COMPLEX ANALYSIS

SUBJECT CODE : 16SCCMM13

UNIT : III

COMPLEX INTEGRATION

1. Evaluate $\int f(z)\, dz$ where $f(z) = y - x - i3x^2$ and l is segment from $z=0$ to $z=1+i$

Soln :-

the equ of the line segment c joining $z=0$ to $z=1+i$

given by $y = x$

$\therefore$ the Paramtric equ c canbe taken as

$x = t$, $y = t$ where $0 \leq t \leq 1$

$z(t) = x(t) + iy(t)$

$= t + it$

$= t(1+i)$

$z'(t) = (1+i)$

Now: $f(z(t)) = t - t - i3t^2$

W.K.T

$$\int f(z)\, dz = \int_0^1 f(z(t))\, z'(t)\, dt$$

$$= \int_0^1 (-i\,3t^2)(1+i)\,dt$$

$$= -3i(1+i)\int_0^1 t^2\,dt$$

$$= -3i(1+i)\left(\frac{t^3}{3}\right)_0^1$$

$$= -3i(1+i)\left(\frac{1}{3}\right)$$

$$= 1-i$$

2. Prove that $\int_c \frac{dz}{(z-a)^n} = \begin{cases} 0 & \text{if } n \neq 1 \\ 2\pi i & \text{if } n = 1 \end{cases}$ where C is the circle with centre a and radius r and $n \in \mathbb{Z}$.

Soln/.

The parameters equation of the circle C is given by $z - a = re^{it}$, $0 \leq t \leq 2\pi$.

$$z = a + re^{it}, \quad 0 \leq t \leq 2\pi$$

$$dz = ire^{it}\,dt$$

Now,

$n \neq 1$

$$\int_c \frac{dz}{(z-a)^n} = \int_0^{2\pi} \frac{i re^{it}}{(re^{it})^n} dt$$

$$= \frac{i}{r^{n-1}} \int_0^{2\pi} e^{it} \cdot e^{-itn} dt$$

$$= \frac{i}{r^{n-1}} \int_0^{2\pi} e^{i(1-n)t} dt$$

$$= \frac{i}{r^{n-1}} \int_0^{2\pi} e^{i(1-n)t} dt$$

$$= \frac{i}{r^{n-1}} \left[\frac{e^{i(1-n)t}}{i(1-n)} \right]_0^{2\pi}$$

$$= \frac{1}{r^{n-1}(1-n)} (e^{i(1-n)t})_0^{2\pi}$$

$$= \frac{1}{r^{n-1}(1-n)} (e^{i(1-n)2\pi} - e^{i(1-n)0})$$

$$= \frac{1}{r^{n-1}(1-n)} [1-1]$$

$$= 0$$

$$\int_c \frac{dz}{(z-a)^n} = 0$$

(case ii)

if $n = 1$

$$\int_c \frac{dz}{(z-a)^n} = \int_0^{2\pi} \frac{ire^{it}}{re^{it}}\, dt$$

$$= i\int_0^{2\pi} dt = i(t)_0^{2\pi}$$

$$= 2\pi i$$

Prblm : 3

Evaluate $\int_c \frac{e^z}{z^2+4}$ where c is positively oriented circle $|z-i| = 2$

<u>soln</u> :-

$$f(z_0) = \frac{1}{2\pi i}\int_c \frac{f(z)}{z-z_0}\, dz$$

$$\int \frac{f(z)}{z-z_0}\, dz = 2\pi i f(z_0)$$

Here :

$f(z) = e^z \Rightarrow f(2i) = e^{2i}$

$f(-2i) = e^{-2i}$

Now :

$$\frac{1}{z^2+4} = \frac{1}{z^2-(2i)^2} = \frac{1}{(z-2i)(z+2i)}.$$

$$\frac{1}{(z-2i)(z+2i)} = \frac{A}{(z-2i)} + \frac{B}{(z+2i)}$$

$$\frac{1}{(z-2i)(z+2i)} = \frac{A(z+2i)+B(z-2i)}{(z-2i)(z+2i)}$$

$$1 = A(z+2i) + B(z-2i)$$

Put : $z = 2i$

$$1 = A(4i)$$

$$n = 1/4i$$

$$z = -2i$$

$$1 = B(-4i)$$

$$B = 1/4i$$

$$\therefore \frac{1}{z^2+4} = \frac{1}{4i(z-2i)} - \frac{1}{4i(z+2i)}$$

$$= \frac{1}{4i}\left[\frac{1}{z-2i} - \frac{1}{z+2i}\right]$$

Now $2i$ lies inside c and $-2i$ lies integral

$\therefore$ by cauchs Integral

$$\int \frac{e^z}{z^2+4} = \frac{1}{4i}\left[\int \frac{e^z}{z-2i}\,dz - \frac{e^z}{z+2i}\,dz\right]$$

$$= \frac{1}{4i}\left[2\pi i f(2i) - 2\pi i f(-2i)\right]$$

$$= \frac{1}{4i}\left[2\pi i\, e^{2i} - 0\right]$$

$$= \frac{1}{4i}\left[2\pi i\, e^{2i}\right]$$

$$= \tfrac{1}{2}\,\pi\, e^{2i}$$

$$= \tfrac{\pi}{2}\, e^{2i}$$

Prblm : 2

Evaluate $\int_c \frac{z\,dz}{z^2-1}$ where c is the positively oriented circle $|z| = 2$

Proof :

$$\int_c \frac{z\,dz}{z^2-1} = |z| = 2$$

$$f(z_0) = \frac{1}{2\pi i}\int_c \frac{f(z)}{z-z_0}\,dz$$

$$2\pi i f(z_0) = \int_c \frac{f(z)}{z-z_0}\,dz$$

$$f(z) = z$$

$$\frac{1}{z^2-1} = \frac{1}{(z-1)(z+1)}$$

$$\frac{1}{(z-1)(z+1)} = \frac{A}{(z-1)} + \frac{B}{(z+1)}$$

$$1 = A(z+1) + B(z+1)$$

put $z = 1$, $z = -1$

$1 = A(2)$ $\qquad$ $1 = B(-2)$

$A = 1/2$ $\qquad$ $B = -1/2$

$$\frac{1}{z^2-1} = \frac{1}{2(z-1)} - \frac{1}{2(z+1)}$$

$$= \frac{1}{2}\left[\frac{1}{z-1} - \frac{1}{z+1}\right]$$

1 and -1 lies in interior of c

$$\int \frac{z\,dz}{z^2-1} = \frac{1}{2}\left[\int_c \frac{z\,dz}{(z-1)} - \int_c \frac{z\,dz}{z+1}\right]$$

$$= \frac{1}{2}\left[2\pi i f(1) - 2\pi i f(-1)\right]$$

$$= \frac{1}{2}(2\pi i) - \frac{1}{2}(-2\pi i)$$

$$= \frac{1}{2} \cdot 4\pi i$$

$$= 2\pi i$$

Prblm : 4

Evaluate $\int_c \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)}$ where c is

a circle $|z| = 3$

<u>proof</u> :

W.K.T.

$$\int_C \frac{f(z)}{z-z_0}\,dz = 2\pi i\, f(z_0)$$

here,

$$f(z) = \sin \pi z^2 + \cos \pi z^2$$

$$\frac{1}{(z-1)(z-2)} = \frac{A}{(z-1)} + \frac{B}{(z-2)}$$

$$1 = A(z-2) + B(z-1)$$

put $z = 2$, $z = 1$

$1 = B$ $\quad$ $1 = -A$

$B = 1$ $\quad$ $A = -1$

$$\frac{1}{(z-1)(z-2)} = \frac{-1}{(z-1)} + \frac{1}{(z-2)}$$

$$= \frac{1}{(z-2)} - \frac{1}{(z-1)}$$

Now: 2 & 1 lies in the interior of c.

$\therefore$ By cauchy's integral

$$\int \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)} dz = \left[\int \frac{\sin \pi z^2 + \cos \pi z^2}{z^2} dz \right.$$

$$\left. \int \frac{\sin \pi z^2 + \cos \pi z^2}{z-1} dz \right]$$

$$\int \frac{f(z)}{z - z_0} dz = 2\pi i \, f(z_0)$$

$$f(z) = \sin 4\pi + \cos 4\pi$$

$$f(z) = \sin 4\pi + \cos 4\pi$$

$$= -2\pi i (\sin \pi + \cos \pi) + 2\pi i$$

$$(\sin 4\pi + \cos 4\pi)$$

$$= -2\pi i(-1) + 2\pi i$$

$$= 4\pi i$$

5. Prblem : 5

let c denote by boundary of the square whose sides the along the lines $x = \pm 2$ and $y = \pm 2$

where c is described in the positive sense.

i) $\int_c \frac{z\,dz}{2z+1}$ ii) $\int \frac{\cos z}{z(z^2+8)}$

<u>Proof:</u>

i) $\int \frac{z\,dz}{z+1/2}$

$f(z) = z$

$= \frac{1}{2}\int \frac{z\,dz}{z+1/2}$

$-1/2$ lies inside the interior of c

By using integral formula

W.K.T. $\int \frac{f(z)}{z-z_0}\,dz = 2\pi i\, f(z_0)$

$\frac{1}{2}\int \frac{z\,dz}{z+1/2} = \frac{1}{2}\, 2\pi i\, f(-1/2)$

$= \frac{1}{2}\, 2\pi i(-1/2)$

$= \pi i/2$

Here, $f(0) = \dfrac{\cos(0)}{0^2+9} = 1/8$

$= 2\pi i f(0)$

$= 2\pi i(1/8)$

$= \pi i/4$

6. Evaluate $\dfrac{z\,dz}{(9-z^2)(z+i)}$ where c is the circle $|z| = 2$ taken in the positive sense

<u>**Proof:**</u>

$$\int \frac{f(z)}{(z-z_0)}\,dz = 2\pi i f(z_0)$$

$f(z) = z$

$$\frac{1}{(9-z^2)(z+1)} = \frac{A}{(z-3)} + \frac{B}{(z+3)} + \frac{C}{z+1}$$

$$\frac{1}{(z-3)(z+3)(z+i)} = \frac{A(z+3)(z+i)+B(z-3)(z+i)+C(z-3)(z+3)}{(z-3)(z+3)(z+i)}$$

$$1 = A(z+3)(z-i) + B(z-3)(z+i) + C(z-3)(z+3)$$

$z = -3$

$1 = B(-6)(-3+i)$

$1 = B(18-6i)$

$B = \dfrac{1}{18-6i}$

$z = 3$

$1 = A(6)(3+i)$

$1 = A(18+6i)$

$A = \dfrac{1}{18+6i}$

put $z = -i$

$1 = C(-i-3)(-i+3)$

$= C(-i^2-3i+3i-9)$

$= C(-1-9)$

$C = 1/_{-10}$

$$\frac{1}{(z-3)(z+3)(z+i)} = \frac{\frac{1}{18+6i}}{z-3} + \frac{1/18+6i}{z+3} + \frac{-1/16}{z+i}$$

Now $-i$ lies in the interior of c

3, -3 does not lies in interior of c

$f(-i) = -i$

$$\int \frac{z\,dz}{(9-z^2)(z+i)} = \int \frac{z\,dz}{(9-z)^2} + \frac{1}{-10} \int \frac{z\,dz}{z+i}$$

$$= 0 + \left(-1/10\right) 2\pi i\, f(-i)$$

$$= 2\pi i \left(-i/10\right).$$

$$= 2\pi \frac{(-1^2)}{-10}$$

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$$= \frac{2\pi}{10}$$

$$= \pi/5$$

7. $$\int_c \frac{\sin 2z}{(z-\pi i/4)^4} dz \quad |z| = 1$$

<u>Soln</u>:

given : $$\int_c \frac{\sin 2z}{\left(z - \frac{\pi i}{4}\right)^4} dz$$

Hence $f(z) = \sin 2z$

$f'(z) = 2\cos 2z$, $f''(z) = -4\sin 2z$

$f'''(z) = -8\cos 2z$

W.K.T

$$\int_c \frac{f(z)}{(z-z_0)^{n+1}} dz$$

$$= \frac{2\pi i}{n!} f^n(z_0)$$

$$\int_c \frac{\sin 2z}{(z-\pi/4)^4} dz = \frac{2\pi i}{3!} f''(\pi i/4)$$

$$= \frac{2\pi i}{6} f'''\left(\frac{\pi i}{4}\right)$$

$$\left[f'''(\pi i/4) = -8\cos 2(\pi i/4)\right.$$

$$= -8\cos \pi i/2$$

$$\left.= -8\cosh \pi/2\right]$$

$$= \frac{\pi i}{3} \cdot 8\cosh(\pi/2)$$

$$= \frac{8\pi i}{3}\cosh(\pi/2)$$

$\int_c \frac{(e^z + z\sinh z)}{(z-ni)^2}\,dz$ where c is circle $|z| = 4$

soln:

given:

$$\int_c \frac{(e^z + z\sinh z}{(z-ni)^2}\,dz$$

Here $f(z) = e^z + z \sinh z$

W.K.T

by higher derivative formula

$$\int_c \frac{f(x)}{(z-z_0)^{n+1}} dz = \frac{2\pi i}{n!} f^n(z_0)$$

there $f'(z) = e^z + z\cosh z + \sinh z$

$f'(\pi i) = e^{\pi i} + \pi i \cosh \pi i + \sinh \pi i$

Now πi lies inside on c

$$\int_c \frac{(e^z + z \sinh z)}{(z-\pi i)^2} dz = \frac{2\pi i}{1!} f'(z_0)$$

$$= \frac{2\pi i}{1!} f'(\pi i)$$

$= 2\pi i (e^{\pi i} + \pi i \cosh \pi i \cdot$

$\sinh \pi i)$

$= -2\pi i (1 + \pi i)$