DIFFERENTIAL EQUATION AND LAPLACE TRANSFORMS

subject code : 16SCCMM3

unit : V

Laplace Transforms.

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LAPLACE TRANSFORMS :

Let $F(t)$ be a of the variable "t" which is defined for all the values of "t". Let "s" be the a constant.

If $\int_0^\infty e^{-st} . f(t)\, dt$ exists and is equal to $F(s)$ then.

$$F(s) = \int_0^\infty e^{-st} f(t)\, dt$$

$F(s)$ is called the laplaces transfer of (t) and it denoted by the symbol $t\,(f(t))$

$$L[f(at)] = 1/a\, F(s/a)$$

NOTE 1 :

Here the operation "t" is called which transform the function $F(t)$

NOTE 2 :

$$s \xrightarrow{\lim} \alpha \quad F(s) = 0$$

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1. Laplace transforms of Elementray Function

Laplace transform of [1]

(or)

$L[F(t)] = 1$

$F[t] = 1$

$$L[F(t)] = \int_0^\infty e^{-st} F(t)\, dt$$

$$= \int_0^\infty e^{-st} (1)\, dt$$

$$= \int_0^\infty e^{-st}\, dt$$

$$= \left[\frac{e^{-st}}{-s}\right]_0^\infty$$

$$= \left[\frac{e^{-s\alpha}}{-s} - \frac{e^0}{-s}\right]$$

$$= \frac{e^{-s\alpha}}{-s} + \frac{e^0}{s}$$

$$= 0 + 1/s$$

$$F(s) = 1/s$$

$L[1] = 1/s$ if $s > 0$

2. Laplace transforms of [t]

$$F[t] = t$$

$$L[F(t)] = \int_0^\infty e^{-st} f(t)\,dt$$

$$= \int_0^\infty e^{-st}\, t\,dt$$

$$\int u\,dv = uv - \int v\,du$$

$$u = t,\ \int dv = \int e^{-st}\,dt\ ;\ du = dt\ ;\ v = \frac{e^{-st}}{-s}$$

$$= \left[\frac{t e^{-st}}{-s}\right]_0^\infty - \int_0^\infty \frac{e^{-st}}{-s}\,dt$$

$$= 0 + 1/s \int_0^\infty e^{-st}\,dt$$

$$= 0 + 1/s \left[\frac{e^{-st}}{s}\right]_0^\infty$$

$$= 1/s\left[0 + 1/s\right]$$

$$= 1/s \cdot 1/s$$

$$= 1/s^2$$

$$L(t) = 1/s^2 \text{ if } s > 0$$

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3. Laplace transforms of $[t^3]$

$$F[t] = t^3$$

$$L\left[f(t)\right] = \int_0^\infty e^{-st} f(t)\, dt$$

$$= \int_0^\infty e^{-st} t^3\, dt$$

$$\int u\,dv = uv - \int v\,du$$

$$u = t^3 \; ; \; dv = e^{-st} dt \; ; \; du = 3t^2 dt \; ; \; v = \frac{e^{-st}}{-s}$$

$$= \left[t^3 \frac{e^{-st}}{-s}\right]_0^\infty - \int_0^\infty e^{-st}\, 3t^2\, dt$$

$$= 0 + 3/s \int_0^\infty e^{-st}\, t^2\, dt$$

We know that,

$$= \int_0^\infty t^2\, dt = 2/s^2$$

$$= 3/s \left[\frac{2}{s^3}\right]$$

$$L\left[t^3\right] = \frac{6}{s^4}$$

$$L\left[t^3\right] = \frac{3!}{s^4} \text{ if } s > 0$$

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4. Laplace transform of $[t^n]$

$$f(t) = t$$

$$L[f(t)] = \int_0^\infty e^{-st} f(t)\, dt$$

$$= \int_0^\infty e^{-st}\, t\, dt$$

$$\int u\,dv = uv - \int v\,du$$

$u = t$; $dv = e^{-st} dt$; $du = dt$; $v = \frac{e^{-st}}{-s}$

$$= \left[t \frac{e^{-st}}{-s}\right]_0^\infty - \int_0^\infty \frac{e^{-st}}{s} dt$$

$$= 0 + 1/s \left[\frac{e^{-st}}{-s}\right]_0^\infty$$

$$= 0 + 1/s\,[1/s]$$

$$L(t) = 1/s^2$$

$$L(t^2) = \frac{2!}{s^3}$$

$$L[t^3] = \frac{3!}{s^4}$$

$$L[t^n] = \frac{n!}{s^{n+1}}$$

5 Laplace transform of $[e^{-at}]$

$$F(t) = e^{-at}$$

$$L[f(t)] = \int_0^\infty e^{-st} f(t)\,dt$$

$$= \int_0^\infty e^{-st} e^{-at}\,dt$$

$$= \int_0^\infty e^{-(s+a)t}\,dt$$

$$= \left[\frac{e^{-(s+a)t}}{-(s+a)}\right]_0^\infty$$

$$= \frac{e^{(s+a)\infty}}{-(s+a)} + \frac{e^{-(s+a)}}{s+a}$$

$$= 0 + \frac{e^0}{s+a}$$

$$L[e^{-at}] = \frac{1}{s+a} \text{ if } s<0$$

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FORMULA :

1 $\int e^{ax} \sin bx = \frac{e^{ax}}{a^2+b^2}\left[a \sin bx - b\cos bx\right]$

2 $\int e^{ax} \cos bx = \frac{e^{ax}}{a^2+b^2}\left[a \cos bx + b\sin bx\right]$

6 Laplace transform of sinat

$$F(t) = \sin at$$

$$L\left[f(t)\right] = \int_0^\infty e^{-st} f(t)\, dt$$

$$= \int_0^\infty e^{-st} \sin at\, dt$$

$$\int e^{ax} \sin bx = \frac{e^{ax}}{a^2+b^2}\left[a \sin bx - b\cos bx\right]$$

$a = -s$; $b = a$; $x = t$

$$\int_0^\infty e^{-st} \sin at\, dt = \frac{e^{-st}}{s^2+a^2} - \left[s \sin at - a\cos at\right]_0^\infty$$

$$= 0 - \frac{e^0}{s^2+a^2}\left[-s\sin a(0) - a\cos a(0)\right]$$

$$= -\frac{1}{s^2+a^2}\left[-a(1)\right]$$

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$$L(\sin at) = \frac{a}{s^2 + a^2} \text{ if } s > 0$$

7 Laplace transform of cost at

$$f(t) = \cos at$$

$$L[F(t)] = \int_0^\infty e^{-st} f(t)\, dt$$

$$= \int_0^\infty e^{-st} \cos at\, dt$$

$$= \int_0^\infty e^{-st} \left[\frac{e^{at} + e}{2}\right]^{-at} dt$$

$$= \frac{1}{2}\int_0^\infty e^{-st} (e^{at} + e^{-at})\, dt$$

$$= \frac{1}{2}\left[\int_0^\infty e^{-st} e^{at} + \int_0^\infty e^{-st} e^{-at}\, dt\right.$$

$$= \frac{1}{2}\left[\int_0^\infty e(s-a)t + \int_0^\infty e^{-(s+a)t}\, dt\right.$$

$$= \frac{1}{2}\left[\frac{1}{s-a} + \frac{1}{s+a}\right]$$

$$= \frac{1}{2}\left[\frac{s+a+s-a}{s^2 - a^2}\right]$$

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$$= \frac{1}{2}\left[\frac{2s}{s^2 - a^2}\right]$$

$$L[\cos at] = \frac{s}{s^2 - a^2} \quad \text{if } s > 0$$

8 Laplace transform of $\cos^3 at$

$$f(t) = \cos^3 t$$

$$\cos 3t = 4\cos^3 t - 3\cos t$$

$$4\cos^3 t = \cos 3t + 3\cos t$$

$$\cos^3 t = \frac{\cos 3t + 3\cos t}{4}$$

$$L[\cos^3 t] = L\left[\frac{\cos 3t + 3\cos t}{4}\right]$$

$$= \frac{1}{4} L(\cos 3t + 3\cos t)$$

$$= \frac{1}{4}\left[\frac{s}{s^2 + 3^2} + \frac{3s}{s^2 + 1^2}\right]$$

$$= \frac{1}{4}\left[\frac{s}{s^2 + 9} + \frac{3s}{s^2 + 1^2}\right]$$

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$$L(\cos^3 t) = \frac{3}{4(s^2+9)} + \frac{3s}{4(s^2+1)}$$

9 Laplace transform of $L\left[e^{2t} + 3e^{-5t}\right]$

$$F(t) = e^{2t} + 3e^{-5t}$$

$$= \int_0^\infty e^{-st} e^{2t}\,dt + 3\int_0^\infty e^{-st}\, e^{-5t}\,dt$$

$$L\left[e^{2t} + 3e^{-5t}\right] = \frac{1}{s-2} + \frac{3}{s+5}$$

10 Find the Laplace tranform of $L\left[\sin(\omega t + \alpha)\right]$ is a constant of $(0, \alpha)$

$$f(t) = \sin(\omega t + 2)$$

$$\sin(A+B) = \sin A \cos B + \cos A \sin B$$

$$\sin(A+B) = \sin \omega t \cos \alpha + \cos \omega t \sin \alpha$$

$$L\left[\sin(\omega t + \alpha)\right] = L\left[\sin \omega t \cos \alpha + \cos \omega t \sin \alpha\right]$$

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$$= \cos\alpha \, L\left[\sin\omega t\right] + \sin L\,(\cos\omega t)$$

$$= \cos\alpha \left(\frac{\omega}{s^2+\omega^2}\right) + \sin\alpha \left(\frac{s}{s^2+\omega^2}\right)$$

$$L\left[\sin(\omega t+\alpha)\right] = \frac{\omega\cos\alpha}{s^2+\omega^2} + \frac{s\sin\alpha}{s^2+\omega^2}$$

11. $L\left[\sin 3t \cdot \cos t\right]$

$$\sin A\cos B = 1/2\left[\sin(A+B) + \sin(A-B)\right]$$

$$\sin 3t\cos t = 1/2\left[\sin(3t+t) + \sin(3t-t)\right]$$

$$= 1/2\left(\sin 4t + \sin 2t\right)$$

$$L(\sin 3t \cdot \cos t) = 1/2\int_0^{\infty} e^{-st}\sin 4t\,dt + 1/2\int_0^{\infty} e^{\sin 2t}\,dt$$

$$= 1/2\left(\frac{4}{s^2+a^2}\right) + 1/2\left(\frac{2}{s^2+4}\right)$$

$$L(\sin 3t \cdot \cos t) = \frac{2}{s^2+16} + \frac{1}{s^2+4}$$

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TYPE - II :

1. $L\left[f(t)\right] = f(s)$ then $L\left[e^{-at} f(t)\right] = f(s+a)$

$$L\left[f(t)\right] = \int_0^\infty e^{-st} f(t)\,dt$$

$$= f(s)$$

$$L\left[e^{at} f(t)\right] = \int_0^\infty e^{-st} e^{at} f(t)\,dt$$

$$= \int_0^\infty e^{-(s-a)t} f(t)\,dt$$

$$L\left[e^{at} f(t)\right] = f(s-a)$$

2. $L\left[e^{t} \sin t \cos t\right]$

$$\sin t \cos t = \tfrac{1}{2}\left[\sin 2t + \sin 0\right]$$

$$= \tfrac{1}{2} \sin 2t$$

$$L\left[\sin t \cos t\right] = \tfrac{1}{2} L(\sin 2t)$$

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$$L(\sin t \cos t) = \frac{1}{2}\left(\frac{2}{s^2+4}\right)$$

$$L(e^t \sin t \cos t) = \frac{1}{(s^2-1)+4}$$

TYPE - III

1. $L\left[f(t)\right] = \begin{cases} 0, & 0 < t < 1 \\ t, & 1 < t < 2 \\ 0, & t < 2 \end{cases}$

$$L\left[f(t)\right] = \int_0^\infty e^{-st} f(t)\,dt$$

$$= \int_0^1 e^{-st}(0)\,dt + \int_1^2 e^{-st} t\,dt + \int_0^2 e^{-st}(0)\,dt$$

$$= 0 + \int_1^2 e^{-st} t\,dt + 0$$

$$= \int_1^2 e^{-st} t\,dt$$

$u = t \quad ; \quad dv = e^{-st} \quad ; \quad du = dt \quad ; \quad v = \frac{e^{-st}}{-s}$

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$$= \left[\frac{t e^{-st}}{-s}\right]_1^2 - \int_1^2 \frac{e^{-st}}{s}\,dt$$

$$= \frac{2e^{-2s}}{-s} + \frac{e^{-s}}{s} + 1/s\left[\frac{e^{-st}}{-s}\right]_1^2$$

TYPE - <u>IV</u>

1. $L[t^2 e^t \sin t]$

$$L[\sin t] = \frac{1}{s^2+1}$$

$$L[e^t \sin t] = \frac{1}{(s-1)^2+1}$$

$$= \frac{1}{s^2+1-2s+1}$$

$$= \frac{1}{s^2-2s+2}$$

$$L[t^2 e^t \sin t] = \frac{d^2}{ds^2}\left[\frac{1}{s^2-2s+2}\right]$$

$$= \frac{d}{ds}\left[\frac{d}{ds}\ \frac{1}{s^2+2s+2}\right]$$

$$= \frac{d}{ds}\left[\frac{s^2 - 2s + 2(0) - 1(2s-2)}{(s^2-2s+2)^2}\right]$$

$$= \frac{d}{ds}\left[\frac{-2s+2}{(s^2-2s+2)^2}\right]$$

$u = -2s+2$; $du = -2$; $v = (s^2-2s+2)^2$;

$dv = 2(s^2-2s+2)(2s-2)$

$$v^2 = (s^2-2s+2)^4$$

$$= \frac{(s^2-2s+2)^2(-2) - (-2s+2)(2)(s^2-2s+2)(2s-2)}{(s^2-2s+2)^4}$$

$$= \frac{(s^2-2s+2)(-2) - 2(-2s+2)(2s-2)}{(s^2-2s+2)^3}$$

$$= \frac{-2s^3+4s-4-2(-4s^2+4s+4s-4)}{(s^2-2s+2)^3}$$

$$= \frac{6s^2-12s+4}{(s^2-2s+2)^3}$$

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$$L(t^2 e^t \sin t) = \frac{2[3s^2 - 6s + 2]}{(s^2 - 2s + 2)^3}$$

TYPE - V

1. Find $L\left[\frac{e^{3t} - e^{-2t}}{t}\right]$

$$L\left[e^{3t} - e^{-2t}\right] = \frac{1}{s-3} - \frac{1}{s+2}$$

$$L\left[\frac{e^{3t} - e^{-2t}}{t}\right] = \int_s^\infty \left[\frac{1}{s-3} - \frac{1}{s+2}\right] dt$$

$$= \left[\log(s-3) - \log(s+2)\right]_s^\infty$$

$$= \log\left[\left(\frac{s-3}{s+2}\right)\right]_s^\infty$$

$$= -\log\left(\frac{s-3}{s+2}\right)$$

$$= -\left[\log(s-3) - \log(s+2)\right]$$

$$L\left[\frac{e^{3t} - e^{-2t}}{t}\right] = \log\left[\frac{s+2}{s-3}\right]$$

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2. $L\left[\frac{\sin 3t \cos t}{t}\right]$

$$\sin A \cos B = \frac{1}{2}\left[\sin(A+B) + \sin(A-B)\right]$$

$$\sin 3t \cos t = \frac{1}{2}\left[\sin 4t + \sin 2t\right]$$

$$L\left[\sin 3t \cos t\right] = \frac{1}{2}\left[\frac{4}{s^2+16} + \frac{2}{s^2+4}\right]$$

$$L\left[\frac{\sin 3t \cos t}{t}\right] = \frac{1}{2}\int_s^\infty \left[\frac{4}{s^2+16} + \frac{2}{s^2+4}\right] ds$$

$$= \frac{2}{2}\int_s^\infty \left[\frac{2}{s^2+16} + \frac{1}{s^2+4}\right] ds$$

$$\int \frac{1}{x^2+a^2} dx = \frac{1}{a}\tan^{-1}(x/a)$$

$$= \int_0^\infty \left[\frac{s}{s^2+16} + \frac{1}{s^2+4}\right] ds$$

$$= \left[2\left[\frac{1}{4}\tan(s/4)\right] + \frac{1}{2}\tan^{-1}(s/2)\right]_s^\infty$$

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$$= \frac{1}{2}\tan^{-1}\left(\frac{\alpha}{4}\right) - \frac{1}{2}\tan^{-1}\left(\frac{s}{4}\right) + \frac{1}{2}\tan^{-1}\left(\frac{s}{2}\right)$$

$$= \frac{1}{2}\left[\frac{\pi}{2} + \frac{\pi}{2}\right] - \frac{1}{2}\tan^{-1}\left(\frac{s}{4}\right) + \frac{1}{2}\left(\frac{\pi}{4}\right) - \frac{1}{2}\tan^{-1}\left(\frac{s}{2}\right)$$

$$= \frac{1}{2}\left[\frac{2\pi}{2}\right] - \frac{1}{2}\tan^{-1}\left(\frac{s}{4}\right) - \frac{1}{2}\tan^{-1}\left(\frac{s}{2}\right)$$

$$= \frac{1}{2}\left[\pi - \tan^{-1}\left(\frac{s}{4}\right) - \tan^{-1}\left(\frac{s}{2}\right)\right]$$

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Laplace Inverse transforms:

1) $L^{-1}\left[\frac{1}{s-a}\right] = e^{at}$

2) $L^{-1}\left[\frac{1}{s+a}\right] = e^{-at}$

3) $L^{-1}\left[\frac{1}{s^2+a^2}\right] = \frac{\sin at}{a}$

4) $L^{-1}\left[\frac{s}{s^2+a^2}\right] = \cos at$

5) $L^{-1}\left[\frac{s}{s^2-a^2}\right] = \cosh at$

6) $L^{-1}\left[1/s\right] = 1$

7) $L^{-1}\left[\frac{1}{s^2-a^2}\right] = \frac{\sinh at}{a}$

8) $L^{-1}\left[\frac{1}{s^2}\right] = t$

9) $L^{-1}\left[1/s^3\right] = t^2/2!$

10) $L^{-1}\left[\frac{1}{s^6}\right] = t^5/5!$

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11) $L^{-1}\left[\frac{1}{s^n+1}\right] = \frac{t^n}{n!}$

Find $L^{-1}\left[\frac{1}{s-3} + 1/s + \frac{s}{s^2-4}\right]$

Soln:

$$L^{-1}\left[\frac{1}{s-3} + \frac{1}{s} + \frac{s}{s^2-4}\right] = L^{-1}\left(\frac{1}{s-3}\right) + L^{-1}(1/s) + L^{-1}\left(\frac{s}{s^2-4}\right)$$

$$= e^{3t} + 1 + \cosh 2t$$

Find $L^{-1}\left[\frac{1}{s^2} + \frac{1}{s+4} + \frac{1}{s^2+4} + \frac{s}{s^2-9}\right]$

Soln:

$$= L^{-1}\left[\frac{1}{s^2}\right] + L^{-1}\left[\frac{1}{s+4}\right] + L^{-1}\left[\frac{1}{s^2+4}\right] + L^{-1}\left[\frac{s}{s^2-9}\right]$$

$$= t + e^{-4t} + \frac{\sin 2t}{2} + \cosh 3t$$

$L^{-1}\left[\frac{1}{(s+1)^2}\right]$

Soln:

$$L^{-1}\left[\frac{1}{(s+1)^2}\right] = e^{-t} L^{-1}\left[\frac{1}{s^2}\right]$$

$$= e^{-t} t$$

$$L^{-1}\left[\frac{1}{(s+1)^2+1}\right] = e^{-t} L^{-1}\left[\frac{1}{s^2+1}\right]$$

$$= e^{-t} \sin t$$

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$$L^{-1}\left[\frac{3}{(s+2)^2+1}\right]$$

Soln:

$$L^{-1}\left[\frac{3}{(s+2)^2+1}\right] = L^{-1}\left[\frac{s+2-2}{(s+2)^2+1}\right]$$

$$= L^{-1}\left[\frac{s+2}{(s+2)^2+1}\right] - 2L^{-1}\left[\frac{1}{(s+2)^2+1}\right]$$

$$= L^{-1}\left[\frac{s+2}{(s+2)^2+1}\right] - 2L^{-1}\left[\frac{1}{(s+2)^2+1}\right]$$

$$= L^{-1}\left[\frac{s+2}{(s+2)^2+1}\right] - 2L^{-1}\left[\frac{1}{(s+2)^2+1}\right]$$

$$= e^{-2t} L^{-1}\left[\frac{s}{s^2+1}\right] - 2e^{-2t}\left[\frac{1}{s^2+1}\right]$$

$$= e^{-2t}\cos t - 2e^{-2t}\sin t.$$

Find $L^{-1}\left[\frac{s-3}{(s-3)^2+4}\right]$

Soln:

$$L^{-1}\left[\frac{s-3}{(s-3)^2+4}\right] = e^{3t} L^{-1}\left[\frac{3}{s^2+4}\right]$$

$$= e^{3t}\cos 2t$$

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$$e^{-t}\int_0^t \frac{\sin t}{t}\,dt$$

Soln:

$$L[\sin t] = \frac{1}{s^2+1}$$

$$L\left[\frac{\sin t}{t}\right] = \int_s^\infty \frac{1}{s^2+1}\,ds$$

$$\int \frac{dx}{a^2+x^2} = \tan^{-1}(x)$$

$$= \left[\tan^{-1}(1)\right]_s^\infty$$

$$= \tan^{-1}(\infty) - \tan^{-1}(s)$$

$$= \pi/2 - \tan^{-1}(s)$$

$$= \cot^{-1}(s)$$

$$\int_0^t \frac{\sin t}{t}\,dt = 1/s\ \cot^{-1}(s)$$

$$e^{-t}\int_0^t \frac{\sin t}{t}\,dt = \frac{1}{s+1}\left[\cot^{-1}(s+1)\right]$$

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Find $L^{-1}\left[\frac{1}{s^2+4s+13}\right]$

Soln:

$$\frac{1}{s^2+4s+13} = L^{-1}\left[\frac{1}{s^2+4s+13+4-4}\right]$$

$$= L^{-1}\left[\frac{1}{(s+2)^2+9}\right]$$

$$= e^{-2t}\, L^{-1}\left[\frac{1}{s^2+9}\right]$$

$$= e^{-2t}\left[\frac{\sin 3t}{3}\right]$$

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$$L^{-1}\left[\frac{1}{s^2-2s+5}\right]$$

Soln:

$$L^{-1}\left[\frac{1}{s^2-2s+5}\right] = \frac{1}{s^2-2s+5+1-1}$$

$$= \frac{1}{(s-1)^2+4}$$

$$= e^t L^{-1} \frac{1}{s^2+4}$$

$$= e^t \left[\frac{\sin 2t}{2}\right]$$

TYPE - II

$L[f(t)] = F(s)$ then $L[t f(t)] = \frac{-d}{ds}$

$$L[t\, f(t)] = -F'(s)$$

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Find $L^{-1}\left[\frac{s}{(s^2+a^2)^2}\right]$

soln:

$$L^{-1}[F'(s)] = -t\, L^{-1}[F(s)]$$

$$F'(s) = \frac{s}{(s^2+a^2)^2}$$

$$\frac{d}{ds} F(s) = \frac{s}{(s^2+a^2)^2}$$

Take integration on both sides,

$$F(s) = \int \frac{s}{(s^2+a^2)} ds$$

put,

$$u = s^2+a^2$$

$$du = 2s\,ds$$

$$\frac{du}{2} = s\,ds$$

$$F(s) = \int \frac{du/2}{u^2}$$

$$= \frac{1}{2}\int \frac{du}{u^2}$$

$$= \frac{1}{2} \times -\frac{1}{u}$$

$$= -\frac{1}{2u}$$

$$F(s) = \frac{-1}{2(s^2+a^2)}$$

$$L^{-1}[F(s)] = -t\, L^{-1}[F(s)]$$

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$$= -t\, L^{-1}\left[\frac{-1}{2(s^2+a^2)}\right]$$

$$= -t/2\; L^{-1}\left[\frac{-1}{s^2+a^2}\right]$$

$$= -t/2\; L^{-1}\left[\frac{-1}{s^2+a^2}\right]$$

$$= t\,\frac{\sin at}{2a}$$

) $L^{-1}\left[\frac{s}{(s^2-a^2)^2}\right]$

Soln:

$$L^{-1} F'(s) = -t\; L^{-1}[F(s)]$$

$$F'(s) = \frac{s}{(s^2-a^2)^2}$$

$$\frac{d}{ds} F(s) = \frac{s}{(s^2-a^2)^2}$$

Taking integration,

$$F(s) = \int \frac{s}{(s^2-a^2)^2}\, ds$$

put, $u = s^2-a^2$

$$du = 2s\,ds$$

$$\frac{du}{2} = s\,ds$$

$$F(s) = \int \frac{du/2}{u^2}$$

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$= \frac{1}{a} \times -\frac{1}{4}$

$= -\frac{1}{2u}$

$$F(s) = \frac{-1}{2(s^2-a^2)}$$

$$L^{-1}[F'(s)] = -t\, L^{-1}[F(s)]$$

$$= -t\, L^{-1}\left[\frac{-1}{2(s^2-a^2)}\right]$$

$$= \frac{t}{2}\, L^{-1}\left[\frac{1}{s^2-a^2}\right]$$

$$L^{-1}\left[\frac{s}{(s^2-a^2)^2}\right] = \frac{t \sinh at}{2a}$$

$$L^{-1}\left[\frac{3}{2s^2+8s+10}\right]$$

Soln:

$$L^{-1}\left[\frac{3}{2s^2+8s+10}\right] = L^{-1}\left[\frac{3}{2(s^2+4s+5)}\right]$$

$$= \frac{3}{2}\, L^{-1}\left[\frac{1}{s^2+4s+5}\right]$$

$$= \frac{3}{2}\, L^{-1}\left[\frac{1}{s^2+4s+5+4-4}\right]$$

$$= \frac{3}{2}\, L^{-1}\left[\frac{1}{(s+2)^2+1}\right]$$

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$= 1/2 \cdot -1/4$

$F(s) = -1/24$

$L^{-1}[F'(s)] = -t\, L^{-1}[F(s)]$

$= -t\, L^{-1}\left[\frac{-1}{2(s^2+6s+13)}\right]$

$= t/2\, L^{-1}\left[\frac{1}{(s^2+6s+13)}\right]$

$= t/2\, L^{-1}\left[\frac{1}{s^2+6s+13+9-9}\right]$

$= t/2\, L^{-1}\left[\frac{1}{(s+3)^2+4}\right]$

$= t/2\, e^{-3t}\, L^{-1}\left[\frac{1}{s^2+4}\right]$

$= t/2\, e^{-3t}\, \frac{\sin 2t}{2}$

Note : 3 Type : 3

$L^{-1}[sF(s)] = \frac{d}{dt} L^{-1} F(s)$

Solve $L^{-1}\left[\frac{s}{(s+2)^2+4}\right]$

soln

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$$L^{-1}\left[\frac{s}{(s+2)^2+4}\right] = L^{-1}\left[s \cdot \frac{1}{(s+2)^2+4}\right]$$

$$= \frac{d}{dt} L^{-1}\left[\frac{1}{(s+2)^2+4}\right]$$

$$= \frac{d}{dt} e^{-2t} L^{-1}\left[\frac{1}{s^2+4}\right]$$

$$= \frac{d}{dt} e^{-2t}\left[\frac{\sin 2t}{2}\right]$$

$$= \frac{1}{2} \frac{d}{dt}\left[e^{-2t} \sin 2t\right]$$

$$= \frac{1}{2}\left[e^{-2t} \cos 2t \cdot 2 + \sin 2t \, e^{-2t}(-2)\right]$$

$$= \frac{2e^{-2t}}{2}\left[\cos 2t - \sin 2t\right]$$

$$L^{-1}\left[\frac{s}{(s+2)^2+4}\right] = e^{-2t}\left[\cos 2t - \sin 2t\right]$$

$$L^{-1}\left[\frac{s}{(s+2)^2}\right]$$

Soln:

$$L^{-1}\left[\frac{s}{(s+2)^2}\right] = L^{-1}\left[s \cdot \frac{1}{(s+2)^2}\right]$$

$$= \frac{d}{dt} L^{-1}\left[\frac{1}{(s+2)^2}\right]$$

$$= \frac{d}{dt} e^{-2t} L^{-1}\left(\frac{1}{s^2}\right)$$

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$$= \frac{d}{dt} e^{-2t} t$$

$$= e^{-2t} - 2t e^{-2t}$$

$$= e^{-2t} [1-2t]$$

$$L^{-1} \left[\frac{s^2}{(s^2+a^2)^2} \right.$$

Soln:

$$L^{-1} \left[\frac{s^2}{(s^2+a^2)^2} \right] = L^{-1} \left[s \cdot \frac{1}{(s^2+a^2)^2} \right]$$

$$= \frac{d}{dt} L^{-1} \left[\frac{s}{(s^2+a^2)} \right]$$

Take $L^{-1} \left[\frac{s}{(s^2+a^2)} \right]$

$$L^{-1} [F'(s)] = -t L^{-1} [F(s)]$$

$$F'(s) = \frac{s}{(s^2+a^2)^2}$$

$$\frac{d}{ds} F(s) = \frac{s}{(s^2+a^2)^2}$$

Integrating on both sides, we get

$$F(s) = \int \frac{s}{(s^2+a^2)^2} ds$$

$$u = s^2 + a^2$$

$$du = 2s\, ds$$

$$du/2 = s\, ds$$

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$$F(s) = \int \frac{du/2}{u^2} \Rightarrow \frac{1}{2}\int du/u^2$$

$$= \frac{1}{2}\left[-\frac{1}{u}\right]$$

$$F(s) = \frac{-1}{2(s^2+a^2)}$$

$$L^{-1}[F'(s)] = -t\, L^{-1}[F(s)]$$

$$= -t\, L^{-1}\left[\frac{1}{s^2+a^2}\right]$$

$$= \frac{t}{2}\, \frac{\sin at}{2}$$

$$L^{-1}\left[\frac{s}{(s^2+a^2)^2}\right] = \frac{t \sin at}{2a}$$

$$= \frac{d}{dt}\, L^{-1}\left[\frac{s}{(s^2+a^2)^2}\right]$$

$$= \frac{d}{dt}\left[\frac{t \sin at}{2a}\right]$$

$$= \frac{1}{2a}\, \frac{d}{dt}\left[t \sin at\right]$$

$$= \frac{1}{2a}\left[t \cos at \cdot a + \sin at(1)\right]$$

$$= \frac{1}{2a}\left[at \cos at + \sin at\right]$$

$$L^{-1}\left[\frac{s^2}{(s^2+a^2)^2}\right] = \frac{at \cos at + \sin at}{2a}$$

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TYPE - 4

$$L^{-1}\left[\frac{1}{s}F(s)\right] = \int_0^t L^{-1}[F(s)]\,dt$$

$$L[F(s)] = L^{-1}\left[\int_0^t \frac{1}{s}F(s)\,ds\right]$$

Find $L^{-1}\left[\frac{1}{s(s+3)}\right]$

Soln:

$$L^{-1}\left[\frac{1}{s(s+3)}\right] = L^{-1}\left[\frac{1}{s}\cdot\frac{1}{s+3}\right]$$

$$= \int_0^t L^{-1}\left[\frac{1}{s+3}\right] dt$$

$$= \int_0^t e^{-3t}\,dt$$

$$= \left[\frac{e^{-3t}}{3}\right]_0^t dt$$

$$= -\frac{1}{3}\left[e^{-3t} - e^0\right]$$

$$= \frac{1}{3}\left[-e^{-3t} + 1\right]$$

$$L^{-1}\left[\frac{1}{s(s+3)}\right] = \frac{1}{3}\left[1 - e^{-3t}\right]$$

Type - V:

$$L^{-1}[F(s)] = -\frac{1}{t}\,L^{-1}[F'(s)]$$

$$L^{-1}\left[\frac{s+2}{(s^2+4s+5)^2}\right]$$

Soln:

$$L^{-1}[F'(s)] = -t\,L^{-1}[F(s)]$$

$$F'(s) = \frac{s+2}{(s^2+4s+5)^2}$$

$$\frac{d}{ds}F(s) = \frac{s+2}{(s^2+4s+5)^2}$$

Integration on both sides,

$$\int \frac{d}{ds} F(s) = \int \frac{s+2}{(s^2+4s+5)^2}\, ds$$

$$F(s) = \int \frac{s+2}{(s^2+4s+5)^2}\, ds$$

$$u = s^2+4s+5$$

$$du = 2s+4\, ds$$

$$du = 2(s+2)\, ds$$

$$\frac{du}{2} = (s+2)\, ds$$

$$F(s) = \int \frac{du/2}{u^2}$$

$$= \frac{1}{2}\int \frac{du}{u^2}$$

$$= \frac{1}{2} \times -\frac{1}{u}$$

$$= -\frac{1}{2u}$$

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$$F(s) = \frac{-1}{2(s^2+1)}$$

$$L^{-1}\left[F'(s)\right] = -t\,L^{-1}\left[F(s)\right]$$

$$F'(s) = \frac{s}{(s^2+1)^2}\,ds$$

$$= -t\,L^{-1}\left[\frac{-1}{2(s^2+4s+5)}\right]$$

$$= \frac{t}{2}\,L^{-1}\left[\frac{1}{(s+2)^2+1}\right]$$

$$= \frac{t}{2}\,L^{-1}\left[\frac{1}{(s^2+4s+4)+1}\right]$$

$$= \frac{t}{2}\,L^{-1}\left[\frac{1}{(s+2)^2+1}\right]$$

$$= \frac{t}{2}\,e^{-2t}\,L^{-1}\left[\frac{1}{s^2+1}\right]$$

$$L^{-1}\left[\frac{s+2}{(s^2+4s+5)^2}\right] = \frac{t}{2}\,e^{-2t}\sin t$$

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$$L^{-1}\left[\frac{s^2+9s+2}{(s+2)(s-1)^2}\right]$$

Soln:

$$\left[\frac{s^2+9s+2}{(s+2)(s-1)^2}\right] = \frac{A}{s+2} + \frac{B}{s-1} + \frac{C}{(s-1)^2}$$

$$\frac{s^2+9s+2}{(s+2)(s-1)^2} = \frac{A}{s+2} + \frac{B}{s-1} + \frac{C}{(s-1)^2} \longrightarrow (1)$$

$$\frac{s^2+9s+2}{(s+2)(s-1)^2} = \frac{A(s-1)^2 + B(s-1)(s+2) + C(s+2)}{(s+2)(s-1)^2}$$

$$s^2+9s+2 = A(s-1)^2 + B(s-1)(s+2) + C(s+2)$$

put $s = -2$

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$$4 + 9(-2) + 2 = A(-2-1)^2 + B(-2-1)(-2+2) + C(-2+2)$$

$$4 - 18 + 2 = A(-3)^2$$

$$-12 = 9A$$

$$A = 12/9$$

$$A = -4/3$$

Put $s = 1$

$$1 + 9(1) + 2 = A(1-1)^2 + B(1-1)(1+2) + C(1+2)$$

$$1 + 9 + 2 = C(3)$$

$$12 = 3C$$

$$C = 12/3$$

$$C = 4$$

Put $s = 0$

$$2 = A(0-1)^2 + B(0-1)(0+2) + C(0+2)$$

$$2 = A(1) + B(-1)(2) + C(2)$$

$$2 = A - 2B + 2C$$

$$2 = -4/3 + 8 - 2B$$

$$2 = \frac{-4+24}{3} - 2B$$

$$2 = 20/3 - 2B$$

$$2 - 20/3 = -2B$$

$$-14/3 = -2B$$

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$14/3 = 2B$

$B = \frac{14}{3 \times 2}$

$B = 7/3$

$$\frac{s^2+9s+2}{(s+2)(s-1)^2} = \frac{-4/3}{s+2} + \frac{7/3}{s-1} + \frac{4}{(s-1)^2}$$

$$L^{-1}\left[\frac{s^2+9s+2}{(s+2)(s-1)^2}\right] = L^{-1}\left[\frac{-4/3}{s+2} + \frac{7/3}{s-1} + \frac{4}{(s-1)^2}\right]$$

$$= L^{-1}\left[\frac{-4/3}{s+2}\right] + L^{-1}\left[\frac{7/3}{s-1}\right] + L^{-1}\left[\frac{4}{(s-1)^2}\right]$$

$$= -4/3\, e^{-2t} + 7/3\, e^t + 4e^t L^{-1}\left[1/s^2\right]$$

$$L^{-1}\left[\frac{s^2+9s+2}{(s+2)(s-1)^2}\right] = \frac{-4}{3}e^{-2t} + 7/3\, e^t + 4e^t t$$

Laplace transform of derivatives:

If $L[f(t)] = F(s)$ then

i) $L[f'(t)] = sF(s) - f(0)$

ii) $L[f''(t)] = s^2F(s) - sf(0) - f'(0)$

$L[f'(t)] = sF(s) - f(0)$

Soln:

$$L[f(t)] = \int_0^\infty e^{-st} f(t)\,dt$$

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$L[f'(t)] = \int_0^\infty e^{-st} f'(t)\,dt$

Take $u = e^{-st}$; $\int dv = f'(t)$

$du = e^{-st}(-s)\,dt$ $v = f(t)$

$du = -se^{-st}\,dt$

$\Rightarrow uv - \int v\,du$

$= \left[e^{-st} f(t)\right]_0^\infty - \int_0^\infty e^{-st}(-s) f(t)\,dt$

$= 0 - f(0) + s\int_0^\infty e^{-st} f(t)\,dt$

$= -f(0) + sF(s)$

$L[F'(t)] = sF(s) - f(0) \longrightarrow (1)$

$L[F''(t)] = s^2 F(s) = sF(0) - f'(0)$

<u>proof</u>:

$L[F(t)] = \int_0^\infty e^{-st} f(t)\,dt$

$L[F''(t)] = \int_0^\infty e^{-st} f''(t)\,dt$

Take $u = e^{-st}$; $\int dv = f''(t)$

$du = e^{-st}(-s)\,dt$ $v = f'(t)$

$du = -se^{-st}\,dt$

$= \left[e^{-st} f'(t)\right]_0^\infty - \int_0^\infty -se^{-st} f'(t)\,dt$

$= 0 - f'(0) + s\int_0^\infty e^{-st} f'(t)\,dt$

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$$= -f'(0) + s\int_0^\infty e^{-st} f'(t)\,dt$$

$$= -f'(0) + s\left[sF(s) - f(0)\right]$$

$$= -f'(0) + s^2F(s) - sf(0)$$

$$L\left[F''(t)\right] = s^2F(s) = sF(0) - f'(0)$$

solving differential equation using laplace transforms of derivation.

$$L\left[F'(t)\right] = sL\left[F(t)\right] - f(0)$$

$$L\left[F''(t)\right] = s^2F(s) - sf(0)$$

$$L\left[F''(t)\right] = s^2F(s) - sf(0) - f'(0)$$

(or)

$$L(y') = sL(y) - y(0)$$

$$L(y'') = s^2L[y] - sy(0) - y'(0)$$

$$L(y''') = s^3L(y) - s^2y(0) - y''(0)$$

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Solve : $y'' + 2y' - 3y = \sin t$ given $y(0)=0$.
$y'(0)=0$

Soln:

given:

$$y'' + 2y' - 3y = \sin t$$

$$L[y''+2y'-3y] = L[\sin t]$$

$$s^2 L[y] - sy[2y'] - 3L[y] = L[\sin t]$$

$$s^2 L[y] - sy[0] - y'[0] + 2sL[y] - 2y[0] - 3L[y].$$

$$\frac{1}{s^2+1}$$

$$L[y][s^2+2s-3] - sy[0] - y'[0]2y[0] = \frac{1}{s^2+1}$$

$$L[y][s^2+2s-3] - 0 - 0 - 0 = \frac{1}{s^2+1}$$

$$L[y][s^2+2s-3] = \frac{1}{s^2+1}$$

$$L[y] = \frac{1}{(s^2+1)(s^2+2s-3)}$$

$$L[y] = \frac{1}{(s^2+1)(s-1)(s+3)}$$

$$\frac{1}{(s^2+1)(s-1)(s+3)} = \frac{A}{s-1} + \frac{B}{s+3} + \frac{Cs+D}{s^2+1} \longrightarrow (1)$$

$$\frac{1}{(s^2+1)(s-1)(s+3)} = \frac{A(s+3)(s^2+1) + B(s-1)(s^2+1) + Cs+D(s-1)(s+3)}{(s^2+1)(s-1)(s+3)}$$

$$1 = A(s+3)(s^2+1) + B(s-1)(s^2+1) + Cs+D(s-1)(s+3)$$

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put $s=1$,

$5+2(1) = A(1-1)(1+5)+B(1)(1+5)+C(1)(1-1)$

$7 = 6B$

$B = 7/6$

put $s=-5$

$5-10 = C(-5)(-5-1)$

$-5 = 30C$

$C = -5/30$

$C = -1/6$

$$\frac{5+2s}{s(s-1)(s+5)} = -1/s + \frac{7/6}{s-1} - \frac{1/6}{s+5}$$

$$\frac{5+2s}{s(s-1)(s+5)} = -1/s + \frac{7/6}{s-1} - \frac{1/6}{s+5}$$

$$L[y] = -1/s + \frac{7/6}{s-1} - \frac{1/6}{s+5}$$

$$y = L^{-1}\left[-1/s + \frac{7/6}{s-1} - \frac{1/6}{s+5}\right]$$

$$y = -1 + 7/6\, e^{t} - 1/6\, e^{-5t}$$

$$y = -1 + \frac{7e^{t}}{6} - \frac{1e^{-5t}}{6}$$

$$y = -1 + \frac{7e^{t}}{6} - \frac{e^{-5t}}{6}$$

$$y = \frac{7e^{t} - e^{-5t} - 6}{6}$$

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