Intrinsic properties of a surface

Defn (surface).

A surface is the locus of a point whose P.V. is a function of two independent parameter u & v.

Implicit (or) constraint equation

The co-ordinates x, y, z of P satisfy a relation of the form $F(x,y,z)=0$ is called the implicit (or) constraint eqns of the surfaces.

Parametric (or) freedom Equation

The eqns of a surfaces are of the form $x=f(u,v)$, $y=g(u,v)$, $z=h(u,v)$ where u, v are parameters of a real values. The funs f, g, h are single-valued & continuous surface of class r.

If the funs. f, g, h are posses to cont. partial derivatives of rth order then the surface is said to be ~~of~~ a surface of class r.

Note :

For any point (x, y, z) on the surface the values of u & v are uniquely determined & that the point is referred to as the point (u, v). Then the parameter u & v are often called curvilinear co-ordinates of point.

Results :

The parametric eqns of a surface are not unique. For example.

2m

Consider the surface given by the parametric Eqn,

$x = u+v$; $y=u-v$; $z=4uv$ $\longrightarrow (1)$

Eliminating u & v the constraint eqn is $x^2-y^2=z$, which represent a whole of constraint certain hyperbolic, parabolic & also parametric eqns of another curve,

$x = u,\ y = v,\ z = u^2 - v^2$ → (2)

Eliminating u & v the constraint eqn is,

$x^2 - y^2 = z$ → (3)

Represent the same paraboloid thus both (1) & (2) are parametric eqns of the hyperbolic paraboloid (3).

Hence, parametric eqns are not unique.

Parameter transformation :

2m Two representations of the same surface are related by the parameter transformation of the form,

$$u' = \phi(u,v),\ v' = \psi(u,v).$$

Proper parameter Transformation :

The parameter transformation is said to be proper if, i) ϕ, ψ are single valued fun. &

ii) have non-vanishing Jacobian $\frac{\partial(\phi,\psi)}{\partial(u,v)} \neq 0$

In some domain D.

Note :

i) If D' is the domain of u', v' corresponding to D, condition $\frac{\partial(\phi,\psi)}{\partial(u,v)} \neq 0$ is necessary & sufficient that the transf. $u' = \phi(u,v)$; $v' = \psi(u,v)$ can be inverted near any point of D'.

ii) The p.v. $\bar{r} = (x,y,z)$ of a point on the surface is a ! of u & v with the same continuity & differentiabilit properties as f, g, h partial diff. w.r. to u & v will b denoted by, $\bar{r}_1 = \frac{\partial \bar{r}}{\partial u}$ & $\bar{r}_2 = \frac{\partial \bar{r}}{\partial v}$.

Ordinary point : 2m

If $\bar{r}_1 \times \bar{r}_2 \neq 0$ (ie) rank $\begin{pmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{pmatrix} = 2$ at a poin on a surface then the point is called an ordinary

Note

An ordinary point by a proper parameter transf is

$$\bar{r}_1 \times \bar{r}_2 \neq 0 \Rightarrow \left(\frac{\partial \bar{r}}{\partial u'}\phi_1 + \frac{\partial \bar{r}}{\partial v'}\psi_1\right)\cdot\left(\frac{\partial \bar{r}}{\partial u'}\phi_2 + \frac{\partial \bar{r}}{\partial v'}\psi_2\right) \neq 0$$

$$\Rightarrow \phi\left(\frac{\phi, \psi}{u, v}\right)\cdot\left(\frac{\partial \bar{r}}{\partial u'} + \frac{\partial \bar{r}}{\partial v}\right) \neq 0$$

then,

$$\frac{\partial \bar{r}}{\partial u'} \times \frac{\partial \bar{r}}{\partial v} \neq 0.$$

Singularity : 2M

A point which is not an ordinary point is called a singularity.

Note :

Some singularities are essential.

Eg : 1

Such a singularity as a vertex of alone other singularity are artificial.

Eg : 2

Origin of polar-co-ordinates in the plane for if, $\bar{r}(u\cos v, u\sin v, 0)$ then $\bar{r}_1 \times \bar{r}_2 \neq 0$ is not satisfy when $u=0$.

Defn :-

A representation R of a surface S of class r in E_3 is a set of point in E_3 covered by a system of overlapping parts $\{V_j\}$ each parts V_j being given by parametric eqn of class r.

Each point lying in the overlap of two parts V_i, V_j is then ∃ (there exist) change of parameters from those of one part to those of the other part is proper & of class r.

Defn: (r-equivalent)

Two representations R & R' are said to be r-equivalent, if the composite family of parts $\{V_j, V_j'\}$ satisfies the cond. (condition) that at each point P lying in

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the overlap of any two parts, the change of points from those of one part to those of another is proper Ep of class r.

Defn :

A surface S of class r in E_3 is an r-equivalent class of representation.

Curves on a surface :

Defn : [parametric curve] 2M

Let $\bar{r} = \bar{r}(u, v)$ is the eqn of the surface the curve lying on the surface the curve which are obtain by keeping either u (or) v constant are of particular important are called parametric curves.

Eg: If $v = c$, then the p.v. $\bar{r}$ be a fun. of single parameter u & hence $\bar{r} = \bar{r}(u, c)$ is a curve lying lying on the surface $\bar{r} = \bar{r}(u, v)$. This curve is called the parametric curve, v is constant. Hly, for u is constant. (ie) $u = v =$ constant

Defn :

Let $u = c_1$ & $v = c_2$ then the constant c_1 & c_2 vary the whole surface is cover with a set of parametric curves two of which passes through every point u, v are called curvilinear co-ordinates of p. The parametric curves are called co-ordinates curves.

Orthogonal : 2M

The two parametric curves through a plane p are orthogonal if $\bar{r}_1 \cdot \bar{r}_2 = 0$ at p. If this cond. is satisfied at every point. (ie) $\forall$ u, v in the domain D, the two system of parametric curves are orthogonal.

Note: For any general curves given by $u = u(t), v = v(t)$ the tangent is in the direction.

$$\frac{d\bar{r}}{dt} = \bar{r}_1 \cdot \frac{du}{dt} + \bar{r}_2 \cdot \frac{dv}{dt}$$

since $\bar{r}_1$ & $\bar{r}_2$ are non-zero & independent.

Tangent plane: 2M

Let $\bar{r}_1$ & $\bar{r}_2$ are non-zero, independent, if the tangent to the curves (on the surface) through a point P lie in the plane which contains the 2 vectors $\bar{r}_1$ & $\bar{r}_2$ at P. This plane is called the tangent plane at P.

Normal:

The normal to the surface at P is the normal to the tangent plane at P & is therefore ⊥r to $\bar{r}_1$ & $\bar{r}_2$. If $\bar{N}$ is the unit normal vector then,

$$\bar{N} = \frac{\bar{r}_1 \times \bar{r}_2}{H}, \quad H = |\bar{r}_1 \times \bar{r}_2| \neq 0.$$

Note:

i) If N' is the new normal vector of parametric transf. then N' is the direction, $\frac{\partial \bar{r}}{\partial u'} \times \frac{\partial \bar{r}}{\partial v'}$.

ii) N & N' are the same vector if $\frac{\partial(\phi, \psi)}{\partial(u, v)} > 0$ & are opposite if $\frac{\partial(\phi, \psi)}{\partial(u, v)} < 0$.

Surface of Revolution:

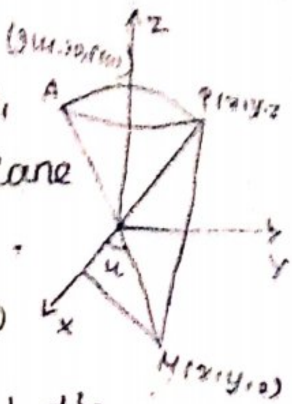
Defn (sphere): 5M

When the polar angles (ie) the co-~~latit~~ longitude u & the longitude v, are taken as parameters on a sphere of centre O & radius 'a' the p.v. is,

$$\bar{r} = a(\sin u \cos v, \sin u \sin v, \cos u).$$

The poles $u = 0$ & $u = \pi$ are singularities & the domain of u, v is $0 < u < \pi$, $0 \le v \le 2\pi$ the parametric curves v = const are the meridians & u = constant are the parallels.

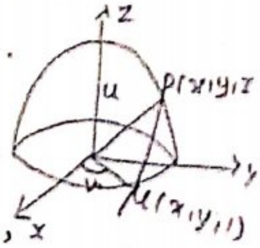
The general surface of revolution :



Let z-axis is the axis of revolution, Let the generating curve in the xz-plane be given by the parametric eqns,

$x = g(u) \; ; \; y = 0 \; ; \; z = f(u).$ $\longrightarrow (1)$

If v is the angle of rotation about the z-axis the p.v. of the point (u,v) is,



$\bar{r} = (g(u)\cos v, \; g(u)\sin v, \; f(u))$

& the domain of u,v is $0 < v \leq 2\pi$ together, with the range of u.

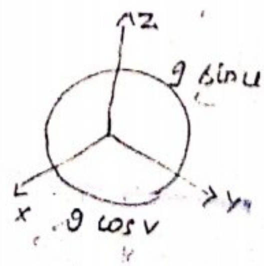
(A surface generated by a rotation of a plane curve about an axis in its plane is called a "surface of revolution".)

Note :

The parametric curves $v =$ constant are the meridian given by the various position of the generating curves & $u =$ constant are the parallels.

(ie) Circles in the plane parallel to the xy-plane.

Verification :



Since $\bar{r} = (g\cos v, \; g\sin v, \; f)$

$\bar{r}_1 = \dfrac{\partial \bar{r}}{\partial u} = (g'\cos v, \; g'\sin v, \; f')$

$\bar{r}_2 = \dfrac{\partial \bar{r}}{\partial v} = (-g\sin v, \; g\cos v, \; 0)$

$\bar{r}_1 \cdot \bar{r}_2 = (g'\cos v, \; g'\sin v, \; f') \cdot (-g\sin v, \; g\cos v, 0)$

$= -gg'\sin v\cos v + g'g\sin v\cos v + 0$

$\bar{r}_1 \cdot \bar{r}_2 = 0 \; , \forall u, v.$

$\therefore$ The parametric curves are orthogonal.

The normal $\bar{N}$ is found to be,

$\bar{N} = \dfrac{\bar{r}_1 \times \bar{r}_2}{H}$ $\longrightarrow (1)$

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$$\vec{r}_1 \times \vec{r}_2 = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ g'\cos v & g'\sin v & f' \\ -g\sin v & g\cos v & 0 \end{vmatrix}$$

$$= \vec{i}\,[0 - f'g\cos v] - \vec{j}\,[0 + f'g\sin v] + \vec{k}\,[gg'\cos^2 v + gg'\sin^2 v]$$

$$= (-f'g\cos v,\ -f'g\sin v,\ gg')$$

$$|\vec{r}_1 \times \vec{r}_2| = \sqrt{f'^2 g^2 \cos^2 v + f'^2 g^2 \sin^2 v + g^2 g'^2}$$

$$= \sqrt{g^2 f'^2 + g^2 g'^2}$$

$$= g\sqrt{f'^2 + g'^2}$$

$$\therefore H = |\vec{r}_1 \times \vec{r}_2| = g\,[f'^2 + g'^2]^{1/2}$$

$$\therefore (1) \Rightarrow \bar{N} = \frac{g(-f'\cos v, -f'\sin v, g')}{g\,[f'^2 + g'^2]^{1/2}} = \frac{(-f'\cos v, -f'\sin v, g')}{(f'^2 + g'^2)^{1/2}}$$

$$\bar{N} = \frac{-f'\cos v, -f'\sin v, g'}{(f'^2 + g'^2)^{1/2}}$$

Note :

Let $g(u) = u$ then the eqn of the surface of revolution is $x = u\cos v$, $y = u\sin v$, $z = f(u)$.

In the right circular cone of semi-vertical angle α is given by $g(u) = u$; $f(u) = u\cot\alpha$.

(*) 2m

5m, 2m

The anchor ring:

Anchor ring is a surface generated by rotating a circle of radius "a" about a line in its plane & at a distance "b (>a)" from its centre.

$\therefore$ The eqn of a anchor ring are,

$g(u) = b + a\cos u$

$f(u) = a\sin u$

& the domain of u, v is $0 < u < 2\pi$, $0 < v < 2\pi$.

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Helicoid

Defn (axis): 2m

A helicoid is a surface generated by the screw motion of a curve about the fixed line, the axis.

Screw motion (or) Helicoidal motion:

The curve which is simultaneously rotated with fixed axis & translated in the direction of the axis with the velocity proportional to the angular velocity of the rotation such a motion of a curve is called screw (or) helicoidal motion.

pitch:

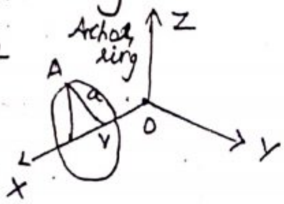
The distance translated in one complete revolution is called the pitch of the helicoid & it is the constant $2\pi a$ where $\boxed{a = \lambda / v}$

$v \rightarrow$ angle of rotation about the axis.
$\lambda \rightarrow$ distance.

Right helicoid 2m

This is the helicoid generated by a straight line which meets the axis at right angles.

If Z-axis is the axis then the p.v. is,



$r = (u\cos v, u \sin v, av)$

where u is the distance from the axis & v is the angle of rotation.

Note:

The curve v = constant are the generators & u = constant are circular helices. Since $\bar{r}_1 \cdot \bar{r}_2 = 0$ the helices are orthogonal to the generators. (ie) $\bar{r}_1 = (\cos v, \sin v, 0)$; $\bar{r}_2 = (-u\sin v, u\cos v, a)$

$\bar{r}_1 \cdot \bar{r}_2 = -u\sin v\cos v + u\cos v \sin v + 0 = 0.$

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General helicoid (Q) 2m

In the case of general helicoid the meridians that is the sections of the surfaces by planes containing the axis are congruent plane curves & the surface is generated by the screw motion of any one of these curves.

$\therefore$ The generating curve is assumed to be a plane curve given by eqn of the form,

$$x = g(u) \;;\; y = 0 \;;\; z = f(u)$$

The p.v. of a point on the surface is then,

$$\bar{r} = (g(u)\cos v,\ g(u)\sin v,\ f(u) + av)$$

Note:

The curves v = constant are the varies positions of the generating curve & u = constant are circular helixes.

Now, $\bar{r}_1 = g'\cos v,\ g'\sin v,\ f'$

$\bar{r}_2 = -g\sin v,\ g\cos v,\ a$

$$\bar{r}_1 \cdot \bar{r}_2 = (g'\cos v,\ g'\sin v,\ f') \cdot (g\sin v,\ g\cos v,\ a)$$

$$= -gg'\cos v\sin v + gg'\sin v\cos v + f'a$$

$$= a f'(u)$$

The parametric curves are orthogonal if either $f'(u) = 0$ [since $\bar{r}_1 \cdot \bar{r}_2 = 0$]

$f(u)$ = constant.

In which case the surface is a right helicoid (or) $a = 0$ which gives a surface of revolution.

(Y) VVI Ex: 4.1 5m

A helicoid is generated by the screw motion of a st. line skew to the axis. Find the curve co-planar with the axis which generates the same helicoid.

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Soln :- Let c is the shortest distance & α be the angle b/n the axis & the given skew line.

$\therefore$ The given eqn of the skew line taken as;

$x = c$; $y = u\sin\alpha$, $z = u\cos\alpha$ where u is parameter.

Rotating through an angle v about the z-axis & translating a distance av parallel to the axis. The p.r. of a point on the helicoid is,

$$\bar{r} = (c\cos v - u\sin\alpha\sin v,\ c\sin v + u\sin\alpha\cos v,\ u\cos\alpha + av) \rightarrow (1)$$

The required plane curve is the section of this surface by the plane $y=0$.

$$\therefore c\sin v + u\sin\alpha\cos v = 0$$

(ie) $u\sin\alpha\cos v = -c\sin v$

$$u = -c\frac{\tan v}{\sin\alpha}$$

(ie) $u\sin\alpha = -c\tan v$.

$(1) \Rightarrow \bar{r} = (c\cos v + c\tan v\sin v;\ c\sin v - c\tan v\cos v,\ -c\tan v\cot\alpha + av)$

$$= \left(c\cos v + c\frac{\sin^2 v}{\cos v},\ 0,\ av - c\tan v\cot\alpha\right)$$

$$= \left(c\left(\frac{\cos^2 v + \sin^2 v}{\cos v}\right),\ 0,\ av - c\tan v\cot\alpha\right)$$

$$\bar{r} = (c\sec v,\ 0,\ av - c\tan v\cot\alpha)$$

where v is the parameter, which is the required eqn of the curve.

Metric (Defn) :

Suppose $r = r(u,v)$ is the eqn of surface. Let

$E = r_1^2 = r_1 \cdot r_1$; $F = r_1 \cdot r_2 = r_2 \cdot r_1$; $G = r_2^2 = r_2 \cdot r_2$

The quadratic differential form, $E\,du^2 + 2F\,du\,dv + G\,dv^2$ in du, dv is called the first fundamental form (or) metric of the surface & the quantities by E, F, G are called the first fundamental co-efficients (or) fundamental magnitudes.

Geometrical Interpretation:

On a given surface $\bar{r} = \bar{r}(u,v)$ consider the curve defined by $u = u(t), v = v(t)$.

Then $\bar{r}$ is a fun. of t along the curve & the arc length s is related to "t".

$$\left(\frac{ds}{dt}\right)^2 = \left(\frac{dr}{dt}\right)^2 = \left(r_1 \frac{du}{dt} + r_2 \frac{dv}{dt}\right)^2$$

$$= r_1^2 \left(\frac{du}{dt}\right)^2 + 2 r_1 r_2 \frac{du}{dt} \cdot \frac{dv}{dt} + r_2^2 \left(\frac{dv}{dt}\right)^2$$

$$= E\left(\frac{du}{dt}\right)^2 + 2F \frac{du}{dt} \cdot \frac{dv}{dt} + G\left(\frac{dv}{dt}\right)^2$$

where $E = r_1^2$; $F = r_1 \cdot r_2$; $G = r_2^2$ $\longrightarrow$ (1)

(ie) $ds^2 = E\,du^2 + 2F\,du\,dv + G\,dv^2$ $\longrightarrow$ (2)

* ds can be interpreted as the "infinitesimal distance" from (u,v) to the point $(u+du, v+dv)$.

Since $(\bar{r}_1 \times \bar{r}_2)^2 = (\bar{r}_1 \times \bar{r}_2) \cdot (\bar{r}_1 \times \bar{r}_2)$

$$= \bar{r}_1^2 \cdot \bar{r}_2^2 - (\bar{r}_1 \cdot \bar{r}_2)^2.$$

Thus, $H^2 = EG - F^2$.

The co-efficients of (2) satisfy, $E > 0$, $G > 0$.

$\therefore$ $\boxed{H^2 = EG - F^2 > 0}$ (*)

These inequalities shows that the metric (2) is a +ve definite quadratic form in du, dv.

Note: 1

$H^2 = EG - F^2 \Rightarrow H = \sqrt{EG - F^2}$.

Note: 2

Obtain the I fundamental form & P.T. it is +ve definite quadratic in du, dv.

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soln: Rewrite the metric defn & interpretation we have,

$$E du^2 + 2F du dv + G dv^2 = \frac{1}{E}\left[E^2 du^2 + 2EF du dv + EG dv^2\right] \quad (\because E \neq 0)$$

$$= \frac{1}{E}\left[(E du + F dv)^2 + (EG - F^2) dv^2\right]$$

$$\geq 0, \ \forall du, dv \quad (\because EG - F^2 > 0 \ \& \ E > 0)$$

also,

$$E du^2 + 2F du dv + G dv^2 = 0$$

$\Rightarrow (E du + F dv)^2 = 0$ & $(EG - F^2) dv^2 = 0$.

$\Rightarrow E du + F dv = 0$ & $dv = 0$ $\quad (\because EG - F^2 \neq 0)$

$\Rightarrow E du = 0$ & $dv = 0$

$\Rightarrow du = 0$ & $dv = 0$ $\quad (\because E \neq 0)$

Hence, metric is a +ve definite quadratic form in du, dv.

Eg: 5.1 2m

2m Calculate E, F, G, H for the paraboloid $x = u$, $y = v$, $z = u^2 - v^2$.

soln :-

the eqn of the given surface is,

$$\bar{r} = (u, v, u^2 - v^2)$$

$$\bar{r}_1 = \frac{dr}{du} = (1, 0, 2u)$$

$$\bar{r}_2 = \frac{dr}{dv} = (0, 1, -2v)$$

$$\therefore E = r_1^2 = 1 + 0 + 4u^2 = 1 + 4u^2$$

$$F = r_1 \cdot r_2 = (0 + 0 - 4uv) = -4uv.$$

$$G = r_2^2 = 0 + 1 + 4v^2 = 1 + 4v^2 \ \&$$

hence, $H = \sqrt{EG - F^2} = \left[(1 + 4u^2)(1 + 4v^2) - (-4uv)^2\right]^{1/2}$

$$= (1 + 16u^2v^2 + 4v^2 + 4u^2 - 16u^2v^2)^{1/2}$$

$$\therefore H = \sqrt{1 + 4(u^2 + v^2)}$$

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Angle between parametric curves : 2m @ 5m

Let r_1 & r_2 be the parametric directions of the parametric curves.

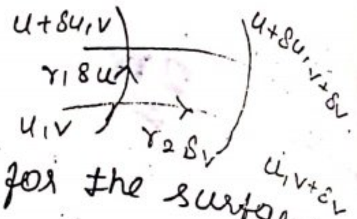
If $\omega (0 < \omega < \pi)$ is the angle b/n parametric curves

then $\cos \omega = \dfrac{r_1 \cdot r_2}{|r_1| \cdot |r_2|} = \dfrac{F}{\sqrt{EG}}$, $\sin \omega = \dfrac{|r_1 \times r_2|}{|r_1|\,|r_2|} = \dfrac{H}{\sqrt{EG}}$

Element of area :

consider with vertices (u,v), $(u+\delta u, v)$, $(u+\delta u, v+\delta v)$ & $(u, v+\delta v)$ joined by parametric curves.

when δu & δv are small & +ve then it is a parallelogram with adjacent sides given by the vectors $r_1 \delta u$, $r_2 \delta v$ & area is,



$$|r_1 \delta u \times r_2 \delta v| = H \delta u \delta v$$

$\therefore$ The element of area ds for the surface is,

$$ds = H\, du\, dv.$$

5m Ex : 5.2 5m

calculate the first fundamental co-efficient & the area corresponding to the domain $0 \le u \le 2\pi$, $0 \le v \le 2\pi$ for the anchor ring.

Soln :- $g(u) = b + a\cos u$, $f(u) = a \sin u$

The eqn of the given surface (anchor ring) is,

$$r = (g(u)\cos v, g(u)\sin v, f(u))$$

$$r = ((b + a\cos u)\cos v, (b + a\cos u)\sin v, a \sin u)$$

$$r_1 = \frac{\partial r}{\partial u} = (-a \sin u \cos v, -a \sin u \sin v, a \cos u)$$

$$r_2 = \frac{\partial r}{\partial v} = (-(b + a\cos u)\sin v, (b + a \cos u)\cos v, 0)$$

$$\therefore E = r_1^2 = a^2 \sin^2 u \cos^2 v + a^2 \sin^2 u \sin^2 v + a^2 \cos^2 u$$

$$= a^2 \sin^2 u + a^2 \cos^2 u$$

$$= a^2.$$

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$$F = r_1 \cdot r_2 = (a(b+a\cos u)\sin u \sin v \cos v)\,.\,(-a(b+a\cos u)\sin u \sin v \cos v + 0)$$

with $a(b+a\cos u)$ written above as $ab+a^2\cos u$, and $-a(b+a\cos u)$ written above as $-ab-a^2\cos u$.

$$= ab\cos u \sin u \sin v \cos v + a^2 \cos u \sin u \sin v \cos v - ab \sin u \sin v \cos v - a^2 \cos u \sin u \cos v \sin v$$

$$= 0$$

$$G = r_2^2 = (b+a\cos u)^2 \sin^2 v + (b+a\cos u)^2 \cos^2 u$$
$$= (b+a\cos u)^2$$

Now, $H = \sqrt{EG-F^2} = \sqrt{a^2(b+a\cos u)^2 - 0} = a(b+a\cos u)$

$$H = a(b+a\cos u).$$

To find surface areas:

Since the element of area is $ds = H\,du\,dv$.

Thus, the whole anchor ring corresponds to the domain $0 \le u \le 2\pi$, $0 \le v \le 2\pi$ is, $S = \int_0^{2\pi}\int_0^{2\pi} H\,du\,dv$.

(ie) $S = \int_0^{2\pi}\int_0^{2\pi} a(b+a\cos u)\,du\,dv$

$$= \int_0^{2\pi} [a(bu + a\sin u)]_0^{2\pi}\,dv$$

$$= \int_0^{2\pi} ab\,2\pi\,dv = 2ab\pi\,(v)_0^{2\pi}$$

$\therefore S = 4\pi^2 ab.$

Property of the metric:

The metric is invarient under a transf. of parameters.

Proof:

Let the parametric transf. be $u' = \phi(u,v)$; $v' = \psi(u,v)$.

$\therefore$ The eqn of the surface is $r = r(u', v')$.

Then, $r_1' = \dfrac{\partial r}{\partial u'} = \dfrac{\partial r}{\partial u} \cdot \dfrac{\partial u}{\partial u'} + \dfrac{\partial r}{\partial v} \cdot \dfrac{\partial v}{\partial u'}$

$$= r_1 \cdot \frac{\partial u}{\partial u'} + r_2 \cdot \frac{\partial v}{\partial u'} \longrightarrow (1)$$

$r_2' = \dfrac{\partial r}{\partial v'} = \dfrac{\partial r}{\partial u} \cdot \dfrac{\partial u}{\partial v'} + \dfrac{\partial r}{\partial v} \cdot \dfrac{\partial v}{\partial v'}$

$$= r_1 \cdot \frac{\partial u}{\partial v'} + r_2 \cdot \frac{\partial v}{\partial v'} \longrightarrow (2)$$

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also, $du = \frac{\partial u}{\partial u'} du' + \frac{\partial u}{\partial v'} dv'$ $\longrightarrow (3)$

$dv = \frac{\partial v}{\partial u'} du' + \frac{\partial v}{\partial v'} dv'$ $\longrightarrow (4)$

If E', F' & G' are the first fundamental coeff. in the parametric transformation then,

$E'du'^2 + 2F'du'dv' + G'dv'^2$

$= r_1'^2 du'^2 + 2r_1' r_2' du'dv' + r_2'^2 dv'^2$

$= (r_1' du' + r_2' dv')^2$

$= \left\{ \left(r_1 \frac{\partial u}{\partial u'} + r_2 \frac{\partial v}{\partial u'}\right) du' + \left(r_2 \frac{\partial v}{\partial v'} + r_1 \frac{\partial u}{\partial v'}\right) dv' \right\}^2$ (by (1) & (2))

$= \left\{ r_1 \left[\frac{\partial u}{\partial u'} du' + \frac{\partial u}{\partial v'} dv'\right] + r_2 \left[\frac{\partial v}{\partial u'} du' + \frac{\partial v}{\partial v'} dv'\right] \right\}^2$ (by (3) & (4))

$= (r_1 du + r_2 dv)^2$

$= r_1^2 du^2 + r_2^2 dv^2 + 2r_1 r_2 du dv.$

$= E du^2 + 2F du dv + G dv^2$

This shows that the metric is invariant under a parameter transformation.

Direction co-efficients 2M

Defn [Normal, Tangential component].

At a point P of a surface there are three independ vectors N, r_1 & r_2.

$\therefore$ Every vector 'a' at P can be expressed as,

$a = a_n N + \lambda r_1 + \mu r_2.$

where the scalars a_n, λ, μ are defined by this relation uniquely.

This gives 'a' as the sum of two vectors & normal to the surface & $\lambda r_1 + \mu r_2$ is the tangent plane at P.

Then the scalar a_n is called the normal component of 'a' & $a_n = a \cdot N$.

Note - 1 :

1.) The vector 'a' lies in the tangent plane iff $a_n = 0$.

2.) The vector $\lambda r_1 + \mu r_2$ is called the tangential part of "a" & λ, μ are the tangential components of "a".

Note - 2 :

λ & μ are depend only upon the tangential part of "a" & are both zero iff 'a' is normal to the surface.

Note - 3 :

If 'a' is the vector (λ, μ) then,

$|a| = |\lambda \bar{r}_1 + \mu \bar{r}_2| = (E\lambda^2 + 2F\lambda\mu + G\mu^2)^{1/2}$.

This is the magnitude of a tangential vector.

Defn (Direction co-efficients).

A direction in the tangent plane at p is described by the components of the unit vector in this direction. These components are called the direction co-eff. & are written as (l, m).

Note :

1.) The direction cosines (l, m, n) are satisfies $l^2 + m^2 + n^2 = 1$.

2.) (l, m) has unit magnitude, which satisfies

$El^2 + 2Flm + Gm^2 = 1$.

Find the angle b/n two directions on the surface at p having direction co-efficients (l, m) & (l', m') :

SM

Res Let (l, m) & (l', m') are two direction co-efficients at a point p on the surface $r = r(u, v)$.

Let 'a' & a' be the unit vectors of the directions then,

$$a = l\bar{r}_1 + m\bar{r}_2 \text{ \& } a' = l'\bar{r}_1 + m'\bar{r}_2 \longrightarrow (1)$$

Let θ be the angle b/n the two directions,

$$\therefore a \cdot a' = \cos\theta \text{ \& } a \times a' = N \sin\theta \longrightarrow (2)$$

Now, $a\cdot a' = (l\bar{r}_1 + m\bar{r}_2)\cdot(l'\bar{r}_1 + m'\bar{r}_2)$

$= ll'\bar{r}_1^2 + mm'\bar{r}_2^2 + [lm'+l'm]\bar{r}_1\bar{r}_2$

$= Ell' + F(lm'+l'm) + Gmm'$

(ie) $\boxed{\cos\theta = Ell' + F(lm'+l'm) + Gmm'}$

& $a\times a' = (l\bar{r}_1 + m\bar{r}_2)\times(l'\bar{r}_1 + m'\bar{r}_2)$

$= lm'(\bar{r}_1\times\bar{r}_2) + ml'(\bar{r}_2\times\bar{r}_1)$

$a\times a' = (lm'-l'm)(\bar{r}_1\times\bar{r}_2) \longrightarrow (3)$

$[\because \bar{r}_1\times\bar{r}_1 = 0 = \bar{r}_2\times\bar{r}_2$

$\bar{r}_1\times\bar{r}_2 = -(\bar{r}_2\times\bar{r}_1)$

since $N = \frac{\bar{r}_1\times\bar{r}_2}{H}$, $NH = \bar{r}_1\times\bar{r}_2$

(3) $\Rightarrow a\times a' = (lm'-l'm)NH$

$N\sin\theta = (lm'-l'm)NH$

$\boxed{\sin\theta = (lm'-l'm)H.}$

Note :

1.) If the two directions are orthogonal then,

$\cos\theta = 0$ (ie) $Ell' + F(lm'+l'm) + Gmm' = 0.$

2.) $\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{H(lm'-l'm)}{Ell' + F(l'm+lm') + Gmm'}$

3.) $0 \le \theta \le \pi$, $\cos\theta = a\cdot a'$ & $\sin\theta = |a\times a'|$

(ie) $\sin\theta = H\cdot|lm'-l'm|$

Defn (Direction Ratio's)

Suppose (l,m) are the direction co-eff. of a directi on the surface. The scalars λ,μ which are proporti to l,m are called direction ratio's of that direction.

Find the direction co-efficients of a direction whose direction ratio's are (λ,μ).

Let (l,m) be the direction co-eff. & direction ratio's are (λ,μ).

Then, $\frac{l}{\lambda} = \frac{m}{\mu} = k$

$\therefore l = \lambda k \; ; \; m = \mu k$

since (l, m) are directions co-efficients, then we have

$El^2 + 2Flm + Gm^2 = 1$

(ie) $E(\lambda^2 k^2) + 2F(\lambda\mu k^2) + G(\mu^2 k^2) = 1$

$k^2(E\lambda^2 + 2F\lambda\mu + G\mu^2) = 1$

$$\therefore k^2 = \frac{1}{E\lambda^2 + 2F\lambda\mu + G\mu^2}$$

$$\text{(ie) } k = \frac{1}{(E\lambda^2 + 2F\lambda\mu + G\mu^2)^{1/2}}.$$

since by defn,

$(l, m) \propto (\lambda, \mu)$

$\therefore (l, m) = k(\lambda, \mu).$

$$\text{(ie) } \boxed{(l, m) = \frac{(\lambda, \mu)}{(E\lambda^2 + 2F\lambda\mu + G\mu^2)^{1/2}}}$$

Note :

The condition for orthogonal direction is $\cos\theta = 0$

(ie) $E\lambda\lambda' + F(\lambda\mu' + \mu\lambda') + G\mu\mu' = 0$

(in terms of direction ratio's).

Note :

The vectors r_1 & r_2 have components $(1, 0)$ & $(0, 1)$ Those are the direction ratio's of the parametric directions.

(ie) $(\lambda, \mu) = (1, 0)$ & $(\lambda, \mu) = (0, 1)$.

$\therefore$ direction co-eff. are,

$$(l, m) = \frac{(1, 0)}{(E(1)^2 + 0)^{1/2}} = \left(\frac{1}{\sqrt{E}}, 0\right) \;\&$$

$$(l, m) = \frac{(0, 1)}{(0 + 0 + G)^{1/2}} = (0, 1/\sqrt{G})$$

Eg:- 6.1 ✱ 5m, 10m

Find the co-eff. of the direction which makes an angle $\pi/2$ with the direction whose co-eff. are (l, m)

soln :

Let (l', m') be the co-eff of the direction which makes an angle $\pi/2$ with the given direction (l, m)

W.K.T. if θ is the angle b/n (l, m) & (l', m') then,

$$\cos\theta = Ell' + F(lm' + l'm) + Gmm' \longrightarrow (1)$$

$$\sin\theta = H(lm' - l'm) \longrightarrow (2)$$

by given $\theta = \pi/2$

$Ell' + F(lm' + l'm) + Gmm' = 0$

(1) $\Rightarrow$ $Ell' + F(lm' + l'm) = 0$

$l'(El + Fm) + m'(Fl + Gm) = 0$

$$\frac{-l'}{Fl + Gm} = \frac{m'}{El + Fm} = \alpha \text{ (say)}$$

$$\therefore l' = -\alpha(Fl + Gm) \; ; \; m' = \alpha(El + Fm) \longrightarrow (3)$$

$$\longrightarrow (4)$$

(2) $\Rightarrow$ $H(lm' - l'm) = 1$

sub (3) in (4),

$H[\alpha l(El + Fm) + \alpha m(Fl + Gm)] = 1$

$H\alpha(El^2 + Fml + Fml + Gm^2) = 1.$

$H\alpha(El^2 + 2Fml + Gm^2) = 1$

(ie) $H\alpha = 1$; $[El^2 + 2Fml + Gm^2 = 1]$.

[For (l, m) actual direction co-eff.]

$\therefore \alpha = 1/H$

(3) $\Rightarrow$ $l' = \frac{-1}{H}[Fl + Gm]$

$m' = \frac{1}{H}[El + Fm]$

(ie) $l' = \dfrac{-[Fl + Gm]}{H}$, $m' = \dfrac{El + Fm}{H}$.

Verification:

If l', m' are direction co-eff., then it must satisfy

$El'^2 + 2Fl'm' + Gm'^2 = 1.$

For, $El'^2 + 2Fl'm' + Gm'^2 = E\left(\frac{1}{H^2}(Fl + Gm)^2\right) + 2F$

$$\left[-\frac{1}{H^2}(Fl + Gm)(El + Fm)\right] + G \cdot \frac{1}{H^2}(El + Fm)^2.$$

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$$El'^2 + 2Fl'm' + Gm'^2 = \frac{1}{H^2}\Big[E(Fl+Gm)^2 - F(Fl+Gm)(El+Fm) - F(Fl+Gm)(El+Fm) + G(El+Fm)^2\Big]$$

$$= \frac{1}{H^2}\Big\{(Fl+Gm)\big[E(Fl+Gm) - F(El+Fm)\big] + (El+Fm)\big[G(El+Fm) - F(Fl+Gm)\big]\Big\}$$

$$= \frac{1}{H^2}\Big\{\big[(Fl+Gm)(EG-F^2)m\big] + \big[(El+Fm)(EG-F^2)l\big]\Big\}$$

$$= \frac{1}{H^2}\{EG-F^2\}.\{El^2+2Flm+Gm^2\}$$

$$= \frac{1}{H^2}\cdot H^2(1) = 1$$

Thus the (l', m') are required direction co-efficients.

Note:

i) $(-l, -m)$ is the direction opposite (l, m).

ii) since formula is needed to distinguish b/n required direction & its opposite.

iii) for a given curve, $u = u(t)$, $v = v(t)$ the p.v. is $r = r(u, v) = r(t)$.

$$\therefore \frac{dr}{dt} = \dot{r} = \dot{u}r_1 + \dot{v}r_2$$

$(\dot{u}, \dot{v})$ are the components of $\dot{r}$ which are the direction ratios for the tangent to the curve.

$\therefore$ The unit tangent vector is written as,

$$\frac{dr}{ds} = \frac{du}{ds}\cdot r_1 + \frac{dv}{ds} r_2$$ & the direction co-eff., are

$$l = \frac{du}{ds}, \quad m = \frac{dv}{ds}$$

since du & dv are proportional to l, m.

$\therefore (du, dv)$ is the direction ratio.

For eg,

In the direction of the curve v = constant.

$\therefore dv = 0 \Rightarrow m = 0$

$$l = \frac{du}{\sqrt{E\cdot du^2}} = \frac{du}{du\sqrt{E}} = \frac{1}{\sqrt{E}}$$

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For a given curve by implicit eqn $\phi(u,v)=0$

diff.,
$$\frac{\partial\phi}{\partial u}du+\frac{\partial\phi}{\partial v}dv=0$$

(i.e) $\phi_1 du+\phi_2 dv=0$

$$\therefore \frac{du}{dv}=-\frac{\phi_2}{\phi_1}$$

$\therefore (-\phi_2,\phi_1)$ is the direction ratios of the tange[nt] to $\phi(u,v)=0$.

Families of curves :

Defn :-

Let $\phi(u,v)$ be a single-valued fun. of u,v posse[ssing] cont. partial derivatives ϕ_1,ϕ_2 which donot vanish [simultaneously]

Then the implicit eqn $\phi(u,v)=c$, c is real para[meter] on the surface $\bar{r}=\bar{r}(u,v)$ gives a family of curves on the surface.

Thm :-

The curve of the family $\phi(u,v)=$ constant are [the] soln of the diff. eqn $\phi_1 du+\phi_2 dv=0$ & conversely [a] first order diff., eqn of the form,

$P(u,v)du+Q(u,v)dv=0$ where P & Q are di[ff.] fun. which donot vanish simultaneously defi[nes] a family of curves.

Proof :

consider the diff. ~~curves~~ eqn.,

$$\phi_1 du+\phi_2 dv=0 \longrightarrow (1)$$

W.K.T. $\phi_1=\frac{\partial\phi}{\partial u}$; $\phi_2=\frac{\partial\phi}{\partial v}$

$(1) \Rightarrow \frac{\partial\phi}{\partial u}du+\frac{\partial\phi}{\partial v}dv=0$

(i.e) $d\phi(u,v)=0$

$\phi(u,v)=$ constant $=c$

This gives the family of curves which is the soln of eqn (1).

conversely,

consider the eqn $P(u,v)du + Q(u,v)dv = 0$ ——(2)
unless the eqn is exact.

It is not possible to find a single fun. $\phi(u,v)$ $\ni$ $\phi_1 = P$ & $\phi_2 = Q$.

$\therefore$ we can find an integrating factor $\lambda(u,v)$,
$\lambda P = \phi_1$ & $\lambda Q = \phi_2$.

$$\therefore P = \frac{\phi_1}{\lambda} \quad \& \quad Q = \frac{\phi_2}{\lambda}$$

$$P = \lambda_1 \phi_1 \quad \& \quad Q = \lambda_1 \phi_2 \quad \text{where } \lambda_1 = \frac{1}{\lambda}.$$

$$\therefore (2) \Rightarrow \lambda_1 \phi_1 du + \lambda_1 \phi_2 dv = 0$$

$$\Rightarrow \phi_1 du + \phi_2 dv = 0$$

$$d\phi(u,v) = 0$$

The soln is $\phi(u,v) =$ constant.

Note:

For the curve is given by,
$P(u,v)du + Q(u,v)dv = 0$.

The tangent vector at (u,v) is given by direction ratios $(-Q, P)$.

[since these are proportional to (du, dv)].

Defn [Orthogonal Trajectories] 2M

Let $\phi(u,v) = c$ is the eqn of the family of curves on the surface such that at each point of the two curves one from each family are orthogonal, then the family of curves is called the orthogonal trajectories of $\phi(u,v) = 0$.

Basic defn [For a given family of curves there always exists a 2nd family the orthogonal traj. such that at each point of the two curve, one from each family are orthogonal]

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Thom:

Every family of curves on a surface posses orthogonal trajectories. 8m

Proof:

Let the eqn of the surface be,

$$\bar{r} = r(u,v) \longrightarrow (1)$$

Let $\phi(u,v) = c$ be a family of curves lying on the surface,

where ϕ has cont. derivative ϕ_1 & ϕ_2 which donot vanish together.

Let $\phi_1 = P$; $\phi_2 = Q$ in $P(u,v)du + Q(u,v)dv = 0$.

$$\phi_1 du + \phi_2 dv = 0 \longrightarrow (2)$$

$$\frac{du}{dv} = -\frac{\phi_2}{\phi_1} = \frac{-Q}{P}$$

$\therefore$ $(-Q, P)$ are direction ratio's of tangent of curve of $Pdu + Qdv = 0$ if du, dv are taken be different in a orthogonal direction.

(ie) $(-Q, P)$ & (du, dv) are orthogonal.

W.K.T. if two directions (λ, μ) & (λ', μ') are orthogonal then,

$$E\lambda\lambda' + F(\lambda\mu' + \lambda'\mu) + G\mu\mu' = 0$$

$$E(-Q)du + F(-Qdv + Pdu) + G \cdot P dv = 0$$

$$(FP - EQ)du + (GP - FQ)dv = 0 \longrightarrow (3)$$

$\because$ The co-eff. du & dv are cont & donot vanish together since $EG \neq F^2$ & P and Q donot vanish together

$\therefore$ (3) is the diff. eqn of the orthogonal traj of the given family of curves & (3) is integrable.

If integral is $\phi(u,v) = \text{constant}$ $\longrightarrow$ (4)

Thus (4) is the eqn of the orth. traj. of given family of curves.

Hence, every family of curves on a surface posses orthogonal trajectories.

Thm :

The parameters on a surface can always be chosen so that the curves of a given family & their orthogonal trajectories become parametric curves.

proof :

Let the given family $\phi(u,v)=c$ of curves given by diff. eqn., $P\,du + Q\,dv = 0$ $\longrightarrow$ (1)

$\therefore$ There exists an integrating factor, $\lambda = \lambda(u,v) \neq 0$

$$P = \lambda\phi_1 \;\&\; Q = \lambda\phi_2$$

the orthogonal family $\psi(u,v) = \text{constant}$ of the given family is the soln of,

$$(FP - EQ)\,du + (GP - FQ)\,dv = 0.$$

$\exists$ a function $\mu(u,v) \neq 0 \ni FP - EQ = \mu\psi_1 ;\; GP - FQ = \mu\psi_2$

where $\psi_1 = \frac{\partial \psi}{\partial u}$; $\psi_2 = \frac{\partial \psi}{\partial v}$.

$$\partial\left(\frac{\phi,\psi}{u,v}\right) = \begin{vmatrix} \frac{\partial\phi}{\partial u} & \frac{\partial\phi}{\partial v} \\ \frac{\partial\psi}{\partial u} & \frac{\partial\psi}{\partial v} \end{vmatrix} = \begin{vmatrix} P/\lambda & Q/\lambda \\ \frac{FP-EQ}{\mu} & \frac{GP-FQ}{\mu} \end{vmatrix}$$

$$= \frac{1}{\lambda\mu}\{GP^2 - FPQ - FPQ + EQ^2\}.$$

$$= \frac{1}{\lambda\mu}\{EQ^2 - 2FPQ + GP^2\} \longrightarrow (2)$$

$\partial\left(\frac{\phi,\psi}{u,v}\right) \neq 0$ [$\because$ Quadratic in the bracket is +ve definite & P and Q donot vanish together].

Thus the transformation,

$u' = \phi(u,v)$ & $v' = \psi(u,v)$ is a proper transf. which gives the family of curves $\phi(u,v) = $ constant & their orthogonal trajectories $\psi(u,v) = $ constant become parametric curves.

$\therefore u' = $ constant $= v'$

Hence the result.

Ex: 1 10m, 5m

On the paraboloid $x^2 - y^2 = z$ find the orthogonal traj of the sections by the planes $z = $ constant.

Soln :-

The parametric eqns of the surface $x^2 - y^2 = z$ is $x = u$; $y = v$; $z = u^2 - v^2$.

If $\vec{r}$ be the p.v. then

$\bar{r} = (u, v, u^2 - v^2)$

$\bar{r}_1 = (1, 0, 2u)$

$\bar{r}_2 = (0, 1, -2v)$

$\therefore E = \bar{r}_1^{\,2} = 1 + 4u^2$

$F = \bar{r}_1 \cdot \bar{r}_2 = -4uv$

$G = \bar{r}_2^{\,2} = 1 + 4v^2$

since by given $z = $ constant.

(ie) $u^2 - v^2 = $ constant.

Let $\phi(u,v) = u^2 - v^2 = $ constant $\longrightarrow (1)$

The tangential direction at any point on the surface (1) is $(-\phi_2, \phi_1) = (v, u)$

(since $u^2 - v^2 = $ constant

Diff., $u\,du - v\,dv = 0$

$$\frac{du}{dv} = \frac{v}{u} \Rightarrow \frac{-\phi_2}{\phi_1} = \frac{v}{u}$$

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If (du, dv) is the direction ratio of the orthogonal direction at (u, v) [(l', m')].

Then from the condition of orthogonality

is $Ell' + F(lm' + l'm) + Gmm' = 0$

(ie) $Ev\,du + F(u\,du + v\,dv) + Gu\,dv = 0 \longrightarrow (2)$

(ie) $(1 + 4u^2)v\,du + (u\,du + v\,dv)(-4uv) + (1 + 4v^2)u\,dv = 0$

$v\,du + \cancel{4u^2v\,du} - \cancel{4u^2v\,du} - \cancel{4uv^2dv} + u\,dv + \cancel{4uv^2dv} = 0$

$\therefore v\,du + u\,dv = 0$

$d(uv) = 0$

$\int$ ing, $uv = \text{constant}$.

which is the eqn of orthogonal trajectories of (1).

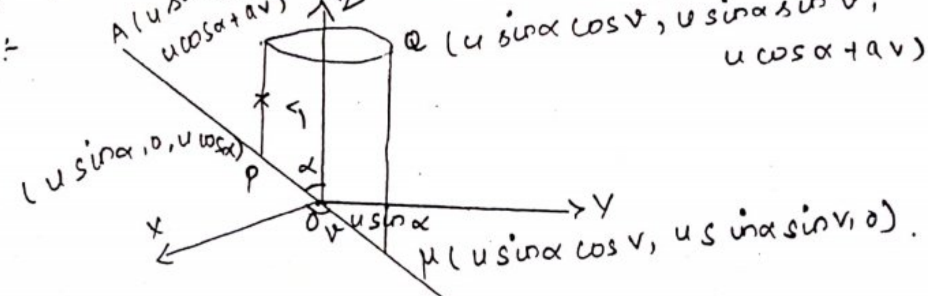
But $u = x$; $v = y$.

Thus they are the sections of the paraboloid by the hyperbolic cylinder $xy = \text{constant}$.

Ex : 7.2 ✗ 10m

✗ ✗ 10m A helicoid is generated by the screw motion of a st. line which meets the axis at an angle α. Find the orthogonal trajectories of the generators. Find also the metric of the surface referred to the generator & their orthogonal trajectories as parametric curves.

Soln :-



Take the axis of helicoid as the z-axis.

Let the generating line on be initially in the zx-plane.

The line on makes an angle α with oz. The co-ordinates of any point p on this line are $(u \sin\alpha, 0, u\cos\alpha)$.

Suppose the line PA translates through a distance λ parallel to OZ & then revolves through an angle v about OZ where $\lambda/v = a$

Let Q be the final position of P. Since the

$\therefore$ co-ordinates of Q are $(u \sin\alpha \cos v,$
$u \sin\alpha \sin v, u\cos\alpha + av)$

Thus, p.v. of Q on the helicoid is,

$r = (u\sin\alpha\cos v, u\sin\alpha \sin v, u\cos\alpha + av)$

with $g = u\sin\alpha$; $f = u\cos\alpha$.

diff $u \to$ $r_1 = (\sin\alpha\cos v, \sin\alpha \sin v, \cos\alpha)$

diff $v \to r_2 = (-u\sin\alpha \sin v, u\sin\alpha\cos v, a)$

$\therefore E = r_1^2 = \cos^2 v \sin^2\alpha + \sin^2\alpha \sin^2 v + \cos^2\alpha = 1$

$F = r_1 \cdot r_2 = -u\sin^2\alpha\cos v \sin v + u\sin^2\alpha \cos v \sin v + a\cos\alpha$

$F = a\cos\alpha.$

$G = r_2^2 = u^2\sin^2\alpha\sin^2 v + u^2\sin^2\alpha\cos^2 v + a^2$

$G = u^2\sin^2\alpha + a^2.$

The generators are given by $v =$ constant & have direction ratios $(1,0)$ [v, u]

Let (du, dv) is orthogonal to $(1,0)$ then

$E du + F(0 + dv) + G \cdot 0 = 0$

i.e. $E du + F dv = 0$

$du + a\cos\alpha\, dv = 0$

Ing. $u + a\cos\alpha\, v = \text{constant}.$

i.e. $u + av\cos\alpha = \text{constant}$

which is the orthogonal trajectories of the generators.

To find Metric.:

If the generators & their orthogonal traj. are taken as parametric curves, then the new parameter u', v' are,

$$u' = u + av\cos\alpha, \quad v' = v$$

(i.e) $u = u' - av\cos\alpha, \quad v' = v.$

Now, $r_1' = \frac{\partial r}{\partial u'} = \frac{\partial r}{\partial u}\cdot\frac{\partial u}{\partial u'} + \frac{\partial r}{\partial v}\cdot\frac{\partial v}{\partial u'}.$

$$= \frac{\partial r}{\partial u}\cdot 1 + 0$$

$$r_1' = \frac{\partial r}{\partial u} = r_1$$

Also,

$$r_2' = \frac{\partial r}{\partial v'} = \frac{\partial r}{\partial u}\cdot\frac{\partial u}{\partial v'} + \frac{\partial r}{\partial v}\cdot\frac{\partial v}{\partial v'}$$

$$= \frac{\partial r}{\partial v} = r_2 = \frac{\partial r}{\partial u}(-a\cos\alpha) + \frac{\partial r}{\partial v}(1)$$

$$r_2' = r_2 - r_1(a\cos\alpha).$$

$$E' = r_1'^2 = r_1^2 = 1$$

$$F' = r_1'\cdot r_2' = r_1(r_2 - a\cos\alpha) = a\cos\alpha - a\cos\alpha$$
$$= 0$$

$$G' = r_2'^2 = r_2^2 + a^2\cos^2\alpha\, r_1^2 - 2a\cos\alpha\, r_1\cdot r_2$$

$$= u^2\sin^2\alpha + a^2 + a^2\cos^2\alpha - 2a^2\cos^2\alpha$$

$$= u^2\sin^2\alpha + a^2 - a^2\cos^2\alpha$$

$$= u^2\sin^2\alpha + a^2\sin^2\alpha$$

$$= \sin^2\alpha(u^2 + a^2)$$

$$G' = \sin^2\alpha\{a^2 + (u' - av'\cos\alpha)^2\}$$

Thus the metric referred to new parameter is,

$$ds'^2 = E'du'^2 + 2F'du'dv' + G'dv'^2$$

$$ds'^2 = du'^2 + \sin^2\alpha\,[a^2 + (u' - a\cos\alpha\, v')^2]dv'^2$$

Double family of curves:

If P, Q & R are continuous funs. of u & v which do not vanish together the Quadratic diff. eqn.

$Pdu^2 + 2Q\,du\,dv + R\,dv^2 = 0$

represents 2 families of curves on the surface provided $Q^2 - PR > 0$.

Note :

The diff. eqn. for the seperate families are found by solving this eqn as a quadratic in du/dv i.e by solving the eqn,

$$\boxed{P\left(\frac{du}{dv}\right)^2 + 2Q\frac{du}{dv} + R = 0}$$

Thrm :

The two families are orthogonal iff $ER - 2FQ + GP = 0$.

10m, 5m

Proof :

The given quadratic diff. eqn is,

$Pdu^2 + 2Q\,du\,dv + R\,dv^2 = 0$ $\longrightarrow$ (1)

Let (l, m) & (l', m') are the direction co-eff for the two tangents at a point where l/m & l'/m' are the roots of quadratics in du/dv.

i.e l/m & l'/m' are the roots of $\longrightarrow$ (2)

$$P\left(\frac{du}{dv}\right)^2 + 2Q\left(\frac{du}{dv}\right) + R = 0$$

$\therefore$ sum of roots $= -\frac{b}{a} = -\frac{2Q}{P}$

i.e $\frac{l}{m} + \frac{l'}{m'} = -\frac{2Q}{P}$

$$\frac{lm' + l'm}{mm'} = \frac{-2Q}{P}$$

i.e $\frac{lm' + l'm}{-2Q} = \frac{mm'}{P}$ $\longrightarrow$ (3)

& product = [illegible] $= \frac{R}{P}$

(i) $\frac{ll'}{mm'} = \frac{R}{P}$

(ii) $\frac{ll'}{\frac{mm'}{R}} = \frac{mm'}{P}$ $\longrightarrow$ (4)

from (3) & (4),

$$\frac{ll'}{R} = \frac{mm'}{P} = \frac{lm' + l'm}{-2Q}$$

W.K.T. the directions (l,m) & (l',m') are orthogonal iff $E\,ll' + F(lm' + l'm) + G\,mm' = 0$

$E\,\frac{R}{P}\,mm' + F\left(-\frac{2Q}{P}\,mm'\right) + G\,mm' = 0$

$\Leftrightarrow ER - 2FQ + GP = 0$

which is the necessary & sufficient condition for the two families of curves to be orthogonal.

corollary:

The necessary & sufficient condition for parametric curves to be orthogonal is that F must be zero.

proof:

The diff. eqn of parametric curves is

$du \cdot dV = 0$.

$P\,du^2 + 2Q\,du\,dv + R\,dv^2 = 0$ is the eqn of parametric curves iff $P = 0, Q \neq 0, R = 0 \longrightarrow$ (1)

since the condition for orthogonality is,

$ER - 2FQ + GP = 0$.

(ie) $0 - 2FQ + GP = 0$ (by (1))

$FQ = 0$

$\therefore F = 0$ $(\because Q \neq 0)$

Hence it is the condition for parametric curves to be orthogonal.

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P.T. if θ is the angle at (u,v) b/n the two directions.

i.e Double family of curves, $P\,du^2 + 2Q\,du\,dv + R\,dv^2 = 0$

Then, $\tan\theta = \dfrac{2H(Q^2-PR)^{1/2}}{ER-2FQ+GP}$.

The given quadratic is,

$P\,du^2 + 2Q\,du\,dv + R\,dv^2 = 0.$

Let (l,m) & (l',m') be two direction co-eff. of the double family of curves at (u,v) in the tangential direction.

$\therefore$ l/m & l'/m' are the roots of the eqn.

$$P\left(\frac{du}{dv}\right)^2 + 2Q\left(\frac{du}{dv}\right) + R = 0$$

$$\therefore \left(l/m + l'/m'\right) = -b/a = -\frac{2Q}{P}$$

$$\left(\frac{ll'}{mm'}\right) = \frac{c}{a} = \frac{R}{P}.$$

Since, if θ is the angle b/n any two directions on a surface then,

W.K.T. $\tan\theta = \dfrac{H(lm'-l'm)}{Ell' + F(lm'+l'm) + Gmm'}$

divide N_r & D_r by mm'.

$$\tan\theta = \frac{H\left(l/m - l'/m'\right)}{E\frac{ll'}{mm'} + F\left(l/m + l'/m'\right) + G}$$

$$= \frac{H\left\{\left(l/m + l'/m'\right)^2 - \frac{4ll'}{mm'}\right\}^{1/2}}{E\frac{ll'}{mm'} + F\left(\frac{l}{m} + \frac{l'}{m'}\right) + G}$$

$$= \frac{H\left\{\frac{4Q^2}{P^2} - \frac{4R}{P}\right\}^{1/2}}{E\cdot\frac{R}{P} + F\left(-\frac{2Q}{P}\right) + G}$$

$$= \frac{H/P\,\{4Q^2 - 4RP\}^{1/2}}{ER - 2FQ + GP/P}$$

$2H\{Q^2 - RP\}^{1/2}$

... proved.

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Isometric Correspondence

Defn :- 1

Two surfaces S & S' are said to be isometric (or) applicable if ∃ a correspondence $u' = \phi(u,v)$, $v' = \psi(u,v)$ b/n their parameters, having same arc length where ϕ, ψ are single valued function & $\frac{\partial(\phi,\psi)}{\partial(u,v)} \neq 0$. such that the metric of S transform into S' such a correspondence is called a isometry.

Defn :- 2

If surface S & S' are isometric there exists a correspondence b/n their parameters where ϕ & ψ are single valued & have non-vanishing Jacobian such that the metric of S transforms into the metric of S'.

Note :

(i) If the point (u',v') on S' corresponds to (u,v) on S. then u', v' are single valued fun. of u & v.

$u' = \phi(u,v)$, $v' = \psi(u,v)$.

If S & S' are of class r & r' assume that ϕ & ψ are fun. of class min (r,r') with Jacobian $\frac{\partial(\phi,\psi)}{\partial(u,v)} \neq 0$ in the domain of u, v.

(ii) Consider a curve of class ≥1 passing through point lying on S is given by parametrically $u = u(t)$ & $v = v(t)$.

If S' is rated to S by u'I.

$u' = \phi(u,v)$; $v' = \psi(u,v)$ then C will map into a curve C' on S' passing through P' with parametric eqn.

$u' = \phi\{u(t), v(t)\}$

$v' = \psi\{u(t), v(t)\}$ & the direction of the tangent to c at p will map into a define direction at p' (ie) tangent to c' is given by $(\dot{u}', \dot{v}')$ where

$$\dot{u}' = \frac{\partial \phi}{\partial u}\cdot \dot{u} + \frac{\partial \phi}{\partial v}\cdot \dot{v}$$

$$\dot{v}' = \frac{\partial \psi}{\partial u}\cdot \dot{u} + \frac{\partial \psi}{\partial v}\cdot \dot{v}$$

solving this,

$$\dot{u} = \left(\dot{u}'\frac{\partial \psi}{\partial v} - \dot{v}'\frac{\partial \phi}{\partial v}\right) / J$$

$$\dot{v} = \left(\dot{v}'\frac{\partial \phi}{\partial u} - \dot{u}'\frac{\partial \psi}{\partial u}\right) / J \qquad \text{where } J \neq 0$$

$$J = \frac{\partial(\phi, \psi)}{\partial(u, v)}.$$

Defn :

If every point of the plane has a nbd. which is isometric with a region of the cylinder is called locally isometric which isometric

Defn :

For an isometric the length of any arc in S must be equal to the length of corr. arc in S'.

(ie) $ds = ds'$

(ie) the metric of S, $\therefore$ transforms into the metric of S'.

U.Q (x) Ex : 8.1 (x) . 10m, 5m

Find the surface of revolution which is isometric with a region of the right helicoid.

<u>Soln</u> :- W.K.T.

the surface of revolution is,

$\bar{r} = (g(u)\cos v, g(u)\sin v, f(u))$

for some fun. f & g.

$\therefore$ Its metric is,

$$ds^2 = E\,du^2 + 2F\,du\,dv + G\,dv^2.$$

$$E = \bar{r}_1^2 = (g'\cos v, g'\sin v, f')\cdot(g'\cos v, g'\sin v, f')$$
$$= g'^2 + f'^2$$

$$F = \bar{r}_1\cdot\bar{r}_2 = (g'\cos v, g'\sin v, f')\cdot(-g'\sin v, g'\cos v, 0)$$

$$G = \bar{r}_2^2 = g^2.$$

$$ds^2 = (g'^2 + f'^2)\,du^2 + g^2 dv^2 \longrightarrow (1)$$

If the pitch of the right helicoid is $2\pi a$. &

its eqn is,

$$\bar{r} = (u'\cos v', u'\sin v', av')$$
$$\bar{r}_1' = (\cos v', \sin v', 0)$$
$$\bar{r}_2' = (-u'\sin v', u'\cos v', a)$$

$$E' = \bar{r}_1'^2 = 1;\quad G' = \bar{r}_2'^2 = a^2 + u^2;\quad F' = \bar{r}_1'\cdot\bar{r}_2' = 0$$

$\therefore$ Its metric is,

$$ds'^2 = du'^2 + (u'^2 + u^2)\,dv'^2 \longrightarrow (2)$$

To find a transformation:

from $(u,v) \longrightarrow (u',v')$ which makes these two matrices identical.

Taking $u' = \phi(u)$, $v' = v$.

$$du' = \phi'\,du,\quad dv' = dv.$$

Now,

the matrices are identical if,

$$\phi'^2 = g'^2 + f'^2 \text{ \& } \phi^2 + a^2 = g^2.$$

If we take $\phi = a\sinh u$, $g = a\cosh u$ then,

$\phi^2 + a^2 = g^2$ is satisfied when from $\phi'^2 = g'^2 + f'^2$

$$\phi' = \frac{\partial\phi}{\partial u} = a\cosh u \Rightarrow \phi'^2 = a^2\cosh^2 u$$

$$g' = \frac{\partial g}{\partial u} = a\sinh u \Rightarrow g'^2 = a^2\sinh^2 u$$

$$\therefore \phi'^2 - g'^2 = f'^2$$
$$f'^2 = a^2(\cosh^2 u - \sinh^2 u)$$
$$f'^2 = a^2$$
$$f' = a$$

(ii) $\left(\frac{d\phi(u)}{du}\right)^2 = a^2$

$d\phi(u) = a\, du$

Sing, $\phi(u) = au$ is its soln.

Hence the right helicoid is isometric with the surface obtained by revolving the curve $x = a \cosh u$, $y = 0$, $z = au$ about the z-axis.

Thus, the surface of revolution isometric to right helicoid is $\bar{r} = (a \cosh u, 0, au)$

The generating curve is the catenary $x = a \cosh(z/a)$ with parameter a & direction the z-axis & the surface of revolution isometric ~~to ri~~ is catenoid.

Note :

i) The correspondence $u' = a \sinh u$, $v' = v$ s.t. the generator $v' =$ constant on the helicoid correspond to the meridians $v =$ const on the catenoid & the helices, $u' =$ constant corresponds to parallel s. $u =$ constant.

ii) On the helicoid u' & v' can take all values but on the catenoid $0 \leq v \leq 2\pi$.

The correspondence is therefore isometric only for that region of the helicoid for which $0 \leq v' \leq 2\pi$ (ie) one period.

iii) Hence one period of a right helicoid of pitch $2\pi a$ corresponds isometrically to the whole catenoid of parameter a.

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Ex: 8.1

P.7. The region of the right helicoid is $|u'| < \frac{a}{\sqrt{p^2-1}}$, $0 < v' < 2p\pi$, $p>1$ corresponds isometrically on the surface.

$x = ap\cosh u$, $y = 0$, $z = \int_0^a (\cosh^2 t - p^2 \sinh^2 t)^{1/2} dt$

given by, $|u| < \cosh^{-1}(p/\sqrt{p^2-1})$ about the z-axis.

Soln :-

Making the transf. $u' = \phi(u)$, $v' = pv$

The metric of the right helicoid becomes,

$$\phi'^2 du^2 + (\phi^2 + a^2) p^2 dv^2 \quad \text{——— (1)}$$

by (8.1) (1) =>

$$ds^2 = (g'^2 + f'^2) du^2 + g^2 dv^2 \quad \text{——— (2)}$$

comparing (1) & (2) we get,

$$f'^2 + g'^2 = \phi'^2 \quad \& \quad g^2 = \phi^2 + a^2.$$

Now, choose $\phi(u) = a\sinh u$ we find

$g(u) = ap\cosh u$.

Hence the curve in the xoz plane is

$x = ap\cosh u$; $y = 0$ & we determine $f(u)$ by integration

$$f'^2(u) = \phi^2 - g'^2 = a^2(\cosh^2 u - p^2 \sinh^2 u)$$

so that, $f(u) = a\int_0^a (\cosh^2 t - p^2 \sinh^2 t)^{1/2} dt$.

using the variation of u' & v', let us find the variation of u, v.

From the hypothesis, $u' \leq \frac{a}{\sqrt{p^2-1}}$, $0 < v' < 2p\pi$.

$$u' = \phi(u) = a\sinh u < \frac{a}{\sqrt{p^2-1}}$$

from the basic relation,

$$a^2\cosh^2 u = a^2 + a^2\sinh^2 u.$$

we find $a^2\cosh^2 u \leq a^2 + \frac{a^2}{p^2-1} \leq \frac{a^2p^2}{(p^2-1)}$

so that, $\cosh u \leq \frac{p}{p^2-1}$ (or) $v = \cosh^{-1}(p/\sqrt{p^2-1})$

$$|v| = \cosh^{-1}(p/\sqrt{p^2-1})$$

about the z-axis & $0 < v' < 2p\pi$.

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Unit - IV

P. Nandhini
II - M.Sc (Maths)

The second Fundamental Form

Local Non - Intrinsic properties of a surface :

Defn: [Intrinsic and Non-intrinsic properties]

Any formula or property of a surface which can be deduced only from the metric of the surface without knowing its eqn is called an intrinsic property of the surface.

The properties of the surface which are not intrinsic are called non-intrinsic properties of the surface

Second Fundamental Form:

Derivation:

5M

Let $r = r(u,v)$ be the eqn of the surface and P be any point (u,v) on it.

If $r = r(s)$ is a curve through P on this surface then the normal curvature k_n of the curve at P is,

$$k_n = N \cdot r''$$

Where $\bar{N}$ is the unit normal vector to the surface at P. we have

$$r' = \frac{dr}{ds} = \frac{dr}{du} \cdot \frac{du}{ds} + \frac{dr}{dv} \cdot \frac{dv}{ds}$$

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$$r'' = r_1 u'' + u' \frac{dr_1}{ds} + r_2 v'' + v' \frac{dr_2}{ds}$$

$$r'' = r_1 u'' + r_2 v'' + \left(\frac{dr_1}{du} \cdot \frac{du}{ds} + \frac{dr_1}{dv} \cdot \frac{dv}{ds}\right) u' + \left(\frac{dr_2}{du} \cdot \frac{du}{ds} + \frac{dr_2}{dv} \cdot \frac{dv}{ds}\right) v'$$

$$= r_1 u'' + r_2 v'' + r_{11} u'^2 + r_{12} u' v' + r_{21} u' v' + r_{22} v'^2$$

(i.e) $r'' = r_1 u'' + r_2 v'' + r_{11} u'^2 + 2 r_{12} u' v' + r_{22} v'^2 \quad (\because r_{1} = r_{2})$

$\therefore k_n = r'' \cdot N = (r_1 u'' + r_2 v'' + r_{11} u'^2 + 2 r_{12} u' v' + r_{22} v'^2) \cdot N$

But $r_1 \cdot N = 0$; $r_2 \cdot N = 0$

$\therefore k_n = (N \cdot r_{11}) u'^2 + 2 (N \cdot r_{12}) u' v' + (N \cdot r_{22}) v'^2$

$$k_n = L \frac{du^2}{ds^2} + 2M \frac{du}{ds} \cdot \frac{dv}{ds} + N \frac{dv^2}{ds^2}$$

(i.e) $k_n = \dfrac{L du^2 + 2M du dv + N dv^2}{ds^2}$

$$k_n = \frac{L du^2 + 2M du dv + N dv^2}{E du^2 + 2F du dv + G dv^2}$$

Where L, M, N are defined by,

$L = N \cdot r_{11}$; $M = N \cdot r_{12}$, $N = N \cdot r_{22}$

$\boxed{L du^2 + 2M du dv + N dv^2}$ is called the "second fundamental form" and the functions of u & v denoted by L, M, N are called the "second fundamental" co-efficients.

It follows from k_n that all curves having the same direction at P have the same normal curvature.

Hence, normal curvature is a property of a surface and a direction at a point on the surface.

MEUSNIER'S THEOREM: Statement (2m)

5m

If ϕ denotes the angle between the principal normal $\vec{n}$ to a curve on the surface and the surface normal N at P, then

$$k \cos\phi = \vec{k_n}$$

Prf:

Let r'' is the curvature vector of given curve at P, then

$$r'' = k \bar{n}$$

$$\therefore r'' \cdot N = k (\bar{n} \cdot N)$$

$$r'' \cdot N = k \cos\phi \qquad \because nN = |n|$$

But for all

The valu

normal

of r''

curve

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But for all curves having the same direction at P, the value of $r''.N$ is fixed and is equal to the normal curvature k_n in that direction at P, the value of $r''.N$ is fixed and is equal to the normal curvature k_n in that direction

$$\therefore k_n = k \cos\phi$$

Note:

If $k_n = k\cos\phi$, then $k_n = k$ iff $\phi = 0$. Thus the curvature of a curve at P is equal to the normal curvature at P in the direction of that curve iff the principal normal to the curve is along the surface normal at that point.

Defn: (Type of points)

Since $$k_n = \frac{L du^2 + 2M du dv + N dv^2}{E du^2 + 2F du dv + G dv^2}$$

The denominator is +ve definite sign of ~~the~~ k_n depends only upon the sign of numerator.

(i) If at a point P on the surface $LN - M^2 > 0$ then the k_n maintains same sign for all directions at P. Then P is called an "elliptic point"

(ii) If $LN - M^2 = 0$, then k_n retains the same sign for all directions through P except one for which the curvature is zero. Then P is called a "Parabolic point."

(iii) If $LN - M^2 < 0$ then k_n is +ve for directions lying within a certain angle, negative for directions lying outside this angle and zero along the directions which form the angle; then P is called a "hyperbolic" and the critical directions are called "asymptotic directions".

2m

Example 1.1

Show that when the parameters are transformed the discriminant $LN - M^2$ is multiplied by the square of the Jacobian determinant of the transformation and deduce that the conditions for an elliptic, Parabolic or hyperbolic point are thus independent of the particular parametric representation chosen.

Soln:

Let the transformation of parameters form u, v to u', v' be given by the relations

$$u' = \phi(u, v) \; ; \; v' = \psi(u, v)$$

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$r'' = r_{11}''$ ... of the transformation is,

$$J = \frac{\partial(u', v')}{\partial(u, v)} = \begin{vmatrix} \frac{\partial u'}{\partial u} & \frac{\partial u'}{\partial v} \\ \frac{\partial v'}{\partial u} & \frac{\partial v'}{\partial v} \end{vmatrix}$$

$$= \frac{\partial u'}{\partial u}\cdot\frac{\partial v'}{\partial v} - \frac{\partial u'}{\partial v}\cdot\frac{\partial v'}{\partial u}$$

let $r_1 = \frac{\partial r}{\partial u}$; $r_2 = \frac{\partial r}{\partial v}$; $N_1 = \frac{\partial N}{\partial u}$; $N_2 = \frac{\partial N}{\partial v}$ also

let $r_1' = \frac{\partial r}{\partial u'}$; $r_2' = \frac{\partial r}{\partial v'}$; $N_1' = \frac{\partial N}{\partial u'}$; $N_2' = \frac{\partial N}{\partial v'}$

Let L, M, N are the second fundamental co-efficients when the parameters are u, v and let L', M', N' be their values when the parameters are u', v'. we have

$$r_1 = \frac{\partial r}{\partial u} = \frac{\partial r}{\partial u'}\cdot\frac{\partial u'}{\partial u} + \frac{\partial r}{\partial v'}\cdot\frac{\partial v'}{\partial u}$$

$$r_1 = r_1'\frac{\partial u'}{\partial u} + r_2'\frac{\partial v'}{\partial u}$$

$$r_2 = \frac{\partial r}{\partial v} = r_1'\frac{\partial u'}{\partial v} + r_2'\frac{\partial v'}{\partial v}$$

$$\therefore r_1 \times r_2 = \left(r_1'\frac{\partial u'}{\partial u} + r_2'\frac{\partial v'}{\partial u}\right) \times \left(r_1'\frac{\partial u'}{\partial v} + r_2'\frac{\partial v'}{\partial v}\right)$$

$$= \left\{\frac{\partial u'}{\partial u}\cdot\frac{\partial v'}{\partial v} - \frac{\partial v'}{\partial u}\cdot\frac{\partial u'}{\partial v}\right\}(r_1' \times r_2')$$

since $r_1 \times r_1 = 0$; $r_2' \times r_2' = 0$; $r_1' \times r_2' = -r_2' ...$ Prf

$$r_1 \times r_2 = J(r_1' \times r_2')$$

$|||^{ly}$ $N_1 \times N_2 = J(N_1' \times N_2')$

Now

$$LN - M^2 = (-N_1 \cdot r_1)(-N_2 \cdot r_2) - (-N_2 \cdot r_1)(-N_1 \cdot r_2)$$

$$= (r_1 \cdot N_1)(r_2 \cdot N_2) - (r_1 \cdot N_2)(r_2 \cdot N_1)$$

$$= (r_1 \times r_2)(N_1 \times N_2) \quad | \because \text{by lagrange's identity}$$

$$= J(r_1' \times r_2') \cdot \{J(N_1' \times N_2')\}$$

$$= J^2\{(r_1' \times r_2') \cdot (N_1' \times N_2')\}$$

$$= J^2(L'N' - M'^2)$$

$$LN - M^2 = J^2(L'N' - M'^2)$$

Now in a proper transformation $J \neq 0$

$\therefore LN - M^2 > 0, = 0, < 0$

$\Rightarrow L'N' - M'^2 > 0, = 0, < 0$

Thus the nature of points does not change under a change of parameters.

Geometrical Intrepretation:

Exercise : 1.2

S.T the anchor ring contains all the types of points.

Prf:

The eqn of the given surface is.

$$r = \{(b+a\cos u)\cos v,\ (b+a\cos u)\sin v,\ a\sin u\}$$

$$r_1 = (-a\sin u\cos v,\ -a\sin u\sin v,\ a\cos u)$$

$$r_2 = (-(b+a\cos u)\sin v,\ (b+a\cos u)\cos v,\ 0)$$

$$r_1 \times r_2 = (b+a\cos u)(-a\cos u\cos v,\ -a\cos u\sin v,\ -a\sin u)$$

also $r_{11} = (-a\cos u\cos v,\ -a\cos u\sin v,\ -a\sin u)$

$$r_{12} = (a\sin u\sin v,\ -a\sin u\cos v,\ 0)$$

$$r_{22} = (-(b+a\cos u)\cos v,\ -(b+a\cos u)\sin v,\ 0)$$

$E = r_1^2 = a^2$; $F = r_1 \cdot r_2 = 0$, $G = r_2^2 = (b+a\cos u)^2$

$\therefore H^2 = EG - F^2 = a^2(b+a\cos u)^2$

$H = a(b+a\cos u)$

also, $LH = r_{11} \cdot (r_1 \times r_2) = (b+a\cos u)a^2$

$MH = r_{12} \cdot (r_1 \times r_2) = 0$

$NH = r_{22} \cdot (r_1 \times r_2) = (b+a\cos u)^2 a\cos u$

$$\therefore L = \frac{a^2(b+a\cos u)}{a(b+a\cos u)} = a$$

$M = 0$; $N = (b+a\cos u)\cos u$

Now

$$LN - M^2 = a(b+a\cos u)\cos u$$

The domain of u is $0 < u < 2\pi$ also $b > a$

$\therefore b + a\cos u$ is +ve $\forall$ a in its domain when $\pi/2 < u < 3\pi/2$, $\cos u$ is -ve & so $LN - M^2$ is -ve.

(i.e) $LN - M^2 < 0$

$\therefore$ all points in this region are hyperbolic points

When $u = \pi/2, 3\pi/2$, $\cos u = 0$ and so $LN - M^2 = 0$

$\therefore$ all Points in this region are parabolic when $u = \pi/2$ or $3\pi/2$.

When $0 < u < \pi/2$ (or) when $3\pi/2 < u < 2\pi$ $\cos u$ is +ve and therefore $LN - M^2 > 0$

$\therefore$ all such points are elliptic points.

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Another expressions for L, M & N

since $N \cdot r_1 = 0$

Diff $N_1 r_1 + N \cdot r_{11} = 0 \longrightarrow (1)$

$N_2 r_1 + N \cdot r_{12} = 0 \longrightarrow (2)$

IIIly

$N \cdot r_2 = 0 \Rightarrow$

diff $N_2 \cdot r_2 + N r_{22} = 0 \longrightarrow (3)$

$N_1 r_2 + N \cdot r_{21} = 0 \longrightarrow (4)$

$(1) \Rightarrow N \cdot r_{11} = -N_1 r_1 \longrightarrow (5)$

$(2) \Rightarrow N \cdot r_{12} = -N_2 \cdot r_1 \longrightarrow (6)$

$(3) \Rightarrow N \cdot r_{22} = -N_2 \cdot r_2 \longrightarrow (7)$

$(4) \Rightarrow N \cdot r_{21} = -N_1 \cdot r_2 \longrightarrow (8)$

From (5) & (8)

$$N \cdot r_{12} = N \cdot r_{21} = -N_2 \cdot r_1 = -N_1 \cdot r_2$$

We know that

$$L = N \cdot r_{11} ; \quad M = -N_1 \cdot r_2 = -N_2 \cdot r_1$$

$$L = -N_1 r_1 ; \quad M = -N_1 \cdot r_2 = -N_2 \cdot r_1$$

$$N = -N_2 \cdot r_2 \quad (\because \text{by } (5), (6), (7) \,\&\, (8))$$

Principal curvatures.

To find the eqn giving the principal curvatures at a point.

WKT, the normal curvature at P in a direction whose co-efficients (l, m) are given by,

$$k = Ll^2 + 2Mlm + Nm^2 \longrightarrow (1)$$

Where

$$El^2 + 2Flm + Gm^2 = 1 \longrightarrow (2)$$

since L, M, N are fixed at P the value of curvature at P depends upon the value of l, m.

To find the extreme values of k :

We shall use the lagrange's multiplier write $k = Ll^2 + 2Mlm + Nm^2 - \lambda(El^2 + 2Flm + Gm^2)$

when k is stationary.

$$\therefore \frac{\partial k}{\partial l} = 2Ll + 2Mm - 2\lambda El - 2Flm = 0$$

$$\frac{1}{2}\frac{\partial k}{\partial l} = Ll + Mm - \lambda El - \lambda Fm = 0 \longrightarrow (3)$$

and $\frac{\partial k}{\partial m} = 2Ml + 2Nm - 2\lambda Fl - 2\lambda Gm = 0$

$$\frac{1}{2}\frac{\partial k}{\partial m} = Ml + Nm - \lambda Fl - \lambda Gm = 0 \longrightarrow (4)$$

multiply (3) by l, (4) by m and adding we get,

$$Ll^2 + Mlm - \lambda El^2 - \lambda Flm + Mlm + Nm^2 - \lambda lFm - \lambda Gm^2 = 0$$

$Ll^2 + 2Mlm + Nm^2$ ... $= 0$

(ie) $k - \lambda = 0 \Rightarrow \boxed{\lambda = k}$

$\therefore$ (3) $\Rightarrow$

$$Ll + Mm - k(El + Fm) = 0 \longrightarrow (5)$$

(4) $\Rightarrow$

$$Ml + Nm - k(Fl + Gm) = 0 \longrightarrow (6)$$

Now, we shall eliminate l, m between (5) & (6)

$$(5) \Rightarrow (L - kE)l = (kF - M)m \longrightarrow (7)$$

$$\Rightarrow (kF - M)l = (N - Gk)m \longrightarrow (8)$$

$$\frac{(7)}{(8)} \Rightarrow \frac{L - kE}{kF - M} = \frac{kF - M}{N - Gk}$$

$$(L - kE)(N - Gk) = (kF - M)(kF - M)$$

$$LN - LkG - kNE + k^2EG = k^2F^2 - 2FkM + M^2$$

$$k^2(EG - F^2) - k(LG + EN - 2FM) + LN - M^2 = 0$$

This gives the maximum (or) minimum values of normal curvature at P.

The roots of this eqn are called the "Principal curvature" is denoted by k_a, k_b

We've

$$k_a + k_b = \frac{EN + GL - 2FM}{EG - F^2}$$

$$k_a k_b = \frac{LN - M^2}{EG - F^2}$$

Def:

(i) **First curvature:**

The sum of the principal curvature k_a & k_b at a point is called the first curvature at that point and is denoted by J.

(ie) $J = k_a + k_b = \dfrac{EN + GL - 2FM}{EG - F^2}$

(ii) **Mean curvature (or) Mean normal curvature:**

2M. The arithmetic mean of the principal curvatures k_a & k_b at a point is called the mean curvature at that point and is denoted by μ.

Thus $\mu = \frac{1}{2}(k_a + k_b) = \dfrac{EN + GL - 2FM}{2(EG - F^2)}$

Note:

Some author's denote mean normal curvature by B. Thus $B = \frac{1}{2}(k_a + k_b)$ and amplitude of normal curvature is A by $A = \frac{1}{2}(k_b - k_a)$. obviously $k_a = B - A$; $k_b = B + A$.

(iii) **Gaussian curvature:**

The product of the principal curvature k_a and k_b at a point is called the Gaussian curvature at that point and is denoted by K

Thus $K = k_a \cdot k_b = \dfrac{LN - M^2}{EG - F^2}$

It is also called [illegible]

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Since the normal curvature at a point P on a surface has different values in different directions. The directions at P in which normal curvature has maximum (or) minimum values are called principal directions at P

To find the eqn giving the principal directions at a point on the surface,

WKT,

the normal curvature at P in a direction co-efficient (l, m) is

$$K = Ll^2 + 2Mlm + Nm^2$$

where

$$El^2 + 2Flm + Gm^2 = 1$$

The principal directions at P are there in which normal curvature has max or min values.

Write $K = Ll^2 + 2Mlm + Nm^2 - \lambda(El^2 + 2Flm + Gm^2 - 1)$.

then when K is stationary

$$\frac{1}{2}\frac{\partial K}{\partial l} = Ll + Mm - \lambda(El + Fm) = 0 \longrightarrow (1)$$

$$\frac{1}{2}\frac{\partial K}{\partial m} = Ml + Nm - \lambda(Fl + Gm) = 0 \longrightarrow (2)$$

Eliminating λ from (1) & (2)

$$(1) \Rightarrow \lambda(El + Fm) = Ll + Mm \longrightarrow (3)$$

$$(2) \Rightarrow \lambda(Fl + Gm) = Ml + Nm \longrightarrow (4)$$

$$\frac{(3)}{(4)} \Rightarrow \frac{El + Fm}{Fl + Gm} = \frac{Ll + Mm}{Ml + Nm}$$

$$\Rightarrow (El + Fm)(Ml + Nm) = (Ll + Mm)(Fl + Gm)$$

$$EMl^2 + ENlm + FMlm + FNm^2 = FLl^2 + LGlm + MFlm + MGm^2$$

$$EMl^2 + ENlm + FMlm + FNm^2 - FLl^2 - LGlm - MFlm - MGm^2 = 0$$

$$l^2(EM - FL) + lm(EN + \cancel{FM} - LG - \cancel{FM}) + m^2(FN - MG) = 0$$

$$(EM - FL)l^2 + (EN - LG)lm + (FN - MG)m^2 = 0 \longrightarrow (5)$$

$\therefore$ The discriminant of this eqn is,

$$(EN - GL)^2 - 4(EM - FL)(FN - GM)$$

(5) gives the principal directions at P, the discriminant is identically equal to,

$$4\left(\frac{EG - F^2}{E^2}\right)(EM - FL)^2 + \left\{EN - GL - \frac{2F}{E}(EM - FL)\right\}^2$$

since $EG - F^2 > 0$ it follows that the roots of (5) are real and distinct, provided that the co-efficients E, F, G and L, M, N are not proportional.

Where these are proportional then the principal directions are indeterminate, and the normal curvature is the same in all directions.

Umbilic:

A point on a surface is called an umbilic if at that point we've $\frac{L}{E} = \frac{M}{F} = \frac{N}{G}$ and at an umbilic the normal curvature is the same in all directions.

P.T at a point which is not an umbilic the principal directions are orthogonal:

Equation of principal directions at p is

$$[EM-FL]l^2 + (EN-GL)lm + (FN-GM)m^2 = 0 \longrightarrow (1)$$

If (du,dv) are ratio's of the direction whose co-efficients are (l,m) then $\frac{l}{du} = \frac{m}{dv}$.

$$\therefore (1) \Rightarrow (EM-FL)du^2 + (EN-GL)dudv + (FN-GM)dv^2 = 0 \longrightarrow (2)$$

It is quadratic in $\frac{du}{dv}$

WKT,

the two directions given by

$Pdu^2 + Qdudv + Rdv^2 = 0 \longrightarrow (3)$ are orthogonal iff $ER - FQ + GP = 0$. Hence from (2) & (3)

$P = EM-FL$; $Q = EN-GL$; $R = FN-GM$

Now,

$$ER - FQ + GP = E(FN-GM) - F(EN-GL) + G(EM-FL)$$

$$= EFN - EGM - FEN + FGL + EGM - FGL$$

$$= 0$$

$\therefore$ The two directions given by (2) are orthogonal.

Thus principal directions are orthogonal.

Agal
3. Lines of curvature:

Def:

A curve on a surface whose tangent at each point is along a principal direction is called a line of curvature.

Equations of lines of curvature:

Suppose (du,dv) is direction of a line of curvature through the point $P(u,v)$, then (du,dv) is a principal direction at p.

If k is the principal curvature at p, then we have

$$(L-kE)du + (M-kF)dv = 0$$

$$(M-kF)du + (N-kG)dv = 0$$

5m, 10m These are the eqns of a line of curvature.

Thm: (Rodrigues formula):

(1) A necessary and sufficient condition that a curve on a surface be a line of curvature is that $kd\vec{r} + d\vec{N} = 0$ at each point ...

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(du, dv) be a line of curvature on the surface. Then the direction (du, dv) is a principal direction at the point (u, v)

$$(L - \kappa E)du + (M - \kappa F)dv = 0 \longrightarrow (1)$$

and $$(M - \kappa F)du + (N - \kappa G)dv = 0 \longrightarrow (2)$$

where κ is one of the principal curvature at (u,v). (1) can be written as,

$$(L\,du + M\,dv) - \kappa(E\,du + F\,dv) = 0 \longrightarrow (3).$$

Put $L = -N_1 r_1$; $M = -N_2 r_1$; $E = r_1 \cdot r_1$; $F = r_1 \cdot r_2$

$\therefore (3) \Rightarrow$

$$-(N_1 r_1 du + N_2 r_1 dv) + \kappa(r_1 r_1 du + r_1 r_2 dv) = 0$$

$$(N_1 du + N_2 dv) r_1 + \kappa(r_1 du + r_2 dv) r_1 = 0$$

$$r_1 dN + (\kappa dr) r_1 = 0$$

since $dN = \dfrac{\partial N}{\partial u} \cdot du + \dfrac{\partial N}{\partial v} \cdot dv$ & $dN = N_1 du + N_2 dv$,

$dr = \dfrac{\partial r}{\partial u} \cdot du + \dfrac{\partial r}{\partial v} \cdot dv$ $\quad dr = r_1 du + r_2 dv$

(i.e) $(\kappa\, dr + dN) r_1 = 0 \longrightarrow (4)$

|||ly, put $M = -N_1 r_2$; $N = -N_2 \cdot r_2$

$F = r_1 \cdot r_2$; $G = r_2 \cdot r_2$ in (2)

we get,

$$(\kappa\, dr + dN) \cdot r_2 = 0 \longrightarrow (5)$$

N is a vector of constant modulus.

$\therefore dN$ is $\perp^r$ to N and so dN is tangential to the surface also dr is tangential vector to the surface.

$\therefore \kappa\, dr + dN$ lies in the plane of r_1 & r_2

We claim that $\kappa\, dr + dN = 0$

For if $\kappa\, dr + dN \neq 0$ then from (4) & (5)

$\kappa\, dr + dN$ is parallel to $r_1 \times r_2$

(ie) parallel to N.

Which is a $\Rightarrow\Leftarrow$

$\therefore \kappa\, dr + dN = 0.$

Conversely,

Suppose that, $\kappa\, dr + dN = 0$ along a curve for function κ. then along that curve, we've,

$$(\kappa\, dr + dN) \cdot r_1 = 0 \text{ and } (\kappa\, dr + dN) \cdot r_2 = 0$$

$\therefore$ If (du, dv) is the direction of that at (u, v) then we find

$$(L - \kappa E)du + (M - \kappa F)dv = 0$$

$$(M - \kappa F)du + (N - \kappa G)dv = 0$$

also $\kappa\, dr + dN = 0$

$$\kappa\, dr = -dN.$$

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$\Rightarrow K(r_1 du + r_2 dv) = -(N_1 du + N_2 dv)$

$K(r_1 du + r_2 dv) \cdot r_1 du + r_2 dv = -(N_1 du + N_2 dv) \cdot (r_1 du + r_2 dv)$

$K(E\,du^2 + 2F\,du\,dv + G\,dv^2) = L\,du^2 + 2M\,du\,dv + N\,dv^2$

$$K = \frac{L\,du^2 + 2M\,du\,dv + N\,dv^2}{E\,du^2 + 2F\,du\,dv + G\,dv^2}$$

$\therefore K$ is the normal curvature at (u,v) in (du,dv).

$\therefore$ The direction at each point of the curve is a principal direction and so the curve is a line of curvature on the surface)

Thm: (~~Rodrigues formula~~) (X) 8m, 10m

A necessary and sufficient conditions that parametric curves be lines of curvature are $F=0$; $M=0$

Pf:

Let the eqn of the surface be $r = r(u,v)$

Then the differential eqn of lines of curvature is,

$(EM-FL)du^2 + (EN-GL)du\,dv + (FN-GM)dv^2 = 0 \rightarrow (1)$

also the diff. eqn of parametric curve is $du\,dv = 0 \rightarrow (2)$

First suppose that the lines of curvature are taken as the parametric curves.

Then the diff eqn (1) & (2) must be identical

$EM - FL = 0 \rightarrow (3)$

Now we know that the two families of lines of curvature are orthogonal.

Since the lines of curvature are taken as the parametric curves

$\therefore$ The parametric curves are orthogonal. Hence we've $F=0$

$\therefore (3) \Rightarrow EM = 0 \Rightarrow M = 0 \quad (\because E \neq 0)$

Thus $F=0$; $M=0$ are necessary conditions for parametric curves to be lines of curvature.

Conversely,

if $F=0$; $M=0$

then (1) of lines of curvature reduces to

$(EN-GL)du\,dv$

(i.e) $du\,dv = 0$

Thus the lines of curvature are the parametr[ic curves].

Hence the theorem.

Euler's Theorem (X) 5m

Let k_a, k_b be the principal curvatures o[f] a surface at any point P on it. Then the n[ormal] curvature k_n at P in the direction making...

angle ψ with that principal direction through p in which the normal curvature k and is given by $k = k_a \cos^2\psi + k_b \sin^2\psi$.

Prf:

Let the lines of curvature be taken as parametric curves. Then $F=0$; $M=0$

Let p be any point on the surface. Then the principal direction through p are nothing but the directions of the two parametric curves passing through p.

If k_a, k_b are principal curvatures at p then k_a, k_b are the normal curvatures at p in the directions of the parametric curves through p.

Let k_a = normal curvature at p in the direction of the curve u = constant whose direction co-efficients are $(0, 1/\sqrt{G})$ and

k_b = normal curvature at p in the direction of the curve v = constant whose direction co-efficients are $(1/\sqrt{E}, 0)$

Let (l, m) be the direction co-efficients of the direction through p making an angle ψ with the direction of the curve u = constant.

direction, then

$$k = Ll^2 + 2Mlm + Nm^2 = Ll^2 + Nm^2 \longrightarrow (1)$$

$\therefore$ from (1)

$$k_a = L\cdot 0 + N\left(\frac{1}{\sqrt{G}}\right)^2 = \frac{N}{G} \quad (\because l=0 ; m=$$

& $$k_b = L\left(\frac{1}{\sqrt{E}}\right)^2 + N\cdot 0 = \frac{L}{E} \quad (\because 1/\sqrt{E}, m=0)$$

Now ψ is the angle b/w (l,m) & $(0, 1/\sqrt{G})$

$$\therefore \cos\psi = E(l\cdot 0) + G(m \cdot 1/\sqrt{G})$$

$$\cos\psi = m(\sqrt{G})$$

also the principal directions at p are at angles.

$\therefore \pi/2 - \psi$ is the angle b/w the directions (l,m) & $(1/\sqrt{E}, 0)$

$$\therefore \cos(\pi/2 - \psi) = \sin\psi = E(l \cdot 1/\sqrt{E}) + 0$$

$$= l(\sqrt{E})$$

from these, we get

$$l = \frac{\sin\psi}{\sqrt{E}} ; \quad m = \frac{\cos\psi}{\sqrt{G}}$$

$$\therefore (1) \Rightarrow k = \frac{L}{E}\sin^2\psi + \frac{N}{G}\cos^2\psi$$

$$= k_b \sin^2\psi + k_a \cos^2\psi$$

$$\therefore k = k_a \cos^2\psi + k_b \sin^2\psi$$

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...: (The Dupin Indicatrix)

Suppose 'o' is a point on the given surface. The section of the surface by a place $\parallel^{ed}$ to the tangent plane at O and infinitely choose to it called the Dupin indicatrix at o.

Derive the eqn of Dupin indicatrix. 1pm (8)

: Suppose O is the given point on the surface.
Let Q be a point on the Dupin indicatrix of o.
If h is the length of the $\perp^r$ QM from Q to the tangent plane at O. then

$$2h = L\,du^2 + 2M\,du\,dv + N\,dv^2 \longrightarrow (1)$$

If we take the lines of curvatures as parametric curves then $F = 0$; $M = 0$

$$\therefore (1) \Rightarrow 2h = L\,du^2 + N\,dv^2 \longrightarrow (2)$$

The normal curvature k at o in the direction (du, dv) is given by,

$$k = \frac{L du^2 + 2M du dv + N dv^2}{E du^2 + 2F du dv + G dv^2}$$

$$= \frac{L du^2 + N dv^2}{E du^2 + G dv^2} \quad \Big| \because F = 0 \; ; M = 0$$

Since lines of curvature taken as parametric curves, therefore the direction of parametric curves

v = constant & u = constant are principal directions at o.

Ratios (1, 0) & (0, 1)

If k_a & k_b are the ~~principle~~ principal curvatures at o then from (3) we get,

$$k_a = \frac{L}{E} \; ; \; k_b = N/G \longrightarrow (4)$$

substituting the values of L and N from (4) in (2) we have,

$$\therefore 2h = k_a E du^2 + k_b G dv^2 \longrightarrow (5)$$

If ds_1, ds_2 denotes the elements of arc length of the curves v = constant and u = constant at o, then,

$$ds_1^2 = E du^2 \; ; \; ds_2^2 = G dv^2$$

$$\therefore (5) \Rightarrow 2h = k_a ds_1^2 + k_b ds_2^2 \longrightarrow (6)$$

Now,

Let us take the point o as origin ox, oy along the principal directions at o and oz along the normal to the surface at o.

If the co-ordinates of Q on the indicatrix are (x, y, z) then

$$z = 2h \; ; \; x = ds_1 \; ; \; y = ds_2 \qquad \Big| \because R_a = \frac{1}{k_a}, \; R_b = \frac{1}{k_b}$$

$$\therefore (6) \Rightarrow z = 2h \; : \; 2h = k_a x^2 + k_b y^2$$

$$\Rightarrow z = 2h \; ; \; \frac{x^2}{R_a} + \frac{y^2}{R_b} = 2h \longrightarrow (7)$$

where R_a, R_b be the reciprocals of k_a, k_b. This conic is known as Dupin's Indicatrix.

Case (i)

If O is an elliptic point, then at O, k_a, k_b have the same sign

$\therefore$ the conic is an ellipse with semiaxes of lengths $(2hR_a)^{1/2}$; $(2hR_b)^{1/2}$. Thus at an elliptic point the indicatrix is an ellipse.

The ellipse is real or imaginary according to sign of h.

Case (ii)

If O is an hyperbolic point than at O, k_a, k_b are the different signs.

$\therefore$ The indicatrix at O is one of the two conjugate hyperbolas according to the sign of h.

In this case the directions of the asymptotes at O are called asymptotic directions at O.

Case (iii)

If O is a parabolic point, then at 'O' one of the principal curvature is zero (ie) either $k_a = 0$ (or) $k_b = 0$.

Thus at a parabolic point indicatrix is a pair of parallel lines.

Two directions at P are said to be conjugate if the corresponding diameters of the Dupin indicatrix are conjugate

Defn: (Asymptotic line)

An asymptotic line is a curve whose direction at every point is asymptotic

The eqn of asymptotic lines is

$$\frac{dr}{ds} \cdot \frac{\delta N}{\delta s} = 0$$

(ie) $L\,du^2 + 2M\,du\,dv + N\,dv^2 = 0$

$\therefore$ asymptotic lines are self-conjugate

Thm:

The two directions (l_1, m_1) & (l_2, m_2) at a point on the surface are conjugate then,

$$L l_1 l_2 + M(l_1 m_2 + l_2 m_1) + N m_1 m_2 = 0$$

Prf:

WKT,

the eqn of the Dupin Indicatrix is,

$$\frac{x^2}{R_a} + \frac{y^2}{R_b} = 2h \text{ and } z = 2h \longrightarrow (1)$$

Let (du, dv) and $(\delta u, \delta v)$ be the conjugate directions at P. Let θ_1 and θ_2 be the angle made by conjugate direction with the principal direction at

$\therefore m_1 =$

from (1)

also if th

m_1

(ie) tan

But k_a

$\therefore$ ta

θ_1

$(l_1, m$

and

$\therefore m_1 = \tan\theta_1$; $m_2 = \tan\theta_2$

from ①, $a^2 = 2R_a h = \frac{2h}{k_a}$

$\therefore b^2 = 2R_b h = \frac{2h}{k_b}$

also if the directions are conjugate then

$m_1 m_2 = -\frac{b^2}{a^2}$

(ie) $\tan\theta_1 \cdot \tan\theta_2 = \frac{-2h/k_b}{2h/k_a} = \frac{-k_a}{k_b}$

But $k_a = \frac{L}{E}$ and $k_b = \frac{N}{G}$

$\therefore \tan\theta_1 \cdot \tan\theta_2 = \frac{-L/E}{N/G} = \frac{-LG}{NE} \longrightarrow ②$

find $\tan\theta_1$ & $\tan\theta_2$:

"θ_1" is the angle b/w the direction $(\frac{1}{\sqrt{E}}, 0)$ & (l, m) which corresponds to (du, dv)

since $F=0$; WKT,

$$\tan\theta = \frac{H(lm' - l'm)}{Ell' + Gmm'} = \frac{H(l(0) - \frac{1}{\sqrt{E}}m)}{El\frac{1}{\sqrt{E}} + Gm(0)}$$

$$= \frac{H(-\frac{m}{\sqrt{E}})}{E\frac{l}{\sqrt{E}}} = \frac{-Hm}{El} = \frac{-(\sqrt{EG - F^2})m}{El}$$

$$= \frac{-\sqrt{E}\sqrt{G}m}{El} \quad (\because F=0)$$

$$\tan\theta_1 = \frac{-\sqrt{G}}{\sqrt{E}} \cdot \frac{m}{l} = -\sqrt{\frac{G}{E}} \cdot \frac{dv}{du}$$

Illy,

$$\tan\theta_2 = -\sqrt{\frac{G}{E}} \cdot \frac{\delta v}{\delta u}$$

$$② \Rightarrow \frac{G}{E} \cdot \frac{dv}{du} \cdot \frac{\delta v}{\delta u} = -\frac{L}{N} \cdot \frac{G}{E}$$

$$\therefore \frac{dv}{du} \cdot \frac{\delta v}{\delta u} = -\frac{L}{N} \Rightarrow \frac{dv}{du} = -\frac{L}{N} \cdot \frac{\delta u}{\delta v}$$

(ie) $N dv \delta v = -L du \delta u$

$\Rightarrow L du \delta u + N dv \delta v = 0 \longrightarrow ③$

since $\frac{du}{ds} = l_1$; $\frac{dv}{ds} = m_1$; $\frac{\delta u}{\delta s} = l_2$; $\frac{\delta v}{\delta s} = m_2$.

$\therefore ③ \Rightarrow L l_1 l_2 + N m_1 m_2 = 0 \longrightarrow ④$

If $M \neq 0$, then $M du \delta v = M dv \delta u = 0$

$④ \Rightarrow L l_1 l_2 + M(du\delta v + dv \delta u) + N m_1 m_2 = 0$

since the principal axes are orthogonal we've

$du \delta v = 0$ and $dv \delta u = 0$

which is the required condition.

Corollary:

The parametric curves are conjugate if $M=0$

Prf:

The directions of the parametric curves $v=$ constant and $u=$ constant are $(\overset{l_1}{\frac{1}{\sqrt{E}}}, \overset{m_1}{0})$ & $(\overset{l_2}{0}, $

WKT,

the condition for conjugate directions is,

$$L l_1 l_2 + M(l_1 m_2 + l_2 m_1) + N m_1 m_2 = 0$$

$(\sqrt{E}\ \sqrt{G}\ \ldots)\ N(u)=0$

$$M\left(\frac{1}{\sqrt{E}}\cdot\frac{1}{\sqrt{G}}\right)=0$$

Since $E>0$ and $G>0$. From this, $M=0$.

4. Developables? 2M

The envelope of one parameter family of plane is called a Developable surface (or) Developable. The equation of such a family is given by the equation $\bar{r}\cdot\bar{a}=p$, where $\bar{a}$ and p are functions of a real parameter u.

Defn: Characteristic line? 2M

The line of intersection of the two consecutive planes is called as characteristic lines.

Characteristic point:

When the planes $f(u)=0$; $f(v)=0$; $f(w)=0$ intersects at a point the limiting position of the point of intersection of the planes as $v\to u$ & $w\to u$ is called the characteristic point corr.- to the plane u.

Edge of regression: 2M

The locus of the characteristic point is called the edge of regression of the developable.

5M In general, a developable ... of two sheets which are tangent to the edge of regression along a sharp edge.

Prf:

Suppose C is the edge of regression of the developement. Let O be the point $s=0$ on C.

Let OX, OY, OZ be a set of rectangular cartesian axes along $\bar{t}, \bar{n}, \bar{b}$ at O.

If R is the position vector of any point (x,y,z) on the developable then $R=x\bar{t}+y\bar{n}+z\bar{b}$

also $R=\bar{r}+v\bar{t}$

$\therefore R(s)=r(s)+v\cdot t(s)$

$$= s\bar{t}+\tfrac{1}{2}s^2k\bar{n}+\tfrac{1}{6}s^3(k'\bar{n}+k\tau\bar{b}-k^2\bar{t})+O(s^4)$$

$$+v\{\bar{t}+sk\bar{n}+\tfrac{1}{2}s^2(k'\bar{n}+k\tau\bar{b}-k^2\bar{t})+O(s^3)\}$$

Comparing the co-efficients of $\bar{t}$ from ① & ② we've

$$x=s-\tfrac{1}{6}s^3k^2+O(s^4)+v\left\{1-\frac{s^2}{2}k^2+O(s^3)\right\}$$

$\therefore$ the normal plane $x=0$ meets the surface where $v=-s-\tfrac{1}{3}s^3k^2+O(s^4)$

Substitute v in ② and comparing the co-efficients of $\bar{n}$ & $\bar{b}$ from ① & ② we get

$$y = -\tfrac{1}{2} k s^2 + O(s^3)$$

$$z = -\tfrac{1}{3} k (\tau s^3) + O(s^4)$$

Eliminating s from these, we get,

$$z^2 = -\frac{8}{9}\frac{\tau^2}{k} y^3$$

It follows that the developable cuts the normal plane to the edge of regression in a whose tangent is along the principal normal.

$\therefore$ The two sheets of the developable are thus tangent to the edge of regression along a sharp edge.

5. **Developables associated with space curves:**

Def: (Tangential Developable (or) Osculating Developable)

The envelope of the family of osculating plane of a space curve is called an osculating developable.

Its characteristic line are tangent to the curve and hence this is also called tangential developable.

Polar developable:-

The envelope of a family of normal plane to a skew curve form the polar developable

Rectifying developables:

The envelope of the family of rectifying planes of a space curve is called rectifying developables

Thm:

P.T the edge of regression is the curve itself

Prf:

Let the eqn of the space curve be $r = r(s)$

let R be the position vector of any point on the osculating plane.

Then the eqn of the osculating plane is

$$f(s) = (R-r)\cdot\vec{b} = 0 \longrightarrow ①$$

Diff 1- w.r.to 's'

$$f'(s) = (R-r)b' - b\cdot r' = 0$$

Using serret frenet formula,

$$(R-\bar{r})\cdot(\tau\bar{b} - k\bar{t}) - \bar{n}\cdot\bar{t} = 0$$

since $\bar{n}\cdot\bar{t} = 0$

$$\therefore (R-\bar{r})(\tau\bar{b} - k\bar{t}) = 0$$

since $k \neq 0$ & using ②,

$$(R-r)\cdot\bar{t} = 0 \longrightarrow ③$$

The point of intersection of eqns ①, ② the characteristic point and its locus is the e regression.

∴ the edge of regression is the given curve

Thm:

P.T the edge of regression of the polar developable is the locus of center of spherical curvature of the given curve.

Prf:

Let the eqn of the given space curve be,

$r = r(s) \longrightarrow (1)$

∴ The eqn of the normal plane of (1) at any point $r(s)$ on it is,

$(R-r).t = 0 \longrightarrow (2)$

(2) is the eqn of the family of planes containing a single parameter s.

The envelope of (2) is the polar developable of (1).

diff (2) partially w.r.to 's'.

$(R-r)t' - r'.t = 0$

$(R-r)kn - t.t = 0$

$(R-r)k.n = 1$

$(R-r).n = 1/k = \rho \longrightarrow (3)$

diff w.r. to 's'.

$(R-r)n' - r'.n = \rho'$

$(R-r)(\tau\bar{b} - k\bar{t}) - t.n = \rho'$

$(R-r).\tau b = \rho'$ $\quad (\because t.n = 0 \;\&\; (R-r).t = 0$

The point of intersections of the planes (5) is the char- point and the locus of the char- point is the edge of regression of the developable of (1).

(2) ⇒ $(R-r)$ is $\perp^r$ to t.

∴ $(R-r)$ lies in the plane of n & b

So let $R-r = \lambda\bar{n} + \mu\bar{b} \longrightarrow (5)$

Taking scalar product by $\bar{n}$, we get,

$(R-r)\bar{n} = \lambda\bar{n}.\bar{n} + \mu\bar{b}.\bar{n}$

$\rho = \lambda$ $\quad \because \bar{b}.\bar{n} = 0$; (R-

again scalar product by $\bar{b}$,

$(R-r)\bar{b} = \lambda\bar{n}.\bar{b} + \mu\bar{b}.\bar{b}$

$(R-r).\bar{b} = \mu$

$\sigma\rho' = \mu$ $\quad | \because \bar{n}.\bar{b} = 0$ & by (4)

∴ (5) ⇒ $R - r = \rho\bar{n} + \sigma\rho'\bar{b}$

$R = \bar{r} + \rho\bar{n} + \sigma\rho'\bar{b}$

∴ R is the position vector of the of spherical curvature.

Hence edge of regression of the po developable is the locus of the centre of sp curvature.

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P.T the edge of regression of the rectifying developable has equation $R = \bar{\gamma} + k\left(\frac{\tau\bar{t}+k\bar{b}}{k'\tau - k\tau'}\right)$

Prf:

Let the eqn of the given space curve be $\gamma = \gamma(s) \longrightarrow (1)$

The eqn of the rectifying plane of (1) at any point $\gamma(s)$ on it is $(R-\bar{\gamma})\bar{n} = 0 \longrightarrow (2)$

$\therefore$ The envelope of (2) is the rectifying developable of (1).

Diff (2) w.r.to 's' we get

$(R-\gamma)\cdot n' - \gamma' n = 0$

$(R-\bar{\gamma})(\tau\bar{b} - k\bar{t}) - \bar{t}\bar{n} = 0$

$(R-\bar{\gamma})(\tau\bar{b} - k\bar{t}) = 0 \longrightarrow (3) \quad | \because \bar{t}\cdot\bar{n} = 0$

diff w.r.to 's'.

$(R-\gamma)(\tau b' + \tau' b - kt' - k't) - \gamma'(\tau b - kt) = 0$

$(R-\gamma)(-\tau^2 n + \tau' b - k^2 n - k't) - t(\tau b - kt) = 0$

$(R-\gamma)(\tau' b - k't) + k = 0 \longrightarrow (4)$ $\quad b' = -\tau n, \; t' = kn$

The edge of regression is the locus of the point intersection of the planes (2) (3), (4) from (2) & (3)

$(R-\gamma)$ is $\perp^r$ to both $\bar{n}$ & $\tau\bar{b} - k\bar{t}$

$\therefore (R-\gamma)$ is $\parallel^{el}$ to $\bar{n}\times(\tau\bar{b} - k\bar{t}) = \tau(\bar{n}\times\bar{b}) - k(\bar{n}\times\bar{t})$

$= \tau\bar{t} + k\bar{b}$

$(R-\gamma) = \lambda(\tau\bar{t} + k\bar{b}) \longrightarrow (5)$ for some scalar

Taking scalar product of (5) with $\tau'\bar{b} - k't'$

$(R-\bar{\gamma})(\tau' b - k't) = \lambda(\tau\bar{t} + k\bar{b})(\tau' b - k'\bar{t})$

$-k = \lambda(0 - \tau k' + k\tau')$

$\lambda = \frac{-k}{-k'\tau + k\tau'}$ $\quad$ | by (4)

$\lambda = \frac{k}{-k\tau' + k'\tau}$

(5) $\Rightarrow \quad (R-\bar{\gamma}) = \frac{k(\tau\bar{t} + k\bar{b})}{k'\tau - k\tau'}$

$R = \bar{\gamma} + \frac{k(\tau\bar{t} + k\bar{b})}{k'\tau - k\tau'}$ as the eqn of the regression of the rectifying developable.

Thm : 5.1 (a) 10m, 5m

The necessary and sufficient condition for a surface to be a developable surface is that its Gaussian curvature shall be zero.

Prf:

Let the eqn of the edge of regression on developable be $\gamma = \gamma(s)$.

Let P be any point on the ...

... is the char. line of the developable

Then the position vector R can be taken as

$R(s,v) = r(s) + v\,t(s)$ —— ①

diff. ① p w.r.to 's'

$R_1 = \frac{\partial R}{\partial s} = \frac{dr}{ds} + v\frac{dt}{ds}$

$= \bar{t} + vk\bar{n}$

diff ① p w.r.to 'v'

$R_2 = \frac{\partial R}{\partial v} = \bar{t}$

$R_{11} = \frac{dt}{dv} + vk\bar{n}' + vk'\bar{n}$

$R_{11} = k\bar{n} + vk(\tau\bar{b} - k\bar{t}) + vk'\bar{n}$

$R_{12} = k\bar{n}$

$R_{21} = t' = k\bar{n}$

$R_{22} = 0$

$E = R_1 \cdot R_1 = (\bar{t} + vk\bar{n})\cdot(\bar{t} + vk\bar{n}) = 1 + v^2k^2$

$F = R_1 \cdot R_2 = (\bar{t} + vk\bar{n})\cdot\bar{t} = 1$

$G = R_2 \cdot R_2 = \bar{t}\cdot\bar{t} = 1$

$H^2 = EG - F^2 = 1 + v^2k^2 - 1 = v^2k^2$

$N = \frac{R_1 \times R_2}{H} = \frac{(t + vk\bar{n})\times\bar{t}}{vk} = \frac{-vkb}{vk} = -b$

$L = N\cdot R_{11} = -\bar{b}(k\bar{n} + vk(\tau\bar{b} - k\bar{t}) + vk'\bar{n}) = vk\tau$

$M = N\cdot R_{12} = -\bar{b}\cdot k\bar{n} = 0$

$N = N\cdot R_{22} = -(-\bar{b})0 = 0$

the ...

$k = \frac{LN - M^2}{EG - F^2} = 0$

(ie) $k = 0$

Conversely,

assume that, $k = 0$

To prove it is a developable surface. To prove this we have to s.t $r = r(u,v)$ is generated by a single parameter family of planes.

WKT,

$L = -r_1\cdot N_1$, $M = -r_1\cdot N_2$; $N = -r_2\cdot N_2$; M...

$LN - NM = (r_1\cdot N_1)(r_2\cdot N_2) - (r_1\cdot N_2)(r_2\cdot N_1)$

$LN - M^2 = (r_1\times r_2)\cdot(N_1\times N_2)$

since $r_1\times r_2 = HN$

$\therefore LN - M^2 = HN(N_1\times N_2)$

$= H[N\ N_1\ N_2]$

as $H \neq 0$ & $k = 0 \Rightarrow LN - M^2 = 0 \Rightarrow [N\ N_1\ N_2]$...

since $N\cdot N = 1 \Rightarrow N\cdot N_1 = 0$ & $N\cdot N_2 = 0$

$\therefore N$ is $\perp^r$ to both N_1 & N_2

$\therefore N$ is parallel to $N_1\times N_2$

$\therefore [N\ N_1\ N_2] = N\cdot(N_1\times N_2) \neq 0$.

unless $N_1 = 0$ (or) $N_2 = 0$ (or) $N_1 \parallel N_2$

$[N\ N_1\ N_2]$ ...

$N_1 =$ ...

The ...

dif...

...

$N_1 = 0$ (or) $N_2 = 0$

The eqn of the tangent plane is,

$(R-r)N = 0$

diff. p w r to 'v'

$\frac{\partial}{\partial v}\{(R-r) - N\} = 0$

$(R-r).N_2 - r_2.N = 0$

Now, $N_2 = 0$ and since r_2 is tangential to the [s]urface.

$r_2 . N = 0$ $\therefore \frac{\partial}{\partial v}\{(R-r).N\} = 0$

$\therefore (R-r).N$ is independent of v.

$\therefore$ The eqn of tangent plane contains only one [par]ameter s.

$\therefore$ The surface is the envelope of a single [par]ameter family of planes.

iii)

Let $N_1 = k.N_2$.

[C]onsider a suitable change of parameters from [u, v] to u', v'

Let the transformation be,

$u = u'+v'$; $v = u'-kv'$

Now, $N_1' = \frac{\partial N}{\partial u'} = \frac{\partial N}{\partial u}.\frac{\partial u}{\partial u'} + \frac{\partial N}{\partial v}.\frac{\partial v}{\partial u'}$

$= N_1 + N_2 \pm n$

$N_2' = N_1 - kN_2$

This show that N_1' & N_2' are not parallel since $N_2' = 0$ as in case (i)

The tangent plane at P is a single parameter family of planes

$\therefore$ The given surface is a developable surface.

6. Developables associated with curves on surfaces:

5. Monge's Theorem: 7(a) 10m

A necessary and sufficient condition that a curve on a surface be a line of curvature is that the surface normals along the curve form a developable.

Prf:

Let $r = r(s)$ be a curve lying on the surface $r = r(u, v)$

Let N be the unique surface normal at $r = r(s)$ on the curve.

$\therefore$ N is a function of 's'. We proves the thm by two steps:

step: 1

To prove the normals to the surface $r = r(u[,v])$ along the curve $r = r(s)$ form a developable iff $[t \; N \; N'] = 0$

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$R(s,v) = r(s) + v N(s)$

Where r is the distance b/w P & Q.

By thm,

WKT the surface generated by the surface normal is a developable iff its Gaussian curvature is zero Thm 5.1

$\Rightarrow LN - M^2 = 0$ at every point

To find L, M, N. Now,

$$R_1 = \frac{\partial R}{\partial s} = \frac{dr}{ds} + v\frac{dN}{ds}$$

$$R_1 = t + vN'$$

$$R_{11} = t' + vN'' = k\vec{n} + vN''$$

$$R_2 = \frac{dR}{dv} = N$$

$$R_{22} = 0 \; ; \; R_{12} = N' \; ; \; R_{21} = N'$$

Now,

$$L = -R_{11} N = -(kn + vN'')N \neq 0$$

WKT

$$HM = [R_{12} \; R_1 \; R_2]$$

$$= [N' \; t + vN' \; N]$$

$$= [N' t N] + [N' \; vN' \; N]$$

$$HM = [N' \; t \; N] + 0$$

$$M = \frac{1}{H}[N' \; t \; N]$$

since $R_{22} = 0$ & $H \neq 0$

we get $HN = [R_{22} \; R_1 \; R_2] = 0$

$\therefore N = 0$

as $L \neq 0$ & $N = 0 \Rightarrow LN - M^2 = 0$ iff $M = 0$

$$\Rightarrow \frac{1}{H}[t \; N \; N'] = 0 \Rightarrow [t \; N \; N'] = 0$$

step : 2

Here we prove that $[t \; N \; N'] = 0$ is necessary and sufficient condition for $r = r(s)$ to be a line of curvature.

Assume that $r = r(s)$ is the line of curvature.

$\therefore$ By Rodrigue's formula we get,

$$k\frac{dr}{ds} + \frac{dN}{ds} = 0 \Rightarrow kr' = -N' \Rightarrow kt = -N'$$

(i.e) $kt = -N'$

$$\therefore [t \; N \; N'] = [t \; N \; -kt] = 0$$

The surface normal along the curve $r = r(s)$ for a developable surface.

Conversly,

assume that $[t \; N \; N'] = 0$

To prove : $r = r(s)$ is the line of curvature on the surface.

$$\therefore [t \; N \; N'] = 0 \Rightarrow [t \; N' N] = 0$$

$$\Rightarrow [t \times N'] \cdot N = 0$$

also, $N \neq 0$ and $N^2 = N \cdot N = 1$; $\therefore N' \perp r$

(i.e) N' is in the tangent ...

$\therefore (t \times N')$ is parallel to N.

If $t \times N' \neq 0$, $(t \times N').N \neq 0$

We conclude that, $(t \times N').N = 0$

$\Rightarrow (t \times N') = 0$

which is true iff one vector is a scalar multiple of the other.

We can take $N' = -kt$ for some k

$\Rightarrow N' + kt = 0 \Rightarrow k\frac{dr}{ds} + \frac{dN}{ds} = 0$

which gives the Rodrigues formula $r = r(s)$ is the line of curvature.

Minimal surfaces:

Def:

If the mean curvature $\mu = \frac{1}{2}(k_a + k_b)$ is 0 at all points of the surface then the surface is called the minimal surface.

By defn,

$$\mu = \frac{EN + GL - 2FM}{2(EG - F^2)}$$

Since $EG - F^2 \neq 0$. $\therefore$ The conditions for the minimal surface is $EN + GL - 2FM = 0$.

Thm: SM

If there is a surface of minimum area passing through a closed curve then it is necessarily a minimal surface in the space that it is of zero mean curvature.

Prf:

Let Σ be a surface bounded by closed curve c'. Let Σ' be another surface derived from Σ by a small displacement.

In the direction of the normal, assume that,

$\frac{\partial \epsilon}{\partial u} = \epsilon_1$ & $\frac{\partial \epsilon}{\partial v} = \epsilon_2$ are small

$\epsilon_1 = O(\epsilon)$; $\epsilon_2 = O(\epsilon)$ as $\epsilon \to 0$

If R denotes the position vector of the displaced surface.

$R = r + \epsilon N$

$R_1 = r_1 + \epsilon_1 N + N_1 \epsilon$

$R_2 = r_2 + \epsilon_2 N + N_2 \epsilon$

Let E^*, F^*, G^* be the point fundamental co-effi[cients] of Σ'.

Then $E^* = R_1 . R_1 = (r_1 + \epsilon_1 N + N_1 \epsilon).(r_1 + \epsilon_1 N + N_1 \epsilon)$

$= (r_1 . r_1) + \epsilon_1 (r_1 . N) + (r_1 . N_1)\epsilon + \epsilon_1 (N . r_1) +$

$\epsilon_1^2 (N.N) + \epsilon_1 \epsilon (N.N_1) + (N_1 . r_1)\epsilon + (N_1 . N_1)\epsilon^2$

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$= r_1 \cdot r_1 + 2\epsilon \mu N \cdot r_1 + 2\epsilon\mu (r_1 \cdot N_1) + o(\epsilon^2)$

Since $r_1 \cdot N = 0$ & $L = -r_1 \cdot N_1$. we've

$$E^* = r_1^2 + 0 - 2\epsilon L + o(\epsilon^2)$$

$$E^* = E - 2\epsilon L + o(\epsilon^2)$$

|||ly,

$$F^* = R_1 \cdot R_2 = F - 2\epsilon M + o(\epsilon^2)$$

$$G^* = R_2 \cdot R_2 = G - 2\epsilon N + o(\epsilon^2)$$

$$\therefore H^{*2} = E^* G^* - F^{*2}$$

$$= [E - 2\epsilon L + o(\epsilon^2)][G - 2\epsilon N + o(\epsilon^2)] - [F - 2\epsilon M + o(\epsilon^2)]^2$$

$$= (EG - F^2) - 2\epsilon (EN - 2FM + GL) + o(\epsilon^2)$$

$$= H^2 - 2\epsilon (2H^2 \mu) + o(\epsilon^2)$$

$$H^{*2} = H^2 - 4\epsilon H^2 \mu + o(\epsilon^2)$$

$$\therefore H^* = \{H^2 - 4\epsilon H^2 \mu + o(\epsilon^2)\}^{1/2}$$

$$= \{H^2(1 - 4\epsilon\mu) + o(\epsilon^2)\}^{1/2}$$

$$= H(1 - 4\epsilon\mu)^{1/2} + o(\epsilon^2)$$

$$H^* = H(1 - 2\epsilon\mu) + o(\epsilon^2) \text{ as } \epsilon \to 0$$

Let A & A^* denote the area of the surfaces Σ & Σ'.

WKT

$$A = \int_{\Sigma} H \, du\,dv \quad \& \quad A^* = \int_{\Sigma'} H^* \, du\,dv.$$

$$A^* = \int_{\Sigma_1} H(1 - 2\epsilon\mu)\, du\,dv + o(\epsilon^2)$$

$$= \int_{\Sigma_1} H\,du\,dv - \int_{\Sigma'} 2\epsilon\mu H\,du\,dv + o(\epsilon^2)$$

$$A^* = A - \int_{\Sigma'} 2\epsilon\mu H\,du\,dv + o(\epsilon^2)$$

$$A^* - A = -\int_{\Sigma'} 2\epsilon\mu H\,du\,dv + o(\epsilon^2)$$

Let $A^* - A = \delta A$

$$\therefore \delta A = -\int_{\Sigma'} 2\epsilon \mu H\,du\,dv$$

Where $\epsilon \to 0$. we can ~~can~~ omit $o(\epsilon^2)$.

δA is first variation of the area enclosed by fixed curve c.

$\therefore \delta A$ vanishes iff $\mu = 0$

(i.e) the mean curvature vanishes. So that the surface is a minimal surface.

Ex. C.T.1

Show that the lines of curvature also an isothermal net on a minimal surface.

Prf:

WKT, a surface is minimal if

$$EN - 2FM + GL = 0 \longrightarrow (1)$$

Let us take the lines of curvature

...ves. Then

$\therefore (1) \Rightarrow$

(i.e) L

Then $F = 0$

$L_2 =$

$N_1 =$

$\therefore$

...iction

curves. Then $F = 0 = M$

$\therefore$ (1) $\Rightarrow EN + GL = 0$

(i.e) $L/E + N/G = 0 \longrightarrow$ (2)

When $F = 0 = M$

$L_2 = -\frac{1}{2} E_2 \left(L/E + N/G \right)$

$N_1 = -\frac{1}{2} G_1 \left(L/E + N/G \right)$

$\therefore L_2 = 0 \quad N_1 = 0$ [$\because$ by (2)]

$\therefore L$ is a function of u only and N is a function of v only.

$$\frac{\partial^2}{\partial u \partial v} \log \frac{E}{G} = \frac{\partial^2}{\partial u \partial v} \log\left(-L/N\right)$$

$$= \frac{\partial^2}{\partial u \partial v} \log(-L) - \log(N) = 0$$

Thus we've $F = 0$; $\frac{\partial^2}{\partial u \partial v} \log E/G = 0$

Hence the parametric curves i.e) lines of curvature and isometric lines.

Thus the lines of curvature form an isothermal net on a minimal surface.

8. Ruled Surfaces:

Def:

A surface generated by the motion of straight line with one degree of freedom is called ruled surface.)

The various positions of straight lines of the family are called the generator of ruling.

Base curve:

Any curve c on a ruled surface with ~~the family are called the generators of~~ ~~[illegible]~~ ruling property precisely only one is called a base curve or directrices.

Equation of a ~~[illegible]~~ ruled surface:

Let $r = r(u)$ be the position vector of a point p on the base curve of a ruled surface. Let $g(u)$ be the unit vector along the generator at

Let R be the position vector of any point. Then $R = r(u) + v g(u)$

where v is the distance of Q from p in the direction of g is the equation of the ruled surface.

Thm: (X) 5m

For any ruled surface $R = r(u) + v g(u)$ the Gaussian curvature $K = -[\dot{r} \; g \; \dot{g}]^2 / H^4$.

Prf:

Given ruled surface is

$$R = r(u) + v g(u).$$

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WKT, the Gaussian curvature $K = \frac{LN - M^2}{H^2}$

To find L, M, N & H.

Now ~~[illegible]~~, $R = r + v g$

$R_1 = \frac{\partial R}{\partial u} = \frac{dr}{du} + v \cdot \frac{dg}{du}$

$R_1 = \frac{\partial R}{\partial u} = \dot r + v\dot g$

$R_2 = \frac{\partial R}{\partial v} = g$

$E = R_1 \cdot R_1 = (\dot r + v\dot g)(\dot r + v\dot g)$

$= \dot r \cdot \dot r + v(\dot r \cdot \dot g) + v(\dot g \cdot \dot r) + v^2(\dot g \cdot \dot g)$

$\boxed{= \dot r^2 + 2v\dot g \dot r + v^2 \dot g^2}$

$F = R_1 \cdot R_2 = (\dot r + v\dot g)(g)$

$= (\dot r \cdot g) + v(\dot g \cdot g)$

$\boxed{F = \dot r \cdot g}$ $\quad \because \dot g \cdot g = 0$

$G = R_2 \cdot R_2 = g\, g = 1$

$H^2 = EG - F^2$

$= (\dot r^2 + 2v\dot r \dot g + v^2 \dot g^2) - \dot r^2 g^2$

Now the unit normal N is given by,

$HN = R_1 \times R_2 = (\dot r + v\dot g) \times g \longrightarrow (1)$

Now,

$R_{11} = \ddot r + v\ddot g$ $\qquad N = r_1 \times r_2$

$R_{12} = \dot g$

$R_{21} = \dot g$

$R_{22} = 0$

$\therefore HL = [R_{11}\ R_1\ R_2] = [\ddot r + v\ddot g \quad \dot r + v\dot g \quad g]$

$= [\ddot r\ \dot r\ g] + v[\ddot r\ \dot g\ g] + v[\ddot g\ \dot r\ g] + v^2[\ddot g \ldots$

$HM = [R_{12}\ R_1\ R_2] = [\dot g \quad \dot r + v\dot g \quad g]$

$= [\dot g\ \dot r\ g] + v[\dot g\ \dot g\ g]$

$HM = [\dot g\ \dot r\ g] \Rightarrow M = \frac{[\dot g\ \dot r\ g]}{H}$

$\therefore M^2 = \frac{[\dot g\ \dot r\ g]^2}{H^2}$

$HN = [R_{22}\ R_1\ R_2] = 0$

since $HN = 0$; $H \neq 0$ we've $N = 0$.

$\frac{LN - M^2}{H^2}$

$K = -\frac{M^2}{H^2} = \frac{-\frac{1}{H^2}[\dot g\ \dot r\ g]^2}{H^2}$

$K = -\frac{[\dot r\ g\ \dot g]^2}{H^4}$

13/10/15 Tuesday

Note: 1

Since $K = 0$ is the necessary and su[fficient] condition for a surface to be a developable, the necessary and sufficient condition f[or a] ruled surface to be a developable is t[hat] $[\dot r\ g\ \dot g] = 0$.

Note: 2

The necessary and sufficient con[dition for a] ruled surface is skew is that ...

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Asymptotic lines on a ruled surface:

Prove that the cross ratio of the four points in which a generator is cut by four given curved asymptotic lines is the same for all generators.

Prf:

First we should calculate second fundamental co-efficient for a ruled surface.

The equation of the asymptotic line on a surface is $\boxed{L\,du^2 + 2M\,du\,dv + N\,dv^2 = 0}$

since for a ruled surface $N = 0$

$$\Rightarrow du\,(L\,du + 2M\,dv) = 0$$

Thus one family of asymptotic lines on a ruled surface is given by $du = 0$.

(i.e) The family of parametric curves $u = $ const which are the generators of the ruled surface.

Thus the generators of a ruled surface constitute one family of asymptotic lines.

The other family of asymptotic lines is given by,

$$L\,du + 2M\,dv = 0$$

$$L\,du = -2M\,dv$$

$$dv/ = -L/$$

$$\frac{dv}{du} = \frac{-2}{[\dot{r}\ g\ \dot{g}]}\left\{[\ddot{r}\ \dot{r}\ g] + [\ddot{g}\ \dot{r}\ g]v + [\dot{r}\ \dot{g}\ g] + [\dot{g}\ \dot{g}\ g]v^2\right\}$$

This is of the form of Riccati type.

$\frac{dv}{du} = A + Bv + Cv^2$, A, B, C are functions of u alone and general soln of this eqn is $v = \frac{cP+Q}{cR+S} \longrightarrow (1)$

P, Q, R, S are functions of u & c is an arbitrary constant.

This gives the family of curves asymptotic lines. Let four asymptotic lines of this family by c_1, c_2, c_3, c_4 and let these lines be met by the generator $u = u_0$ in four points v_1, v_2, v_3 & v_4 of parameters

$\therefore (1) \Rightarrow v_1 = \frac{c_1P+Q}{c_1R+S}$; $v_2 = \frac{c_2P+Q}{c_2R+S}$; $v_3 = \frac{c_3P+Q}{c_3R+S}$

$v_4 = \frac{c_4P+Q}{c_4R+S}$.

$$\therefore v_1 - v_2 = \frac{c_1P+Q}{c_1R+S} - \frac{c_2P+Q}{c_2R+S}$$

$$= \frac{(c_1P+Q)(c_2R+S) - (c_2P+Q)(c_1R+S)}{(c_1R+S)(c_2R+S)}$$

$$= \frac{c_1c_2PR + c_1PS + c_2QR + QS - c_1c_2PR \ldots}{(c_1R+S)(c_2R+S)}$$

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$$= \frac{(c_1-c_2)PS + (c_2-c_1)QR}{(c_1R+S)(c_2R+S)}$$

$$v_1 - v_2 = \frac{(c_1-c_2)(PS-QR)}{(c_1R+S)(c_2R+S)}$$

$|||^{ly}$ we can calculate $v_3 - v_4$, $v_1 - v_3$, $v_2 - v_4$

$$\therefore \frac{(v_1-v_2)(v_3-v_4)}{(v_1-v_3)(v_2-v_4)} = \frac{(c_1-c_2)(c_3-c_4)}{(c_1-c_3)(c_2-c_4)}$$

which is independent of u_0 and is therefore the same for all generators. Thus the result.

Parameter distribution :-

The function $p(u) = \frac{[\dot{r}\ g\ \dot{g}]}{\dot{g}^2}$ is called the Parameter distribution of a ruled surface.

Properties:

(i) $p(u)$ has constant value at each point of a generators.

Proof:

since $p(u) = [\dot{r}\ g\ \dot{g}] / \dot{g}^2$

and WKT, $\dot{r}$, g & $\dot{g}$ are all functions of u only. (i.e) They are all independent of v.

$\therefore p(u)$ is a function of 'u' only and it does not dependent upon v.

since along a generator u is constant

$\therefore$ The functions $p(u)$ is constant along ruled surf generator. Hence the result.

(ii) The parameter distribution $p(u)$ is independent the base curve :-

soln:

since $p(u) = \frac{[\dot{r}\ g\ \dot{g}]}{\dot{g}^2}$

If we replace r by $r+wg$

$$\therefore \frac{[\dot{r}+w\dot{g}\ \ g\ \ \dot{g}]}{\dot{g}^2} = \frac{[\dot{r}\ g\ \dot{g}] + w[\dot{g}\ g\ \dot{g}]}{\dot{g}^2}$$

$$= \frac{[\dot{r}\ g\ \dot{g}]}{\dot{g}^2}$$

This shows that $p(u)$ is independent of particular base curve and it is also indepen of the choice of parameter u.

(iii) The developable surface is a ruled su which the parameter of distribution is

prf:

WKT

$$p(u) = \frac{[\dot{r}\ g\ \dot{g}]}{\dot{g}^2}$$

$$[\dot{r}\ g\ \dot{g}] = \dot{g}^2\, p(u)$$

ruled surface is Gaussian curvature is of a
All also the

$$K = \frac{-[\dot{r}\ g\ \dot{g}]^2}{H^4}$$

$$K = \frac{-p^2 \dot{g}^4}{H^4}.$$

Since $K=0$ for a developable surface.

$\therefore p=0$ (i.e) $p(u)$ is identically zero

Hence the result.

central point:

Let P & Q be two points on a space curve of a ruled surface. let the common $\perp^r$ to the generator through P & Q meets the generator at P_1 and Q_1 as $Q \to P$. The point tends to a definite point c on the generator through the point c, the point 'c' is called central point.

line of striction:

The locus of the central point of all the generators in a definite curve lying on the ruled surface is called the line of striction of the ruled surface.

Thm:

Obtain a formula for the position of the central point on each generators.

Prf:

Let P, Q be the two points on some base curve c.

Let the common $\perp^r$ to the generating lines through P and Q as $Q \to P$, the point P_1 will tends to some point called One central point.

The limiting direction of the vector P_1Q_1 must lie in the surface.

$\therefore$ It is $\perp^r$ to N also it must be $\perp^r$ to the generator through P.

$\therefore P_1Q_1$ in the limit is parallel to $g \times N$ as P_1Q_1 is $\perp^r$ to both g & N.

Now taking limit as $Q \to P$ and using this fact that,

$\overline{P_1Q_1}$ is parallel to $g \times N$.

(i.e) $(g \times N)\cdot \dot{g} = 0$

$(\dot{g} \times g)\cdot N = 0 \longrightarrow (1)$

also $HN = (\dot{r} + v\dot{g}) \times g$

$\therefore (1) \Rightarrow \frac{1}{H}(\dot{g} \times g)\cdot[(\dot{r}+v\dot{g}) \times g] = 0$

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$(\dot g \times g)\,[(\dot r \times g) + v(\dot g \times g)] = 0$

$\Rightarrow (\dot g \times g).(\dot r \times g) + v\,(\dot g \times g)^2 = 0 \longrightarrow (2)$

WKT,

$(a \times b)^2 = a^2b^2 - (a.b)^2$

$\therefore (\dot g \times g)^2 = \dot g^2 g^2 - (\dot g . g)^2$

$(2) \Rightarrow (\dot g \times g).(\dot r \times g) + v\{\dot g^2 g^2 - \underbrace{(\dot g . g)^2}_{0}\} = 0$

$(\dot g \times g)(\dot r \times g) + v\dot g^2 g^2 = 0 \longrightarrow (3)$

Now,

$\underset{a\ \ b}{(\dot g \times g)}\ \underset{c\ \ d}{(\dot r \times g)} = (\dot g . \dot r)(g.g) - (\dot g . g)(g . \dot r)$

$= (\dot g . \dot r)$

$(3) \Rightarrow (\dot g . \dot r) + v\dot g^2 g^2 = 0$

$$v = \frac{-(\dot g . \dot r)}{\dot g^2 g^2}$$

WKT, The position vector of a central point be

$R = r(u) + v g(u)$

$= r(u) + \left(\frac{-\dot g . \dot r}{\dot g^2 g^2}\right) g(u) = r(u) - \frac{(\dot g . \dot r)}{\dot g^2 g^2} g(u)$

$= r(u) - \frac{(\dot g . \dot r)}{\dot g^2} g(u) \qquad g^2 = g.g = 1$

The tangent plane at any central of a generator is called the central plane generation.

1. P.T $N = \frac{P}{(P^2+v^2)^{1/2}} a + \frac{v}{(P^2+v^2)^{1/2}} a \times g$; a is unit vector along $\dot g$.

Let the eqn of the ruled surface be,

$R = r(u) + v g(u) \longrightarrow (1)$

Let us take the line of striction as d

Then $\dot g . \dot r = 0$

Since g is a vector of constant mod

$\therefore g.g = 1$

$\therefore \dot g$ is parallel to $\dot r \times g$

Let $\dot r \times g = \alpha \dot g$ for some scalar α. (A)

Taking scalar product by $\dot g$, we get

$(\dot r \times g).\dot g = \alpha \dot g . \dot g$

$[\dot r\ g\ \dot g] = \alpha \dot g^2$

$\therefore P\,\dot g^2 = \alpha \dot g^2$

$\boxed{P = \alpha}$

from (1), we've

$R_1 = \dot r + v\dot g$; $R_2 = g$

$\therefore HN = R_1 \times R_2 = (\dot r + v\dot g) \times g = \dot r \times g$

$HN = P\dot g + v\,(\dot g \times g)$

$P = \frac{[\dot r\ g\ \dots}{\dot g \dots}$

$P\dot g^2 = [\dots$

$P\dot g^2 = \dots$

$P\dot g = \dots$

$$(HN)^2 = H^2 = p^2\dot{g}^2 + 2pv\,\dot{g}\,(\dot{g}\times g) + v^2(\dot{g}\times g)^2 \rightarrow (4)$$

But $\dot{g}\cdot(\dot{g}\times g) = [\dot{g}\ \dot{g}\ g] = 0$

also $(\dot{g}\times g)^2 = \dot{g}^2 g^2 - (\dot{g}\cdot g)^2 = \dot{g}^2$

$(4) \Rightarrow \therefore H^2 = p^2\dot{g}^2 + v^2\dot{g}^2 = (p^2+v^2)\dot{g}^2$

$$\dot{g}^2 = \frac{H^2}{p^2+v^2}$$

$$|\dot{g}| = \frac{H}{(p^2+v^2)^{1/2}}.$$

let 'a' be unit vector along $\dot{g}$ then

$|\dot{g}|a = \dot{g}$ (i.e) $\dot{g} = \dfrac{H}{(p^2+v^2)^{1/2}}\cdot a$

$(3) \Rightarrow HN = p\cdot\dfrac{H}{(p^2+v^2)^{1/2}}a + v\left(\dfrac{H}{(p^2+v^2)^{1/2}}a\times g\right)$

$$N = \frac{p}{(p^2+v^2)^{1/2}}a + \frac{v}{(p^2+v^2)^{1/2}}(a\times g).$$

This gives us normal N to the surface t a point of generator distance v from the ntral point.

roblem:

Find the inclination of the tangent plane t any point of generator to the tangent plane at the central point ...

At the central point $v=0$

If N_0 is the normal at the central point of the generator, then

$N_0 = a$; $\left[\because N = \dfrac{p}{(p^2+v^2)^{1/2}}a + \dfrac{v}{(p^2+v^2)^{1/2}}(a\times g)\right]$

let ϕ denote the angle b/w the directions of N at points on a generator distance v & 0 from the central point.

$$\therefore N\cdot N_0 = \frac{p}{(p^2+v^2)^{1/2}}a\cdot a$$

$|N|\,|N_0|\cos\phi = \dfrac{p}{(p^2+v^2)^{1/2}}$ $\quad|\because a$ is unit vector

$\cos\phi = \dfrac{p}{(p^2+v^2)^{1/2}}$; $\sin\phi = \dfrac{v}{(p^2+v^2)^{1/2}}$

$$\tan\phi = \frac{\sin\phi}{\cos\phi} = \frac{v}{(p^2+v^2)^{1/2}}\cdot\frac{(p^2+v^2)^{1/2}}{p}$$

$$\tan\phi = v/p$$

Thus the tangent of the angle through which the normal N rotates as the p moves a a generator varies directly with the distance moved from the central point.

... increases from $-\infty$ to ∞ the angle θ increases ~~fom~~ from $-\pi/2$ to $\pi/2$ if $p>0$ and decreases from $\pi/2$ to $-\pi/2$ if $p<0$.

Note:

since $H^2 = (p^2+v^2)\dot{g}^2$

$\therefore$ The gaussian curvature in terms of p & v.

$$K = \frac{-p^2\dot{g}^4}{H^4} \Rightarrow K = \frac{-p^2\dot{g}^4}{(p^2+v^2)\dot{g}^4}$$

$$= \frac{-p^2}{(p^2+v^2)^2}$$

Thm.

If d is the length of the common $\perp^r$ b/w two successful generators then $d = s\,p(u)|\dot{g}|$.

Prf:

Let P be $r = r(u)$ & Q be $r = r(u+\delta u)$.

then $\overline{PQ}$ is $r(u+\delta u) - r(u) = \frac{s}{1!}\dot{r}(u) + \frac{s^2}{2!}\ddot{r}(u) + \dots$

$\because \delta u = s$

By neglecting the higher powers of s. we get,

$$\overline{PQ} = s\,\dot{r}(u)$$

Let $g(u)$ & $g(u+\delta u)$ be unit vectors along the generators through P & Q.

the generators. It is parallel to $g \times g(u+\delta u)$

$\therefore$ The shortest distance is parallel to,

$$g \times (\overline{g} + \dot{g}\,\delta u) = g \times \dot{g}\,\delta u$$

$\therefore$ We can take the direction of the shortest distance to be $g \times \dot{g}$.

If d is the shortest distance b/w generators. d = projection of PQ on $P|Q|$.

$$= s\cdot\dot{r}(u)\cdot\frac{g\times\dot{g}}{|g\times\dot{g}|}$$

But $|g\times\dot{g}| = |g||\dot{g}|\sin\pi/2 = |\dot{g}|$ $\quad | \because |g|$

$$d = s\,\dot{r}(u)\cdot\frac{g\times\dot{g}}{|\dot{g}|}$$

$$= \frac{s}{|\dot{g}|}[\dot{r}\ g\ \dot{g}]$$

$$= \frac{s}{|\dot{g}|}p(u)\,|\dot{g}|^2 \quad | \because p(u)$$

$$d = s\,p(u)|\dot{g}|$$

$$\therefore d = s\,p|\dot{g}|.$$

Ex: 8.1

S.T the ruled surface generated by binomials of skew curve has the curve as line of striction.

Let the eqn of the skew curve be.

$r = r(s) \longrightarrow (1)$

Let $\bar{t}, \bar{n}, \bar{b}$ be its unit tangent, principal normal and binormal respectively.

For the ruled surface generated by the binormals of (1) take the curve (1) itself as the directrix.

Since the binormals of (1) are the generators of the rule surface.

$\therefore$ the unit vector $g(s)$ along the generator of the ruled surface passing through the point $\bar{r}(s)$ of the directrix is given by $g = \bar{b}$.

$$\therefore g' = \frac{d\bar{b}}{ds} = -\tau\bar{n}$$

Thus the directrix itself is the line of striction. Hence the (1) itself is the line of striction of the ruled surface generated by its binormals.

:8.2

S.T the parameter of distribution of the ruled surface generated by the principal normals of a skew curve is equal to $\tau(\tau^2+k^2)^{-1}$

Where k & τ are the curvature and torsion of the curve.

Soln:

Let the eqns of the skew curve be,

$\bar{r} = \bar{r}(s) \longrightarrow (1)$

Let $\bar{t}, \bar{b}$ be its unit tangent principal binormal respectively. For the ruled surface generated by the principal normals of (1). take the curve (1) itself as the directrix.

Since the principal normals of (1) are the generators of the ruled surface.

$\therefore$ The unit vector $g(s)$ along the generator the ruled surface passing through the point $\bar{r}(s)$ of the directrix is $g = \bar{n}$.

$$\therefore g' = \bar{n}' = \tau\bar{b} - k\bar{t}$$

The parameter of distribution p is given

$$p = \frac{[r' \; g \; g']}{g'^2} = \frac{[\bar{t} \; \bar{n} \; \tau\bar{b} - k\bar{t}]}{(\tau\bar{b} - k\bar{t})^2}$$

$$= \frac{[\bar{t} \; \bar{n} \; \tau\bar{b}]}{\tau^2\bar{b}^2 + k^2\bar{t}^2}$$

$$= \frac{(t \times \bar{n}) \cdot \tau\bar{b}}{\tau^2 + k^2} \qquad \because \bar{b}\cdot\bar{b} = \bar{t}\cdot\bar{t} = 1$$

UNIT-I

The theory of space curves:-

Introduction:-

In the theory of plane curves, a curve is usually specified either by means of a single equation or else by a parametric representation.

Example:-

The circle centre (0,0) & radius 'a' is specified in cartesian co-ordinates (x,y) by the single equation $x^2+y^2=a^2$.

(or) by a parametric representation $x=a\cos u$, $y=a\sin u$, $0\leq u\leq 2\pi$.

Also in three-dimensional euclidean space E_3 a single equation generally represents a surface & two equation are needed to specify a curve.

Thus the curve appears as the intersection of the two surfaces represented by the two eqns.

In cartesian co-ordinates of the eqn.,

$x=x(u)$, $y=y(u)$, $z=z(u)$ $\longrightarrow$ ①

where x, y, z are real-valued function of the real parameter u which is restricted to some interval.

In vector notation, the curve is specified by a vector valued function,

$r=R(u)$.

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2m Function of class m :-

Let I be a real interval & m be a (+)ve integer. A real-valued fun. f defined on I is said to be of class m (or) to be a C^m fun. if f has an m^{th} derivative at every point of I & if this derivative is continuous of I.

Note :-

* When a fun. is infinitely differentiable we say it is of class ∞ (or) C^∞ function & when a function is analytic we say it is class ω (or) C^ω function.

* A vector valued function $R = (x, y, z)$ defined on I is said to be of class m if it has an mth derivative at every point & if this derivative is continuous on I (or) if each of its components x, y, z is of class m.

* the cartesian components are $x = x(u)$, $y = y(u)$, $z = z(u)$.

Regular function

2m If the derivative $\frac{dR}{du} = \dot{r}$ never vanishes on I. equivalently if $\dot{x}, \dot{y}, \dot{z}$ never vanish simultaneously the function is said to be regular.

Path :-

2m A regular vector valued funt of class m is called a "path" of class m.

Equivalent :-

Two paths R_1, R_2 of the same class m on I_1, I_2 are called equivalent if there exist a strictly increasing funt ϕ of class m which maps I_1 onto I_2 & is such that $R_1 = R_2 \phi$.

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Arc length

10 m

Let $r = R(u)$ be a path corresponding two number a, b $(a<b)$ in the range of the parameter then the path $r = R(u)$ $(a \le u < b)$ is an arc of the original path joining the points corresponding to a & b.

To any subdivision Δ of the interval (a, b), Them - 1 $a = u_0 < u_1 < \dots < u_n = b$, the length is $\boxed{L_\Delta = \sum_{i=1}^{n} |R(u_i) - R(u_{i-1})|}$

(Q) If $r = R(u)$ is the parametric representation of a curve where $u \in [a, b]$ the length of the curve

$$s = \int s(u) = \int_a^u |\dot{R}(u)|\, du \longrightarrow (1)$$

For a subdivision $\Delta = \{a = u_0 < u_1 < u_2 \cdots < u_n = b\}$

we've

$$L(\Delta) = \sum_{i=1}^{n} |R(u_i) - R(u_{i-1})| \longrightarrow (2)$$

since R is atleast of class C' we've

$$|R(u_i) - R(u_{i-1})| = \left| \int_{u_{i-1}}^{u_i} \dot{R}(u)\, du \right| \longrightarrow (3)$$

using (3) in (2),

$$L(\Delta) = \sum_{i=1}^{n} \left| \int_{u_{i-1}}^{u_i} \dot{R}(u)\, du \right|.$$

By Schwartz inequality we get,

$$\sum_{i=1}^{n} \left| \int_{u_{i-1}}^{u_i} \dot{R}(u)\, du \right| \le \sum_{i=1}^{n} \int_{u_{i-1}}^{u_i} |\dot{R}(u)|\, du$$

$$= \int_a^b |\dot{R}(u)|\, du \longrightarrow (4)$$

so that we've,

$$L(\Delta) \le \int_a^b |\dot{R}|\, du.$$

since the right hand side of (4) is finite & independent of Δ, the set $\{L(\Delta)\}$ for all possible subdivisions Δ of $[a,b]$ is a bounded set of real numbers & it is bounded above.

So the l.u.b. of $\{L(\Delta)\}$ exists as a finite quantity.

Next we've shall S.T. this upper bound is actually θ given in the thm.

If $s = s(u)$ denoted the arc length from u_0 to u then $s(u)-s(u_0)$ gives the arc length b/n u_0 & u.

Since we've defined the arc length as the l.u.b of $\{L(\Delta)\}$ we've from (4),

$$s(u)-s(u_0) \leq \int_{u_0}^{u} |\dot{R}(u)|\,du \longrightarrow (5)$$

since the length of the chord joining $R(u)$ & $R(u_0)$ is less than the arcual length, we've

$$|R(u)-R(u_0)| \leq s(u)-s(u_0) \longrightarrow (6)$$

from (5) & (6) we've,

$$\frac{|R(u)-R(u_0)|}{u-u_0} \leq \frac{s(u)-s(u_0)}{u-u_0}$$

$$\leq \frac{1}{u-u_0} \int_{u_0}^{u} |\dot{R}(u)|\,du \longrightarrow (7)$$

Taking limit as $u \to u_0$, (7)

$$|\dot{R}(u_0)| \leq \dot{s}(u_0) \leq |\dot{R}(u_0)| \longrightarrow (8)$$

Hence $\dot{s}(u_0)$ exists & has the value $\dot{s}(u_0) = |\dot{R}(u_0)|$

Since ⑧ is equally true for any parameter u_0 in I, we conclude from ⑧.

i) s is a fun. of the same class as the curve.

ii) As $s(a) = 0$, $s(u) = s = \int_a^u |\dot{R}(u)|\, du \longrightarrow$ ⑨

where ϕ denotes the length of the curve from a to u

Corollary :-

Interms of Cartesian parameter representation

$$s = \int_a^u \sqrt{\dot{x}^2+\dot{y}^2+\dot{z}^2}\, du$$

Proof

In cartesian parametric representation.

Let $R(u) = x(u)\vec{i} + y(u)\vec{j} + z(u)\vec{k}$

$$\dot{R} = \dot{x}\vec{i} + \dot{y}\vec{j} + \dot{z}\vec{k}$$

$$|\dot{R}| = \sqrt{\dot{x}^2+\dot{y}^2+\dot{z}^2}$$

using this in eqn ⑨ we get,

$$s = \int_a^u \sqrt{\dot{x}^2+\dot{y}^2+\dot{z}^2}\, du.$$

Further since $\dot{s} = |\dot{R}|$, $\dot{s}^2 = \dot{x}^2+\dot{y}^2+\dot{z}^2$ which gives interms of differential,

$$ds^2 = dx^2+dy^2+dz^2.$$

Note :- since $\dot{s} \neq 0$ we can take s as a new para. The change of parameter from s to u is given by $s(u)$ in ⑨, from ⑨ we can obtain $u = \phi(s)$ so that the curve parametric w.r.to s is $r = R[\phi(s)]$.

1) Obtain the eqns of the circular helix
$r = (a\cos u, a\sin u, bu), -\infty < u < \infty, a > 0.$

Sm

$r = (a\cos u, a\sin u, bu)$

$\dot{r} = (-a\sin u, a\cos u, b)$

$\dot{x} = -a\sin u,\ \dot{y} = a\cos u,\ \dot{z} = b$

$r = R(u) \Rightarrow \dot{r} = \dot{R}(u)$

$\therefore s = s(u) = \int_a^u |\dot{R}(u)| du$

$s = \int_0^u \sqrt{a^2\sin^2 u + a^2\cos^2 u + b^2}\, du$

$= \sqrt{a^2+b^2} \int_0^u du$ where $c = \sqrt{a^2+b^2}$

$= c(u)_0^u$

$s = cu$

The range of u corresponding to one complete turn of the helix is $u_0 \leq u \leq u_0 + 2\pi$.

$\therefore s = \int_{u_0}^{u_0+2\pi} c\, du$; $c = \sqrt{a^2+b^2}$

$s = c[u]_{u_0}^{u_0+2\pi} = c[u_0 + 2\pi - u_0] = c2\pi.$

$\therefore$ The length is $2\pi c$.

2) Find the length of the curve given as the intersection of the surfaces $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$
$x = a\cos(z/a)$ from $(a, 0, a)$ to (x, y, z).

Sm

The eqn of parametric form is given by

$x = a\cosh u,\ y = b\sinh u,\ z = au.$

$\dot{x} = a\sinh u, \dot{y} = b\cosh u, \dot{z} = a$

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$$s = \int_0^u |\dot{R}(u)|\, du$$

$$= \int_0^u \sqrt{a^2\sinh^2 u + b^2\cosh^2 u + a^2}\; du$$

$$= \int_0^u \sqrt{a^2(1+\sinh^2 u) + b^2\cosh^2 u}\; du$$

$$= \int_0^u \sqrt{a^2\cosh^2 u + b^2\cosh^2 u}\; du$$

$$= \int_0^u \sqrt{(a^2+b^2)\cosh^2 u}\;.\, du$$

$$= \sqrt{a^2+b^2} \int_0^u \cosh u\, du$$

$$= \sqrt{a^2+b^2}\; \sinh u$$

$$\therefore s = \sqrt{a^2+b^2}\,.(y/b).$$

Tangent Normal & Binormal.

Let μ be a curve of class ≥ 1 & let P, Q be two neighbouring points on the curve μ presents by two eqn $\bar{r} = \dot{r} = \dot{\bar{r}}(u)$ & let P, Q have parametric u_0 & u,

since μ has curve ≥ 1,

$$\bar{r}(u) = \bar{r}(u_0) + (u-u_0)\dot{\bar{r}}(u_0) + 0.(u-u_0) \text{ as } u \to u_0 \longrightarrow ①$$

Hence, $$\lim_{u\to u_0} \frac{\bar{r}(u) - \bar{r}(u_0)}{|\bar{r}(u) - \bar{r}(u_0)|} = \frac{\dot{\bar{r}}(u_0)}{|\dot{\bar{r}}(u_0)|} \longrightarrow ②$$

(ie) the unit vector along the chord $PQ \to$ to a unit vector of P as $Q \to P$.

em

This is called the "unit tangent vector" to μ at P is denoted by t.

$$t = \frac{\dot{\bar{r}}(u_0)}{|\dot{\bar{r}}(u_0)|} = \frac{\dot{\bar{r}}}{\dot{s}} = \frac{dr}{ds} \longrightarrow ③$$

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Tangent line to γ at P :-

The line through a parallel to t is called the tangents line to γ at P, If R is any point on this line the vector from the point of contact P to R is called a "tangent vector" to γ at P.

Osculating plane :-

2M Let γ be a curve of class ≥ 2 & let P, Q be two neighbouring points on γ.

Then the limiting position as $Q \to P$ of that plane which contains the tangent line at P & the point Q is called the "osculating plane" of γ at P.

($[R - r(0), r'(0), r''(0)] = 0$) is the equation of the osculating plane.

~~Theorem~~

Inflexional point :-

2M A point P $r''(0) = 0$ is called point of inflexional & the tangent line at P is called inflexional.

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Theorem:-

10m

(3) Let γ be a curve of class $m \geq 2$ with arc length s as parameter if the point P on γ has parameter "0" the eqn of osculating plane is $[R - r(0) \cdot r'(0), r''(0)] = 0$ where $r''(0) \neq 0$. If $r''(0) = 0$. Let us assume that the curve γ is analytic then the eqn of the plane at a inflexional point is,

$$[R - r(0), r'(0), r^{k}(0)] = 0.$$

proof:-

Using the arc length s as parameter. let 0 & s be the parameter of P & Q. let R be the p.v. of the point on the plane containing the tangents line at P on passing through Q then $\bar{r}$ is the p.v. of P.

then the vector $\bar{R} - \bar{r}(0)$, $t = \bar{r}'(0)$ & $r(s) - r(0)$ are co-planar vectors. Hence the conditions of co-planarity gives the eqn. as,

$$[R - r(0), r'(0), r(s) - r(0)] = 0 \longrightarrow (1)$$

since the curve is of class $m \geq 2$. we've by Taylor's thm, formula La(f)...

$$r(s) = r(0) + s r'(0) + \frac{1}{2} s^2 r''(0) + \text{order of } s^2 \text{ as } s^2 \to 0 \longrightarrow (2)$$

using (1) in (2) we get,

$$[R - r(0), r'(0), s r'(0) + \tfrac{1}{2} s^2 r''(0) + o(s^2)] = 0$$

Neglecting the terms of higher order the above eqn becomes,

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$[R-r(0), r'(0), s r'(0)] + [R-r(0), r'(0), \frac{s^2}{2!} r''(0)]$ → (3)

Since $r'(0) \times r'(0) = 0$ & s is a scalar the Ist term of (3) vanishes & so we get,

$[R-r(0), r'(0), r''(0)] = 0$ → (4)

As the eqn of osculation plane provided the vectors $r'(0)$ & $r''(0)$ are linearly independ so to complete the proof we've to prove $r'(0)$ & $r''(0)$ are linearly independent unless $r''(0) = 0$.

If $r''(0) = 0$ then the point P is an inflexional point so we derive the eqn of osculating plane at an inflexional point with the assumption that the curve γ is analytic.

Diff., $(r')^2 = 1$ we've

$2(r' \cdot r'') = 0 \Rightarrow \boxed{r' \cdot r'' = 0}$

Diff this once again we've,

$r'' \cdot r'' + r' \cdot r''' = 0$

since $r'' = 0$ we get, $r' \cdot r''' = 0$ at P.

Since r' can't be zero r' & r''' are linearly independent unless $r''' = 0$.

Repeating this process of differentiation let us assume that $r^{(k)}$ is Ist non-vanishing derivative of r such that,

$(r' \cdot r^k) = 0$ so if $r^k \neq 0$ we've from Taylor thm,

$$\gamma(s)-\gamma(0) = \gamma'(0)\cdot\frac{s}{1!} + \gamma''(0)\frac{s^2}{2!} + \dots + \gamma^k(0)\cdot\frac{s^k}{k!} + o(s^k)$$
as $s \to 0$ $\longrightarrow$ (5)

using (5) in (1) we get,

$$R-\gamma(0)\cdot\gamma'(0)\cdot\frac{\gamma''(0)}{1!}\cdot s^2 + \frac{s^k}{k!}\gamma^k(0) = 0 \longrightarrow (6)$$

As in the previous case the above eqn reduces to $[R-\gamma(0)\cdot\gamma'(0), \gamma^k(0)] = 0$ as the eqn of the osculating plane at an inflexional point. If $\gamma^k = 0$ for all $k \geq 2$ then since the curve is analytic we inflex that t is constant & $\therefore$ the curve is a straight line.)

corollary

If P is not point of inflexion any vector lying in the osculating plane can be expressed as $\alpha\gamma' + \beta\gamma''$ for some co-efficient α & β.

Proof

Since P is not a point of inflexion $\gamma'' \neq 0$ from (4) γ' & γ'' lies in the osculating plane & pass through Hence any vector in the osulating & is a linear combination of γ' & γ'' so that we can take it as $\alpha\gamma' + \beta\gamma''$ for some constant α & β it is of important to note γ'' lies in this osculating plane.

7. When a curve is analytic a define osculating plane at a point of inflexion p exists unless the curve is a straight line.

IAT-I, IAT-II

soln W.K.T. $\bar{t}^2 = 1$ (ie) $\bar{\gamma}'^2 = \bar{\gamma}'\cdot\bar{\gamma}' = 1$

we diff. this $\bar{\gamma}'\cdot\bar{\gamma}'' = 0$ again diff.

at p we get,

$$\bar{\gamma}'\cdot\bar{\gamma}''' + \bar{\gamma}''\cdot\bar{\gamma}'' = 0$$

$$\bar{\gamma}'\cdot\bar{\gamma}''' = 0 \quad [\because \bar{\gamma}'' = 0] \text{ at } p.$$

The Second Fundamental Form

(ie) $\bar{\gamma}'$ is linearly independent of $\bar{\gamma}'''$ unless $\bar{\gamma}''' = 0$.

Repeating this process $\bar{\gamma}' \cdot \bar{\gamma}^k = 0$ where $\bar{\gamma}^k$ is the 1st non-zero derivative of $\bar{\gamma}$ at $P (k \geq 2)$.

If $\bar{\gamma}^k = 0$, $\forall k \geq 2$ then since the curve is analytic we conclude that $\bar{t}$ is constant & the curve is a straight line.

If $\bar{\gamma}^k \neq 0$ then we're use Taylor's thm

$$\bar{\gamma}(s) - \bar{\gamma}'(0) = s\gamma'(0) + \frac{s^k}{k!}\gamma^k(0) + O(s^k) \text{ as } s \to 0, \text{ \&}$$

the eqn of osculating plane is,

$$[R - \bar{\gamma}(0) \cdot \gamma'(0), \bar{\gamma}^k(0)] = 0$$

2. S.T. at a point of inflexion even a curve of class ∞ need not possess an osculating plane.

SM

Consider the curve ν defined by,

$\bar{\gamma}(u) = (u, e^{-1/u^2}, 0)$; $u < 0$.

$\bar{\gamma}(u) = (u, 0, e^{-1/u^2})$; $u > 0$.

$\gamma(0) = (0, 0, 0)$

Here ν is a curve of class ∞ with $\gamma^k(0)$ for all $k \geq 2$

The osculating plane at all points with parameter $u < 0$ is $z = 0$ while the osculating plane at all points with parameter $u > 0$ is $y = 0$ the osculating plane at $u = 0$ is indeterminate

At a point of inflexion even a curve of class ∞ need not passes an osculating plane.

Example

1) S.T. If a curve is given interms of a general parameter u. Then the eqn of the osculating plane $[R-\bar{r}, \dot{\bar{r}}, \ddot{\bar{r}}] = 0$

soln

W.K.T. the eqn of osculating plane is,

$$[\bar{R} - \bar{r}(0), \bar{r}'(0), \bar{r}''(0)] = 0 \longrightarrow (1)$$

Also we've $r' = \dot{r}/\dot{s}$

$$r'' = -\frac{\ddot{s}\dot{\bar{r}} + \dot{s}\ddot{\bar{r}}}{\dot{s}^2} = \frac{\dot{s}\ddot{\bar{r}} - \ddot{s}\dot{\bar{r}}}{\dot{s}^2}$$

$$= \frac{\ddot{r}}{\dot{s}} - \frac{\ddot{s}\dot{r}}{\dot{s}^2}$$

$$\bar{r}' \times \bar{r}'' = \dot{r}/\dot{s} \times \left(\frac{\ddot{r}}{\dot{s}} - \frac{\ddot{s}\dot{r}}{\dot{s}^2}\right)$$

$$= \dot{\bar{r}}/\dot{s} \times \ddot{r}/\dot{s} - \dot{\bar{r}}/\dot{s} \times \frac{\ddot{s}\cdot\dot{r}}{\dot{s}^2}$$

$$= 1/\dot{s}^2 (\dot{\bar{r}} \times \ddot{r}) - \ddot{s}/s^3 (\dot{\bar{r}} \times \dot{r})$$

$$= 1/\dot{s}^2 (\dot{\bar{r}} \times \ddot{\bar{r}})$$

The osculating plane is,

$$[R(0) - \bar{r}(0) \cdot \bar{r}'(0) \cdot \bar{r}''(0)] = 0$$

$$(R(0) - \bar{r}(0)) \cdot (\bar{r}'(0) \times \bar{r}''(0)) = 0$$

$$(R - \bar{r}) \cdot (\bar{r}' \times \bar{r}'') = 0$$

$$(R - \bar{r}) \cdot \frac{\dot{r} \times \ddot{r}}{\dot{s}^2} = 0$$

$$1/\dot{s}^2 (R - \bar{r}) - (\dot{r} \times \ddot{\bar{r}}) = 0$$

$$((R - \bar{r}), \dot{\bar{r}}, \ddot{\bar{r}}) = 0$$

2) Find the eqn of the osculating plane at a point on the cubic curve given by $\bar{r} = (u, u^2, u^3)$ & S.T. the osculating plane at any 3 points of the meet at a point lying in the plane determined by these 3 points.

The second Fundamental Form

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The eqn of the osculating plane is,

$$\begin{vmatrix} (x-u) & y-u^2 & z-u^3 \\ 1 & 2u & 3u^2 \\ 0 & 2 & 6u \end{vmatrix} = 0$$

$$(x-u)\left[12u^2-6u^2\right]-(y-u^2)\left[6u\right]+\left[z-u^3\right](2)=0$$

$$(x-u)(6u^2)-(y-u^2)(6u)+(z-u^3)2=0$$

$$6xu^2-6u^3-6uy+6u^3+2z-2u^3=0$$

$$6xu^2-6uy+2z-2u^3=0$$

$$3xu^2-3uy+z-u^3=0$$

If u_1, u_2, u_3 are 3 distinct values of the parameter the osculating planes at these points are linearly independent & the planes meet at a point (x_0, y_0, z_0).

The parameter u_1, u_2, u_3 therefore satisfy the conditions,

$$u^3-3u^2x_0+3uy_0-z_0=0 \longrightarrow (1)$$

If $lx+my+nz+p=0$ is the eqn of the plane passing through 3 points,

Then the parameter (u) must also satisfy the condition,

$$lu+mu^2+nu^3+p=0$$

$$l/n\,u+m/n\,u^2+u^3+p/n=0 \longrightarrow (2)$$

Since this eqn. has 3 distinct roots we've $n \neq 0$ comparing co-eff. in the two cubic eqn (1) & (2) gives

$l = 3ny_0$, $m=-3nx_0$, $P=-nz_0$.

The eqn on the plane is,

$$3y_0x-3x_0y+z-z_0=0 \text{ eq}$$

Since this is satisfied by $(x_0\ y_0\ z_0)$ the result

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Helix (or) a circular helix :

7. A curve drawn on a right circular cylinder so as to cut all the generators at the same angle is called a right circular.

2m

If $P(x, y, z)$ is any point on the helix then $x = a\cos u, y = a\sin u, z = bu$ gives the equation of circular helix.

Note :-

If b is +ve then the helix is said to be right handed when it is -ve it is said to be left hand.

vector eqn of the space curve :-

Let $P(x, y, z)$ be any general point on the space curve.

Let $\bar{e}_1 \bar{e}_2 \bar{e}_3$ be the unit vectors along ox, oy, oz

then $\bar{op} = \bar{r} = x\bar{e}_1 + y\bar{e}_2 + z\bar{e}_3$.

If $x = x(u), y = y(u), z = z(u)$ gives the parametric eqn of the space curve.

Then $\bar{x} = \bar{x}(u), \bar{y} = \bar{y}(u), \bar{z} = \bar{z}(u)$ gives the vector eqn of the space curve.

Note :-

1) $x\bar{e}_1, y\bar{e}_2, z\bar{e}_3$ are called the component of $\bar{r}$ along x, y, z - axis.

2.) The magnitude of $\bar{r} = |\bar{r}| = \sqrt{x^2+y^2+z^2}$

3.) In 3-dimensional, the arc length of any space curve is given by,

$$ds^2 = dx^2 + dy^2 + dz^2.$$

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Tangent line :

A tangent line at any point on a curve is the limiting position of a st. line passing through two consecutive points of the curve.

Arc length :

Let $r = R(u)$ be the vector eqn of the space curve. The arc length of the space curve b/n the points a & b is given by,

$$L = R(a) - R(b).$$

Lemma :-1

Derive expression for arc length.

5M

Let $\bar{r} = \bar{R}(u)$ be the vector eqn of the space curve.

Let s be the arc length of the curve b/n the points $A(u_0)$ & $P(u)$.

Let $P(x,y,z)$ be the co-ordinates of P.

By calculus of

$$ds^2 = dx^2 + dy^2 + dz^2$$

(ie) $\left(\frac{ds}{du}\right)^2 = \left(\frac{dx}{du}\right)^2 + \left(\frac{dy}{du}\right)^2 + \left(\frac{dz}{du}\right)^2$.

$$\left(\frac{ds}{du}\right)^2 = \dot{x}^2 + \dot{y}^2 + \dot{z}^2$$

$$\frac{ds}{du} = (\dot{x}^2 + \dot{y}^2 + \dot{z}^2)^{1/2}$$

$\int$ ing,

$$s = \int_{u_0}^{u} \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}\, du \longrightarrow ①$$

$$\bar{r} = x\bar{e}_1 + y\bar{e}_2 + z\bar{e}_3$$

$$\dot{\bar{r}} = \dot{x}\bar{e}_1 + \dot{y}\bar{e}_2 + \dot{z}\bar{e}_3$$

$$|\dot{\bar{r}}| = \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}$$

$$① \Rightarrow s = |\dot{\bar{r}}| = \int_{u_0}^{u} |\dot{\bar{r}}|\, du \longrightarrow ②$$

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also W.K.T

$$\dot{\bar{r}}\cdot\dot{\bar{r}} = |\dot{\bar{r}}|^2 \Rightarrow |\dot{\bar{r}}| = \sqrt{\dot{\bar{r}}\cdot\dot{\bar{r}}}$$

$$② \Rightarrow s = \int_{u_0}^{u} \sqrt{\dot{\bar{r}}\cdot\dot{\bar{r}}}\, du \longrightarrow ③$$

from ① & ② & ③ we've,

the length of the arc,

$$s = \int_{u_0}^{u} \sqrt{\dot{x}^2+\dot{y}^2+\dot{z}^2}\, du = \int_{u_0}^{u} |\dot{\bar{r}}|\, du = \int_{u_0}^{u} \sqrt{\dot{\bar{r}}\cdot\dot{\bar{r}}}\, du$$

Result :- 1

P.T. $\frac{d\bar{r}}{ds}$ is a unit vector.

2m

Let $P(x,y,z)$ be any point of the space curve.

$$\bar{r} = \bar{R}(u)$$

$$\bar{r} = x\bar{e}_1 + y\bar{e}_2 + z\bar{e}_3$$

$$d\bar{r} = dx\,\bar{e}_1 + dy\,\bar{e}_2 + dz\,\bar{e}_3$$

$$d\bar{r}\cdot d\bar{r} = dx^2+dy^2+dz^2$$

$$d\bar{r}^2 = ds^2$$

$$\frac{d\bar{r}^2}{ds^2} = 1$$

$$\frac{d\bar{r}}{ds}\cdot\frac{d\bar{r}}{ds} = 1 \qquad \therefore \left|\frac{d\bar{r}}{ds}\right|^2 = 1$$

$$\frac{d\bar{r}}{ds} = 1$$

$\therefore \frac{d\bar{r}}{ds}$ is a unit vector.

Result :- 2

To S.T $\frac{d\bar{r}}{ds}$ is a unit vector along the tangent as the P to the space curve.

By before result $\frac{d\bar{r}}{ds}$ is a unit vector.

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Let $p(x,y,z)$ & $Q(x+\Delta x,\ y+\Delta y,\ z+\Delta z)$ be any two neighbouring points on the space curve

$$\bar{r} = \bar{R}(u)$$

then $\overline{OP} = \bar{r}$ & $\overline{OQ} = \bar{r} + \Delta\bar{r}$

$$\overline{PQ} = \overline{OQ} - \overline{OP}$$
$$= \bar{r} + \Delta\bar{r} - \bar{r}$$
$$= \Delta\bar{r}.$$

$\therefore$ $\Delta\bar{r}$ gives the chord PQ. If Δs is the arc PQ then

$\dfrac{\Delta\bar{r}}{\Delta s}$ is a vector along any chord PQ.

When $Q \to P$ the chord $PQ \to$ the tangent at p also

$\Delta s \to 0$.

$\lim\limits_{\Delta s \to 0} \dfrac{\Delta\bar{r}}{\Delta s}$ gives the vector along the tangent at p.

$\therefore$ $\dfrac{d\bar{r}}{ds}$ gives a vector along the tangent at p.

$\therefore$ Already we prove $\dfrac{d\bar{r}}{ds}$ is a unit vector.

Note :-

i.) $\bar{t} = r' = \dfrac{d\bar{r}}{ds}$

$\bar{t} = \dfrac{d\bar{r}}{ds}$ = unit tangent vector.

$$|\bar{t}| = \left|\frac{d\bar{r}}{ds}\right| = 1$$

ii) $\dfrac{d\bar{r}}{du} = \dfrac{d\bar{r}}{ds} \cdot \dfrac{ds}{du}$

$$= \frac{d\bar{r}}{du} \cdot \frac{du}{ds}.$$

$$= \frac{d\bar{r}/du}{ds/du}$$

$$\bar{t} = \dot{\bar{r}}/\dot{s}$$

$$\bar{t} = \dot{r}/\dot{s}$$

Ex: 4.5 calculate the curvature & torsion of the cubic curve given by $\bar{r} = (u, u^2, u^3)$.

W.K.T. $\bar{t} = \frac{\dot{\bar{r}}}{\dot{s}}$ But $r = (u, u^2, u^3)$

$t = \frac{\dot{r}}{\dot{s}}$

$r' = t = \dot{r}/\dot{s}$

$t' = \dot{t}/\dot{s}$

$\dot{s}t' = \dot{t}$

$t' = \kappa\bar{n}$

$\bar{t} \times \bar{n} = \bar{b}$

$\bar{n} \times \bar{b} = \bar{t}$

$\bar{b} \times \bar{t} = \bar{n}$

$b' = \frac{\dot{b}}{\dot{s}} = -\tau\bar{n}$

$\dot{\bar{r}} = (1, 2u, 3u^2)$

$\dot{r} = \dot{s}\bar{t}$

$\dot{s}t = \dot{r} = (1, 2u, 3u^2) \longrightarrow ①$

$\ddot{\bar{r}} = \dot{s}\dot{t} + t\ddot{s} \Rightarrow \dot{s}\dot{s}t' + t\ddot{s}$

$\ddot{\bar{r}} = \dot{s}\dot{s}(\kappa\bar{n}) + t\ddot{s} = (0, 2, 6u) \longrightarrow ②$

① × ② ⇒

$(\dot{s}\dot{s}(\kappa\bar{n}) + \bar{t}\ddot{s}) \times (\dot{s}\bar{t}) = (0, 2, 6u) \times (1, 2u, 3u^2)$

$(\dot{s}^2\kappa\bar{n} \times \dot{s}\bar{t}) + (\bar{t}\ddot{s} \times \dot{s}\bar{t}) = (0, 2, 6u) \times (1, 2u, 3u^2)$

$\dot{s}^3(\kappa\bar{n}t) + 0 = (0, 2, 6u) \times (1, 2u, 3u^2)$.

$\kappa\dot{s}^3(-b) = -2(3u^2, -3u, 1)$.

$\dddot{\bar{r}} = \kappa\dot{s}^3 b = 2(3u^2, -3u, 1) \longrightarrow ③$.

Diff. w.r.to u,

$\kappa b\left(3\dot{s}^2 \frac{d\dot{s}}{du}\right) + \kappa\dot{s}^3\dot{b} = 2(6u, -3, 0)$

$3\kappa b\left(\dot{s}^2 \frac{d\dot{s}}{du}\right) + \kappa\dot{s}^3 b'\dot{s} \; 6(2u, -1, 0)$

$3\kappa b\left(\dot{s}^2 \frac{d\dot{s}}{du}\right) + \dot{s}^4\kappa b' = 6(2u, -1, 0)$

$b\frac{d(\dot{s}^3\kappa)}{du} - \dot{s}^4\kappa\tau\vec{n} = 6(2u, -1, 0)$ (Torsion defn $b' = -\tau\bar{n}$)

$b[\dot{s}^3\dot{\kappa} + 3\dot{s}^2\kappa\ddot{s}] - \dot{s}^4\kappa\tau\bar{n} = 6(2u, -1, 0)$

$b[\dot{s}^3\dot{\kappa} + 3\kappa\dot{s}^2\ddot{s}] - \dot{s}^4\kappa\tau\bar{n} = 6(2u, -1, 0) \longrightarrow ④$

from ② · ④ by scalar product,

$[\dot{s}^2\kappa\bar{n} + t\ddot{s}] \cdot \{b[\dot{s}^3\dot{\kappa} + 3\kappa\dot{s}^2\ddot{s}] - \dot{s}^4\kappa\tau\bar{n}\} = (0, 2, 6u).$
$(12, -6, 0)$

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$$-\dot{s}^2 \kappa \tau \dot{s}^4 \kappa = -12$$

$$\dot{s}^6 \kappa^2 \tau = 12 \longrightarrow (5)$$

$(1)^2 \Rightarrow (\dot{s}\,t)^2 = (1, 2u, 3u^2) \cdot (1, 2u, 3u^2) \qquad (\because t \cdot t = 1)$

$$\dot{s}^2 t^2 = (1 + 4u^2 + 9u^4)$$

$$\dot{s}^2 = (1 + 4u^2 + 9u^4) \longrightarrow (6)$$

$(3)^2 \Rightarrow (\kappa \dot{s}^3 b)^2 = 2^2 (3u^2, -3u, 1) \cdot (3u^2, -3u, 1) \longrightarrow (7)$

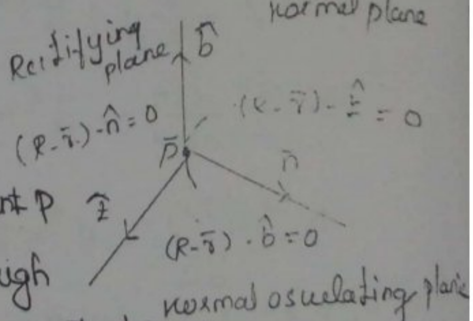
$$\kappa^2 = \frac{4(9u^4 + 9u^2 + 1)}{(\dot{s}^2)^3}$$

$$\kappa^2 = \frac{4(9u^4 + 9u^2 + 1)}{(1 + 4u^2 + 9u^4)^3} \quad [\text{by } (6)]$$

$(5) \Rightarrow$
$$\tau = \frac{12}{\dot{s}^6 \kappa^2}$$

$$= \frac{12 \cdot (1 + 4u^2 + 9u^4)^3}{4(9u^4 + 9u^2 + 1) \cdot (1 + 4u^2 + 9u^4)^3}$$

$$\therefore \boxed{\tau = \frac{3}{(9u^4 + 9u^2 + 1)}}$$



The Normal plane:

The normal plane at a point p on a curve is that plane through p which is orthogonal to the tangent at p.

Principal normal :-

The principal normal at p is the line of intersection of the normal plane & the osculating plane at p.

Note :-

A unit vector along the principal normal is denoted by $\bar{n}$.

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Curvature

2m The arc rate at which the tangent changes direction as P moves along the curve is the "curvature" of the curve & is denoted by κ (kappa).

Note :-

By definition we've $|\bar{\kappa}| = |\bar{t}'|$ & we usually take $\bar{t}' = \kappa\bar{n}$ & also $\bar{t}' = \bar{r}''$ is called the curvature vector

P.T. A necessary & sufficient condition that a curve be a straight line is that $\kappa = 0$ at all points.

2m Assume that the curve is a straight line.

We know any eqn of the st. line is of the form,

$$\bar{r} = \bar{a}\bar{s} + \bar{b}$$

$$\bar{t} = \bar{r}' \Rightarrow \bar{t} = \bar{a} \ \& \ \bar{t}' = 0 \ \& \ \kappa = 0$$

conversely,

if $\kappa = 0$ then we've $\bar{r}'' = 0$

$$\bar{r}'' = 0 \Rightarrow \bar{r} = \bar{a}\bar{s} + \bar{b}$$

which is the eqn of the st. line.

Binormal line :

The binormal line at P is the normal in a direction orthogonal to the osculating plane.

Note :

i) The unit vector along the binomial & usually $\bar{b}$ is choose $\ni$ $\bar{b} = \bar{t} \times \bar{n}$.

ii) The osculating plane passes through the tangent & the principal normal.

iii) The normal plane passes through the principal normal & Binomial.

iv) The rectifying plane through the binomial & tangent.

Torsion :

2m

As P moves along a curve the arc rate at which the osculating plane turns about the tangent is called the torsion of the curve & is denoted by τ.

Note :

i) Since $\bar{b}^2 = 1 = \bar{b}\cdot\bar{b}$. It follows that,

$b\cdot\bar{b}' = \bar{b}\cdot\bar{b}' + \bar{b}'\cdot\bar{b} = 0 \Rightarrow 2\bar{b}\cdot\bar{b}' = 0 \Rightarrow \bar{b}\cdot\bar{b}' = 0$

& $\bar{b}'$ lies in the osculating plane.

Also $\bar{b}\cdot\bar{t} = 0 \Rightarrow \bar{b}\cdot\bar{t}' + \bar{b}'\cdot\bar{t} = 0$ but as,

$\bar{b}(\kappa\bar{n}) + \bar{b}'\cdot\bar{t} = 0$

It follows that, $\bar{b}'$ is orthogonal to $\bar{t}$.

But as $\bar{b}'$ lies in the osculating plane it must be parallel to $\bar{n}$. Thus the eqn. $|\bar{b}'| = |\tau|$ follows

$|\tau| \Rightarrow$ absolute Magnitude of the torsion.

ii) The torsion τ is determined both in magnitude & sign where as the curvature $\bar{n}$ is determined only in magnitude.

2m

Lemma :

Let γ be a curve for which $\bar{b}$ varies differentiably with arc length. Then a necessary & sufficient condition that γ be a plane curve is that $\tau = 0$ at all points.

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Proof :-

Assume that r is a plane curve.

W.K.T. The osculating plane curve r is the plane containing the curve & is therefore fixed.

hence, $\tau = 0$.

conversely, if $\tau = 0$ then $\bar{b}$ be must be a constant the identity $\bar{r}' \cdot \bar{b}' = 0 \Rightarrow (\bar{r} \cdot \bar{b})' = 0$ from $r \cdot b = \text{constant}$

This shows that the <u>curve is plane</u>.

1. S.T. $[r', r'', r'''] = k^2 \tau$.

$r' = t$
$r'' = t'$
$\bar{b} = \bar{t} \times \bar{n}$
$\bar{b}' = -\tau \bar{n}$

$|k| = |\bar{t}'|$

W.K.T. $\bar{t}' = k\bar{n}$ & $\bar{t}' = \bar{r}''$

$$\bar{r}' \times \bar{r}'' = \bar{t} \times \bar{t}' = \bar{t} \times k\bar{n}.$$

$$= k(\bar{t} \times \bar{n})$$

$$\bar{r}' \times \bar{r}'' = k \cdot \bar{b} \longrightarrow (1)$$

Diff,

$a \times b + c = a + c) \times (b + c)$

$$\bar{r}' \times \bar{r}''' + \underbrace{\bar{r}'' \times \bar{r}''}_{0} = k\bar{b}' + \bar{b}k'$$

$$\bar{r}' \times \bar{r}''' = k'\bar{b} - k\tau\bar{n} \longrightarrow (2)$$

the scalar product eqn (2) with $\bar{r}''$:-

$$\bar{r}'' \cdot (\bar{r}' \times \bar{r}''') = \bar{r}'' \cdot (k'\bar{b} - k\tau\bar{n})$$

$$= \bar{t}' \cdot (k'\bar{t}\bar{n} - k\tau\bar{n})$$

$$= k\bar{n}\,(k'\bar{t} - k\tau\bar{n})$$

$$= kk'(\bar{n} \cdot \bar{t}) - k^2\tau(\bar{n} \cdot \bar{n})$$

$$= 0 - k^2\tau(1) = -k^2\tau$$

$$\bar{r}'' \cdot (\bar{r}' \times \bar{r}''') = -k^2\tau \Rightarrow (\bar{r}'', \bar{r}', \bar{r}''') = -k^2\tau$$

$$-(\bar{r}', \bar{r}'', \bar{r}''') = -k^2\tau.$$

$$\Rightarrow (\bar{r}', \bar{r}'', \bar{r}''') = k^2\tau.$$ Hence proved.

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Ex 4.4 S.T $[\dot{\bar{r}}, \ddot{\bar{r}}, \dddot{\bar{r}}] = 0$ is a necessary & sufficient conditions that the curve be plane, Evidently

$[r', r'', r'''] = [\dot{r}\dot{u}, \ddot{r} u'^2 + \dot{r} u'', \dddot{r} u'^3 + 3\ddot{r} u' u'' + \dot{r} u''']$

$= u'^6 [\dot{\bar{r}}, \ddot{\bar{r}}, \dddot{\bar{r}}]$

Proof:

Let the curve be represented by,

$\bar{r} = \bar{r}(u)$

$\bar{r}' = \bar{r}'(u) . u' = \dot{\bar{r}} u'$

$\bar{r}'' = \dot{\bar{r}} u'' + u' \ddot{\bar{r}} u' = \dot{\bar{r}} u'' + u'^2 \ddot{\bar{r}}$

$\bar{r}''' = \dot{\bar{r}} u''' + u'' \ddot{\bar{r}} u' + 2u'u'' \ddot{\bar{r}} + u'^2 \dddot{\bar{r}} u'$

$= \dot{\bar{r}} u''' + 3u'u'' \ddot{\bar{r}} + u'^3 \dddot{\bar{r}}$

$[\bar{r}', \bar{r}'', \bar{r}'''] = [\dot{\bar{r}} u', \dot{\bar{r}} u'' + u'^2 \ddot{\bar{r}}, \dot{\bar{r}} u''' + 3u'u'' \ddot{\bar{r}} + u'^3 \dddot{\bar{r}}]$

$= [\dot{\bar{r}} u', \dot{\bar{r}} u'', \dot{\bar{r}} u'''] + [\dot{\bar{r}} u', \dot{\bar{r}} u'', 3u'u'' \ddot{\bar{r}}] +$

$[\dot{\bar{r}} u', \dot{\bar{r}} u'', u'^3 \dddot{\bar{r}}] + [\dot{\bar{r}} u', u'^2 \ddot{\bar{r}}, \dot{\bar{r}} u'''] +$

$[\dot{\bar{r}} u', u'^2 \ddot{\bar{r}}, 3u'u'' \ddot{\bar{r}}] + [\dot{\bar{r}} u', u'^2 \ddot{\bar{r}}, u'^3 \dddot{\bar{r}}]$

$= u' . u'^2 . u'^3 [\dot{\bar{r}}, \ddot{\bar{r}}, \dddot{\bar{r}}]$

$= u'^6 [\dot{\bar{r}}, \ddot{\bar{r}}, \dddot{\bar{r}}]$

$(\because K^2 \tau = 0)$

$[\bar{r}', \bar{r}'', \bar{r}'''] = K^2 \tau$

$K^2 \tau = u'^6 (\dot{\bar{r}}, \ddot{\bar{r}}, \dddot{\bar{r}})$

$\therefore (\dot{\bar{r}}, \ddot{\bar{r}}, \dddot{\bar{r}}) = 0$

$\therefore K^2 \tau = 0, \tau = 0$

Hence the curve be plane conversely, if curve is a plane curve, we've $\tau = 0$.

$\therefore u'^6 [\dot{\bar{r}}, \ddot{\bar{r}}, \dddot{\bar{r}}] = 0$

$[\dot{\bar{r}}, \ddot{\bar{r}}, \dddot{\bar{r}}] = 0$ //

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1) Find the curvature & the torsion of the curve

$r = \{a(3u-u^3), 3au^2, a(3u+u^3)\}$.

$\dot{r} = \bar{t}\dot{s}$

$\dot{s}\bar{t} = \{a(3-3u^2), 6au, a(3+3u^2)\} \longrightarrow ①$

$\ddot{r} = \ddot{s}\bar{t} + \dot{s}\bar{t}'\dot{s} = \ddot{s}\bar{t} + \dot{s}^2\bar{t}'$.

$\{a(-6u), 6a, a(6u)\} = \ddot{s}\bar{t} + \dot{s}^2\bar{t}'$.

$\{-6au, 6a, 6au\} = \ddot{s}\bar{t} + \dot{s}^2\bar{t}'$.

$6a\{-u, 1, u\} = \ddot{s}\bar{t} + \dot{s}^2 \kappa\bar{n} \longrightarrow ②$

② × ① ⇒

$(\ddot{s}\bar{t} + \dot{s}^2\kappa\bar{n}) \times (\dot{s}\bar{t}) = 6a\{-u, 1, u\} \times 3a\{(1-u^2), 2u, (1+u^2)\}$

$= \vec{i}\{(-6au)(6au)\} - \vec{j}\{(-6au)[3a(1+u^2)]\} - \vec{k}\{(6a)[3a(1-u^2)] + \vec{i}[(6au)(6au)]\}$.

$= \vec{k}\{-36a^2u^2 - 18a^2 + 18a^2u^2\} + \vec{j}\{18a^2u + 18a^2u^3 + 18a^2u - 18a^2u^3\} + \vec{i}\{18a^2 + 18a^2u^2 - 36a^2u^2\}$

$= \vec{k}\{-18a^2u^2 - 18a^2\} + \vec{j}\{36a^2u\} + \vec{i}\{18a^2 - 18a^2u^2\}$

$-\kappa\dot{s}^3\bar{b} = \{(18a^2 - 18a^2u^2), 36a^2u, -(18a^2u^2 + 18a^2)\}$

$-\kappa\dot{s}^3\bar{b} = -18a^2\{(u^2-1), -2u, +(u^2+1)\}$

$\kappa\dot{s}^3\bar{b} = 18a^2\{(u^2-1), -2u, (u^2+1)\} \longrightarrow ③$

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Diff. w.r.to "u",

$$\bar{b}.\frac{d(\kappa\dot{s}^3)}{du} + (-\dot{s}^4\kappa\tau n) = 18a^2(2u,-2,2u)$$
$$= 36a^2(u,-1,u)$$

$$\bar{b}[\dot{s}^3\kappa' + 3\kappa\dot{s}^2\ddot{s}] - \bar{n}(\dot{s}^4\kappa\tau) = 36a^2(u,-1,u) \longrightarrow (4)$$

(2) (4) $\Rightarrow$

$$(\ddot{s}\bar{t} + \dot{s}^2\bar{n}\kappa)\{[\bar{b}(\dot{s}^3\kappa') + 3\kappa\dot{s}^2\ddot{s} - \bar{n}(\dot{s}^4\kappa\tau)]\}$$
$$= 6a(-u,1,u)\cdot 36a^2(u,-1,u)$$

$$-\dot{s}^2\kappa\dot{s}^4\kappa\tau = (-6au)(36a^2u) + 6a(-36a^2) + (6au)(36a^2u)$$

$$-\dot{s}^6\kappa^2\tau = -216a^3u^2 - 216a^3 + 216a^3u^2.$$

$$\dot{s}^6\kappa^2\tau = 216a^3 \longrightarrow (5)$$

$(1)^2 \Rightarrow (\dot{s}\bar{t})^2 = 3a\{(1-u^2), 2u, (1+u^2)\}.$

$$3a\{(1-u^2), 2u, (1+u^2)\}$$

$$\dot{s}^2\bar{t}^2 = 9a^2\{(1-u^2)^2 + 4u^2 + (1+u^2)^2\}$$
$$= 9a^2\{1+u^4-2u^2+4u^2+1+u^4+2u^2\}$$

$$\dot{s}^2 = 9a^2\{2+2u^4+4u^2\} \qquad (\because \bar{t}=1)$$

$$\dot{s}^2 = 18a^2\{1+u^4+2u^2\} \longrightarrow (6)$$

$(3)^2 \Rightarrow (\kappa\dot{s}^3\bar{t})^2 = (18a^2)^2[(u^2-1)^2 + (-2u)^2 + (u^2+1)^2]$

$$\kappa^2\dot{s}^6\bar{t}^2 = 324a^4[u^4+1-2u^2+4u^2+1+u^4+2u^2]$$

$$\kappa^2\dot{s}^6 = 324a^4(2u^4+4u^2+2)$$

$$\kappa^2\dot{s}^6 = 648a^4(1+u^4+2u^2)$$

$$\kappa^2 = \frac{648a^4(1+u^4+2u^2)}{(18a^2)^3(1+u^4+2u^2)^3}$$

$$K^2 = \frac{1}{9a^2} \cdot \frac{1}{1+u^4+2u^2}$$

curvature, $K = \frac{\dot{r} \times \ddot{r}}{|\dot{r}|^3}$

Torsion, $\tau = \frac{\dot{r}\ \ddot{r}\ \dddot{r}}{|\dot{r} \times \ddot{r}|^2}$

$$K = \frac{1}{3a} \cdot \left(\frac{1}{(1+u^2)^2}\right)$$

(5) $\Rightarrow$ $$\tau = \frac{216a^3}{\dot{s}^6 K^2}$$

$$= \frac{216\, a^3 \cdot 9a^2 (u^2+1)^4}{(18a^2)^3 [(1+u^2)^2]^3}$$

$$= \frac{216\, a^3 \cdot 9a^2 (u^2+1)^4}{18^3 \cdot a^6 \cdot (1+u^2)^6}$$

$$\therefore \tau = \frac{1}{3a} \cdot \left(\frac{1}{(u^2+1)^2}\right)$$

Theorem :- (Serret - Frenet formulae)

(*) Statement

10m

If $(\bar{t}, \bar{n}, \bar{b})$ is the moving orthogonal triad of unit vectors at a point P on space curve γ.

5m

then i) $\frac{d\bar{t}}{ds} = K\bar{n}$ ii) $\frac{d\bar{n}}{ds} = \tau\bar{b} - K\bar{t}$ iii) $\frac{d\bar{b}}{ds} = -\tau\bar{n}$.

(*)

Proof :-

10m

To prove i) diff. $\bar{t} \cdot \bar{t} = 1$ w.r.to "s", at a point P we get,

$$\bar{t} \cdot \bar{t}' + \bar{t} \cdot \bar{t}' = 0$$

$$2\bar{t}\,\bar{t}' = 0$$

$$\boxed{\bar{t} \cdot \bar{t}' = 0}$$

$\therefore \bar{t}'$ is perpendicular to $\bar{t}$.

$\because \bar{t} = \bar{r}'$; $\bar{t}' = \bar{r}''$

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As $\bar{r}''$ lies in the osculating plane $\bar{t}'$ also lies in the osculating plane.

$\bar{t}'$ is a vector ⊥r to $\bar{t}$ & lies in the osculating plane

Hence, $\bar{t}'$ is parallel to the principal normal.

By defn., $\bar{t}' = |\kappa|$

since w.r.t. the magnitude κ & the direction $\bar{n}$ of $\bar{t}'$ we can write,

$$\bar{t}' = \pm \kappa \bar{n}.$$

By convention, we take $\bar{t}' = \kappa \bar{n}$.

iii) diff., $\bar{b}\cdot\bar{b} = 1$ w.r.to "s",

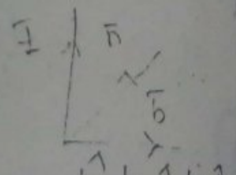
$$\bar{b}\cdot\bar{b}' + \bar{b}'\cdot\bar{b} = 0$$

$$2\bar{b}\cdot\bar{b}' = 0$$

$$\boxed{\bar{b}\cdot\bar{b}' = 0}$$

∴ $\bar{b}'$ is perpendicular to $\bar{b}$.

Hence $\bar{b}'$ lies in the osculating plane.



$\hat{t}$ ⊥r $\hat{n}$ ⊥r $\hat{b}$

since $\bar{b}\cdot\bar{t} = 0$

Diff., w.r.to "s",

$$\bar{b}\cdot\bar{t}' + \bar{b}'\cdot\bar{t} = 0$$

$$\bar{b}\kappa\bar{n} + \bar{b}'\cdot\bar{t} = 0$$

$$\kappa(\bar{b}\cdot\bar{n}) + \bar{b}'\bar{t} = 0 \Rightarrow 0 + \bar{b}'\bar{t} = 0$$

$$\boxed{\bar{b}'\cdot\bar{t} = 0}$$

∴ $\bar{b}'$ is perpendicular to $\bar{t}$ & lies in the osculating plane.

Hence, $\bar{b}'$ is parallel to the principal normal at

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By defn, $|\bar{b}'| = \tau$ being the torsion at P.

Since w.k.t. the magnitude τ & direction $\bar{n}$ of $\bar{b}'$ we can write $\bar{b}' = -\tau\bar{n}$.

where the (-)ve is introduced because as a convention torsion is recorded as +ve when the rotation of the osculating plane as s increases is in the direction of a right-handed screw moving in the direction of $\bar{t}$.

ii) Let us consider $\bar{n} = \bar{b} \times \bar{t}$.

Diff. b.s., of the above vector w.r.to s,

$$\bar{n}' = \frac{d\bar{n}}{ds} = \bar{b}' \times \bar{t} + \bar{b} \times \bar{t}'$$

$$= (-\tau\bar{n}) \times \bar{t} + \bar{b} \times (\kappa\bar{n})$$

$$= -\tau(\bar{n} \times \bar{t}) + \kappa(\bar{b} \times \bar{n})$$

$$= -\tau(-\bar{b}) + \kappa(-\bar{t})$$

$$\therefore \bar{n}' = \tau\bar{b} - \kappa\bar{t}$$

Behaviour of a curve nearer one of its points:

Theorem:-

Let the curve be of class $m \geq 4$ at a point P on the curve let the co-ordinates axes ox, oy, oz be taken along $\bar{t}, \bar{n}, \bar{b}$ respectively.

If x, y, z are the co-ordinates of a neighbouring points Q on the curve then,

$$x = s - \frac{\kappa^2 s^3}{6} - \frac{\kappa\kappa'}{8} s^4 + o(s^4)$$

$$y = \frac{\kappa}{2}s^2 + \frac{\kappa'}{6}s^3 + \frac{\kappa'' - \kappa\tau^2 - \kappa^3 s^4}{24} + o(s^4)$$

$$z = \frac{\kappa\tau}{6}s^3 + \frac{2\kappa'\tau + \kappa\tau'}{24} s^4 + o(s^4) \quad \text{as } s \to 0$$

The second Fundamental form

Proof :-

If the curve is of the class $m \geq 4$ we've by Taylor's theorem,

$$r(s) = r(0) + s r'(0) + \frac{s^2}{2!} r''(0) + \frac{s^3}{3!} r'''(0) + \frac{s^4}{4!} r^{IV}(0) + o(s^4) \text{ as } s \to 0$$

where s - small arc PQ & $r(0) = 0$ $\longrightarrow (*)$

To study the (*) let us find $\bar{r}', \bar{r}'', \bar{r}''', \bar{r}^{IV}$ at the origin "O".

$$(\bar{n}' = \tau \bar{b} - k\bar{t}) \longrightarrow (1)$$

$$r'(0) = \bar{t}$$

$$r''(0) = \bar{t}' = k\bar{n} \longrightarrow (2)$$

$$r'''(0) = k'\bar{n} + k\bar{n}'$$

$$= k'\bar{n} + k(\tau\bar{b} - k\bar{t})$$

$$= k'\bar{n} + k\bar{b}\tau - k^2\bar{t} \longrightarrow (3)$$

$$r^{IV}(0) = \bar{n}k'' + k'\bar{n}' + k\tau\bar{b}' + k'\bar{b}\tau - 2kk'\bar{t} - k^2\bar{t}' + k\bar{b}\tau'$$

$$= k''\bar{n} + k'(\bar{b}\tau - k\bar{t}) + k\tau(-\tau\bar{n}) + k'\bar{b}\tau + k\bar{b}\tau' - 2kk'\bar{t} - k^2(k\bar{n})$$

$$= k''\bar{n} + k'\bar{b}\tau - kk'\bar{t} - k\tau^2\bar{n} + k'\bar{b}\bar{t} - 2kk' - k^3\bar{n} + k\bar{b}\tau'$$

$$= -3kk'\bar{t} + 2k'\bar{b}\tau + k''\bar{n} - k\tau^2\bar{n} - k^3\bar{n} - k\bar{b}\tau'$$

$$r^{IV}(0) = \bar{t}(-3kk') + \bar{b}(2k'\tau + k\tau') + \bar{n}(k'' - k\tau^2 - k^3) \longrightarrow (4)$$

sub in (*)

$$r(s) = 0 + s\bar{t} + \frac{s^2}{2!} k\bar{n} + \frac{s^3}{3!}(k'\bar{n} + k\bar{b}\tau - k^2\bar{t}) + \frac{s^4}{4!}(-3kk'\bar{t} + \bar{b}(2k'\tau + k\tau') + \bar{n}(k'' - k\tau^2 - k^3)) + o(s^4)) \text{ as } s \to 0 \longrightarrow (5)$$

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Since $x(0)=0$ at P & gathering the co-efficients of $\bar{b}, \bar{n}, \bar{t}$ in ⑤ we get,

$$\bar{r}(s) = \bar{t}\left(s - \kappa^2 \frac{s^3}{3!} - 3\kappa\kappa' \frac{s^4}{4!}\right) + \bar{n}\left(\frac{s^2}{2!}\kappa + \frac{s^3}{3!}\kappa' + (\kappa'' - \kappa\tau^2 - \kappa^3)\frac{s^4}{4!}\right) + \bar{b}\left(\frac{s^3}{3!}\kappa\tau + \frac{s^4}{4!}(2\kappa'\tau + \kappa\tau') + o(s^4)\right) \text{ as } s \to 0.$$

If x, y, z are the co-ordinates of the neighbouring point Q with +ve vector $\bar{r}(s)$ with reference to the co-ordinate system $O(x, y, z)$ in the direction of $\bar{t}, \bar{n}, \bar{b}$ then,

$$x = s - \frac{\kappa^2 s^3}{6} - \frac{\kappa\kappa'}{8} \cdot s^4 + o(s^4)$$

$$y = \frac{\kappa}{2}s^2 + \frac{\kappa'}{6}s^3 + \frac{\kappa'' - \kappa\tau^2 - \kappa^3}{24} \cdot s^4 + o(s^4)$$

$$z = \frac{\kappa\tau}{6}s^3 + \frac{2\kappa'\tau + \kappa\tau'}{24}s^4 + o(s^4)$$

express the co-ordinates x, y, z in terms of the arcual length $PQ = s$.

(x, y, z) is called serret-frenet approximation of the curve.

Note:-

using the above eqn we've the following deduction

i) $2y/x^2 \sim \kappa$ as $s \to 0$ & $\frac{3z}{xy} \sim \tau$ as $s \to 0$.

Proof:

$$x = s - \frac{\kappa^2 s^3}{6} - \frac{\kappa\kappa'}{8} s^4 + o(s^4) \longrightarrow ①$$

$$y = \frac{\kappa}{2}s^2 + \frac{\kappa'}{6}s^3 + \frac{\kappa'' - \kappa\tau^2 - \kappa^3}{24} s^4 + o(s^4) \longrightarrow ②$$

$$z = \frac{\kappa\tau}{6} \cdot s^3 + \frac{2\kappa'\tau + \kappa\tau'}{24} s^4 + o(s^4) \longrightarrow ③$$

From ① neglecting the powers of s^3 we've $x \sim s \longrightarrow ④$

The second Fundamental Form

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from ② neglecting the powers of s^3 we've,

$$y \sim \frac{k}{2} s^2$$

$$y \sim \frac{k}{2} x^2 \quad (\text{from } ④)$$

$$\frac{2y}{x^2} \sim k \longrightarrow ⑤$$

from ③, neglecting the powers of s^4 we've,

$$z \sim \frac{k\tau s^3}{6} \Rightarrow z \sim \frac{k\tau x^3}{6}$$

$$z \sim \frac{2y x^3 \tau}{x^2 \cdot 6} \Rightarrow z \sim \frac{\cancel{2} x y \tau}{\cancel{6}\,3}$$

$$\Rightarrow \frac{3z}{xy} \sim \tau$$

ii) The above formula for curvature k resembles Newton's formula for curvature of plane curve.

The chord $PQ = (x^2+y^2+z^2)^{1/2} \sim s\left(1-\frac{k^2 s^2}{24}\right)$.

Proof:

From ①,②,③ neglecting the power series $x^{6}s^4$

we get the co-ordinates as,

$$x \sim s - \frac{k^2 s^3}{6}$$

$$y \sim \frac{k}{2} s^2 + \frac{k'}{6} s^3$$

$$z \sim \frac{k\tau}{6} \cdot s^3$$

$$x^2+y^2+z^2 \sim \left(s - \frac{k^2 s^3}{6}\right)^2 + \left(\frac{k}{2}s^2 + \frac{k'}{6}s^3\right)^2 + \left(\frac{k\tau}{6}s^3\right)^2$$

$$\sim \left(s^2 + \frac{k^4 s^6}{36} - 2 \cdot \frac{s^4 k^2}{6}\right) + \frac{k^2 s^4}{4} + \frac{k'^2 s^6}{36} + \frac{2kk's^5}{12} + \frac{k^2\tau^2 s^6}{36}$$

$$x^2+y^2+z^2 \sim s^2+\frac{\kappa^4 s^6}{36}-\frac{s^4\kappa^2}{3}+\frac{s^4\kappa^2}{4}+\frac{\kappa'^2 s^6}{36}+$$

$$\frac{\kappa\kappa' s^5}{6}+\frac{\kappa^2\tau^2 s^6}{36}$$

$$x^2+y^2+z^2 \sim s^2+\frac{\kappa^4 s^6}{36}-\frac{s^4\kappa^2}{12}+\frac{\kappa'^2 s^6}{36}+\frac{\kappa\kappa' s^5}{6}+\frac{\kappa^2\tau^2 s^6}{36}$$

Neglecting the degree ≥ 5 we've

$$x^2+y^2+z^2 \sim s^2-\frac{s^4\kappa^2}{12} \sim s^2\left(1-\frac{s^2\kappa^2}{12}\right).$$

$$(x^2+y^2+z^2)^{1/2} \sim s\left(1-\frac{s^2\kappa^2}{12}\right)^{1/2}$$

Using the binomial expansion in R.H.S.

$$(x^2+y^2+z^2)^{1/2} \sim s\left[1-\tfrac{1}{2}\cdot\tfrac{1}{12}\,s^2\kappa^2+\dots\right]$$

$$\sim s\left[1-\frac{s^2\kappa^2}{24}\right] \text{ (neglecting higher powers)}$$

which shows that ~~that~~ the curvature $\kappa \neq 0$ the arc length PQ differs from the chord PQ by the term of a 3rd order in s.

Rectifying plane :

2M The plane containing the tangent & binomial is called rectifying plane from this Rectifying plane we've the eqn of the Rectifying plane if $\bar{r}=\bar{r}(u)$ is a point on the curve & R is the position vector of any point of the rectifying plane. Then $R-\bar{r}$ is in the rectifying plane & orthogonal to $\bar{n}$. Hence,

$(R-\bar{r})\cdot\bar{n}=0$ is the eqn of the rectifying plane

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Ex :- 4.6 S.T. the projection of the curve near p on the osculating plane is app. the curve $Z=0$, $Y=\frac{1}{2}Kx^2$, its projection on the rectifying plane is app. $Y=0$, $Z=\frac{1}{6}K\tau x^3$ & its projection on the normal plane is app.; $X=0$, $Z^2=\frac{2}{9}(\tau^2/K)\,Y^3$.

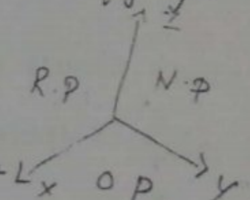
Proof :-

W.K.T. ①, ②, ③ of the projection of curve on the 3 planes in each case we obtain the lowest powers of x^3 & eliminate S.

The projection of the curve on the osculating plane is $x\sim S$ & $Y\sim \frac{K}{2}x^2$ & so $Z=0$

The projection of the curve on the rectifying plane is,

$$Z\sim \frac{K\tau x^3}{6},\quad x\sim S,\quad Y=0$$



The projection of the curve on the normal plane is,

$$x=0,\ Y\sim \frac{K}{2}x^2,\ Z\sim \frac{K\tau x^3}{6}$$

$$Z^2\sim \frac{K^2\tau^2x^6}{36} \Rightarrow Z^2\sim \frac{K^2\tau^2(2Y/K)^3}{36}$$

$$Z^2\sim \frac{8Y^3K^2\tau^2}{K^3\,36}$$

$$\Rightarrow \boxed{Z^2\sim \frac{2}{9}\cdot\frac{Y^3\tau^2}{K}}$$

Ex :- 4.7 S.T. the length of the common ⊥r d of the tangent at two near points distance S apart is app. given by, $d=K\tau S^3/12$.

Proof :-

Let P, Q have parameter 0 & S respectively

The unit tangent vectors of P & Q are

$\gamma'(0)$, $\gamma'(s)$ so the unit vector of the common $\perp$ is along $\gamma'(s) \times \gamma'(0)$.

The projection of the vector $(r(s) - r(0))$ in this direction is equal to d.

$$d = [\gamma(s) - \gamma(0)] \cdot \frac{\gamma'(s) \times \gamma'(0)}{|\gamma'(s) \times \gamma'(0)|}$$

$$d = \gamma(s) \cdot \frac{\gamma'(s) \times \gamma'(0)}{|\gamma'(s) \times \gamma'(0)|} \qquad (\because \gamma(0) = 0)$$

$$d = \frac{[r(s), \gamma'(s), \gamma'(0)]}{|\gamma'(s) \times \gamma'(0)|} \longrightarrow (1)$$

since we like to find the app. value of above expansions we shall use the Taylor series expansion of $\bar{r}(s)$ & $\bar{r}'(s)$ including the terms of 3rd degree in s.

From the previous them,

$$x \sim s - \frac{k^2 s^3}{6}$$

$$y \sim \frac{k s^2}{2} + \frac{k' s^3}{6}$$

$$z \sim k\tau s^3/6 \qquad (x\bar{i} + y\bar{j} + z\bar{k}) \cdot (t\bar{i} + n\bar{j} + b\bar{k})$$

$$\text{Hence } \gamma(s) = \left(s - \frac{k^2 s^3}{6}\right)\bar{t} + \left(\frac{k s^2}{2} + \frac{k' s^3}{6}\right)\bar{n} + \frac{k\tau s^3}{6}\bar{b} \longrightarrow (2)$$

Since $\bar{t}, \bar{n}, \bar{b}$ given the fixed direction of the co-ordinate axes to "o".

Diff., the above eqn (2) w.r.to 's',

$$\gamma'(s) = \left(1 - \frac{3s^2 k^2}{6}\right)\bar{t} + \left(\frac{2ks}{2} + \frac{3s^2 k'}{6}\right)\bar{n} + \left(\frac{3s^2 k\tau}{6}\right)\bar{b}$$

$$\gamma'(s) = \left(1 - \frac{s^2 k^2}{2}\right)\bar{t} + \left(ks + \frac{s^2 k'}{2}\right)\bar{n} + \left(\frac{s^2 k\tau}{2}\right)\bar{b}.$$

$$\boxed{\gamma'(0) = \bar{t}}$$

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$$\gamma'(s) \times \gamma''(0) = \left[\left(1-\frac{s^2k^2}{2}\right)\bar{t} + \left(ks+\frac{s^2k'}{2}\right)\bar{n} + \left(\frac{s^2k\tau}{2}\right)\bar{b}\right] \times \bar{t}k$$

$$= \left(ks+\frac{s^2k'}{2}\right)(-\bar{b}) + \left(\frac{s^2k\tau}{2}\right)(+\bar{n})$$

$$\gamma'(s) \times \gamma'(0) = \bar{n}\left(\frac{s^2k\tau}{2}\right) - \bar{b}\left(ks+\frac{s^2k'}{2}\right)$$

$$|\gamma'(s) \times \gamma'(0)|^2 = \bar{n}^2 \frac{s^4k^2\tau^2}{4} - \bar{b}^2\left(ks+\frac{s^2k'}{2}\right)^2$$

$$= \frac{k^2s^4\tau^2}{4} + k^2s^2 + \frac{s^4k'^2}{4} + \frac{2kk's s^2}{2}$$

$$|\gamma'(s) \times \gamma'(0)|^2 = \frac{k^2s^4\tau^2}{4} + k^2s^2 + \frac{s^4k'^2}{4} + kk's^3 \longrightarrow (3)$$

Neglecting the terms of higher order ≥ 4.

$$\therefore |\gamma'(s) \times \gamma'(0)|^2 = k^2s^2 + kk's^3$$

$$= k^2s^2\left[1+\frac{k's}{k}\right]$$

$$|\gamma'(s) \times \gamma'(0)| = ks\left[1+\frac{k'}{k}s\right]^{1/2} \longrightarrow (4)$$

$$[\gamma(s), \gamma'(s), \gamma''(0)] = \gamma(s) \cdot [\gamma'(s) \times \gamma'(0)]$$

$$= \left[\left(s-\frac{s^3k^2}{6}\right)\bar{t} + \left(\frac{ks^2}{2}+\frac{k's^3}{6}\right)\bar{n} + \left(\frac{k\tau s^3}{6}\right)\bar{b}\right] \cdot \left[\bar{n}\left(\frac{s^2k\tau}{2}\right) - \bar{b}\left(ks+\frac{s^2k'}{2}\right)\right]$$

$$= \left(\frac{ks^2}{2}+\frac{k's^3}{6}\right)\cdot\left(\frac{s^2k\tau}{2}\right) - \left(\frac{k\tau s^3}{6}\right)\left(ks+\frac{s^2k'}{2}\right)$$

$$= \frac{k^2s^4\tau}{4} + \frac{kk's^5\tau}{12} - \frac{k^2s^4\tau}{6} - \frac{kk's^5\tau}{12}$$

$$= \frac{3k^2s^4\tau - 2k^2s^4\tau}{12}$$

$$[\gamma(s), \gamma'(s), \gamma'(0)] = \frac{k^2s^4\tau}{12} \longrightarrow (5)$$

From ①, $d = \dfrac{\frac{\kappa^2 s^4 \tau}{12}}{\kappa s\left[1+\frac{\kappa' s}{\kappa}\right]^{1/2}}$

$\frac{③}{④} \Rightarrow$

$= \dfrac{\kappa^2 s^4 \tau}{12 \kappa s}\left[1+\dfrac{\kappa' s}{\kappa}\right]^{-1/2}$

$= \dfrac{\kappa s^3 \tau}{12}\left[1-\dfrac{1}{2}\dfrac{\kappa' s}{\kappa}+\dfrac{1}{2}\left(\dfrac{1}{2}+1\right)\left(\dfrac{\kappa' s}{\kappa}\right)^2+\cdots\right]$

$= \dfrac{\kappa s^3 \tau}{12}\left[1-\dfrac{1}{2}\dfrac{\kappa' s}{\kappa}\right]$

$\therefore \boxed{d = \dfrac{\kappa s^3 \tau}{12}}$ (app) [∵ neglecting higher powers]

Curvature & torsion of a curve given as the intersection of two surfaces:

If a curve is given as the intersection of two surfaces $f(x,y,z)=0$, $g(x,y,z)=0$ & if a sets of parametric eqns for the curve can't be obtained then the curvature & torsion of the curve may be calculated by the following method.

Proof :-

Let the curve of intersection be represented by the eqn $\bar{r} = \bar{r}(u)$, & the two surfaces be given by $f(\bar{r})=0$, $g(\bar{r})=0$.

Now, the unit tangent vector to the curve is orthogonal to the normal of both surface.

Thus if $\nabla f = \left(\dfrac{\partial f}{\partial x}, \dfrac{\partial f}{\partial y}, \dfrac{\partial f}{\partial z}\right)$ it follows that $\bar{t}$ is parallel to $\nabla f \times \nabla g$

$\lambda \bar{r}' = \nabla f \times \nabla g$

(ie) $\boxed{h = \lambda \bar{r}'}$

Let us assume $h = (h_1, h_2, h_3)$ since,

$$\nabla f \times \nabla g = h = \lambda \bar{r}' \longrightarrow (1)$$

$$h = \lambda \bar{t} \longrightarrow (2)$$

$$|h| = \sqrt{\lambda^2} \Rightarrow |h| = \lambda$$

In the above eqn (1) one should note the R.H.S given interms of the dash derivatives, where in the L.H.S is given interms of partial derivatives.

Hence let us find the relation b/n these two

$$h = \lambda \frac{d\bar{r}}{ds} = \lambda \left[\frac{dr}{dx}\frac{dx}{ds} + \frac{dr}{dy}\frac{dy}{ds} + \frac{dr}{dz}\frac{dz}{ds} \right]$$

$$= \lambda \left[x' \frac{\partial}{\partial x} + y' \frac{\partial}{\partial y} + z' \frac{\partial}{\partial z} \right] r$$

$$= (h_1, h_2, h_3)$$

Since $\frac{\partial r}{\partial x} = (1,0,0)$; $\frac{\partial r}{\partial y} = (0,1,0)$; $\frac{\partial r}{\partial z} = (0,0,1)$

we obtain, from the above eqn,

$$(\lambda x', \lambda y', \lambda z') = (h_1, h_2, h_3)$$

Let Δ be the operator defined by,

$$\therefore \Delta = \lambda \frac{d}{ds} = \left(h_1 \frac{\partial}{\partial x} + h_2 \frac{\partial}{\partial y} + h_3 \frac{\partial}{\partial z} \right) \longrightarrow (3)$$

Hence by the defn of operator $\Delta r = h \longrightarrow (4)$

operating on b.s of (2),

$$\therefore \Delta h = \Delta(\lambda \bar{t}) = \lambda \frac{d}{ds}(\lambda \bar{t})$$

$$= \lambda [\lambda \bar{t}' + \lambda' \bar{t}] = \lambda [\lambda \kappa \bar{n} + \lambda' \bar{t}]$$

$$\Delta h = \lambda^2 \kappa \bar{n} + \lambda \lambda' \bar{t} \longrightarrow (5)$$

Taking the vector product on (2) & (5),

$$h \times \Delta h = \lambda \bar{t} \times (\lambda^2 \kappa \bar{n} + \lambda \lambda' \bar{t}) = \lambda^3 \kappa \bar{b} \longrightarrow (6)$$

Take $h \times \Delta h = k$

$$\therefore k = \lambda^3 \kappa \bar{b} \longrightarrow (7)$$

$|\kappa| = \sqrt{\lambda^6 \kappa^2} \Rightarrow |\kappa| = (\lambda^6 \kappa^2)^{1/2}$

$|\kappa| = (\lambda^3 \kappa)^{2/2} = \lambda^3 \kappa \longrightarrow (8)$

from which $\kappa = \frac{|\kappa|}{\lambda^3}$ where $\kappa = h \times \Delta h \longrightarrow (*)$

operating on b.s. by (7) with Δ,

$$\Delta \kappa = \Delta(\lambda^3 \kappa \bar{b}) = \lambda \frac{d}{ds}(\lambda^3 \kappa \bar{b})$$

$$= \lambda^4 \frac{d}{ds}(\kappa \bar{b}) = \lambda^4 [\kappa \bar{b}' + \kappa' \bar{b}]$$

$$= \lambda^4 [\kappa(-\tau \bar{n}) + \kappa' \bar{b}]$$

$$\Delta \kappa = \lambda^4 [\kappa' \bar{b} - \kappa \tau \bar{n}] \longrightarrow (9)$$

Taking scalar product on (5) & (9),

$$\Delta h \cdot \Delta \kappa = (\lambda^2 \kappa \bar{n} + \lambda \lambda' \bar{t}) \cdot [\lambda^4 (\kappa' \bar{b} - \kappa \tau \bar{n})]$$

$$= -\lambda^2 \kappa \lambda^4 \kappa \tau$$

$$\Delta h \cdot \Delta \kappa = -\lambda^6 \kappa^2 \tau \longrightarrow (10)$$

$$\tau = -\frac{\Delta h \cdot \Delta \kappa}{\lambda^6 \kappa^2} = -\frac{\Delta h \cdot \Delta \kappa}{(\lambda^3 \kappa)^2}$$

$$\Rightarrow \tau = -\frac{\Delta h \cdot \Delta K}{|\kappa|^2} \quad [\because \text{from } (8)]$$

(*) Ex: 5.1 obtain the curvature & torsion of the curve of intersection of the two quadratic surfaces $ax^2 + by^2 + cz^2 = 1$; $a'x^2 + b'y^2 + c'z^2 = 1$.

10 m

Important.

Proof: Let $f = \frac{1}{2}(ax^2 + by^2 + cz^2 - 1)$

$$g = \frac{1}{2}(a'x^2 + b'y^2 + c'z^2 - 1)$$

$$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$$

$$\nabla f = (ax, by, cz) \longrightarrow (1)$$

$$\text{Ily, } \nabla g = (a'x, b'y, c'z) \longrightarrow (2)$$

The second Fundamental Form

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$(1)\times(3) \Rightarrow \nabla f\times\nabla g = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ ax & by & cz \\ a'x & b'y & c'z \end{vmatrix}$

$$= \bar{i}[bc'yz - cb'yz] - \bar{j}[ac'xz - xca'z] + \bar{k}[ab'xy - a'bxy]$$

$$= \bar{i}[(bc'-cb')yz] - \bar{j}[(ac'-ca')xz] + \bar{k}[(ab'-a'b)xy]$$

Take $bc'-cb' = A$; $ca'-ac' = B$; $ab'-a'b = C$

$\therefore \nabla f\times\nabla g = \bar{i}Ayz + \bar{j}Bxz + \bar{k}Cxy$

(ii) $\nabla f\times\nabla g = (Ayz, Bxz, Cxy) \longrightarrow (3)$

$\nabla f\times\nabla g = xyz\left(\frac{A}{x}, \frac{B}{y}, \frac{C}{z}\right) \longrightarrow (4)$

Since $\nabla f\times\nabla g$ is parallel to $\bar{t}$.

But $\bar{t}$ $\frac{d\bar{r}}{ds}$ we take,

$\lambda\bar{t} = \lambda\frac{d\bar{r}}{ds} = (xyz)\left(\frac{A}{x}, \frac{B}{y}, \frac{C}{z}\right) = h \longrightarrow (5)$

Thus, $h_1 = \frac{A}{x}$, $h_2 = \frac{B}{y}$, $h_3 = \frac{C}{z}$

$\lambda\bar{t} = (A/x, B/y, C/z)$

$\lambda^2 = (A/x)^2 + (B/y)^2 + (C/z)^2$

$\lambda^2 = \Sigma(A/x)^2 \longrightarrow (6)$

Hence, $\Delta = \left(h_1\frac{\partial}{\partial x} + h_2\frac{\partial}{\partial y} + h_3\frac{\partial}{\partial z}\right)$

$\Delta = A/x\cdot\partial/\partial x + B/y\cdot\partial/\partial y + C/z\cdot\partial/\partial z$

$\Delta h = (A/x\cdot\partial/\partial x + B/y\,\partial/\partial y + C/z\cdot\partial/\partial z)\cdot(A/x, B/y; C/z)$

$= A/x\,\partial/\partial x(A/x) + A/x\cdot\partial/\partial x(B/y) + A/x\cdot\partial/\partial x(C/z),$

$B/y\,\partial/\partial y(A/x) + B/y\cdot\partial/\partial y(B/y) + B/y\cdot\partial/\partial y(C/z),$

$C/z\,\partial/\partial z(A/x) + C/z\,\partial/\partial z(B/y) + C/z\cdot\partial/\partial z(C/z)$

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$$\Delta h = A/x \cdot \partial/\partial x (A/x),\ B/y \cdot \partial/\partial y (B/y),\ C/z \cdot \partial/\partial z (C/x)$$

$$= A^2/x\ \partial/\partial x (1/x),\ B^2/y \cdot \partial/\partial y (1/y),\ C^2/z \cdot \partial/\partial z (1/x)$$

$$= A^2/x\ (-x^{-2}),\ B^2/y (-y^{-2}),\ C^2/z (-z^{-2})$$

$$\Delta h = -\left[\frac{A^2}{x^3}, \frac{B^2}{y^3}, \frac{C^2}{z^3}\right] \longrightarrow (7)$$

But $K = h \times \Delta h$

$$h \times \Delta h = \begin{vmatrix} \bar{i} & \bar{n} & \bar{b} \\ A/x & B/y & C/z \\ -A^2/x^3 & -B^2/y^3 & -C^2/z^3 \end{vmatrix}$$

$$= \bar{i}\left[-B/y \cdot C^2/z^3 + C/z \cdot B^2/y^3\right] - \bar{n}\left[-A/x \cdot C^2/z^3 + C/z \cdot A^2/x^3\right]$$

$$+ \bar{b}\left[-A/x \cdot B^2/y^3 + B/y \cdot A^2/x^3\right]$$

$$= \bar{i}\left[\frac{B^2C}{zy^3} - \frac{BC^2}{yz^3}\right] - \bar{n}\left[\frac{A^2C}{zx^3} - \frac{AC^2}{xz^3}\right] + \bar{b}\left[\frac{A^2B}{yx^3} - \frac{AB^2}{xy^3}\right]$$

$$= \bar{i}\ \frac{BC}{y^3z^3}\left[Bz^2 - Cy^2\right] - \bar{n}\left[Az^2 - Cx^2\right]\frac{AC}{x^3z^3} + \bar{b}\ \frac{AB}{x^3y^3}$$

$$K = h \times \Delta h = \bar{i}\ \frac{BC}{y^3z^3}\left[Bz^2 - Cy^2\right] + \bar{n}\left[Cx^2 - Az^2\right]\frac{AC}{x^3z^3} + \bar{b}\ \frac{AB}{x^3y^3}\left[Ay^2 - Bx^2\right]$$

Now,

$$Bz^2 - Cy^2 = (ca' - ac')z^2 - (ab' - a'b)y^2 \longrightarrow (8)$$

$$= ca'z^2 - ac'z^2 - ab'y^2 + a'by^2$$

$$= a'(cz^2 + by^2) - a(c'z^2 + b'y^2)$$

$$= a'(1 - ax^2) - a(1 - a'x^2) \quad \text{[from given]}$$

$$= a' - aa'x^2 - a + a'ax^2$$

$$Bz^2 - Cy^2 = a' - a$$

$$Cx^2 - Az^2 = (ab' - a'b)x^2 - (bc' - cb')z^2$$

$$= ax^2b' - a'bx^2 - bc'z^2 + cb'z^2$$

$$= b'(ax^2 + cz^2) - b(a'x^2 + b'z^2)$$

$$= b'(1 - by^2) - b(1 - b'y^2) \Rightarrow b' - bb'y^2 - b + bb'y^2$$

$$Cx^2 - Az^2 = b' - b$$

$$Ay^2 - Bx^2 = (bc' - cb')y^2 - (ca' - ac')x^2$$
$$= bc'y^2 - cb'y^2 - ca'x^2 + ac'x^2$$
$$= c'(by^2 + ax^2) - c(b'y^2 + a'x^2)$$
$$= c'(1 - cz^2) - c(1 - c'z^2)$$
$$= c' - cc'z^2 - c + cc'z^2$$

$$Ay^2 - Bx^2 = c' - c$$

(8) $\Rightarrow h \times \Delta h = \bar{I} \cdot \dfrac{BC(a'-a)}{y^3 z^3} + \bar{n} \cdot \dfrac{AC(b'-b)}{x^3 z^3} + \bar{b}\, \dfrac{AB(c'-c)}{x^3 y^3}$

$|a|^2 = (\bar{a})^2 = a$

$$K = |h \times \Delta h|^2 = \left[\frac{B^2C^2(a'-a)^2}{y^6 z^6} + \frac{A^2C^2(b'-b)^2}{x^6 z^6} + \frac{A^2B^2(c'-c)^2}{x^6 y^6}\right]$$

$$K = h \times \Delta h = \left[\frac{BC(a'-a)}{y^3 z^3}, \frac{AC(b'-b)}{x^3 z^3}, \frac{AB(c'-c)}{x^3 y^3}\right] \longrightarrow (9)$$

$$|k|^2 = |h \times \Delta h|^2 = \frac{A^2B^2C^2}{x^6 y^6 z^6}\left[\frac{x^6(a'-a)^2}{A^2} + \frac{y^6(b'-b)^2}{B^2} + \frac{z^6(c'-c)^2}{C^2}\right]$$

$$= \frac{A^2B^2C^2}{x^6 y^6 z^6} \sum_{y,z} \frac{x^6(a'-a)^2}{A^2} \longrightarrow (10)$$

W.K.T. $K = \dfrac{|k|}{h^3} = \dfrac{|k|}{|h|^3}$

$$K^2 = \frac{|k|^2}{|h|^6} = \frac{\dfrac{A^2B^2C^2}{x^6 y^6 z^6} \cdot \sum \dfrac{x^6}{A^2}(a'-a)^2}{(\sum A/x)^6} \longrightarrow (11)$$

To find τ let us find Δh:-

from $K = \lambda^3 K \bar{b}$ $\rightarrow$ previous thm.

$$\lambda^3 K \bar{b} = K = \frac{ABC}{x^3 y^3 z^3} \sum \frac{x^3}{A}(a'-a), \frac{y^3}{B}(b'-b), \frac{z^3}{C}(c'-c)$$

$$\frac{x^3 y^3 z^3}{ABC} \lambda^3 K \bar{b} = \left[\frac{x^3}{A}(a'-a), \frac{y^3}{B}(b'-b), \frac{z^3}{C}(c'-c)\right] \longrightarrow (12)$$

$$\text{let } \mu = \frac{x^3 y^3 z^3}{ABC} \lambda^3 K \longrightarrow (13)$$

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from (12) $\Rightarrow$ $\mu\bar{b} = \left[\frac{x^3}{A}(a'-a), \frac{y^3}{B}(b'-b), \frac{z^3}{C}(c'-c)\right]$ $\longrightarrow$ (13)

operating with Δ on b.s. on (13) we've,

$$\Delta\mu\bar{b} = \Delta\left[\frac{x^3}{A}(a'-a), \frac{y^3}{B}(b'-b), \frac{z^3}{C}(c'-c)\right]$$

$$\lambda\frac{d}{ds}(\mu\bar{b}) = \left(\frac{A}{x}\frac{\partial}{\partial x} + \frac{B}{y}\frac{\partial}{\partial y} + \frac{C}{z}\frac{\partial}{\partial z}\right)\cdot\left(\frac{x^3}{A}(a'-a), \frac{y^3}{B}(b'-b), \frac{z^3}{C}(c'-c)\right)$$

$$\lambda[\mu\bar{b}' + \mu'\bar{b}] = \frac{A}{x}\frac{\partial}{\partial x}\left(\frac{x^3}{A}(a'-a)\right), \frac{B}{y}\frac{\partial}{\partial y}\left(\frac{y^3}{B}(b'-b)\right), \frac{C}{z}\frac{\partial}{\partial z}\left(\frac{z^3}{C}(c'-c)\right).$$

$$\lambda[\mu(-\tau\bar{n}) + \mu'\bar{b}] = \left[\frac{A(a'-a)}{Ax}\cdot\frac{\partial}{\partial x}(x^3), \frac{B(b'-b)}{By}\frac{\partial}{\partial y}(y^3), \frac{C(c'-c)}{Cz}\cdot\frac{\partial}{\partial z}(z^3)\right]$$

$$= \left[\frac{(a'-a)}{x}\cdot 3x^2, \frac{(b'-b)}{y}\cdot 3y^2, \frac{(c'-c)}{z}\cdot 3z^2\right]$$

$$\lambda\mu'\bar{b} - \lambda\mu\tau\bar{n} = [3x(a'-a), 3y(b'-b), 3z(c'-c)]$$ $\longrightarrow$ (14)

from (7) of the thm,

$$\lambda\lambda'\bar{t} + \lambda^2\kappa\bar{n} = \Delta h = -\left(\frac{A^2}{x^3}, \frac{B^2}{y^3}, \frac{C^2}{z^3}\right)$$ $\longrightarrow$ (15).

Taking scalar product of (14) $\times$ (15),

$$(\lambda\mu'\bar{b} - \lambda\mu\tau\bar{n})\cdot(\lambda\lambda'\bar{t} + \lambda^2\kappa\bar{n}) = \{3x(a'-a)\bar{t} + 3y(b'-b)\bar{n} + 3z(c'-c)\bar{b}\}\cdot\left(-\frac{A^2}{x^3}\bar{t} - \frac{B^2}{y^3}\bar{n} - \frac{C^2}{z^3}\bar{b}\right).$$

$$(-\lambda\mu\tau)(\lambda^2\kappa) = -\frac{3x(a'-a)A^2}{x^3} - \frac{3y(b'-b)B^2}{y^3} - \frac{3z(c'-c)C^2}{z^3}$$

$$-\lambda^3\kappa\mu\tau = -3\left[\frac{(a'-a)A^2}{x^2} + \frac{(b'-b)B^2}{y^2} + \frac{(c'-c)C^2}{z^2}\right]$$

$$\lambda^3\kappa\mu\tau = 3\sum\frac{A^2}{x^2}(a'-a)$$ $\longrightarrow$ (16).

Sub. the value of μ & simplify we get,

$$\lambda^3\kappa\tau\cdot\underbrace{\frac{x^3y^3z^3}{ABC}}_{\mu\text{-value}}\lambda^3\kappa = 3\sum\frac{A^2}{x^2}(a'-a)$$

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$$\lambda^6 K^2 \tau = \frac{3ABC}{x^3y^3z^3} \Sigma \frac{A^2}{x^2}(a'-a)$$

$$\tau = \frac{3ABC}{x^3y^3z^3} \Sigma \frac{A^2}{x^2}(a'-a) \cdot \frac{1}{\lambda^6 K^2}$$

$$\tau = \frac{\frac{3ABC}{x^3y^3z^3} \cdot \Sigma \frac{A^2}{x^2}(a'-a)}{\frac{A^2B^2C^2}{x^6y^6z^6} \Sigma \frac{x^6}{A^2}(a'-a)^2}$$

$$\boxed{\tau = 3\frac{x^3y^3z^3}{ABC} \Sigma \frac{A^2(a'-a)/x^2}{x^6(a'-a)^2/A^2}}$$

Hence proved.

20/1/2020

I - IA1

Contact between curves & surfaces:

Let γ be a curve.

$\bar{r}(u) = \{f(u), g(u), h(u)\}$ & let S be a surface $F(x, y, z) = 0$.

Let us assume that the curve γ & the surface S are of high class in the sense that $\dot{r}(u)$ & $F(x, y, z)$ have continuous derivative of sufficiently high order from the eqn of the curve we take $x = f(u), y = g(u), z = h(u)$ if this point lies on the surface. we've,

$$F(f(u), g(u), h(u)) = 0$$

which is an eqn in u giving the points of intersection of the curve & the surface depending upon the nature of the roots of the eqn we shall define the contact between curves & surfaces, as follows.

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Let u_0 be one such zero of $F(u)=0 \longrightarrow (1)$

$\because F(u)$ posses the derivative of sufficiently higher order $F(u)$ has the following power series representation in the neighbourhood of $u=u_0$,

$$F(u) = F(u_0) + \frac{(u-u_0)}{1!} F'(u_0) + \cdots + \frac{(u-u_0)^n}{n!} F^n(u_0) + O(h)^{n+1}$$

Defn : 1 If $F'(u_0)=0$ then u_0 is a simple zero of $F(u)=0$ Then the curve γ & the surfaces s is said to have simple intersection at $\bar{r}(u_0)$.

Defn : 2 If $F'(u_0)=0$ & $F''(u_0)\neq 0$, u_0 is a double zero of $F(u)$ & $F(u)$ is a 2nd order of h, then the curve γ & surfaces are said to have two point to contact.

Defn : 3 If $F'(u_0) = 0 = F''(u_0)$ & $F'''(u_0) \neq 0$ then the curve γ & surfaces are said to have three point contact at $u=u_0$. under there condition u_0 is a triple zero of $f(u)$.

In general $F'(u_0) = F'(u_0) = \cdots = F^{n-1}(u_0) = 0$ & $F^n(u_0)\neq 0$ then the curve γ & the surface S are said to have n point contact at $u=u_0$.

Theorem :-

The condition of a surface having n-point contact with the curve γ are invariant over the change of parameter.

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Proof: Let $u = \phi(t)$ be the given parameter transformation.

Since it is regular we've [ϕ^k of u] $\phi^k(u) \neq 0$ for $k \geq 1$ corresponding to the point $u = u_0$ we've $u_0 = \phi(t_0)$ at $t = t_0$.

Now $F(u) = F(\phi(t)) = f(t)$

where f is a function of t only.

$$\dot{f}(t) = \frac{d}{dt}F(u) = \frac{d}{du}F(u)\cdot\frac{du}{dt} = F'(u)\dot{\phi}(t) \rightarrow (1)$$

$$\ddot{f}(t) = \frac{d}{dt}\left(F'(u)\cdot\dot{\phi}(t)\right) = F''(u)[\dot{\phi}(t)]^2 + F'(u)\ddot{\phi}(t) \rightarrow (2)$$

If $F'(u) = 0$ then $\dot{f}(t) = 0$ as $\dot{\phi}(t) \neq 0$

If $F'(u) = 0$ & $F''(u) \neq 0$ from (1) & (2) we get,

$$\dot{f}(t) = 0 \text{ \& } \ddot{f}(t) \neq 0$$

Since $\dot{\phi}(t),\ \ddot{\phi}(t) \neq 0$

Thus if the surface S is given by $F(u)$ has two point contact with the curve γ at $\bar{r}(u)$ then the surface S given by $f(t)$ has two point contact γ at $\bar{r}(\phi(t_0))$.

Diff. (2) again we get,

$$\dddot{f}(t) = \frac{d}{dt}(\ddot{f}(t)) = F'''(u)[\dot{\phi}(t)]^3 + 2F''(u)\dot{\phi}(t)\ddot{\phi}(t) + F''(u)\dot{\phi}\ddot{\phi}(t) + F'(u)\dddot{\phi}(t)$$

$$= F'''(u)[\dot{\phi}(t)]^3 + 3F''(u)\dot{\phi}(t)\ddot{\phi}(t) + F'(u)\dddot{\phi}(t)$$

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If $F'(u)=0$, $F''(u)=0$ & $F'''(u)\neq 0$.

Then, from ③ $\Rightarrow$ $\dot{f}(t)=0$, $\ddot{f}(t)=0$ & $\dddot{f}(t)\neq 0$

as $\phi(t)$ is regular.

Hence the surface S given by $f(t)$ has <u>three point contact</u> with the curve at $\bar{r}(\phi(t_0))$.

proceeding like this $F(u)=F''(u)=\cdots=F^{n-1}(u_0)=0$ & $F^{n}(u)\neq 0$ at $u=u_0$ then,

$\dot{f}(t)=\ddot{f}(t)=\cdots=f^{n-1}(t)=0$ & $f^{n}(t)\neq 0$ at $\bar{r}(\phi(t_0))$.

Thus the surface having <u>n-point contact</u> with the curve ν are invariant over a change of parameter.

Hence, we conclude that the property of the curve having n-point contact with S is a property of ν in the sense that any part which represents ν will have those property.

Ex: 6.1 S.T. the osculating plane at P has in general 3 point contact with the curve at P.

<u>solution</u>:

Let P be a point on the curve. Let the arc length measured from P so that $s=0$ at P.

Let the eqn of the curve be $\bar{r}=\bar{r}(s)$.

W.K.T. the eqn of osculating plane is;

$$[\bar{r}(s)-\bar{r}(0),\ \bar{r}'(0),\ \bar{r}''(0)]=0.$$

Let $F(s)=[r(s)-r(0),\ r'(0),\ r''(0)] \longrightarrow ①$

Taylor's formula

$$r(s)=r(0)+\frac{s}{1!}r'(0)+\frac{s^2}{2!}r''(0)+\frac{s^3}{3!}r'''(0)+o(s^4)$$

$$r(s)-r(0)=sr'(0)+\frac{s^2}{2}r''(0)+\frac{s^3}{6}r'''(0)+o(s^4)$$

$$F(s)=[sr'(0)+\frac{s^2}{2}r''(0)+\frac{s^3}{6}r'''(0),\ r'(0),\ r''(0)]+o(s^4)$$

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$$F(s) = s[\gamma'(0), \gamma'(0), \gamma''(0)] + \frac{s^2}{2}[\gamma''(0), \gamma'(0), \gamma''(0)] + \frac{s^3}{6}[\gamma'''(0), \gamma'(0), \gamma''(0)] + o(s^4)$$

$$F(s) = \frac{s^3}{6}[\gamma'(0), \gamma''(0), \gamma'''(0)] + o(s^4)$$

since s is decreasing, so we can take,

$$[\gamma'(0), \gamma''(0), \gamma'''(0)] = K^2\tau$$

$$F(s) = \frac{s^3}{6}K^2\tau + o(s^4) \text{ as } s \to 0$$

$$F'(s) = \frac{3s^2}{6}K^2\tau \quad ; \quad F'(0) = 0$$

$$F''(s) = \frac{6s}{6}K^2\tau \quad ; \quad F''(0) = 0$$

$$F'''(s) = K^2\tau \quad ; \quad F'''(0) = K^2\tau.$$

$\therefore$ the osculating plane has 3 point contact with the curve at p.

Osculating circle & osculating sphere :-

Osculating circle (or) circle of curvature :-

Let γ be a given space curve & p be any point on it the circle having 3 point ~~curve~~ contact with the given space curve at p is called the osculating circle at p.

Centre of curvature :

The radius of the osculating circle is called the radius of curvature of the curve at p. It is denoted by ρ. The centre of the osculating circle is called the centre of curvature at p.

$$\rho = 1/K.$$

Note :- (properties)

i) Since the Osculating plane has also 3 point contact with the curve at p. The osculating circle lies on the osculating plane. If it evident even otherwise, if we define the osculating circle as the curve passing through 3 consecutive points on the curve as we've defined the osculating plane has the plane passing through 3-consecutive points on the curve.

ii) Since the circle of curvature & the curve have the same tangent at p in the osculating plane the centre of the circle lies on the principal normal at P.

Osculating Sphere :-

A sphere having 4 point contact with the curve at a point p is called the osculating sphere at P on the curve.

$$(C-R)^2-\rho^2=0.$$

Radius of the Spherical curvature:

The centre of the osculating sphere is called the centre of spherical curvature & its radius is called the radius of spherical curvature.

Theorem :-

The radius of the osculating circle at p is the reciprocal of the curvature of curve at P & the p.v of its centre of the osculating circle is $c=\bar{r}+\rho\bar{n}$ where $\rho=1/\kappa$.

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Proof :-

Choosing arc length s as parameter. Let C be the P.V. of the centre of the osculating circle.

The centre C is at a distance ρ from P along the principal normal at P.

Hence we've, $C-\bar{r} = \rho\bar{n}$

$$(C-\bar{r})\bar{n} = \rho.$$

To P.T. $\rho = 1/k$:-

Since any point $\bar{r} = \bar{r}(s)$ on the osculating circle satisfying the eqn of sphere.

$$(C-R)^2 = \rho^2.$$

& lies in the osculating plane the osculating circle is the intersection of the osculating plane & the sphere, $(C-R)^2 = \rho^2$.

Where R is the P.V. of any point on the sphere if $\bar{r}(s)$ is any point of intersection of sphere & the curve.

The sphere has 3 point contact with the curve at $\bar{r} = \bar{r}(s)$. Let the point of intersection be $F(s) = (C-\bar{r})^2 - \rho^2$ & the cond. of 3 point contact are $F(s) = 0$; $F'(s) = 0$; $F''(s) = 0$.

$$F(s) = (C-\bar{r})^2 - \rho^2 = 0$$

$$F'(s) = 2(C-\bar{r})\cdot(-\bar{r}') = 0$$

$$= -2(C-\bar{r})\cdot(\bar{t}) = 0$$

$$F'(s) = (C-\bar{r})\cdot\bar{t} = 0.$$

$$F''(s) = (C-\bar{r})\cdot\bar{t}' + \bar{t}(-\bar{r}') = 0$$

$(c-\bar{r})\cdot(k\bar{n}) - \bar{t}^2 = 0$

$(c-\bar{r})\cdot(k\bar{n}) - \bar{t}\,\bar{t} = 0$

$(c-\bar{r})\cdot k\bar{n} - 1 = 0$

$$k\bar{n} = \frac{1}{c-\bar{r}}$$

$$k = 1/\rho \Rightarrow \underline{\rho = 1/k}.$$

Thus we've proved that the centre of the circle of curvature is $c = \bar{r} + \rho\bar{n}$ & the radius of circle of curvature is the reciprocal of the curvature of the curve at P.

Thrm :-

If $\bar{r} = \bar{r}(s)$ is the given curve γ the centre c & the radius R of spherical curvature at a point p on γ are given by,

$$c = \bar{r} + \rho\bar{n} + \sigma\rho'\bar{b} \quad ; \quad R = \sqrt{\rho^2 + \sigma^2\rho'^2}$$

Proof :-

If c is the centre & R is the radius of the osculating sphere. then its eqns is $(c-\bar{R})^2 = R^2$ where $\bar{R}$ is the p.v. of any point on the sphere.

The points of intersection of the curve & the sphere are given by,

$$F(s) = (c-\bar{r})^2 - R^2 = 0$$

$\therefore$ The sphere has 4 point contact with γ at P.

The cond. of 4 point contact are

$F(s) = 0$; $F'(s) = 0$; $F''(s) = 0$; $F'''(s) = 0$ which give rise to the following eqn.

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$$F(s) = (c-\bar{r})^2 - R^2 = 0$$

$$F'(s) = 2(c-\bar{r}) \cdot (-\bar{r}') = 0$$

$$F'(s) = (c-\bar{r}) \cdot \bar{t} = 0 \longrightarrow (1)$$

$$F''(s) = (c-\bar{r}) \cdot \bar{t}' + \bar{t} \cdot (-\bar{r}') = 0$$

$$= (c-\bar{r}) \cdot \bar{t}' - \bar{t} \cdot \bar{t} = 0$$

$$F''(s) = (c-\bar{r}) \cdot k\bar{n} - 1 = 0 \longrightarrow (2)$$

$$F'''(s) = (c-\bar{r}) k\bar{n}' - (c-\bar{r}) k'\bar{n} + k\bar{n}(-\bar{r}') = 0$$

$$= (c-\bar{r}) k (\tau\bar{b} - k\bar{t}) + (c-\bar{r}) k' \bar{n} - \underbrace{k\bar{n}\cdot\bar{t}}_{0} = 0$$

$$F'''(s) = (c-\bar{r}) k\tau\bar{b} - \underbrace{(c-\bar{r}) k^2 \bar{t}}_{0} + (c-\bar{r}) k' \bar{n} = 0 \longrightarrow (3)$$

using (1) & (2) in (3) we get,

$$F'''(s) = (c-\bar{r}) k\tau\bar{b} + (c-\bar{r}) k' \frac{1}{(c-\bar{r})k} = 0$$

$$F'''(s) = (c-\bar{r}) k\tau\bar{b} + \frac{k'}{k} = 0 \longrightarrow (4)$$

Let $\rho = \frac{1}{k}$; $\sigma = \frac{1}{\tau}$ then, $\rho' = -\frac{k'}{k^2}$

$$(4) \Rightarrow F'''(s) \Rightarrow (c-\bar{r}) \cdot \frac{1}{\rho} \cdot \frac{1}{\sigma} \bar{b} + k'\rho = 0$$

$$\Rightarrow (c-\bar{r}) \left(\frac{1}{\sigma}\right) \left(\frac{1}{\rho}\right) \bar{b} - k^2 \rho' \rho = 0$$

$$\Rightarrow (c-\bar{r}) \left(\frac{1}{\sigma}\right) \left(\frac{1}{\rho}\right) \bar{b} - \frac{\rho \cdot \rho'}{\rho^2} = 0$$

$$\Rightarrow (c-\bar{r}) \left(\frac{1}{\sigma}\right) \left(\frac{1}{\rho}\right) \bar{b} - \frac{\rho'}{\rho} = 0$$

$$\Rightarrow (c-\bar{r}) \bar{b} - \sigma\rho' = 0$$

$$F'''(s) \Rightarrow (c-\bar{r}) \bar{b} = \sigma\rho' \longrightarrow (5)$$

From (1), (2), (5) we've

$$(1) \Rightarrow (c-\bar{r}) \cdot \bar{t} = 0$$

$$(2) \Rightarrow (c-\bar{r}) \cdot k\bar{n} = 1 \Rightarrow (c-\bar{r}) \cdot \bar{n} = \frac{1}{k}$$

$$(c-\bar{r}) \cdot \bar{n} = \rho$$

$$(5) \Rightarrow (c-\bar{r}) \cdot \bar{b} = \sigma\rho'$$

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This above eqn shows that $(c-\bar{r})$ lies in the normal plane & its component along the normal & binormal ρ & $\rho'\sigma$ respectively.

We can write, $(c-\bar{r}) = \rho\bar{n} + \sigma\rho'\bar{b}$

① & ② & ⑤ $\Rightarrow$ $c = \bar{r} + \rho\bar{n} + \sigma\rho'\bar{b}$

The radius of the osculating sphere is given by,

$(c-\bar{r})^2 = R^2$

$(c-\bar{r}).(c-\bar{r}) = R.R$

$(\rho\bar{n} + \sigma\rho'\bar{b}).(\rho\bar{n} + \sigma\rho'\bar{b}) = R^2$

$R^2 = \rho^2 + \sigma^2\rho'^2$

$R = \sqrt{\rho^2 + \sigma^2\rho'^2}$

16/07/15

(*) Locus of the centre of spherical curvature :-

10 m

Let c be the given curvature & c be the locus of centre of spherical curvature.

Already we've the relation b/n moving trial $(\bar{t}, \bar{n}, \bar{b})$ on C.

Illy, we can find the relation b/n moving trial $(\bar{t}_1, \bar{n}_1, \bar{b}_1)$ on C.

We express the curvature & torsion on C interms of those of C we shall use the sufficies one for the quantities pertaining to c_1.

To discuss them from the corresponding quantities of C.

Thm :-

Let c be the given curve & c_1 be the locus of its centres of spherical curvature then,

i) $\bar{t}_1 = e\bar{b}$, $\bar{n}_1 = e_1\bar{n}$, $\bar{b}_1 = -ee_1\bar{t}$ whose $e = e_1 = \pm 1$

ii) The product of the torsion at the corresponding pts is equal to the product of curvatures.

(ie) $\tau\tau_1 = \kappa\kappa_1$

Local Non - Intrinsic properties of a ...

proof: The p.v. of $\bar{r}_1$ the centre of spherical curvature is given by,

$$r_1 = \bar{r} + \rho\bar{n} + \rho'\sigma\bar{b} \longrightarrow (1)$$

$$\frac{d\bar{r}_1}{ds} = \frac{d\bar{r}}{ds} + \rho\bar{n}' + \rho'\bar{n} + \rho''\sigma\bar{b} + \rho'\sigma'\bar{b} + \rho'\sigma\bar{b}'$$

$$\frac{d\bar{r}_1}{ds_1}\cdot\frac{ds_1}{ds} = \bar{t} + \rho(\tau\bar{b} - k\bar{t}) + \rho'\bar{n} + \rho''\sigma\bar{b} + \rho'\sigma'\bar{b} + \rho'\sigma(-\tau\bar{n})$$

$$\bar{r}_1'.s_1' = \bar{t} + \rho\left(\frac{\bar{b}}{\sigma} - \frac{\bar{t}}{\rho}\right) + \rho'\bar{n} + \rho''\sigma\bar{b} + \rho'\sigma'\bar{b} - \rho'\sigma\frac{\bar{n}}{\sigma}$$

$$t_1.s_1' = \bar{t} + \frac{\rho\bar{b}}{\sigma} - \bar{t} + \rho'\bar{n} + \rho''\sigma\bar{b} + \rho'\sigma'\bar{b} - \rho'\bar{n}$$

$$t_1 s_1' = \bar{b}(\rho/\sigma + \rho''\sigma + \rho'\sigma') \longrightarrow (2)$$

This shows that $\bar{t}_1$ parallel to $\bar{b}$. C_1 is parameter by s & s_1 is an increasing function of s. so that, s_1' is non-negative.

If we take $t_1 = e\bar{b}$ where $e = \pm 1 \longrightarrow (3)$

$$\frac{dt_1}{ds} = \cancel{e'\bar{b}} + e\bar{b}'$$

$$\frac{d\bar{t}_1}{ds_1}\cdot\frac{ds_1}{ds} = \cancel{e'\bar{b}} + e\bar{b}'$$

$$t_1'.s_1' = \cancel{e'\bar{b}} - e(\tau\bar{n}) \qquad [\because e\bar{b} = e(\bar{t}\times\bar{n}) = 0]$$

$$(k_1\bar{n}_1).s_1' = -\tau\bar{n}e \longrightarrow (4)$$

$\Rightarrow (k_1 e_1 \cancel{\bar{n}})s_1' = -\tau\cancel{\bar{n}}e \Rightarrow k_1 e_1 s_1 = -\tau e \longrightarrow (4A)$

so that, $\bar{n}_1$ is parallel to $\bar{n}$.

$$\bar{b}_1 = \bar{t}_1\times\bar{n}_1$$

$$= e\bar{b}\times e_1\bar{n} = ee_1(\bar{b}\times\bar{n})$$

$$\bar{b}_1 = ee_1(-\bar{t}) = -ee_1\bar{t} \longrightarrow (5)$$

$$\bar{b}_1 = -ee_1\bar{t}$$

Diff. w.r.to "s",

$$\frac{db_1}{ds} = -ee_1\bar{t}'$$

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$$\frac{db_1}{ds_1} \cdot \frac{ds_1}{ds} = -ee_1(\kappa\bar{n})$$

$$b_1' \cdot s_1' = -ee_1\kappa\bar{n}$$

$$-\tau_1\bar{n}_1 \cdot s_1' = -ee_1\kappa\bar{n}$$

$$\tau_1(e_1\bar{n})s_1' = ee_1\kappa\bar{n} \qquad (\because \bar{n}_1 = e_1\bar{n})$$

$$\therefore \tau_1 s_1' = e\kappa \longrightarrow (6)$$

Multiply by $e\tau$ on b.s. of (6),

$$e\tau(\tau_1 s_1') = e\tau(e\kappa)$$

$$= -\kappa_1 \bar{n}_1 \bar{s}_1 (e\kappa) \cdot \frac{1}{\bar{n}}$$

$$\tau\tau_1 = -\kappa\kappa_1 e_1 \qquad (\because e_1 = -1)$$

$$\underline{\underline{\tau\tau_1 = \kappa\kappa_1}}$$

Note :-

If we measure arc length s_1 of C_1 in the direction which makes its units tangent $\bar{t}_1$ have the same direction as $\bar{b}$ then, $\bar{t}_1 = \bar{b}$. we may choose the direction of $\bar{n}_1$ opp. to $\bar{n}$, so that $\bar{n}_1 = -\bar{n}$ with those choose $\bar{b}_1 = \bar{t}$ these are the particular cases of the above them.

Tangent surface, involute & Evolutes :-

Defn (Tangent surface) :-

The tangent surface of a curve c is the surface generated by lines tangent surface is determined by two parameters s & u the p.v. of P can thus be writtenas

$$\boxed{\bar{R}(s,u) = \bar{r}(s) + u\bar{t}(s).}$$

Note :- Any additional relation b/n u & s of the form

$$\boxed{u = \lambda(s)}$$

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Involutes :- 2m

An involutes of c is a curve which lies on the tangent surface of c & intersect the generators orthogonally. It is denoted by $\bar{c}$.

Note :

From the above defn., we can say that the tangent of c are normals to $\bar{c}$. (ie)

The tangent to c at a point is orthogonal to the tangent at corresponding point of $\bar{c}$.

Evolutes :- 2m

If $\bar{c}$ is an involute of a given curve then c is defined be evolute of $\bar{c}$.

Thm :-

5m If $\bar{R}$ is the p.v. of a point p on the involute $\bar{c}$ of c, then $\bar{R} = \bar{r} + (c-s)\bar{t}$ where c is an arbitrary constant, $\bar{r}$ is the p.v. of p on c.

Proof :

Since the involute lies on the tangent surface the p.v. $\bar{R}$ of a point p on the involute can be taken as

$$\bar{R} = \bar{r}(s) + u\bar{t}(s) \longrightarrow (1)$$

We've any addition relation b/n u & s is,

$u = \lambda(s)$ we get,

$$(1) \Rightarrow \bar{R} = \bar{r}(s) + \lambda(s)\,\bar{t}(s) \longrightarrow (2)$$

Diff. w.r.to "s",

$$\frac{d\bar{R}}{ds} = \frac{d\bar{r}}{ds} + \lambda(s)\,\bar{t}'(s) + \bar{t}(s)\,.\,\lambda'(s)$$

$\frac{d\bar{R}}{ds_1} \cdot \frac{ds_1}{ds} = \bar{t} + \lambda'(s) \cdot \bar{t}(s) + \lambda(s) \kappa \bar{n}$ $\quad [\because |\bar{t}_1| = \bar{t}]$

$\bar{t}_1 \cdot \bar{s}_1' = \bar{t} + \lambda'(s) \bar{t}(s) + \lambda(s) \cdot \kappa \bar{n} \longrightarrow (3)$ $\quad [\because R' = \bar{t}_1]$

since the tangent to the involute cuts the generators orthogonally $\bar{t} \cdot \bar{t}_1 = 0$

Now taking of product with $\bar{t}$ of (3),

(3) $\Rightarrow \bar{t} \cdot (\bar{t}_1 \cdot s_1') = \bar{t} [\bar{t} + \lambda'(s) \cdot \bar{t}(s) + \lambda(s) \kappa \bar{n}]$ $\quad [\because \bar{n} \cdot \bar{t} = 0]$

$0 = 1 + \lambda'(s)$

$\lambda'(s) = -1 \Rightarrow \frac{d\lambda}{ds} = -1$

$d\lambda = -ds$

Sing, $\lambda(s) = -s + c$

$\lambda = c - s$

(2) $\Rightarrow \boxed{\bar{R} = \bar{r} + (c-s)\bar{t}}$

Hence proved.

Thrm:

Obtain the eqn of the Evolute in the form,

$\bar{R} = \bar{r} + \rho \bar{n} + \rho \cot(\int \tau ds + c) \bar{b}$.

proof:-

Let P be the point on C corresponding to the point on C then P must lie in the plane through Q normal to $\bar{C}$.

If $\bar{R}$ & $\bar{r}$ denote the points of P & Q.

$\bar{R} = \bar{r} + \lambda \bar{n} + \mu \bar{b} \longrightarrow (1)$

Diff. (1) w.r.to 's',

$\frac{d\bar{R}}{ds} = \frac{d\bar{r}}{ds} + \lambda'(s) \bar{n} + \lambda(s) \cdot \bar{n}' + \mu'(s) \bar{b} + \mu \bar{b}'$.

(Serret-Frenet)

$\frac{d\bar{R}}{ds_1} \cdot \frac{ds_1}{ds} = \bar{t} + \lambda'(s) \bar{n} + \lambda(s) [\tau \bar{b} - \kappa \bar{t}] + \mu' \bar{b} - \tau \bar{n} \mu$.

$\bar{t}_1 \cdot s_1' = \bar{t}(1 - \lambda \kappa) + \bar{n}(\lambda'(s) - \tau \mu) + \bar{b}(\lambda(s) \tau + \mu') \longrightarrow (2)$

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Now, $\frac{d\hat{R}}{ds} = \bar{t}_1 \cdot \frac{ds_1}{ds}$, it is a tangent at p to c

so it is in the normal plane to the $\bar{c}$ at Q.

It is parallel to $\lambda\bar{n} + \mu\bar{b}$

$$\bar{t}_1 \cdot \frac{ds_1}{ds} = \lambda\bar{n} + \mu\bar{b} \longrightarrow (3)$$

comparing (2) & (3),

$$1 - \lambda\kappa = 0 \Rightarrow \lambda = 1/\kappa \longrightarrow (4)$$

$$\lambda'(s) - \tau\mu = \lambda \qquad [\because \rho = 1/\kappa] \longrightarrow (5)$$

$$\lambda\tau + \mu' = \mu \longrightarrow (6)$$

$$(5) \Rightarrow \frac{\lambda' - \tau\mu}{\lambda} = 1$$

$$(6) \Rightarrow \frac{\lambda\tau + \mu'}{\mu} = 1$$

Equating we've

$$\frac{\lambda' - \tau\mu}{\lambda} = \frac{\lambda\tau + \mu'}{\mu}$$

$$\mu\lambda' - \mu^2\tau = \tau\lambda^2 + \lambda\mu'$$

$$\mu\lambda' - \lambda\mu' = \tau\lambda^2 + \mu^2\tau$$

$$\mu\lambda' - \lambda\mu' = \tau(\lambda^2 + \mu^2)$$

since $\mu \neq 0$,

$$\frac{\mu^2}{\mu^2}(\mu\lambda' - \lambda\mu') = \tau(\mu^2 + \lambda^2)$$

$$\mu^2 \frac{d}{ds}\left(\frac{\lambda}{\mu}\right) = \tau(\lambda^2 + \mu^2)$$

$$\tau = \frac{\mu^2}{\lambda^2 + \mu^2} \cdot \frac{d}{ds}(\lambda/\mu)$$

$$= \frac{\mu^2 \frac{d}{ds}(\lambda/\mu)}{\mu^2(1 + \lambda^2/\mu^2)}$$

$$\tau = \frac{d/ds\,(\lambda/\mu)}{(1+\lambda^2/\mu^2)}$$

$$\tau = \frac{d/ds\,(\lambda/\mu)}{1+(\lambda/\mu)^2}$$

$\int$ing, $\int \tau\, ds + c = \tan^{-1}(\lambda/\mu)$

$$\int \tau\, ds + \text{constant} = \tan^{-1}(\lambda/\mu)$$

$$\lambda/\mu = \tan\left[\int \tau\, ds + c\right]$$

$$\mu = \lambda \cot\left[\int \tau\, ds + c\right] \longrightarrow (7)$$

sub., μ, λ value in (1) we get,

$$\bar{R} = \bar{r} + \rho\bar{n} + \rho \cot\left(\int \tau\, ds + c\right)\bar{b}$$

$\therefore$ This is given eqn of evolute of $\bar{c}$.

Intrinsic equations, fundamental existence

20/7/2015

thrm for space curves :-

The eqn expressing K & τ as functions of arc length are called intrinsic natural eqn of the curve.

(ie) $K = f(s)$; $\tau = g(s)$ are called intrinsic eqn.

thrm :-

(X) 2 marks If the curve have the same intrinsic eqn then, they are congruent.

proof:-

let C & C_1 be two curves define in terms of the arc length having equal curvature & equal torsion for the same values of s.

A & A_1 be two points of C & C_1 corresponding to $s=0$.

Now, $\frac{d}{ds}(\bar{t}\cdot\bar{t}_1) = \frac{d\bar{t}}{ds}\cdot\bar{t}_1 + \bar{t}\cdot\frac{d\bar{t}_1}{ds}$

$= \bar{t}'\cdot\bar{t}_1 + \bar{t}\cdot\bar{t}_1'$

$= k\bar{n}\cdot\bar{t}_1 + \bar{t}\cdot k\bar{n}_1 \longrightarrow (1)$

$\frac{d}{ds}(\bar{n}\cdot\bar{n}_1) = \frac{d\bar{n}}{ds}\cdot\bar{n}_1 + \bar{n}\cdot\frac{d\bar{n}_1}{ds}$

$= \bar{n}'\cdot\bar{n}_1 + \bar{n}\cdot\bar{n}_1'$

$= (\tau\bar{b} - k\bar{t})\cdot\bar{n}_1 + \bar{n}(\tau\bar{b}_1 - k\bar{t}_1)$

$\frac{d}{ds}(\bar{n}\cdot\bar{n}_1) = \tau\bar{b}\bar{n}_1 - k\bar{t}\bar{n}_1 + \bar{n}\tau\bar{b}_1 - \bar{n}k\bar{t}_1 \longrightarrow (2)$

$\frac{d}{ds}(\bar{b}\cdot\bar{b}_1) = \bar{b}'\cdot\bar{b}_1 + \bar{b}\cdot\bar{b}_1'$

$= -\tau\bar{n}\cdot\bar{b}_1 + \bar{b}(-\tau\bar{n}_1)$

$= -\tau\bar{n}\bar{b}_1 - \tau\bar{b}\bar{n}_1 \longrightarrow (3)$

$(1)+(2)+(3) \Rightarrow$

$\frac{d}{ds}(\bar{t}\cdot\bar{t}_1) + \frac{d}{ds}(\bar{n}\cdot\bar{n}_1) + \frac{d}{ds}(\bar{b}\cdot\bar{b}_1) =$

$k\bar{n}\bar{t}_1 + k\bar{t}\bar{n}_1 + \tau\bar{b}\bar{n}_1 - k\bar{t}\bar{n}_1 + \bar{n}\tau\bar{b}_1 - \bar{n}k\bar{t}_1 - \tau\bar{n}\bar{b}_1 - \tau\bar{n}\bar{b}$

$\Rightarrow 0$

$\frac{d}{ds}[\bar{t}\cdot\bar{t}_1 + \bar{n}\cdot\bar{n}_1 + \bar{b}\cdot\bar{b}_1] = 0$

Jing, $\bar{t}\cdot\bar{t}_1 + \bar{n}\cdot\bar{n}_1 + \bar{b}\cdot\bar{b}_1 = c \longrightarrow (4)$

$1+1+1 = c$

$\boxed{3 = c}$

Since the scalar product of two unit vector gives the cosines above.

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But the sum of these cosines is 3 only, when each angle is zero. $\cos 0^\circ = 1$

$\therefore$ At all point of the curve $\bar{t} = \bar{t}_1$, $\bar{b} = \bar{b}_1$, $\bar{n} = \bar{n}_1$

further, $\bar{t} - \bar{t}_1 = 0 \implies \bar{r} - \bar{r}_1 = 0$

$$\frac{d}{ds}(\bar{r} - \bar{r}_1) = 0$$

Sing., $\bar{r} - \bar{r}_1 = d$

At $s = 0$, $\bar{r}_1 = \bar{r}$ [Identity]

$\therefore d = 0 \implies \underline{\bar{r} - \bar{r}_1 = 0}$.

$\therefore$ Two curves are congruent.

thm :-

A fundamental existence thm for space curve

~~proof~~ :- 10m, 5m

If $\kappa(s)$, $\tau(s)$ are cont. funs of the real variables s, where $s \geq 0$ then there exists a space curve for which κ is the curvature.

τ is the torsion & s is the arc length measured from some suitable base point such a curve is uniquely determined to within a Euclidean motion.

$[\bar{t}\ \bar{n}\ \bar{b}] = [\bar{\alpha}\ \bar{\beta}\ \bar{\gamma}]$

proof :- consider the differential eqn. - Serret - Frenet formula

$$\frac{d\alpha}{ds} = \kappa\beta \;;\; \frac{d\beta}{ds} = \tau\gamma - \kappa\alpha \;;\; \frac{d\gamma}{ds} = -\tau\beta \longrightarrow (1)$$

where α, β, γ are the unknown funs. of s & κ, τ are given funs.

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The set of eqn ① admits a unique set of soln which assume prescribed values $(\alpha_0, \beta_0, \gamma_0)$ when $s=0$.

Let $(\alpha_1, \beta_1, \gamma_1)$ be one such soln taking prescribed value $\alpha_1(0) = 1, \beta_1(0)=0, \gamma_1(0)=0$ ⟶ ②

lly, we can find to more solns, $(\alpha_2, \beta_2, \gamma_2)$ & $(\alpha_3, \beta_3, \gamma_3)$ having the following cond. $(0,1,0)$ & $(0,0,1)$.

(i) $\alpha_2(0)=0, \ \beta_2(0)=1, \ \gamma_2(0)=0$.

We prove this thrm by following steps.

Step:-1 consider $\alpha_1^2+\beta_1^2+\gamma_1^2=1$

$$\frac{d}{ds}(\alpha_1^2+\beta_1^2+\gamma_1^2) \neq \phi$$

$$\Rightarrow 2\alpha_1\alpha_1' + 2\beta_1\beta_1' + 2\gamma_1\gamma_1' = 0$$

$$\Rightarrow \alpha_1\alpha_1' + \beta_1\beta_1' + \gamma_1\gamma_1' = 0$$

$$\Rightarrow \alpha_1\frac{d\alpha_1}{ds} + \beta_1\frac{d\beta_1}{ds} + \gamma_1\frac{d\gamma_1}{ds} = 0.$$

Since $\alpha_1, \beta_1, \gamma_1$ are the soln of ①, From ①

$$\alpha_1(\kappa\beta_1) + \beta_1(\tau\gamma_1 - \kappa\alpha_1) + \gamma_1(-\tau\beta_1) = 0$$

$$\frac{d}{ds}(\alpha_1^2+\beta_1^2+\gamma_1^2) = 0$$

$$\alpha_1^2+\beta_1^2+\gamma_1^2 = c \text{ (constant)}.$$

using the I.C. $c=1$.

$$\therefore \alpha_1^2+\beta_1^2+\gamma_1^2 = 1.$$

lly, $\alpha_2^2+\beta_2^2+\gamma_2^2=1$

$\alpha_3^2+\beta_3^2+\gamma_3^2=1$

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consider $\alpha_1\alpha_2+\beta_1\beta_2+\gamma_1\gamma_2$,

$$\frac{d}{ds}(\alpha_1\alpha_2+\beta_1\beta_2+\gamma_1\gamma_2) = \alpha_1\frac{d\alpha_2}{ds}+\alpha_2\frac{d\alpha_1}{ds}+\beta_1\frac{d\beta_2}{ds}+\beta_2\frac{d\beta_1}{ds}+\gamma_1\frac{d\gamma_2}{ds}+\gamma_2\cdot\frac{d\gamma_1}{ds}.$$

$$= \alpha_1 K\beta_2+\alpha_2 K\beta_1+\beta_1(\tau\gamma_2-K\alpha_2)+\beta_2(\tau\gamma_1-K\alpha_1)+\gamma_1(-\tau\beta_2)+\gamma_2(-\tau\beta_1).$$

$$\frac{d}{ds}(\alpha_1\alpha_2+\beta_1\beta_2+\gamma_1\gamma_2)=0$$

Ing, $\alpha_1\alpha_2+\beta_1\beta_2+\gamma_1\gamma_2=d$ (constant).

Using I.C. we get,

$$\alpha_1\alpha_2+\beta_1\beta_2+\gamma_1\gamma_2=0$$

$||^{ly}$, $\alpha_2\alpha_3+\beta_2\beta_3+\gamma_2\gamma_3=0$

$$\alpha_3\alpha_1+\beta_3\beta_1+\gamma_3\gamma_1=0$$

Step: 2

To prove $\bar{t}=(\alpha_1,\alpha_2,\alpha_3)$; $\bar{n}=(\beta_1,\beta_2,\beta_3)$; $\bar{b}=(\gamma_1,\gamma_2,\gamma_3)$ are 3 mutually $\perp$r unit vectors.

consider the matrix,

$$A=\begin{bmatrix}\alpha_1 & \beta_1 & \gamma_1\\ \alpha_2 & \beta_2 & \gamma_2\\ \alpha_3 & \beta_3 & \gamma_3\end{bmatrix}$$

The six relation prove in the first ~~set~~ step so that, the matrix A is orthogonal.

(i.e) $AA' = \begin{bmatrix}\alpha_1 & \beta_1 & \gamma_1\\ \alpha_2 & \beta_2 & \gamma_2\\ \alpha_3 & \beta_3 & \gamma_3\end{bmatrix}\cdot\begin{bmatrix}\alpha_1 & \alpha_2 & \alpha_3\\ \beta_1 & \beta_2 & \beta_3\\ \gamma_1 & \gamma_2 & \gamma_3\end{bmatrix}$

Where A' is the transpose of A.

$$AA'=\begin{bmatrix}\alpha_1^2+\beta_1^2+\gamma_1^2 & \alpha_1\alpha_2+\beta_1\beta_2+\gamma_1\gamma_2 & \alpha_1\alpha_3+\beta_1\beta_3+\gamma_1\gamma_3\\ \alpha_1\alpha_2+\beta_1\beta_2+\gamma_1\gamma_2 & \alpha_2^2+\beta_2^2+\gamma_2^2 & \alpha_2\alpha_3+\beta_2\beta_3+\gamma_2\gamma_3\\ \alpha_1\alpha_3+\beta_1\beta_3+\gamma_1\gamma_3 & \alpha_2\alpha_3+\beta_2\beta_3+\gamma_2\gamma_3 & \alpha_3^2+\beta_3^2+\gamma_3^2\end{bmatrix}$$

Local Non - Intrinsic ...

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$$\therefore AA' = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (\text{by step (ii)}).$$

$AA' = I$

The above six relations s.t. $\bar{t} = (\alpha_1, \alpha_2, \alpha_3)$; $\bar{n} = (\beta_1, \beta_2, \beta_3)$; $\bar{b} = (\gamma_1, \gamma_2, \gamma_3)$ are mutually $\perp$ unit vectors.

Step (iii)

To find the p.v. of a point on the curve.

$$\bar{r}(s) = \int_0^s \alpha(s)\, ds.$$

Diff. w.r.to "s" we get,

$$\frac{d\bar{r}}{ds} = \alpha(s) \Rightarrow \bar{t} = \alpha(s) = \alpha \quad \{\because \dot{\bar{r}} = \bar{t}\}$$

This shows that the arc length s & the unit tangent vector of s. $\bar{t} = \alpha(s)$.

Also, $$\frac{d\bar{t}}{ds} = \frac{d\alpha}{ds} = \kappa\beta \quad (\text{using eqn } ①)$$

$$\therefore \frac{d\bar{t}}{ds} = \kappa\bar{n}$$

The unit normal vector $\bar{n}$ is parallel to the unit vector β & we get $\bar{n} = \beta$.

$\|^{ly}$, $\bar{b} = \gamma$

$$\therefore \frac{d\bar{b}}{ds} = -\tau\bar{n}$$

$$\frac{d\bar{b}}{ds} = \frac{d\gamma}{ds} = -\tau\beta$$

$$\therefore \boxed{\bar{b} = \beta}$$

We have, $\tau = \tau(s)$

$$\bar{\gamma}(s) = \int_0^s \alpha(s)\,ds = \bar{\gamma}'(s) = \int_0^s \bar{t}\,ds$$

$\therefore \bar{\gamma}(s)$ is a p.v. of a point on the curve with arc length s as parameter having $(\bar{t}, \bar{n}, \bar{b})$ as curvature & torsion. This gives the existence of the curve.

Helixes 2m

22/7/15

A cylindrical helix is a space curve which lies on a cylinder & cuts the generatores at a constant angle. Its tangent makes a constant angle α with a fixed line known as the axis of the helix.
$\Rightarrow \bar{t}\cdot\bar{a} = \cos\alpha$

Thrm: 10m Key.

5m

The ratio of curvature to the torsion is constant at all pts $\Leftrightarrow$ the curve is a helix. The necessary & sufficient condition for a curve to be a helix is that its curvature & torsion is a constant ratio.

(i.e) κ/τ = constant.)

Proof :-

Let $\vec{a}$ be a unit vectors in the direction of the axis. Since the helixes cuts the generators at a constant angle α.

Let the angle b/n the generator at the tangent & at any point P on the helixes be α.

By conditions of helixes, $\bar{t}\cdot\bar{a} = \cos\alpha$

Diff. w.r. to "t",

$$\bar{t}'\cdot\bar{a} + \cancel{\bar{a}'\bar{t}} = 0 \qquad [\because \bar{a} \text{ is const.}]$$

$$\kappa\bar{n}\cdot\bar{a} = 0.$$

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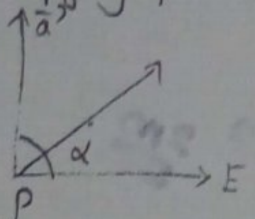
If $\kappa=0$ the curve is a st. line

If $\bar{n}\cdot\bar{a}=0$, then $\bar{a}$ is ⊥r to the normal $\bar{n}$.

(ie) α is ⊥r to normal.

∴ The vector $\bar{a}$ must lie in the rectifying plane



$\bar{a}=\bar{t}\cos\alpha+\bar{b}\sin\alpha$

Diff., $\frac{d\bar{a}}{ds}=\bar{t}'\cos\alpha+\bar{b}'\sin\alpha$

$\bar{a}'=\kappa\bar{n}\cos\alpha-\tau\bar{n}\sin\alpha$ [∵ $\bar{a}$ is constant]

$0=(\kappa\cos\alpha-\tau\sin\alpha)\cdot\bar{n}$, $\bar{n}\neq 0$

$\kappa\cos\alpha-\tau\sin\alpha=0$.

$\kappa\cos\alpha=\tau\sin\alpha$

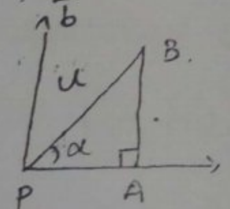
$\frac{\kappa}{\tau}=\tan\alpha$

∴ $\frac{\kappa}{\tau}=$ constant.

Hence proved.

Conversely, Assume that $\frac{\kappa}{\tau}=$ constant $=\lambda$ (say)

to p.T. the curve is helix.



Given any constant λ we can find

~~wrt~~ the smallest angle α ∋, $\tan\alpha=\lambda$

unit vector $\vec{a}$, the length of PB = 1

$\therefore\frac{\kappa}{\tau}=\frac{\sin\alpha}{\cos\alpha}$.

this shows that $\vec{a}$ ⊥r to Z-axis, Hence $\vec{a}$ lies in the R.P. (ie) The plane containing tangent $\bar{t}$ & binormal $\bar{b}$ I.t P are represents the,

By triangle law of vectors we've,

$\kappa\cos\alpha=\tau\sin\alpha$

$P\vec{B}=PA\,\vec{t}+AB\,\vec{b}$.

Since $\bar{n}\neq 0$, $PA=PB\cos\alpha$; $AB=PB\sin\alpha$

∴ $PB=1$, $(\kappa\cos\alpha-\tau\sin\alpha)\cdot\bar{n}=0$

$\kappa\cdot\bar{n}\cos\alpha-\tau\bar{n}\sin\alpha=0$

$\bar{t}'\cdot\cos\alpha+\bar{b}'\sin\alpha=0$

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Sing, $\bar{t}\cos\alpha + \bar{b}\sin\alpha = \vec{a}$ (constant)

$\vec{a}$ is a constant unit vector.

$\therefore$ Hence, $\bar{a}\cdot\bar{t} = \cos\alpha$

$[(\bar{t}\cos\alpha + \bar{b}\sin\alpha)\cdot\bar{t} = \cos\alpha]$

This shows that the tangent makes the constant angle with the generator this shows that, the curve is a helix.

Thrm:- SRD

P.T. $k = k_1\sin^2\alpha$ where k, k_1 are the curvature of the curve C & the projection of curve c_1 (say a).

proof :-

W.K.T. $\bar{r} = \bar{r}_1 + (\bar{a}\cdot\bar{r})\bar{a}$

Diff. w.r.to "s",

$$\frac{d\bar{r}}{ds} = \frac{d\bar{r}_1}{ds} + \left(\bar{a}\cdot\frac{d\bar{r}}{ds}\right)\cdot\bar{a}$$

$$\bar{t} = \frac{d\bar{r}_1}{ds_1}\cdot\frac{ds_1}{ds} + (\bar{a}\cdot\bar{t})\cdot\bar{a}$$

$$\bar{t} = \bar{t}_1\cdot s_1' + (\bar{a}\cdot\bar{t})\cdot\bar{a} \longrightarrow (1).$$

since $\bar{a}\cdot\bar{t} = \overset{\cos\alpha}{\cancel{\sin\alpha}}$; $s_1' = \sin\alpha$

$$(1) \Rightarrow \bar{t} = \bar{t}_1 . s\sin\alpha + \cos\alpha\cdot\bar{a} \longrightarrow (2)$$

Diff., w.r.to "s",

$$\frac{d\bar{t}}{ds} = \frac{d\bar{t}_1}{ds}\sin\alpha + 0$$

$$\bar{t}' = \frac{d\bar{t}_1}{ds_1}\cdot\frac{ds_1}{ds}\cdot\sin\alpha$$

$$k\bar{n} = \frac{d\bar{t}_1}{ds_1}\cdot\frac{ds_1}{ds}\cdot\sin\alpha$$

$$k\bar{n} = \bar{t}_1'\cdot s_1'\sin\alpha \qquad \{\because s_1' = \sin\alpha\}$$

$$k\bar{n} = k_1\bar{n}_1\cdot\sin^2\alpha \longrightarrow (3)$$

Local Non - Intrinsic properties of ...

This S.T. the normal $\bar{n}_1$ to the $\bar{a}$ is parallel to the principal normal $\bar{n}$ at the corresponding points of the helix.

So, taking modulus we get,

$$|k\bar{n}| = |k_1\bar{n}_1 \sin^2\alpha|$$

$$\sqrt{k^2} = (k_1^2 \sin^4\alpha)^{1/2}$$

$$k = k_1 \sin^2\alpha.$$

Hence proved.

Thrm :-

curvature k & torsion τ of a helix c are in a constant rate to a curvature of a plane curve c obtained by projecting c on a plane ⊥r to the axis of the helix. Then P.T. $k = k_1 \sin^2\alpha$ where α is the unit angle at which the helix cuts the generator. (proof in before)

Ex:- 6.1 ✓ 10M, 5M, ⑧.

P.T. the radius of curvature of the locus of the centre of curvature of a curve is given by,

$$\left[\left(\frac{\rho^2\sigma}{R^3}\frac{d}{ds}\left(\frac{\sigma\rho'}{\rho}\right) - \frac{1}{R}\right)^2 + \frac{\rho'\sigma^4}{\rho^2 R^4}\right]^{-1/2}$$

proof :-

Let the p.v. r of centre of curvature,

$$\therefore \bar{r}_1 = \bar{r} + \rho\bar{n}$$

Diff w.r. to "s",

$$\left[\because K = \frac{1}{\rho},\ \tau = \frac{1}{\sigma}\right]$$

$$\frac{d\bar{r}_1}{ds} = \bar{r}_1 + \rho\bar{n}' + \rho'\bar{n}$$

$$\frac{d\bar{r}_1}{ds_1}\cdot\frac{ds_1}{ds} = \bar{t} + \rho(\tau\bar{b} - K\bar{t}) + \rho'\bar{n}$$

$$\bar{r}_1'\cdot s_1' = \bar{t} + \rho\tau\bar{b} - \rho K\bar{t} + \rho'\bar{n} \Rightarrow \bar{t} + \frac{\rho}{\sigma}\bar{b} - \rho(1/\rho)\bar{t} + \rho'\bar{n}$$

$$\bar{t}_1\cdot s_1' = \frac{\rho}{\sigma}\bar{b} + \rho'\bar{n}.$$

Multiply by σ/ρ,

$$\frac{\sigma}{\rho}\bar{t}_1\cdot s_1' = \bar{b} + \frac{\sigma\rho'}{\rho}\bar{n} \longrightarrow (1)$$

$$s_1' = \frac{\rho}{\sigma \cancel{t_1}}\cdot\bar{b}\cdot\bar{t}_1 + \frac{\rho\sigma\rho'}{\rho\sigma t_1}\cdot n \quad s_1' = \left(\frac{\rho\bar{b}}{\sigma} + \frac{\rho'}{t_1}\bar{n}\right)^2$$

$$s_1'^2 = \left(\frac{\rho\bar{b}}{\sigma} + \frac{\rho'}{t_1}\bar{n}\right)^2 = \frac{\rho^2}{\sigma^2} + \frac{\rho'^2}{t_1^2} = \frac{\rho^2+\sigma^2\rho'^2}{\sigma^2}$$

$$s_1'^2 = \frac{\rho^2}{\sigma^2} + \rho'^2 = \frac{\rho^2+\sigma^2\rho'^2}{\sigma^2}$$

$$\therefore s_1'^2 = \frac{R^2}{\sigma^2} \longrightarrow (2)$$

$\{\because R^2 = \rho^2+\sigma^2\rho'^2$ of $\Rightarrow$ osculating sphere the spherical curvature$\}$

Diff. (1) w.r. to "s",

$$(1) \Rightarrow \frac{\sigma}{\rho}s_1'\frac{d\bar{t}_1}{ds} + \bar{t}_1\frac{d}{ds}\left(\frac{\sigma}{\rho}\cdot s_1'\right) = \bar{b}' + \frac{\sigma\rho'}{\rho}\bar{n}' + \frac{d}{ds}\left(\frac{\sigma\rho'}{\rho}\right)\cdot\vec{n}.$$

$\frac{db}{ds}$

$$\frac{\sigma}{\rho}s_1'\frac{d\bar{t}_1}{ds_1}\frac{ds_1}{ds} + \bar{t}_1\frac{d}{ds}\left(\frac{\sigma}{\rho}\cdot s_1'\right) = -\tau\vec{n} + \frac{\sigma\rho'}{\rho}(\tau\bar{b} - K\bar{t}) + \frac{d}{ds}\left(\frac{\sigma\rho'}{\rho}\right)\vec{n}.$$

$$= -\tau\vec{n} + \frac{\sigma\rho'}{\rho}(\overset{1/\sigma}{\tau}\bar{b} - \overset{1/\rho}{K}\bar{t}) + \frac{d}{ds}\left(\frac{\sigma\rho'}{\rho}\right)\vec{n}.$$

$$= -\tau\vec{n} + \frac{\rho'\bar{b}}{\rho} - \frac{\sigma\rho'}{\rho^2}\vec{t} + \frac{d}{ds}\left(\frac{\sigma\rho'}{\rho}\right)\vec{n} \quad (\because \tau = 1/\sigma,\ K = 1/\rho)$$

$$= \left(\frac{d}{ds}\left(\frac{\sigma\rho'}{\rho}\right) - \frac{1}{\sigma}\right)\vec{n} + \frac{\rho'}{\rho}\vec{b} - \frac{\sigma\rho'}{\rho^2}\vec{t}$$

$\frac{d}{ds}\left(\frac{1}{\sigma}\bar{n}\right)$

(iv) $\frac{\sigma}{\rho} s_1' t_1' s_1' + t_1 \frac{d}{ds}\left(\frac{\sigma}{\rho} s_1'\right) = \frac{d}{ds}\left(\left(\frac{\sigma\rho'}{\rho}\right) - \frac{1}{\sigma}\right)\vec{n} + \frac{\rho'}{\rho}\vec{b} - \frac{\sigma\rho'}{\rho^2}\vec{t} \longrightarrow (3)$

(1) × (3) =>

$$\left(\left(\frac{\sigma}{\rho}\right)\vec{t_1} s_1'\right) \times \left[\frac{\sigma}{\rho} s_1'^2 K_1 \vec{n_1} + \vec{t_1}\frac{d}{ds}\left(\frac{\sigma}{\rho} s_1'\right)\right]$$

$$= \left(\vec{b} + \frac{\sigma\rho'}{\rho}\vec{n}\right) \times \left[\frac{d}{ds}\left(\frac{\sigma\rho'}{\rho} - \frac{1}{\sigma}\right)\vec{n} + \frac{\rho'}{\rho}\vec{b} - \frac{\sigma\rho'}{\rho^2}\vec{t}\right]$$

$$\frac{\sigma^2}{\rho^2} s_1'^3 K_1 (\vec{t_1} \times \vec{n_1}) = \frac{d}{ds}\left(\frac{\sigma\rho'}{\rho} - \frac{1}{\sigma}\right)(\vec{b} \times \vec{n}) - \frac{\sigma\rho'}{\rho^2}(\vec{b} \times \vec{t}) + \frac{\sigma\rho'^2}{\rho^2}(\vec{n} \times \vec{b}) - \frac{\sigma^2\rho'^2}{\rho^3}(\vec{n} \times \vec{t})$$

$$\frac{\sigma^2}{\rho^2} s_1'^3 K_1 \vec{b_1} = -\left(\frac{d}{ds}\cdot\frac{\sigma\rho'}{\rho} - \frac{1}{\sigma}\right)\vec{t} - \frac{\sigma\rho'}{\rho^2}\vec{n} + \frac{\sigma\rho'^2}{\rho^2}\vec{t} + \frac{\sigma^2\rho'^2}{\rho^3}\vec{b}$$

$$= -\left(\frac{d}{ds}\frac{\sigma\rho'}{\rho} - \frac{1}{\sigma} + \frac{\sigma\rho'^2}{\rho}\right)\vec{t} - \frac{\sigma\rho'}{\rho^2}\vec{n} + \frac{\sigma^2\rho'^2}{\rho^3}\cdot\vec{b}$$

$$\frac{\sigma^2}{\rho^2} s_1'^3 K_1 \vec{b_1} = -\left(\frac{d}{ds}\cdot\frac{\sigma\rho'}{\rho} - \frac{R^2}{\rho^2\sigma}\right)\vec{t} - \frac{\sigma\rho'}{\rho^2}\vec{n} + \frac{\sigma^2\rho'^2}{\rho^3}\vec{b} \longrightarrow (4)$$

Eqn (4) squaring on b.s.,

$$\frac{\sigma^4}{\rho^4} s_1'^6 K_1^2 = \left(\frac{d}{ds}\frac{\sigma\rho'}{\rho} - \frac{R^2}{\rho^2\sigma}\right)^2 + \frac{\sigma^2\rho'^2}{\rho^4} + \frac{\sigma^4\rho'^4}{\rho^6}$$

$$= \left(\frac{d}{ds}\cdot\frac{\sigma\rho'}{\rho} - \frac{R^2}{\rho^2\sigma}\right)^2 + \frac{\sigma^2\rho'^2}{\rho^4}\left(1 + \frac{\sigma^2\rho'^2}{\rho^2}\right)$$

$$= \left(\frac{d}{ds}\frac{\sigma\rho'}{\rho} - \frac{R^2}{\sigma\rho^2}\right)^2 + \frac{\sigma^2\rho'^2}{\rho^4}\left(\frac{\rho^2 + \sigma^2\rho'^2}{\rho^2}\right)$$

$$\frac{\sigma^4}{\rho^4} \cdot s_1'^6 \cdot K_1^2 = \left(\frac{d}{ds}\frac{\sigma\rho'}{\rho} - \frac{R^2}{\rho^2\sigma}\right)^2 + \frac{\sigma^2\rho'^2}{\rho^6}\cdot R^2 \quad [\text{by } (3)]$$

$$\therefore \kappa_1^2 = \frac{\rho^4}{\sigma^4 s_1'^6}\left(\frac{d}{ds}\frac{\sigma\rho'}{\rho} - \frac{R^2}{\sigma\rho^2}\right)^2 + \frac{\rho^4}{\sigma^4 s_1'^6}\cdot\frac{\sigma^2\rho'^2}{\rho^6}\cdot R^2$$

$s_1'^2 = \frac{R^2}{\sigma^2}$

$$= \frac{\rho^4}{\sigma^4}\cdot\frac{\sigma^6}{R^6}\left(\frac{d}{ds}\cdot\frac{\sigma\rho'}{\rho} - \frac{R^2}{\rho^2\sigma}\right)^2 + \frac{\rho^4}{\sigma^4}\cdot\frac{\sigma^6\sigma^2\rho'^2}{R^6\rho^6}\cdot R^2$$

(by ④)

$$= \sigma^2\left[\frac{\rho^4}{R^6}\left[\left(\frac{d}{ds}\cdot\frac{\sigma\rho'}{\rho} - \frac{R^2}{\sigma\rho^2}\right)^2\right]\right] + \frac{\sigma^4\rho'^2}{\rho^2R^4}$$

$$\kappa_1^2 = \left(\frac{\sigma\rho^2}{R^3}\left(\frac{d}{ds}\frac{\sigma\rho'}{\rho} - \frac{1}{R}\right)\right)^2 + \frac{\sigma^4\rho'^2}{R^4\rho^2}$$

$$\kappa_1 = \left[\left[\frac{\sigma\rho^2}{R^3}\left(\frac{d}{ds}\frac{\sigma\rho'}{\rho} - \frac{1}{R}\right)\right]^2 + \frac{\sigma^4\rho'^2}{R^4\rho^2}\right]^{1/2}$$

$$\rho_1 = \frac{1}{\kappa_1} = \frac{1}{\left[\left(\frac{\sigma\rho^2}{R^3}\cdot\frac{d}{ds}\frac{\sigma\rho'}{\rho} - \frac{1}{R}\right)^2 + \frac{\sigma^4\rho'^2}{R^4\rho^2}\right]^{1/2}}$$

$$\rho_1 = \left[\left(\frac{\sigma\rho^2}{R^3}\cdot\frac{d}{ds}\cdot\frac{\sigma\rho'}{\rho} - \frac{1}{R}\right)^2 + \frac{\sigma^4\rho'^2}{R^4\rho^2}\right]^{-1/2}$$

Ex:- 6.2 10M, kon.

21/7/15 If the radius of the spherical curvature is constant. P.T. the curve either lies on a sphere (or) has constant curvature.

Soln :-

Let the radius of spherical curvature R is consta

(i.e) $R = (\rho^2 + \sigma^2\rho'^2)^{1/2}$ is constant.

$\therefore R^2 = \rho^2 + \sigma^2\rho'^2$

$R^2 = \rho^2 + (\sigma\rho')^2$.

Diff., $0 = 2\rho\rho' + 2(\sigma\rho')\frac{d}{ds}(\sigma\rho')$

$\rho'\left(\rho + \sigma\frac{d}{ds}(\sigma\rho')\right) = 0$

Hence, either $\rho' = 0$ (or) $\rho + \sigma\frac{d}{ds}(\sigma\rho') = 0$

case(i) If $\rho' = 0$ then $\rho = $ constant

$\Rightarrow \frac{1}{\rho} = $ constant

$\Rightarrow \kappa = $ constant.

thus the curve has a constant curvature.

Case(ii) If $\dot{\rho} + \sigma \frac{d}{ds}(\sigma\rho') = 0$

$$\frac{\rho}{\sigma} + \frac{d}{ds}(\sigma\rho') = 0 \longrightarrow (1)$$

To prove, the curve lies on a sphere.

Since R is constant. the radius of the osculating sphere is independent of the position of a point on the curve.

If $\vec{r_1}$ is the centre of spherical curvature.

Then, $\vec{r_1} = \vec{r} + \rho\vec{n} + \sigma\rho'\vec{b}$.

Diff., $$\frac{d\vec{r_1}}{ds} = \frac{d\vec{r}}{ds} + \rho\frac{d\vec{n}}{ds} + \sigma\rho'\frac{d\vec{b}}{ds} + \rho'\vec{n} + \sigma'\rho'\vec{b} + \sigma\rho''\vec{b}$$

$$= \vec{t} + \rho(\tau\vec{b} - \kappa\vec{t}) + \sigma\rho'(-\tau\vec{n}) + \rho'\vec{n} + \frac{d}{ds}(\sigma\rho')\vec{b}$$

put $\kappa = \frac{1}{\rho}$; $\sigma = \frac{1}{\tau}$

$$\frac{d\vec{r_1}}{ds} = \vec{t} + \frac{\rho\vec{b}}{\sigma} - \vec{t} + \rho'(-\vec{n}) + \rho'\vec{n} + \frac{d}{ds}(\sigma\rho')\vec{b}$$

$$= \left[\frac{\rho}{\sigma} + \frac{d}{ds}(\sigma\rho')\right]\vec{b} = 0 \qquad \text{[by (1)]}$$

$$\frac{d\vec{r_1}}{ds} = 0$$

This shows that the centre of osculating sphere is independent of the position of a point on the curve.

Hence, curve lies on a sphere.

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(7) Ex: 6. If a curve lies on a sphere S.T. ρ & σ are related by $\frac{d}{ds}(\sigma\rho') + \frac{\rho}{\sigma} = 0$.

Soln:-

If a curve lies on a sphere then that sphere is the osculating sphere for all point & radius R of the osculating sphere is constant. ($\because$ by above result)

(i.e) $R = \sqrt{\rho^2 + \sigma^2\rho'^2}$ is constant.

$$R^2 = \rho^2 + \sigma^2\rho'^2$$

$$R^2 = \rho^2 + (\sigma\rho')^2$$

Diff., $2\rho\rho' + 2\frac{d}{ds}(\sigma\rho')\cdot\sigma\rho' = 0$

$$\rho + \sigma\frac{d}{ds}(\sigma\rho') = 0$$

$$\frac{\rho}{\sigma} + \frac{d}{ds}(\sigma\rho') = 0.$$ Hence proved.

Ex: 7.2 S.T. the curvature of a circular helix are plane curve

Soln:-

The torsion of the involute is given by,

$$\tau_1 = \frac{\kappa\tau' - \kappa'\tau}{\kappa(c-s)(\kappa^2+\tau^2)}$$

W.K.T. the curvature & torsion for a circular helix are constant.

So, $\kappa' = 0$ & $\tau' = 0$ $\quad\therefore \tau_1 = 0$

Hence the involute must be plane curve.

Ex: 7.3

S.T. the involutes of a curve is equal to,

$$\tau_1 = \frac{\rho(\sigma\rho' - \sigma'\rho)}{(\rho^2+\sigma^2)(c-s)}$$

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Soln :-

The p.v. $\vec{r}_1$ of a current point on the involute given by, $\vec{r}_1 = \vec{r} + (c-s)\vec{t}$ $\longrightarrow$ (1) Theorem of the involute

Diff.,
$$\frac{d\vec{r}_1}{ds} = \frac{d\vec{r}}{ds} + (c-s)\vec{t}' + \vec{t}(-1)$$
$$\frac{d\vec{r}_1}{ds_1} \cdot \frac{ds_1}{ds} = \vec{t} + (c-s)\vec{t}' + \vec{t}(-1)$$
$$\vec{r}_1' \cdot \frac{ds_1}{ds} = \vec{t} + (c-s)\vec{t}' - \vec{t}$$
$$\frac{ds_1}{ds} \cdot \vec{r}_1' = (c-s)\kappa\vec{n}$$
$$\frac{ds_1}{ds} \cdot \vec{t}_1 = (c-s)\cdot\kappa\vec{n} \longrightarrow (2)$$

Eqn (1) s.t. the tangent to the involute is parallel to the principal normal to the given curve be choose the +ve direction along the involute $\ni$,
$$\vec{t}_1 = \vec{n} \longrightarrow (3)$$

(2) $\Rightarrow$ $\frac{ds_1}{ds} \cdot \vec{n} = (c-s)\kappa\vec{n}$
$$\frac{ds_1}{ds} = (c-s)\kappa \longrightarrow (4)$$

Diff., (3)
$$\frac{d\vec{t}_1}{ds} = \vec{n}'$$
$$\frac{d\vec{t}_1}{ds_1} \cdot \frac{ds_1}{ds} = \tau\vec{b} - \kappa\vec{t}$$

Serret Frenet
$$\kappa_1\vec{n}_1 \cdot \kappa(c-s) = \tau\vec{b} - \kappa\vec{t} \quad (\text{by } (4)) \longrightarrow (5)$$

squaring on b.s.,
$$\kappa^2\kappa_1^2\vec{n}_1^2(c-s)^2 = \tau^2 + \kappa^2$$
$$\kappa^2\kappa_1^2(c-s)^2 = \tau^2 + \kappa^2$$
$$\kappa_1^2 = \frac{\tau^2 + \kappa^2}{\kappa^2(c-s)^2} \longrightarrow (6)$$

{Taking modulus on b.s}

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Taking the cross product of (3) & (5) we get,

$$\vec{t_1} \times \kappa\kappa_1 \vec{n_1}(c-s) = \vec{n}(\tau\vec{b} - \kappa\vec{t})$$

$$\kappa\kappa_1(c-s)[\vec{t_1} \times \vec{n_1}] = (\vec{n} \times \vec{b})\tau - \kappa(\vec{n} \times \vec{t})$$

$$\kappa\kappa_1(c-s)\vec{b_1} = \vec{t}\tau + \kappa\vec{b} \longrightarrow (7)$$

Diff. (7),

$$\kappa\kappa_1(c-s)\frac{d\vec{b_1}}{ds_1} \cdot \frac{ds_1}{ds} + \vec{b_1}\frac{d}{ds}(\kappa\kappa_1(c-s))$$
$$= \vec{t}'\tau + \tau'\vec{t} + \kappa\vec{b}' + b\kappa'$$

$$\kappa^2\kappa_1(c-s)^2(-\tau_1\vec{n_1}) + \vec{b_1}\frac{d}{ds}[\kappa\kappa_1(c-s)]$$
$$= \tau\kappa\vec{n} + \vec{t}\tau' - \cancel{\kappa\tau\vec{n}} + b\kappa'$$

$$\kappa^2\kappa_1(c-s)^2(-\tau_1\vec{n_1}) + \vec{b_1}\frac{d}{ds}[\kappa\kappa_1(c-s)] \quad (\text{by } (4) \text{ \& } b_1' = -\tau_1\vec{n_1})$$
$$= \vec{t}\tau' + \vec{b}\kappa' \longrightarrow (8)$$

Taking the ~~cross~~ dot product of (5) & (8),

$$-\tau_1\kappa_1^2\kappa^3(c-s)^3 = \kappa'\tau - \kappa\tau' \qquad (\because \vec{n_1}\cdot\vec{n_1} = 1)$$

$$\tau_1 = \frac{\kappa\tau' - \kappa'\tau}{\kappa_1^2\kappa^3(c-s)^3}$$

$$\tau_1 = \frac{\kappa\tau' - \kappa'\tau\,[\kappa^2(c-s)^2]}{(\tau^2+\kappa^2)\kappa^3(c-s)^3} \quad (\because \text{by } (6))$$

$$= \frac{\kappa\tau' - \kappa'\tau}{\kappa(c-s)(\tau^2+\kappa^2)} \longrightarrow (9)$$

Put $\kappa = \frac{1}{\rho}$, $\kappa' = \frac{-\rho'}{\rho^2}$, $\tau = \frac{1}{\sigma}$, $\tau' = -\frac{\sigma'}{\sigma^2}$ in (9),

$$(9) \Rightarrow \tau_1 = \frac{\frac{1}{\rho}\cdot\left(-\frac{\sigma'}{\sigma^2}\right) + \frac{\rho'}{\rho^2}\frac{1}{\sigma}}{\frac{1}{\rho}(c-s)\left(\frac{1}{\rho^2} + \frac{1}{\sigma^2}\right)} \Rightarrow \frac{(-\sigma'\rho + \rho'\sigma)\rho}{(c-s)(\sigma^2+\rho^2)}$$

$$\frac{\frac{-\sigma'\rho + \rho'\sigma}{\sigma^2\rho\kappa}}{\frac{(c-s)(\sigma^2+\rho^2)}{\sigma^2\rho}}$$

$$\tau_1 = \frac{\rho(\sigma\rho' - \rho\sigma')}{(c-s)(\sigma^2+\rho^2)}$$

Hence proved.

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