{12}v_2 + \cdots + K_{1r}v_r$.

$R_2 = K_{21}v_1 + K_{22}v_2 + \cdots + K_{2r}v_r$.

$\cdots \quad \cdots \quad \cdots$

$\cdots \quad \cdots \quad \cdots$

$R_m = K_{m1}v_1 + K_{m2}v_2 + \cdots + K_{mr}v_r$

where $K_{ij} \in F$.

Equating the i^{th} component of each of the above equations, we get

$a_{1i} = K_{11}b_{1i} + K_{12}b_{2i} + \cdots + K_{1r}b_{ri}$

$a_{2i} = K_{21}b_{1i} + K_{22}b_{2i} + \cdots + K_{2r}b_{ri}$

$\cdots \quad \cdots \quad \cdots$

$\cdots \quad \cdots \quad \cdots$

$a_{mi} = K_{m1}b_{1i} + K_{m2}b_{2i} + \cdots + K_{mr}b_{ri}$.

Hence

$$\begin{bmatrix} a_{1i} \\ \vdots \\ a_{mi} \end{bmatrix} = b_{1i}\begin{bmatrix} K_{11} \\ \vdots \\ K_{m1} \end{bmatrix} + b_{2i}\begin{bmatrix} K_{12} \\ \vdots \\ K_{m2} \end{bmatrix} + \cdots + b_{ri}\begin{bmatrix} K_{1r} \\ \vdots \\ K_{mr} \end{bmatrix}$$

Thus each column of A is a linear combination of r vectors.

Hence the dimension of the column space $\leq r$.

$\therefore$ column rank of $A \leq r =$ row rank of A.

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Similarly, row rank of $A \leq$ column rank of A.

Hence the row rank and the column rank of A are equal.

Definition

The rank of a matrix A is the common value of its row and column rank.

Note 1:

Since the row rank and the column rank of matrix are unaltered by elementary row and column operations, equivalent matrices have the same rank In particular if a matrix A is reduced to its canonica form $\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$, then rank of $A = r$.

Thus to find the rank of a matrix we reduce to the canonical form and find the number of non-zero entries in the diagonal.

Note that in the canonical form of the matrix there exists an $r \times r$

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Note 3 :-

The rank of a matrix is not altered on multiplication by non-sing matrices, since pre multiplication by a non-singular matrix is equivalent to applying elementary row operations and post-multiplication by a non-singular matrix is equivalent to applying elementary column operations

Solved problems:

Problem 1 :-

Find the rank of the matrix

$$A = \begin{bmatrix} 4 & 2 & 1 & 3 \\ 6 & 3 & 4 & 7 \\ 2 & 1 & 0 & 7 \end{bmatrix}$$

$$A = \begin{bmatrix} 4 & 2 & 1 & 3 \\ 6 & 3 & 4 & 7 \\ 2 & 1 & 0 & 7 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 4 & 3 \\ 4 & 3 & 6 & 7 \\ 0 & 1 & 2 & 7 \end{bmatrix} C_1 \leftrightarrow C_3$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 4 & -5 & -10 & -5 \\ 0 & 1 & 2 & 7 \end{bmatrix} \begin{matrix} C_1 \to C_2 - 2C_1 \\ C_3 \to C_3 - 4C_1 \\ C_4 \to C_4 - 3C_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -5 & -10 & -5 \\ 0 & 1 & 2 & 7 \end{bmatrix} R_2 \to R_2 - 4$$

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$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -5 & 0 & 0 \\ 0 & 1 & 0 & 6 \end{bmatrix} \begin{matrix} C_3 \to C_3 - 2C_2 \\ C_4 \to C_4 - C_2 . \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -5 & 0 & 0 \\ 0 & 0 & 0 & 6 \end{bmatrix} R_3 \to R_3 + \frac{1}{5} R_2 .$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -5 & 0 \\ 0 & 0 & 0 & 6 \end{bmatrix} C_2 \longleftrightarrow C_3 .$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{matrix} R_2 \to -\frac{1}{5} R_2 \\ R_3 \to \frac{1}{6} R_3 . \end{matrix}$$

Rank of $A = 3$.

Problem 2.

Find the rank of the matrix $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 4 & 1 & 0 & 2 \\ 0 & 3 & 4 & 2 \end{bmatrix}$ by examing the determinant minors.

$$\begin{vmatrix} 1 & 1 & 1 \\ 4 & 1 & 0 \\ 0 & 3 & 4 \end{vmatrix} = 0 = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 0 & 2 \\ 3 & 4 & 2 \end{vmatrix}$$

$$\begin{vmatrix} 1 & 1 & 1 \\ 4 & 1 & 2 \\ 0 & 3 & 2 \end{vmatrix} = 0 = \begin{vmatrix} 1 & 1 & 1 \\ 4 & 0 & 2 \\ 0 & 4 & 2 \end{vmatrix}$$

$\therefore$ Every 3×3 submatrix of A has determinant zero

Also, $\begin{vmatrix} 1 & 1 \\ 4 & 1 \end{vmatrix} = -3 \neq 0$

$\therefore$ Rank of $A = 2$.

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Unit-5

7.7 Characteristic Equation And Cayley Hamilton Theorem

Definition:-

An expression of the form $A_0 + A_1x + A_2x^2 + \ldots\ldots + A_nx^n$ where $A_0, A_1, \ldots\ldots, A_n$ are square matrices of the same order and $A_n \neq 0$ is called a matrix polynomial of degree n

For example, $\begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix} + \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix} x + \begin{pmatrix} 2 & 0 \\ 3 & 1 \end{pmatrix} x^2$ is a matrix polynomial of degree 2 and it is simply the matrix $\begin{pmatrix} 1+x+2x^2 & 2+x \\ 2x+3x^2 & 3+x+x^2 \end{pmatrix}$

Definition

Let A be any square matrix of order n and let I be the identity matrix of order n. Then the matrix polynomial given by $A - xI$ is called the characteristic matrix of A.

The determinant $|A - xI|$ which is an ordinary polynomial in x of deg

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corollary :-

If $\lambda_1, \lambda_2, \ldots, \lambda_n$ are eigen values of A then $\lambda_1^k, \lambda_2^k, \ldots, \lambda_n^k$ are eigen values of A^k for any positive integer k.

Property 7:

Eigen vectors corresponding to distinct eigen values of a matrix are linearly independent.

Proof :-

Let $\lambda_1, \lambda_2, \ldots, \lambda_k$ be distinct eigen values of a matrix and let X_i be the eigen vector corresponding to λ_i;

Hence $AX_i = \lambda_i X_i \quad (i = 1, 2, \ldots, k)$ → ①

Now, suppose $X_1, X_2, \ldots, X_k$ are linearly dependent. Then there exist real numbers $\alpha_1, \alpha_2, \ldots, \alpha_k$, not all zero, such that $\alpha_1 X_1 + \alpha_2 X_2 + \ldots + \alpha_k X_k = 0$.

Among all such relations, we choose one of shortest length, say j.

By rearranging the vectors

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$$= |P^{-1}P|\,|A-\lambda I|$$
$$= |I|\,|A-\lambda I|$$
$$= |A-I\lambda|$$

∴ The characteristic equations of A and $P^{-1}AP$ are the same.

Property 14 :-

If λ is a characteristic roots of then $f(\lambda)$ is a characteristic root of of the matrix $f(A)$ where $f(x)$ is any polynomial.

Proof :-

Let $f(x) = a_0x^n + a_1x^{n-1} + \ldots + a_{n-1}x + a_n$

where $a_0 \neq 0$ and $a_1, a_2, \ldots, a_n$ are all real numbers.

$$\therefore f(A) = a_0A^n + a_1A^{n-1} + \ldots + a_{n-1}A + a_nI.$$

Since λ is a characteristic root of A, λ^n is a characteristic roots of A^n for any positive integer n (refer property 6)

$$\therefore A^nX = \lambda^nX$$
$$A^{n-1}X = \lambda^{n-1}X$$
$$\ldots\ldots\ldots$$
$$\ldots\ldots\ldots$$
$$AX = \lambda X$$
$$a_0A^nX = a_0\lambda^nX$$
$$a_1A^{n-1}X = a_1\lambda^{n-1}X$$
$$\ldots\ldots\ldots$$

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The characteristic roots equation of A is given by $|A-\lambda I|=0$

$$\therefore \begin{vmatrix} \cos\theta-\lambda & -\sin\theta \\ -\sin\theta & \cos\theta-\lambda \end{vmatrix}=0$$

$$\therefore (\cos\theta-\lambda)^2-\sin^2\theta=0$$

$$\therefore (\cos\theta-\lambda-\sin\theta)(\cos\theta-\lambda+\sin\theta)=0$$

$$\therefore [\lambda-(\cos\theta-\sin\theta)][\lambda-(\cos\theta+\sin\theta)]=0$$

∴ The two charateristic roots, (the two eigen values) of the matrix are $(\cos\theta-\sin\theta)$ and $(\cos\theta+\sin\theta)$

Problem 6 :-

Find the characteristic roots of the matrix $A=\begin{pmatrix} \cos\theta & -\sin\theta \\ -\sin\theta & -\cos\theta \end{pmatrix}$

solution :-

The characteristic equation of A is given by $|A-\lambda I|=0$

i.e) $$\begin{vmatrix} \cos\theta-\lambda & -\sin\theta \\ -\sin\theta & -\cos\theta-\lambda \end{vmatrix}=0$$

$$\therefore -(\cos^2\theta-\lambda^2)-\sin^2\theta=0$$

$$\therefore \lambda^2-(\cos^2\theta+\sin^2\theta)=0$$

$$\lambda^2-1=0$$

∴ The characteristic roots are 1 and −1.

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Problem 7 :-

Find the sum and product of the eigen values of the matrix $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ without finding the roots of the characteristic equation.

Solution :-

Sum of the eigen values of A = trace of $A = a_{11} + a_{22}$.

Product of the eigen values of $A = |A| = a_{11}a_{22} - a_{12}a_{21}$.

Problem 8 :-

Verify the statement that the sum of the elements in the diagonal of a matrix is the sum of the eigen values of the matrix $A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & 2 & 0 \end{bmatrix}$

Solution :-

The characteristic equation of A is $|A - \lambda I| = 0$

i.e) $$\begin{vmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & -\lambda \end{vmatrix} = 0$$

i.e) $(-2-\lambda)[(1-\lambda)(-\lambda) - 12] - 2[-2\lambda - 6] - 3[-4 + (1-\lambda)] = 0$

i.e) $(-2-\lambda)(\lambda^2 - \lambda - 12) + 4(\lambda+3) + 3(\lambda+3) = 0$

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We know that the product of the eigen values is $|A|$.

i.e) $\lambda_1\lambda_2\lambda_3 = \begin{vmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{vmatrix}$

i.e) $16\lambda_3 = 6(9-1)+2(-6+2)+2(2-6)$

$= 48-8-8$

$= 32$

$\lambda_3 = 2$

$\therefore$ The third eigen value is 2.

Also we know that the sum of the eigen values of

A = trace of A.

$= 6+3+3 = 12$.

Problem 10:-

The product of two eigen values of the matrix $A = \begin{pmatrix} 2 & 2 & -7 \\ 2 & 1 & 2 \\ 0 & 1 & -3 \end{pmatrix}$ is -12. Find the eigen values of A.

Solution:-

Let $\lambda_1, \lambda_2, \lambda_3$ be the eigen values of A. Given product of 2 eigen values say λ_1 and λ_2 is -12

$\therefore \lambda_1\lambda_2 = -12 \longrightarrow (1)$

we know that the product of the eigen values is $|A|$.

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$(1-\lambda)(\lambda^2-6\lambda+4)+(\lambda+2)+3(3\lambda-14)=0$

$\lambda^2-6\lambda+4-\lambda^3+6\lambda^2-4\lambda+\lambda+2+9\lambda-42=0$

$\therefore -\lambda^3+7\lambda^2-36=0$. Hence $\lambda^3-7\lambda^2+36=0$

$\therefore (\lambda+2)(\lambda^2-9\lambda+18)=0$

$(a+b)^3 = a^3+3a^2b+3ab^2+b^3$.

Hence $(\lambda+2)(\lambda-6)(\lambda-3)=0$

$\therefore \lambda = -2, 3, 6$ are the three eigen values.

Case (i) Eigen vector corresponding to $\lambda = -2$.

Let $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ be an eigen vector corresponding to $\lambda = -2$.

Hence $AX = -2X$.

i.e) $\begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2x_1 \\ -2x_2 \\ -2x_3 \end{bmatrix}$

$\therefore x_1+x_2+3x_3 = -2x_1$

$x_1+5x_2+x_3 = -2x_2$

$3x_1+x_2+x_3 = -2x_3$

$3x_1+x_2+3x_3 = 0$

$x_1+7x_2+x_3 = 0$

$3x_1+x_2+3x_3 = 0$

clearly this system of three equations reduces to two equations only. From ① and ② we get

$\therefore x_1 = -2K$; $x_2 = 0$; $x_3 = 2K$.

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case (i) $\lambda = 2$.

The eigen vector $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ is got from $AX = 2X$.

$$\therefore 6x_1 - 2x_2 + 2x_3 = 2x_1$$
$$-2x_1 + 3x_2 - x_3 = 2x_2$$
$$2x_1 - x_2 + 2x_3 = 2x_3$$

$$\therefore 4x_1 - 2x_2 + 2x_3 = 0$$
$$-2x_1 + x_2 - x_3 = 0$$
$$2x_1 - x_2 + x_3 = 0$$

The above three equations are equivalent to the single equation

$2x_1 - x_2 + x_3 = 0$

The independent eigen vectors can be obtained by giving arbitrary values to any two of the unknowns x_1, x_2, x_3.

Giving $x_1 = 1$; $x_2 = 2$ we get $x_3 = 0$

Giving $x_1 = 3$; $x_2 = 4$ we get $x_3 =$

∴ Two independent vectors corresponding to $\lambda = 2$ are $(1, 2, 0)$ and $(3, 4, -2)$

case (ii) $\lambda = 8$.

The eigen vector $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ is got form $AX = 8X$.

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$\therefore -\lambda^3 + 2\lambda^2 + 4\lambda - 8 = 0$

Hence $\lambda^3 - 2\lambda^2 - 4\lambda + 8 = 0$

$\therefore (\lambda - 2)(\lambda^2 - 4) = 0$

Hence $(\lambda - 2)(\lambda - 2)(\lambda + 2) = 0$

$\therefore \lambda = 2, 2, -2$. are the three eigen values.

case (i) $\lambda = 2$.

let $X = (x_1, x_2, x_3)$ be an eigen vector corresponding to $\lambda = 2$, X is got from $AX = 2X$

i.e) $\begin{bmatrix} 2 & -2 & 2 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2x_1 \\ 2x_2 \\ 2x_3 \end{bmatrix}$

$\therefore$ The eigen vector corresponding to $\lambda = 2$ is given by the equations.

$$2x_1 - 2x_2 + 2x_3 = 2x_1$$
$$x_1 + x_2 + x_3 = 2x_2$$
$$x_1 + 3x_2 - x_3 = 2x_3$$

i.e) $-x_2 + x_3 = 0 \longrightarrow (1)$

$x_1 - x_2 + x_3 = 0 \longrightarrow (2)$

$x_1 + 3x_2 - 3x_3 = 0 \longrightarrow (3)$

Taking (1) and (2) we get

$$\frac{x_1}{0} = \frac{x_2}{1} = \frac{x_3}{1} = K \text{ (say)}$$

$\therefore x_1 = 0$; $x_2 = K$; $x_3 = K$.

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Taking $K=1$, we get $(0,1,1)$ as an eigen vector corresponding to $\lambda=2$

case (ii) $\lambda=-2$

corresponding to $\lambda=-2$ we have

$$AX=-2X.$$

$$\therefore \quad 2x_1-2x_2+2x_3=-2x_1$$
$$x_1+x_2+x_3=-2x_2$$
$$x_1+3x_2-x_3=-2x_3$$

$$\therefore \quad 2x_1-x_2+x_3=0$$
$$x_1+3x_2+x_3=0$$
$$x_1+3x_2+x_3=0$$

$\therefore$ Taking the first two equations we get

$$\frac{x_1}{-4}=\frac{x_2}{-1}=\frac{x_3}{7}=k \text{ (say)}$$

$$\therefore \quad x_1=-4k\ ;\ x_2=-k\ ;\ x_3=7k$$

Taking $K=1$ we get $(-4,-1,7)$ as an eigen vector corresponding to the eigen value $\lambda=-2$.

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