ODE, PDE, LAPLACE TRANSFORMS & VECTOR CALCULUS

UNIT : IV

SUBJECT : LAPLACE TRANSFORMS

ALLIED [MATHS]

Find $L^{-1}\left[\frac{s}{(s+2)^2+1}\right]$.

Soln :-

$$L^{-1}\left[\frac{s}{(s+2)^2+1}\right] = L^{-1}\left[\frac{s+2-2}{(s+2)^2+1}\right]$$

$$= L^{-1}\left[\frac{s+2}{(s+2)^2+1}\right] - L^{-1}\left[\frac{2}{(s+2)^2+1}\right]$$

$$= e^{-2t}.L^{-1}\left[\frac{s}{s^2+1}\right] - 2e^{-2t}L^{-1}\left[\frac{1}{s^2+1}\right]$$

$$= e^{-2t}\cos t - 2e^{-2t}\sin t$$

$$= e^{-2t}(\cos t - 2\sin t)$$

$$L^{-1}\left[\frac{s}{(s+2)^2+1}\right] = e^{-2t}[\cos t - 2\sin t].$$

Find $L^{-1}\left[\frac{s}{(s+2)^2}\right]$.

Soln :-

$$L^{-1}\left[\frac{s}{(s+2)^2}\right] = L^{-1}\left[\frac{s+2-2}{(s+2)^2}\right]$$

$$= L^{-1}\left[\frac{s+2}{(s+2)^2}\right] - 2L^{-1}\left[\frac{1}{(s+2)^2}\right]$$

$$= L^{-1}\left[\frac{1}{s}\right] - 2L^{-1}\left[\frac{1}{(s+2)^2}\right]$$

$$= \cancel{L^{-1}[} \; e^{-2t} - 2e^{-2t} L^{-1}\left[\frac{1}{s^2}\right].$$

$$= e^{-2t} - 2e^{-2t} \cdot t.$$

$$L^{-1}\left[\frac{s}{(s+2)^2}\right] = e^{-2t}[1-2t].$$

3) Find $L^{-1}\left[\frac{1}{(s+2)^2+16}\right]$.

Soln:-

$$L^{-1}\left[\frac{1}{(s+2)^2+16}\right] = e^{-2t} \cdot L^{-1}\left[\frac{1}{s^2+16}\right]$$

$$= e^{-2t} \cdot \frac{\sin 4t}{4}$$

4) Find $L^{-1}\left[\frac{1}{s^2+4s+13}\right]$.

Soln:-

$$L^{-1}\left[\frac{1}{s^2+4s+13}\right] = L^{-1}\left[\frac{1}{s^2+4s+4-4+13}\right]$$

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$$= L^{-1}\left[\frac{1}{(s+2)^2+9}\right]$$

$$= e^{-2t} L^{-1}\left[\frac{1}{s^2+9}\right].$$

$$= e^{-2t} \cdot \frac{\sin 3t}{3}.$$

1. $L^{-1}\left[\frac{6}{(s+2)^4}\right].$

<u>Soln:-</u>

$$L^{-1}\left[\frac{6}{(s+2)^4}\right] = e^{-2t} \cdot L^{-1}\left[\frac{6}{s^4}\right].$$

$$= 6e^{-2t} \cdot L^{-1}\left[\frac{1}{s^4}\right]$$

$$= 6e^{-2t} \cdot \frac{t^3}{6}$$

$$= e^{-2t} \cdot t^3.$$

2. $L^{-1}\left[\frac{6}{(s-1)^4}\right]$

<u>Soln:-</u>

$$L^{-1}\left[\frac{6}{(s-1)^4}\right] = e^{t} \cdot L^{-1}\left[\frac{6}{s^4}\right]$$

$$L^{-1}\left[\frac{4}{s^2+9}\right].$$

Soln:-

$$L^{-1}\left[\frac{4}{s^2+9}\right] = 4L^{-1}\left[\frac{1}{s^2+9}\right]$$

$$= \frac{4\sin 3t}{3}$$

$$L^{-1}\left[\frac{3}{(2s+5)^3}\right].$$

Soln:-

$$L^{-1}\left[\frac{3}{(2s+5)^3}\right] = 3L^{-1}\left[\frac{1}{\left[2\left[s+5/2\right]\right]^3}\right]$$

$$= \frac{3}{8}L^{-1}\left[\frac{1}{\left[s+5/2\right]^3}\right].$$

$$= \frac{3}{8}e^{-5/2t}\cdot L^{-1}\left[\frac{1}{s^3}\right]$$

$$= \frac{3}{8}\cdot e^{-5/2t}\cdot t^2/2$$

$$= \frac{3}{16}\cdot e^{-5/2t}\cdot t^2.$$

$$L^{-1}\left[\frac{1}{s^2+4s+8}\right].$$

Soln :-

$$L^{-1}\left[\frac{1}{s^2+4s+8}\right] = L^{-1}\left[\frac{1}{s^2+4s+4-4+8}\right]$$

$$= L^{-1}\left[\frac{1}{(s+2)^2+4}\right]$$

$$= e^{-2t}\, L^{-1}\left[\frac{1}{s^2+4}\right]$$

$$= e^{-2t} \cdot \frac{\sin 2t}{2}$$

b) $$L^{-1}\left[\frac{3s-15}{2s^2-4s+16}\right]$$

Soln :-

$$L^{-1}\left[3s-15 \right] = L^{-1}\left[3s-15 \right]$$

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$$= \frac{3}{2} L^{-1}\left[\frac{s}{(s-1)^2+(\sqrt{7})^2} - \frac{5}{(s-1)^2+(\sqrt{7})^2}\right]$$

$$= \frac{3}{2} L^{-1}\left[\frac{s-1+1}{(s-1)^2+(\sqrt{7})^2} - \frac{5}{(s-1)^2+(\sqrt{7})^2}\right]$$

$$= \frac{3}{2} L^{-1}\left[\frac{(s-1)}{(s-1)^2+7} + \frac{1}{(s-1)^2+7} - \frac{5}{(s-1)^2+7}\right]$$

$$= \frac{3}{2} L^{-1}\left[\frac{(s-1)}{(s-1)^2+7} + \frac{1}{(s-1)^2+7} - \frac{5}{(s-1)^2+7}\right]$$

$$= \frac{3}{2} e^{t} \cdot L^{-1}\left[\frac{s}{s^2+7} + \frac{1}{s^2+7} - \frac{5}{s^2+7}\right]$$

$$= \frac{3}{2} e^{t}\left[\cos\sqrt{7}t - \frac{4\sin\sqrt{7}t}{\sqrt{7}}\right]$$

$$L^{-1}\left[\frac{3}{2s^2+8s+10}\right]$$

Soln :-

$$L^{-1}\left[\frac{3}{2s^2+8s+10}\right] = \frac{3}{2} L^{-1}\left[\frac{1}{s^2+4s+4-4+5}\right]$$

$$= \frac{3}{2} L^{-1}\left[\frac{1}{(s+2)^2+1}\right]$$

$$= \frac{3}{2} e^{-2t} \cdot L^{-1}\left[\frac{1}{s^2+1}\right]$$

$$= \frac{3}{2} e^{-2t} \cdot \sin t$$

8) $L^{-1}\left[\dfrac{s-1}{s^2-2s+5}\right]$

Soln :-

$$L^{-1}\left[\frac{s-1}{s^2-2s+5}\right] = L^{-1}\left[\frac{s-1}{s^2-2s+1-1+5}\right]$$

$$= L^{-1}\left[\frac{s-1}{(s-1)^2+4}\right].$$

$$= e^{t} L^{-1}\left[\frac{s}{s^2+4}\right]$$

$$= e^{t} \cdot \cos 2t .$$

9) $L^{-1}\left(\dfrac{3}{s^4}\right)$

Soln :-

$$L^{-1}\left[\frac{3}{s^4}\right] = 3L^{-1}\left[\frac{1}{s^4}\right]$$

$$= \frac{3t^3}{6}$$

$$= \frac{t^3}{2}$$

Type : 2 :-

Method of partial fraction :-

$$L^{-1}\left[\frac{7s-1}{(s+1)(s+2)(s+3)}\right]$$

Soln :-

$$L^{-1}\left[\frac{7s-1}{(s+1)(s+2)(s+3)}\right] = \frac{A}{(s+1)} + \frac{B}{(s+2)} + \frac{C}{s+3}$$

$$7s-1 = A(s+2)(s+3) + B(s+1)(s+3) + C(s+2)(s+1) \rightarrow (1)$$

$$s = -1$$

$$-7-1 = A(-1+2)(-1+3)$$

$$-8 = A(1\times 2)$$

$$-8 = 2A$$

$$A = -4.$$

Put $s = -2$ in (1).

$$-14-1 = B(-2+1)(-2+3)$$

$$-15 = B(-1\times 1)$$

$$-15 = -B$$

$$B = 15$$

put $s = -3$ in eq(1)

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$$-21-1 = c(-3+1)(-3+2)$$

$$-22 = c(-2\times -1)$$

$$-22 = 2c$$

$$c = -11$$

$$L^{-1}\left[\frac{7s-1}{(s+1)(s+2)(s+3)}\right] = L^{-1}\left[\frac{-4}{s+1} + \frac{15}{s+2} - \frac{11}{s+3}\right]$$

$$= L^{-1}\left[\frac{-4}{s+1}\right] + L^{-1}\left[\frac{15}{s+2}\right] - L^{-1}\left[\frac{11}{s+3}\right].$$

$$= -\frac{4L^{-1}}{(s+1)} + 15L^{-1}\left[\frac{1}{s+2}\right] - 11L^{-1}\left[\frac{1}{s+3}\right].$$

$$= -4e^{-t} + 15e^{-2t} - 11e^{-3t}$$

$$L^{-1}\left[\frac{7s-1}{(s+1)(s+2)(s+3)}\right] = -4e^{-t} + 15e^{-2t} - 11e^{-3t}.$$

$$L^{-1}\left[\frac{s^2+9s+2}{(s-1)^2(s+2)}\right].$$

Soln:-

$$\left[\frac{s^2+9s+2}{(s-1)^2(s+2)}\right] = \frac{A}{(s+2)} + \frac{B}{(s-1)} + \frac{C}{(s-1)^2} \rightarrow (*)$$

$$s^2+9s+2 = A(s-1)^2 + B(s-1)(s+2) + C(s+2)$$

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Put $s = -2$

$$(-2)^2 + 9(-2) + 2 = A(-2-1)^2 + 0$$

$$4 - 18 + 2 = 9A$$

$$-12 = 9A$$

$$A = -4/3$$

$s = 1$ in (1)

$$(1)^2 + 9(1) + 2 = C(1+2)$$

$$3C = 12$$

$$C = 4$$

Equating coeff. of s^2

$$1 = A + B$$

$$1 = -4/3 + B$$

$$B = 4/3 + 1$$

$$B = 7/3$$

$$L^{-1}\left[\frac{s^2+9s+2}{(s-1)^2(s+2)}\right] = L^{-1}\left[-\frac{4/3}{s+2} + \frac{7/3}{(s-1)} + \frac{4}{(s-1)^2}\right]$$

$$= -\frac{4}{3} L^{-1}\left[\frac{1}{s+2}\right] + \frac{7}{3} L^{-1}\left[\frac{1}{s-1}\right] + 4L^{-1}\left[\frac{1}{(s-1)^2}\right].$$

$$= -\frac{4}{3}e^{-2t} + \frac{7}{3}e^{t} + 4e^{t} L^{-1}\left[\frac{1}{s^2}\right]$$

$$= -\frac{4}{3}e^{-2t} + \frac{7}{3}e^{t} + 4e^{t}.t$$

$$L^{-1}\left[\frac{s^2+9s+2}{(s-1)^2(s+2)}\right] = -\frac{4}{3}e^{-2t} + \frac{7}{3}e^{t} + 4te^{t}$$

1) $L^{-1}\left[\frac{s+2}{(s-4)(s^2+1)}\right]$

Soln:-

$$L^{-1}\left[\frac{s+2}{(s-4)(s^2+1)}\right] = \frac{A}{(s-4)} + \frac{Bs+C}{(s^2+1)} \rightarrow (*)$$

$$s+2 = A(s^2+1) + (Bs+C)(s-4) \rightarrow (1)$$

$s = 4$ in (1).

$$4+2 = A(4^2+1)$$

$$6 = A(17)$$

$$A = \frac{6}{17}.$$

Equating the coeff. of s^2

$$0 = A + B$$

$$0 = 6/17 + B$$

$$B = -\frac{6}{17}$$

Equating the co-efficient of s

$$1 = C - 4B$$

$$1 = C - 4\left[-6/17\right]$$

$$1 = C + \frac{24}{17}$$

$$1 - \frac{24}{17} = C$$

$$C = \frac{17-24}{17}$$

$$C = -7/17$$

$$L^{-1}\left[\frac{s+2}{(s-4)(s^2+1)}\right] = L^{-1}\left[\frac{6/17}{(s-4)} + \frac{(-6/17)s + (-7/17)}{s^2+1}\right]$$

$$= \frac{6/17}{s-4} - \frac{(6/17)s}{s^2+1} - \frac{7/17}{s^2+1}$$

$$= \frac{6}{17} L^{-1}\left[\frac{1}{s-4}\right] - \frac{6}{17} L^{-1}\left[\frac{1}{s^2+1}\right] -$$

$$\frac{7}{17} L^{-1}\left[\frac{1}{s^2+1}\right].$$

$$= \frac{6}{17} e^{4t} - \frac{6}{17} \frac{\sin at}{a} - \frac{7}{17} \frac{\sin at}{a}$$

$$L^{-1}\left[\frac{s+2}{(s-4)(s^2+1)}\right] = \frac{6}{17} e^{4t} - \frac{6}{17} \frac{\sin at}{a} - \frac{7}{17} \frac{\sin at}{a}$$

3) $L^{-1}\left[\dfrac{-8s^2-5s+9}{(s+1)^2(s-1)(s-2)}\right]$

Soln :-

$$L^{-1}\left[\frac{-8s^2-5s+9}{(s+1)^2(s-1)(s-2)}\right] = \frac{A}{(s-1)} + \frac{B}{(s-2)} + \frac{C}{(s+1)} + \frac{D}{(s+1)^2} \rightarrow (*)$$

$$-8s^2-5s+9 = A(s-2)(s+1)^2 + B(s-1)(s+1)^2 + C(s-1)(s-2)(s+1) + D(s-1)(s-2) \rightarrow ①$$

put $s=1$

$$-8-5+9 = A(-1)(2)^2+0+0+0$$

$$\cancel{+4} = \cancel{+4}A$$

$$A=1$$

put $s=2$

$$-8(2)^2-5(2)+9 = 0+B(1)(3)^2+0+0$$

$$\Rightarrow -32-10+9 = 9B$$

$$-33 = 9B$$

$$B = \frac{-33}{9}$$

Put $s = -1$

$$8+5+9 = 0+0+0 + D(-2)(-3)$$

$$22 = -6D$$

$$-\frac{\cancel{22}^{11}}{\cancel{6}_{3}} = D$$

$$D = -11/3$$

Equating the co-eff. of s^3

$$0 = A+B+C$$

$$0 = 1 + \left(-11/3\right) + C$$

$$0 = 1 - 11/3 + C$$

$$0 = \frac{\cancel{3}-11}{3} + C$$

$$\cancel{-}C = \cancel{+}\frac{8}{3}$$

$$C = 8/3.$$

Subs.. the values of A, B, C, D in (*)

$$L^{-1}\left[\frac{-8s^2-5s+9}{(s+1)^2(s-1)(s-2)}\right] = L^{-1}\left[\frac{1}{(s-1)} + \frac{-\frac{33}{9}}{(s-2)} + \frac{\frac{24}{9}}{(s+1)} + \frac{11/3}{(s+1)^2}\right].$$

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$$= L^{-1}\left[\frac{1}{(s-1)} - \frac{33}{9(s-2)} + \frac{\cancel{24}\,8}{\cancel{9}\,3\,(s+1)} - \frac{11}{3(s+1)^2}\right]$$

$$= e^{t} - \frac{33}{9}e^{2t} + \frac{8}{3}e^{-t} - \frac{11}{3}e^{-t}t.$$

4) $L^{-1}\left[\frac{6s^2-13s+2}{s(s-1)(s-2)}\right]$

Soln:-

$$L^{-1}\left[\frac{6s^2-13s+2}{s(s-1)(s-2)}\right] = \frac{A}{s} + \frac{B}{s-1} + \frac{C}{s-2} \rightarrow (*)$$

$$6s^2-13s+2 = A(s-1)(s-2) + B(s)(s-2) + C(s)(s-1) \rightarrow (1)$$

put $s=1$

$$6-13+2 = 0 + B(1)(1-2) + 0$$

$$-5 = B(1)(-1)$$

$$B = 5$$

put $s=2$

$$6(2)^2 - 13(2) + 2 = 0 + 0 + C(2)(2-1)$$

$$6(4) - 13(2) + 2 = C(2)(1)$$

$$24 - 26 + 2 = 2C$$

$$2C = 0$$

$$C = 0$$

Equating the co-efficient of s^2.

$$6 = A+B+C$$

$$6 = A+5+0$$

$$A+5=6$$

$$A=1$$

Sub the values of A, B, C in (*)

$$L^{-1}\left[\frac{6s^2-13s+2}{s(s-1)(s-2)}\right] = L^{-1}\left[\frac{1}{s}+\frac{5}{(s-1)}+\frac{0}{s-2}\right]$$

$$= L^{-1}\left[\frac{1}{s}\right] + L^{-1}\left[\frac{5}{(s-1)}\right] + 0$$

$$= 1+5e^t$$

$$= 5e^t+1$$

$$L^{-1}\left[\frac{1}{s(s^2+a^2)}\right]$$

Soln:-

$$L^{-1}\left[\frac{1}{s(s^2+a^2)}\right] = \frac{A}{s}+\frac{Bs+C}{(s^2+a^2)} \longrightarrow (*).$$

$$\frac{1}{s\cancel{(s^2+a^2)}} = \frac{A(s^2+a^2)+(Bs+C)s}{s\cancel{(s^2+a^2)}}$$

$$1 = A(s^2+a^2)+(Bs+C)s \longrightarrow (1)$$

put $s=0$ in (1)

$$1 = Aa^2 + 0$$

$$A = 1/a^2$$

Equating the co-efficient of s^2

$$0 = A + B$$

$$0 = \frac{1}{a^2} + B$$

$$B = -1/a^2$$

Equating the co.eff. of s.

$$0 = C$$

$$C = 0$$

subs.. A, B, C in (*)

$$L^{-1}\left(\frac{1}{s(s^2+a^2)}\right) = L^{-1}\left[\frac{1/a^2}{s} + \frac{-1/a^2 s + 0}{s^2+a^2}\right]$$

$$= \frac{1}{a^2} L^{-1}\left[1/s\right] - \frac{1}{a^2} L^{-1}\left[\frac{s}{s^2+a^2}\right].$$

$$= \frac{1}{a^2}(1) - \frac{1}{a^2}\cos at$$

$$= \frac{1}{a^2}\left[1 - \cos at\right)$$

2) $$L^{-1}\left[\frac{7s^2 + 23s + 30}{(s-2)(s^2+2s+5)}\right]$$

$$\frac{}{(s-2)(s^2+2s+5)} \quad \frac{}{s-2} \quad \frac{}{s^2+2s+5}$$

$$7s^2+23s+30 = A(s^2+2s+5)+(Bs+C)(s-2) \longrightarrow (*)$$

put $s=2$

$$7(2)^2+23(2)+30 = A(2^2+2(2)+5)$$

$$28+46+30 = A(4+4+5)$$

$$104 = 13A$$

$$A = \frac{104}{13}$$

$$A = 8$$

Equating the co-eff. of s^2

$$7 = A+B$$

$$7 = 8+B$$

$$7-8 = B$$

$$B = -1$$

Equating the co-eff of s

$$23 = 2A-2B+C$$

$$25 = 2(8)-2(-1)+C$$

$$23 = 16+2+C$$

$23-18 = C$

$C = 5$

Sub A, B, C in (*)

$$L^{-1}\left[\frac{\cancel{7}s^2+23s+30}{(s-2)(s^2+2s+5)}\right] = L^{-1}\left[\frac{8}{(s-2)} + \frac{(-1)s+5}{(s^2+2s+5)}\right]$$

$$= 8L^{-1}\left[\frac{1}{s-2}\right] + L^{-1}\left[\frac{-s+5}{s^2+2s+5}\right].$$

$$= 8e^{2t} + L^{-1}\left[\frac{-s}{(s+1)^2+4}\right] + 5L^{-1}\left[\frac{1}{(s+1)^2+4}\right]$$

$$= 8e^{2t} - L^{-1}\left[\frac{s+1-1}{(s+1)^2+4}\right] + 5L^{-1}\left[\frac{1}{(s+1)^2+4}\right]$$

$$= 8e^{2t} - L^{-1}\left[\frac{(s+1)}{(s+1)+4}\right] + L^{-1}\left[\frac{1}{(s+1)^2+4}\right] +$$

$$5L^{-1}\left[\frac{1}{(s+1)^2+4}\right].$$

$$= 8e^{2t} - e^{-t}.L^{-1}\left[\frac{s}{s^2+a^2}\right] + e^{-t}.L^{-1}\left[\frac{1}{s^2+a^2}\right]$$

$$+5e^{-t}.L^{-1}\left[\frac{1}{s^2+a^2}\right]$$

$$= 8e^{2t} - e^{-t}\cos 2t + e^{-t}\frac{\sin 2t}{2} +$$

$$5e^{-t}\frac{\sin 2t}{2}$$

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$$= 8e^{2t} - e^{-t}\cos 2t + \frac{\cancel{6}^{3} e^{-t}\sin 2t}{\cancel{2}}$$

$$= 8e^{2t} - e^{-t}\cos 2t + 3e^{-t}\sin 2t$$

$$L^{-1}\left[\frac{2(s+1)}{(s^2+2s+2)^2}\right]$$

<u>soln:-</u>

$$F'(s) = \frac{2s+2}{(s^2+2s+2)^2}$$

$$\frac{d}{ds}F(s) = \frac{2s+2}{(s^2+2s+2)^2}$$

$\int$ing on both sides,

$$F(s) = \int \frac{2s+2}{(s^2+2s+2)^2}\,ds$$

Take $u = s^2+2s+2$

$$du = (2s+2)\,ds$$

$$F(s) = \int \frac{du}{u^2} = \frac{-1}{u} = -\frac{1}{s^2+2s+2}$$

$$F(s) = -\frac{1}{s^2+2s+2}$$

Using the Formula,

$$L^{-1}[F'(s)] = -t\,L^{-1}[F(s)]$$

$$= t L^{-1}\left[\frac{1}{s^2+2s+1^2-1^2+2}\right]$$

$$= t L^{-1}\left[\frac{1}{(s+1)^2+1}\right]$$

$$= te^{-t}.L^{-1}\left[\frac{1}{s^2+1}\right]$$

$$= L^{-1}\left[\frac{2s+2}{(s^2+2s+2)^2}\right] = te^{-t}\sin t$$

4) $L^{-1}\left[\frac{s+3}{(s^2+6s+13)^2}\right]$

Soln:-

$$F'(s) = \frac{s+3}{(s^2+6s+13)^2}, \quad \frac{d}{ds}F(s) = \frac{s+3}{(s^2+6s+13)^2}$$

$\int$ing on b/s

$$F(s) = \int \frac{s+3}{(s^2+6s+13)^2}\,ds$$

Take $U = s^2+6s+13$

$$du = (2s+6)\,ds$$

$$= 2(s+3)\,ds$$

$$F(s) = \int\frac{du}{2u^2} = \frac{1}{2}\cdot -\frac{1}{u} = \frac{-1}{2(s^2+6s+13)}$$

$$F(s) = \frac{-1}{2(s^2+6s+13)^2}$$

$$= -t\,L^{-1}\left[\frac{-1}{2(s^2+6s+13)}\right]$$

$$= \frac{1}{2}\,t\,L^{-1}\left[\frac{1}{s^2+6s+9-9+13}\right]$$

$$\Rightarrow \frac{1}{2}\,t\,L^{-1}\left[\frac{1}{(s+3)^2+4}\right] \Rightarrow \frac{1}{2}\,t e^{-3t}\left[\frac{1}{s^2+2^2}\right]$$

$$= \frac{1}{2}\,t e^{-3t}\,\frac{\sin 2t}{2}$$

$$= \frac{1}{4}\,t e^{-3t}\sin 2t$$

$$L^{-1}\left[\frac{2s^2+10s}{(s^2-2s+2)(s+1)}\right]$$

Soln :-

$$L^{-1}\left[\frac{2s^2+10s}{\quad}\right] = \frac{A}{\quad} + \frac{Bs+C}{\quad} \rightarrow (*)$$

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Equating the co-eff. of s^2

$$2 = A + B$$

$$A + B = 2$$

$$-8/5 + B = 2$$

$$B = 2 + 8/5$$

$$B = 18/5$$

Equating the co-eff. of s

$$10 = -2A + B + C$$

$$10 = -2\left[-8/5\right] + B + C$$

$$10 = 16/5 + 18/5 + C$$

$$10 = \frac{34}{5} + C$$

$$\frac{50 - 34}{5} = C$$

$$C = 16/5$$

Subs.. A, B, C in eq (1)

$$L^{-1}\left[\frac{2s^2+10s}{(s^2-2s+2)(s+1)}\right] = L^{-1}\left[\frac{-8/5}{(s+1)} + \frac{\frac{18}{5}s + 16/5}{s^2+2-2s}\right]$$

$$= L^{-1}\left[\frac{-8}{5(s+1)}\right] + L^{-1}\left[\frac{18/5\,s + 16/5}{s^2-2s+2}\right]$$

$$= -\frac{8}{5} L^{-1}\left[\frac{1}{s+1}\right] + \frac{18}{5} L^{-1}\left[\frac{s}{(s^2-2s+2)}\right] + \frac{16}{5} \cdot L^{-1}\left[\frac{1}{s^2-2s+2}\right]$$

$$= -\frac{8}{5} e^{-t} + \frac{18}{5} L^{-1}\left[\frac{s}{s^2-2s+1-1+2}\right] + \frac{16}{5} L^{-1}\left[\frac{1}{s^2-2s+1-1+2}\right]$$

$$= -\frac{8}{5} e^{-t} + \frac{18}{5} L^{-1}\left[\frac{s}{(s-1)^2+1}\right] + \frac{16}{5} L^{-1}\left[\frac{1}{(s-1)^2+1}\right]$$

$$= -\frac{8}{5} e^{-t} + \frac{18}{5} L^{-1}\left[\frac{s-1+1}{(s-1)^2+1}\right] + \frac{16}{5} L^{-1}\left[\frac{1}{(s-1)^2+1}\right]$$

$$= -\frac{8}{5} e^{-t} + \frac{18}{5} L^{-1}\left[\frac{s-1}{(s-1)^2+1}\right] + \frac{18}{5} L^{-1}\left[\frac{1}{(s-1)^2+1}\right] + \frac{16}{5} e^{t}\left[\frac{1}{s^2+1}\right].$$

$$= -\frac{8}{5} e^{-t} + \frac{18}{5} e^{t}\left[\frac{s}{s^2+1}\right] + \frac{18}{5} e^{t}\left[\frac{1}{s^2+1}\right] + \frac{16}{5} e^{t} \sin t.$$

$$= -\frac{8}{5} e^{-t} + \frac{18}{5} e^{t} \cos t + \frac{18}{5} e^{t} \sin t + \frac{16}{5} e^{t} \sin t$$

$$= -\frac{8}{5} e^{-t} + \frac{18}{5} e^{t} \cos t + \left(\frac{18}{5} + \frac{16}{5}\right) \sin t . e^{t}.$$

$$= -\frac{8}{5} e^{-t} + \frac{18}{5} e^{t} \cos t + \frac{34}{5} e^{t} \sin t.$$

3) $L^{-1}\left[\dfrac{1}{s(s^2-2s+3)}\right]$.

<u>Soln</u> :-

$$L^{-1}\left[\frac{1}{s(s^2-2s+3)}\right] = \frac{A}{s} + \frac{Bs+C}{s^2-2s+3} \rightarrow (*)$$

$$1 = A(s^2-2s+3) + (Bs+C)s \rightarrow (1)$$

Put $s=0$.

$$1 = A(0-0+3) + B(0) + C + 0$$

$$1 = A(3)$$

$$A = 1/3$$

Equating the co-effi. of s^2

$$0 = A+B$$

$$0 = 1/3 + B$$

$$B = -1/3$$

Equating the co. eff of s

$$0 = -2A + C$$

$$0 = -2\left(1/3\right) + C$$

$$C = 2/3$$

Sub A, B, C in (*).

$$L^{-1}\left[\frac{1}{s(s^2-2s+3)}\right] = L^{-1}\left[\frac{1/3}{s} + \frac{(-1/3)s + 2/3}{s^2-2s+3}\right]$$

$$= \frac{1}{3}L^{-1}\left[\frac{1}{s}\right] - \frac{1}{3}L^{-1}\left[\frac{s}{s^2-2s+3}\right] +$$

$$\frac{2}{3}\left[\frac{1}{s^2-2s+3}\right]$$

$$= \frac{1}{3}(1) - \frac{1}{3}L^{-1}\left[\frac{s}{s^2-2s+1-1+3}\right] +$$

$$\frac{2}{3}\left[\frac{1}{s^2-2s+1-1+3}\right].$$

$$= \frac{1}{3} - \frac{1}{3}L^{-1}\left[\frac{s}{(s-1)^2+2}\right] + \frac{2}{3}\left[\frac{1}{(s-1)^2+2}\right]$$

$$= \frac{1}{3} - \frac{1}{3}L^{-1}\left[\frac{s-1+1}{(s-1)^2+2}\right] + \frac{2}{3}\left[\frac{1}{(s-1)^2+2}\right]$$

$$= \frac{1}{3} - \frac{1}{3}L^{-1}\left[\frac{s-1}{(s-1)^2+2}\right] - \frac{1}{3}L^{-1}\left[\frac{1}{(s-1)^2+2}\right] +$$

$$\frac{2}{3}e^t\left[\frac{1}{s^2+2}\right].$$

$$= \frac{1}{3} - \frac{1}{3}e^t\left[\frac{s}{s^2+2}\right] - \frac{1}{3}e^t\left[\frac{1}{s^2+2}\right] +$$

$$\frac{2}{3}e^t\left[\frac{1}{s^2+2}\right]$$

$$= \frac{1}{3} - \frac{1}{3}e^t\left[\frac{s}{s^2+(\sqrt{2})^2}\right] - \frac{1}{3}e^t\left[\frac{1}{s^2+(\sqrt{2})^2}\right] +$$

$$\frac{2}{3}e^t\left[\frac{1}{s^2+(\sqrt{2})^2}\right].$$

$$= \frac{1}{3} - \frac{1}{3} e^{t}\cos\sqrt{2}t - \frac{1}{3} e^{t}\sin\sqrt{2}t + \frac{2}{3} e^{t}\frac{\sin\sqrt{2}t}{\sqrt{2}}$$

$$= \frac{1}{3} - \frac{1}{3} e^{t}\cos\sqrt{2}t - \left(\frac{1}{3} - \frac{2}{3}\right) \frac{e^{t}\sin\sqrt{2}t}{\sqrt{2}}$$

$$= \frac{1}{3} - \frac{1}{3} e^{t}\cos\sqrt{2}t - \left[-\frac{1}{3}\right] \frac{e^{t}\sin\sqrt{2}t}{\sqrt{2}}$$

$$= \frac{1}{3} - \frac{1}{3} e^{t}\cos\sqrt{2}t + \frac{1}{3} \frac{e^{t}\sin\sqrt{2}t}{\sqrt{2}}$$

4) $$L^{-1}\left[\frac{1}{(s+1)(s^2+2s+2)}\right]$$

Soln :-

$$L^{-1}\left[\frac{1}{(s+1)(s^2+2s+2)}\right] = \frac{A}{(s+1)} + \frac{Bs+C}{(s^2+2s+2)} \longrightarrow (*)$$

$$1 = A(s^2+2s+2) + (Bs+c)(s+1) \rightarrow (1)$$

$s = -1$

$$1 = A((-1)^2 + 2(-1) + 2) + 0$$

$$1 = A(1-2+2)$$

$$1 = A$$

$$A = 1$$

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Equate the co-eff. of s^2

$$0 = A + B$$

$$0 = 1 + B$$

$$B = -1$$

Equate the co-efficient of s

$$0 = 2A + B + C$$

$$0 = 2(1) + C - 1$$

$$0 = 1 + C$$

$$C = -1$$

Sub. A, B, C in eq (*)

$$L^{-1}\left[\frac{1}{(s-1)(s^2+2s+2)}\right] = L^{-1}\left[\frac{1}{(s+1)} + \frac{(-1)s+(-1)}{s^2+2s+2}\right]$$

$$= L^{-1}\left[\frac{1}{s+1}\right] + L^{-1}\left[\frac{(-1)s}{s^2+2s+2} - \frac{1}{s^2+2s+2}\right]$$

$$= e^{-t} - L^{-1}\left[\frac{s}{s^2+2s+1-1+2}\right] - L^{-1}\left[\frac{1}{s^2+2s+1-1+2}\right]$$

$$= e^{-t} - L^{-1}\left[\frac{s}{(s+1)^2+1}\right] - L^{-1}\left[\frac{1}{(s+1)^2+1}\right]$$

$$= e^{-t} - t^{-1}\left[\frac{s+1-1}{(s+1)^2+1}\right] - L^{-1}\left[\frac{1}{(s+1)^2+1}\right]$$

$$= e^{-t} - t^{-1}\left[\frac{s+1}{(s+1)^2+1}\right] - L^{-1}\left[\frac{-1}{(s+1)^2+1}\right] - e^{-t}\left[\frac{1}{s^2+1}\right]$$

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$$= e^{-t} - e^{-t}\left[\frac{s}{s^2+1}\right] + e^{-t}\left[\frac{1}{s^2+1}\right] - e^{-t}\sin t$$

$$= e^{-t} - e^{-t}\cos t - e^{-t}\sin t - e^{-t}\sin t$$

$$= e^{-t} - e^{-t}\cos t - 2e^{-t}\sin t.$$

) $L^{-1}\left[\frac{s^2}{(s^2-a^2)^2}\right]$.

Soln :-

$$F'(s) = \frac{s^2}{(s^2-a^2)^2}$$

$$\frac{d}{ds}F(s) = \frac{s^2}{(s^2-a^2)^2}$$

$$F(s) = \int \frac{s^2}{(s^2-a^2)^2}\,ds$$

$$u = s^2 - a^2$$

$$du = 2s\,ds$$

$$\frac{du}{2} = s\,ds$$

$$\frac{dus}{2} = s^2 ds$$

$$F(s) = \int \frac{s/2\,du}{u^2}$$

$$= s/2 \int \frac{du}{u^2}$$

$$F(s) = \frac{s}{2}\left(-\frac{1}{4}\right)$$

$$F(s) = \frac{-s}{2(s^2-a^2)}$$

Using the formula,

$$L^{-1}\left[F'(s)\right] = -t\,L^{-1}\left[F(s)\right]$$

$$= -t\,L^{-1}\left[\frac{-s}{2(s^2-a^2)}\right]$$

$$= \frac{t}{2}\,L^{-1}\left[\frac{s}{s^2-a^2}\right]$$

$$= \frac{t}{2}\cosh at.$$

Type : 3

$$L^{-1}\left[F'(s)\right] = -t\,L^{-1}\left[F(s)\right]$$

$$L^{-1}\left[F(s)\right] = -\frac{1}{t}\,L^{-1}\left[F'(s)\right]$$

) Find $L^{-1}\left[\log\left[\frac{s^2+9}{s^2+1}\right]\right]$.

Soln :-

$$F(s) = \log\left(\frac{s^2+9}{s^2+1}\right)$$

$$F(s) = \log(s^2+9) - \log(s^2+1)$$

$$F'(s) = \frac{1}{s^2+9}\,2s - \frac{1}{s^2+1}(2s)$$

$$L^{-1}[F'(s)] = L^{-1}\left[\frac{1}{1+s} - \frac{1}{s}\right]$$

$$= L^{-1}\left[\frac{1}{1+s}\right] - L^{-1}\left[1/s\right]$$

$$= L^{-1}\left[\frac{1}{s+1}\right] - L^{-1}\left[1/s\right]$$

$$= e^{-t}\left[1/s\right] - L^{-1}\left[1/s\right]$$

$$= e^{t} - 1$$

$$L^{-1}[F(s)] = -1/t \; L^{-1}[F'(s)]$$

$$L^{-1}\left[\log\left(\frac{1+s}{s}\right)\right] = 1/t\left[e^{-t} - 1\right]$$

$$= -\frac{e^{-t}}{t} + 1/t$$

$$= \frac{1 - e^{-t}}{t}$$

$$L^{-1}\left[\log\left(\frac{s(s+1)}{s^2+1}\right)\right]$$

Soln :-

$$F(s) = \log\left[\frac{s(s+1)}{s^2+1}\right]$$

$$F(s) = \log(s(s+1)) - \log(s^2+1)$$

$$F'(s) = \frac{1}{s(s+1)} \cdot 2 - \frac{1}{(s^2+1)} 2s$$

$$= 2\left[\frac{s}{s^2+9} - \frac{s}{s^2+1}\right)$$

$$L^{-1}\left[F'(s)\right] = 2L^{-1}\left[\frac{s}{s^2+9} - \frac{s}{s^2+1}\right)$$

$$= 2\left[L^{-1}\left[\frac{s}{s^2+9}\right] - L^{-1}\left[\frac{s}{s^2+1}\right]\right)$$

$$= 2\left[\cos 3t - \cos t\right)$$

Using the formula:

$$L^{-1}\left[F(s)\right] = -\frac{1}{t} L^{-1}\left[F'(s)\right]$$

$$L^{-1}\left[\log\left(\frac{s^2+9}{s^2+1}\right)\right) = -\frac{1}{t}\left[2(\cos 3t - \cos t)\right)$$

$$= \frac{2\left[\cos t - \cos 3t\right)}{t}$$

2) Find $L^{-1}\left[\log\left(\frac{1+s}{s}\right)\right]$.

<u>soln</u>:-

$$F(s) = \log\left[\frac{1+s}{s}\right)$$

$$F(s) = \log(1+s) - \log s$$

$$F'(s) = \frac{1}{1+s} \cdot 1 - \frac{1}{s}$$

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$$= \frac{2s+1}{s^2+s} - \frac{2s}{s^2+1}$$

$$L^{-1}[F'(s)] = L^{-1}\left[\frac{2s+1}{s^2+s} - \frac{2s}{s^2+1}\right]$$

$$= L^{-1}\left[\frac{2s+1}{s^2+s}\right] - L^{-1}\left[\frac{2s}{s^2+1}\right]$$

$$= L^{-1}\left[\frac{2s}{s^2+s}\right] + L^{-1}\left[\frac{1}{s^2+s}\right] - L^{-1}\left[\frac{2s}{s^2+1}\right]$$

$$= L^{-1} \cdot 2\left(\frac{s}{s(s+1)}\right)$$

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Type : 4

Laplace Transform of derivative

$$L[f(t)] = F(s) \text{ then,}$$

i) $L[f'(t)] = sL[f(t)] - f(0)$

ii) $L[f''(t)] = s^2 L[f(t)] - sf(0) - f'(0)$.

iii) $L[f'''(t)] = s^3 L[f(t)] - s^2 f(0) - sf'(0) - f''(0)$.

Solve $\frac{d^2y}{dt^2} - \frac{dy}{dt} - 2y = 0$ given, $y(0) = -2$

$y'(0) = 5$

Soln :-

Given, $\frac{d^2y}{dt^2} - \frac{dy}{dt} - 2y = 0$

$$L(y'') - L(y') - 2L[y] = 0$$

$$s^2 L[y] - sy(0) - y'(0) - [sL(y) - y(0)] - 2L(y) = 0$$

$$s^2 L(y) - sy(0) - y'(0) - sL(y) + y(0) - 2L(y) = 0$$

$$s^2 L(y) - s(-2) - 5 - sL(y) - 2 - 2L(y) = 0$$

$$s^2 L(y) + 2s - 7 - sL(y) - 2L(y) = 0$$

$$L(y)\left[s^2 - s - 2\right] + 2s - 7 = 0$$

$$L(y) = \frac{7-2s}{s^2-s-2}$$

$$y = L^{-1}\left[\frac{7-2s}{s^2-s-2}\right]$$

$$y = L^{-1}\left[\frac{7-2s}{(s-2)(s+1)}\right]$$

$$L^{-1}\left[\frac{7-2s}{(s-2)(s+1)}\right] = \frac{A}{(s-2)} + \frac{B}{(s+1)} \longrightarrow (*)$$

$$7-2s = A(s+1) + B(s-2) \longrightarrow 1$$

put $s = -1$

$$7 - 2(-1) = A(0) + B(-1-2)$$

$$7+2 = -3B$$

$$9 = -3B$$

$$-3B = 9$$

$$B = \cancel{9}^{3}/\cancel{3}$$

$$B = -3$$

put $s=2$

$$7 - 2(2) = A(2+1) + B(0)$$

$$7-4 = A(3)$$

$$3A = 3$$

$$A = 1$$

$$s^2L(y) - 2s - 7 - 4sL(y) + 8 + 5L(y) = \frac{4}{s-3}$$

$$s^2L(y) - 2s - 4sL(y) + 1 + 5L(y) = \frac{4}{s-3}$$

$$L(y)\left[s^2 - 2s + 5\right] = \frac{4}{s-3} + 2s - 1$$

$$L(y)\left[s^2 - 4s + 5\right] = \frac{4 + \cancel{2s}(2s-1)(s-3)}{s-3}$$

$$L(y)\ (s^2 - 4s + 5) = \frac{4 + 2s^2 - 6s - s + 3}{s-3}$$

$$L(y)\ (s^2 - 4s + 5) = \frac{2s^2 - \cancel{5}\,7s + 7}{(s-3)}$$

$$L(y) = \frac{2s^2 - 7s + 7}{(s-3)(s^2 - 4s + 5)}$$

$$y = L^{-1}\left[\frac{2s^2 - 7s + 7}{(s-3)(s^2 - 4s + 5)}\right]$$

$$L^{-1}\left[\frac{2s^2 - 7s + 7}{(s-3)(s^2 - 4s + 5)}\right] = \frac{A}{s-3} + \frac{Bs + C}{s^2 - 4s + 5}$$

$$2s^2 - 7s + 7 = A(s^2 - 4s + 5) + (Bs + C)(s-3)$$

Put $s = 3$

$$2(3)^2 - 7(3) + 7 = A(3^2 - 4(3) + 5) + (B(3) + C)\,0$$

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Sub:. A, B in (*)

$$L^{-1}\left[\frac{7-2s}{(s-2)(s+1)}\right] = L^{-1}\left(\frac{1}{(s-2)} - \frac{3}{(s+1)}\right)$$

$$= L^{-1}\left[\frac{1}{s-2}\right] - L^{-1}\left[\frac{3}{s+1}\right]$$

$$= e^{2t}\left[\frac{1}{s}\right] - e^{-t}\left[\frac{3}{s}\right]$$

$$= e^{2t}(1) - 3e^{-t}(1/s)$$

$$y = e^{2t} - 3e^{-t}.$$

solve $\frac{d^2y}{dt^2} - 4\frac{dy}{dt} + 5y = 4e^{3t}$ given,

$y(0) = 2$, $y'(0) = 7$

<u>soln</u> :-

Given,

$$\frac{d^2y}{dt^2} - 4\frac{dy}{dt} + 5y = 4e^{3t}$$

$$L(y'') - 4L(y') + 5L(y) = 4L(e^{3t})$$

$$s^2L(y) - sy(0) - y'(0) - 4[sL(y) - y(0)].$$

$$5L(y) = 4/s-3$$

$$s^2L[y] - s(2) - 7 - 4sL(y) + 4(2) + 5L(y) =$$

$$4/s-3.$$

$$2\times 9-21+7 = A(9-12+5)$$

$$18-21+7 = A(-12+14)$$

$$-21+25 = A(-12+14)$$

$$+4 = 2A$$

$$2A = 4$$

$$A = \cancel{4}^{2}/\cancel{2}$$

$$A = 2$$

Put $s=0$

$$2(0)-7(0)+7 = A(\cancel{0}-0+5)+(0+C)(0-3)$$

$$7 = 5-3C$$

$$7-5 = -3C$$

$$-\frac{2}{3} = C$$

$$L^{-1}\left(\frac{2s^2-7s+7}{(s-3)(s^2-4s+5)}\right) = \frac{2}{(s-3)} + \frac{Bs+C}{s^2-4s+5}$$

$$2L^{-1}\left[\frac{2}{s-3}\right] + L^{-1}\left[\frac{Bs+C}{s^2\cancel{+4s}-4s+5}\right]$$

$$= 2e^{3t} + L^{-1}\left[\frac{1}{s^2-4s+5}\right]$$

$$= 2e^{3t} + L^{-1}\left[\frac{1}{(s-2)^2+1^2}\right]$$

$$= 2e^{3t} + e^{2t}\sin t$$

$$= \frac{2}{s-3} + \frac{0(s)+1}{(s^2+4s+5)}$$

$$= \frac{2}{s-3} + \frac{1}{s^2+4s+5}$$

$$= 2\left(\frac{1}{s-3}\right) + \frac{1}{s^2-4s+4-4+5}$$

$$= 2e^{3t}\left(\frac{1}{3}\right) + \frac{1}{(s-2)^2+1}$$

$$= 2e^{3t} + \sin at.$$