LAPLACE TRANSFORM

UNIT - III

Laplace Transform:

Let $f(t)$ be a function of the values variables t which is defined for all the values of t let s be a constant I $\int_0^\infty e^{-t} f(t)$ dt exist and is equal to

$f(s) = \int_0^\infty e^{-st} f(dt)$ $F(s)$ is called the laplace transform of $f(t)$ and it is denoted by

$$L[f(t)]$$

$$L[f(t)] = \int_0^\infty e^{-st} f(t)\,dt$$

1) Laplace transform of $L[1]$ dimentory function.

$$L[f(t)] = \int_0^\infty e^{-st} f(t)\,dt$$

$$L[1] = \int_0^\infty e^{-st} \times 1\,dt = \int_0^\infty e^{-st}\,dt \quad e^{-\alpha} = 0$$

$$= \frac{e^{-s(\alpha)}}{-s} - \left(\frac{e^{-s(0)}}{-s}\right)$$

$L(1) = 1/s$ if $s > 0$

2. Laplace Transform of $[L[f(t)]]$

$$= \int_0^\infty e^{-st} f(t)\, dt .$$

Solution:

$$L(f) = \int_0^\infty e^{-st} t\, dt$$

$$\int u\,dv = uv - \int v\,du$$

$u = t,\quad \int dv = \int e^{-st} dt$

$du = dt$

$$v = \frac{e^{-st}}{-s}$$

$$= \int_0^\infty t e^{-st} dt - \left[\frac{t e^{-st}}{-s}\right] - \int_0^\infty \frac{e^{-st}}{-s} dt$$

$$= 1/s \int_0^\infty e^{-st}\, dt = 1/s \int_0^\infty \frac{e^{-st}}{-s} dt$$

$$= 1/s \left[\frac{e^{-st}}{-s}\right] = 1/s \times 1/s = 1/s^2$$

$$L[t] = 1/s^2 \text{ f } s > 0 .$$

3. Laplace Transform of t^2

Soln:

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$$L[f(t)] = \int_0^\infty e^{-st} f(t)\, dt$$

$$L(t^2) = \int_0^\infty e^{-st}\, t^2\, dt$$ (3)

$\int u\,dv = uv - \int v\,du$

$u = t^2$

$du = 2t\,dt$ $\quad dv = e^{-st}$

$v = e^{-st}/-s$

$$L(t^2) = \left(\frac{t^2 e^{-st}}{-s}\right)_0^{\infty} - \int_0^\infty \frac{e^{-st}}{-s} 2t\,dt$$

$$= 2/s \int_0^\infty e^{-st}\,dt = 2/s \times 1/s^2$$

$$= L(t^2) = 2/s^2$$

4. Laplace transform of e^{-at}.

Soln:

$$L[f(t)] = \int_0^\infty e^{-st} f(t)\,dt$$

$$L[e^{-at}] = \int_0^\infty e^{-st} e^{-at}\,dt$$

$$= \int_0^\infty e^{-t}(s+a)\,dt$$

$$= \left[\frac{e^{-(s+a)t}}{-(s+a)}\right]_0^\infty$$

$$= \left[-\frac{e^{-(s+a)0}}{-(s+a)}\right]$$

$$= \frac{e^0}{s+a} = 1/s+a$$

5. Laplace Transform of e^{at}.

Soln:

$$L[f(t)] = \int_0^\infty e^{-st} f(t)\, dt$$

$$L[e^{at}] = \int_0^\infty e^{-st} e^{at}\, dt$$

$$= \int_0^\infty e^{-t(s-a)}\, dt$$

$$= \left[\frac{e^{-(s-a)t}}{-(s-a)}\right]_0^\infty$$

$$= \left[\frac{e^{-(s-a)0}}{-(s-a)}\right]$$

$$= \left[-\frac{1}{-(s-a)}\right]$$

$$= 1/s-a.$$

formula:

$$\int e^{ax} \sin bx\, dx = \frac{e^{ax}}{a^2+b^2}[a \sin bx - b \cos bx]$$

$$\int e^{ax} \cos bx\, dx = \frac{e^{ax}}{a^2+b^2}[a \cos bx - b \sin bx]$$

6) Laplace Transform of $\sin x$.

$$L[f(t)] = \int_0^\infty e^{-st} f(t)\, dt$$

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$$L[\sin at] = \int_0^\infty e^{-st} \sin at \, dt$$

$a = -s, \quad b = a$

$$\int e^{ax} \sin bx \, dx = \frac{e^{ax}}{a^2+b^2} [a \sin bx - b \cos bx]$$

$$L(\sin at) = \left[\frac{e^{-st}}{a^2+b^2}\right] [-s \sin at - a \cos at]_0^\infty$$

$$= \frac{-1}{s^2+a^2} [-(s \sin a(0)) + a \cos a(0)]$$

$$= \frac{-1}{s^2+a^2} (-a) = \frac{a}{s^2+a^2}$$

7) Laplace transform of cos at

$$L[f(t)] = \int_0^\infty e^{-st} f(t) \, dt$$

$$L(\cos at) = \int_0^\infty e^{-st} \cos at \, at$$

$a = -s, \quad b = a$

$$\int e^{ax} \cos bx \, dx = \frac{e^{ax}}{a^2+b^2} (a \cos bx + b \sin bx)$$

$$L(\cos at) = \left[\frac{e^{-st}}{s^2+a^2} (-s \cos at + a \sin at)\right]_0^\infty$$

$$= \frac{-1}{s^2+a^2} [-s \cos at + a \sin at]_0^\infty$$

$$= \frac{-1}{s^2+a^2} [-s \cos a(0) + a \sin(0) t]$$

$$= \frac{-1}{s^2+a^2}(-s) = \frac{s}{s^2+a^2}$$ 6

8) Laplace transform of $\sinh x + a$

Soln:

$$\sin hx = \frac{e^x - e^x}{2}, \quad \cosh x = \frac{e^x + e^x}{2}$$

$$L[f(t)] = \int_0^\infty e^{-st} f(t)\, dt$$

$$L[\sin hx t) = \int_0^\infty e^{-st} \sinh x\, dt$$

$$= \int_0^\infty e^{-st} \left[\frac{e^{at} - e^{-at}}{2}\right] dt$$

$$= \frac{1}{2}\int_0^\infty [e^{-st} e^{at} - e^{-st} - e^{-at}]\, dt$$

$$= \frac{1}{2}\left\{\int_0 e^{-(s-a)t} dt - \int_0^\infty e^{-(s+a)t} dt\right\}$$

$$= \frac{1}{2}\left[\frac{e^{-(s-a)t}}{-(s-a)} - \frac{e^{-(s+a)t}}{-(s+a)}\right]_0^\infty$$

$$= \frac{1}{2}\left[\frac{1}{-(s+a)} + \frac{1}{s+a}\right]$$

$$= \frac{1}{2}\left[\frac{1}{-(s-a)} + \frac{1}{s+a}\right]$$

$$= \frac{1}{2}\left[\frac{1}{s-a} - \frac{1}{s+a}\right]$$

$$= \frac{1}{2}\left[\frac{s+a-s+a}{s^2-a^2}\right]$$

$$= \frac{1}{2}\left[\frac{2a}{s^2-a^2}\right] = \frac{a}{s^2-a^2}$$

9. Laplace transform of $L\,[e^{-4t}]$

$$L\,[f(t)] = \int_0^\infty e^{-st} f(t)\,dt$$

$$L\,(e^{-4t}) = \int_0^\infty e^{-st}\, e^{-4t}\,dt$$

$$= \int_0^\infty e^{-t}\,(s+4)\,dt$$

$$= \int_0^\infty e^{-t}\,(s+4)\,t\; s\,dt$$

$$= \left[\frac{e^{-(s+4)t}}{-(s+4)}\right]_0^\infty$$

$$= \left[-\frac{e^{-(s+a)}}{(s+a)}\right] = 1/s+4$$

Type:

find Laplace transform of

1) $\sin 4t$

ii) $\cos t/2$

iii) e^{-3t}

i) $\sin 4t$

$$L\,[f(t)] = \int_0^\infty e^{-st} f(t)\,dt$$

$L[\sin 4t] = \int_0^{\infty} e^{-st} \sin 4t \, dt$ $a = -s$ (8)
$b = 4$

$\int e^{ax} \sin bx \, dx = \frac{e^{ax}}{a^2+b^2}[a \sin bx - b \cos bx]$

$L[\sin 4t] = \left[\frac{e^{-st}}{s^2+a^2}(-s \sin 4t - 4 \cos 4t)\right]_0^{\infty}$

$\frac{a}{s^2+a^2} = \frac{4}{s^2+16}$

2. $\cos t/2$

Soln:

$\frac{s}{s^2+a^2} = \frac{s}{s^2} + (1/2)^2$

$= \frac{s}{s^2} + 1/4 = \frac{4s}{4s^2+1}$

1. Find the laplace transform of $(\cos^2 2t)$

Soln:

Gn $\cos^2 2t$

$\cos^2 x = \frac{1+\cos 2x}{2}$

$\cos^2 2t = \frac{1+\cos 4t}{2}$

$L(\cos^2 2t) = 1/2 \, L(1+\cos 4t)$

$= 1/2 [1(1) + L(\cos 4t)]$

$= 1/2 [1/s + \frac{s}{s^2} + 16]$

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2. Find the laplace of $\sin^2 3t$

Given $\sin^2 3t$

$$\sin^2 x = \frac{1-\cos 2x}{2}$$

$$\sin^2 3t = \frac{1-\cos 6t}{2}$$

$$L(\sin^2 3t) = \frac{1}{2}\left[L(1-\cos(t))\right]$$

$$= \frac{1}{2}\left[L(1) - [\cos 6t]\right]$$

$$= \frac{1}{2}\left[\frac{1}{3} - \frac{s}{s^2}+36\right]$$

3. Find the laplace of $\cos^3 t$

$$\cos^3 t = \frac{1}{4}\left[\cos 3t + 3\cos t\right]$$

$$L\left[\cos^3 t\right] = \frac{1}{4} L\left[\cos 3t + 3\cos t\right]$$

$$= \frac{1}{4}\left[L\cos 3t + 3L\cos t\right]$$

$$= \frac{1}{4}\left[\frac{s}{s^2}+9+\frac{3s}{s^2}+1\right].$$

4 find $L\left[\cos^2 t\right]$

Given $L\left[\cos^2 t\right]$

$$= L(\cos t)$$

$$= L(\cos^2 t)^2$$

$$= \left(\frac{1+\cos 2t}{2}\right)^2$$

$$= \frac{1}{4}\left[1+\cos^2 2t + 2\cos 2t\right]$$

$= \frac{1}{4}\left[1 + \frac{(1+\cos 4t)}{2} + 2\cos 2t\right]$

$= \frac{1}{4}\left[1 + \frac{(1+\cos 4t)}{2} + 2\cos 2t\right]$

$= \frac{1}{4}\left[\frac{2+1+\cos 4t + 4\cos 2t}{2}\right]$

$= \frac{1}{4}\left[3 + \cos 4t + 4\cos 2t\right]$

$L(\cos^4 t) = \frac{1}{8}\left[3L(1) + L(\cos 4t) + 4L(\cos 2t)\right]$

$= \frac{1}{8}\left[\frac{3}{s} + \frac{s}{s^2+16} + \frac{4s}{s^2} + 4\right]$

5. Find $L(\cos 4t + \sin 3t)$

$L(\cos 4 + \sin 3t)$

$\therefore \cos A \sin B = \frac{1}{2}\left[\sin(A+B) - \sin(A-B)\right]$

$\cos 4t \sin 3t = \frac{1}{2}\left[\sin(4t+3t) - \sin(4t-5t)\right]$

$\cos 4t \sin 3t = \frac{1}{2}\left[\sin 7t - \sin t\right]$

$L(\cos 4 + \sin 3t) = \frac{1}{2}\left[1 - (\sin 7t) - 2(\sin t)\right]$

$= \frac{1}{2}\left[\frac{7}{s^2} + 49 - \frac{1}{s^2} + 1\right]$

6. Find $L[\sin^2 t + \cos^3 t)$

$L[\sin^2 t + \cos^3 t)$

$\sin^2 t = \frac{1 - \cos 2t}{2}$

$\cos 3t = \frac{1}{4}(\cos 3t + 3\cos t]$

$\sin 2t \cos^3 t = \left(\frac{1-\cos 2t}{2}\right) \cdot \frac{1}{4}(\cos t + 3\cos t)$

$= \frac{1}{8}[1-\cos 2t][\cos 3t + 3\cos t)$

$= \frac{1}{8}[\cos 3t + 3\cos t - \cos 2t + \cos 3t - \cos 2t$
$3\cos t]$

$= \frac{1}{8}\left[\cos 3t + 3\cos t - \left[\frac{\cos 5t + \cos t}{2}\right] - 3\left[\frac{\cos 3t + \cos t}{2}\right]\right.$

$= \frac{1}{8}\left[\frac{2\cos 3t + 6\cos t - \cos 5t - \cos t - 3\cos 3t - 3\cos t]}{2}\right.$

$= \frac{1}{16} L[-\cos 3t + 2\cos t - \cos 5t]$

$= \frac{1}{16}[-L(\cos 3t) + 2L(\cos t) - L(\cos 5t)]$

$= \frac{1}{16}\left[\frac{-s}{s^2} + 9 + \frac{2s}{s^2+1} - s/s^2 + 25\right]$

$= L[\sin^2 t + \cos^3 t] = \frac{1}{16}\left[\frac{2s}{s^2+1} - \frac{s}{s^2+9} - \frac{s}{s^2+25}\right]$

Type 2.

1) Find $L(e^{7t}\sin^2 t)$

$$\sin^2 t = \frac{1-\cos 2t}{2}$$

$$L(\sin^2 t) = \frac{1}{2}\left[L(1) - L(\cos 2t)\right]$$

$$= \frac{1}{2}\left[\frac{1}{s} - \frac{s}{s^2+4}\right]$$

Replace s by $s-7$

$$L[e^{7t}\sin^2 t] = \frac{1}{2}\left[\frac{1}{s-7} - \frac{(s-7)}{(s-7)^2+4}\right]$$

2. Find $L[e^{-3t}\cos^2 t]$

$$\cos^2 t = \frac{1+\cos 2t}{2}$$

$$L(\cos^2 t) = \frac{1}{2}\left[L(1) - L(\cos 2t)\right]$$

Replace s by $s+3$

$$L[e^{-3t}\cos^2 t] = \frac{1}{2}\left[\frac{1}{(s+3)} + \frac{s+3}{(s+3)^2+4}\right]$$

1.

S.No	F(t)	$L(f(t))$	$L[e^{at}F(t)]$	$L[e^{-at}f(t)]$
1	1	$\frac{1}{s}$	$\frac{1}{s-a}$	$\frac{1}{s+a}$
2	$\sin at$	$\frac{a}{s^2+a^2}$	$\frac{a}{(s-a)^2+a^2}$	$\frac{a}{(s+a)^2+a^2}$
3	$\cos at$	$\frac{s}{s^2+a^2}$	$\frac{(s-a)}{(s-a)^2+a^2}$	$\frac{(s+a)}{(s+a)^2+a^2}$
4	$\sinh at$	$\frac{a}{s^2-a^2}$	$\frac{a}{(s-a)^2+a^2}$	$\frac{a}{(s+a)^2+a^2}$
5	$\cosh at$	$\frac{s}{s^2-a^2}$	$\frac{s-a}{(s-a)^2-a^2}$	$\frac{s+a}{(s+a)^2-a^2}$

6	t	$1/s^2$	$\frac{1}{s-a^2}$	$\frac{1}{(s+a)2}$
7	t^2	$2/s^3$	$2/(s-a)^3$	$2/(s+a)^2$
8	t^3	$t^3/s4$	$\frac{L3}{(s-a)^4}$	$\frac{L3}{(s+a)^2}$
9	t^n	$\frac{Ln}{s^{n+1}}$	$\frac{Ln}{(s-a)^{n+1}}$	$\frac{Ln}{(s+a)^{n+1}}$

1. $L\left[t^4 - t^2 - t + \sin\sqrt{2}t\right]$

Soln:

$= L\left[t^4 - t^2 - t + \sin\sqrt{2}t\right]$

$= L(t^4) - L(t^2) - L(t) + L(\sin\sqrt{2})]$

$= \frac{L4}{s^3} - \frac{2}{s^3} - 1/s^2 + \frac{\sqrt{2}}{(s^2+\sqrt{2})^2}$

$= \frac{L4}{s^5} - \frac{2}{s^3} - \frac{1}{s^2} + \frac{\sqrt{2}}{s^2} + 2$

2. $L\left[e^{-3t}\cos 2t\right]$

$L\left[e^{-3t}\cos 2t\right] = L\left[e^{-3t}\cos 2t\right]$

$= \frac{s+3}{(s+3)^2} + 4$

3. $e^{-2t}\sin 2t + e^{3t}$

$L[e^{-2t}\sin 2t + e^{3t}) = \frac{2}{(s+2)^2+2^2} + \frac{1}{(s-3)}$

$= \frac{2}{(s+2)^2} + 4 + \frac{1}{(s+3)}$

4) $L(t^5 - 4t^3 + 3]$

$L[t^5 - 4t^3 + 3] = L[t^5) - L[4+3] + 3L(1)$

$= \frac{L5}{s6} - \frac{4\times 3!}{s4} + 3\times \frac{1}{s}$

$= \frac{15}{s6} - \frac{24}{s4} \cdot \frac{3}{s}$

5) $L(\cosh 4t)$

$\cosh at = \frac{a}{s^2-a^2}$

$\cosh 4t = \frac{4}{s^2-4^2}$

$= \frac{4}{s^2} - 16$

6) $L(\sin^4 t)$

$L(\sin^4 t)$

$= L(\sin^2 t)^2$

$= \left[\frac{1-\cos 2t}{2}\right]$

$= \frac{1}{4}\left[1 + \cos^2 2t - 2\cos 2t\right]$

$= \frac{1}{4}\left[1 + \frac{(1+\cos 4t)}{2} - 2\cos 2t\right]$

$= \frac{1}{4}\left[\frac{2+1+\cos 4t - 4\cos 2t}{2}\right]$

$= \frac{1}{8}\left[3t \cos 4t - 4\cos 2t\right]$

$L(\sin^4 t) = \frac{1}{8}\left[3L(1) + L(\cos 4t) - 4L(\cos 2t)\right]$

$= \frac{1}{8}\left[\frac{3}{s} + \frac{s}{s^2+16} - \frac{4s}{s^2+4}\right]$

7) $L(\sin 3t \cdot \cos 2t)$

Gn $L(\sin 3t \cdot \cos 2t)$

$\therefore \sin A \cos B = \frac{1}{2} \sin(A+B) + \sin(A-B)$

$\sin 3t \cdot \cos 2t = \frac{1}{2}\left[\sin 5t + \sin t\right)$

$L[\sin 3t \cos 2t] = \frac{1}{2}\left[L(\sin 5t) + t (\sin t)\right)$

$= \frac{1}{2}\left[\frac{5}{s^2+25}\right] + \frac{1}{(s^2+1)}\Big]$

8) $L(\sin 7t \cos 3t)$

Gn $L(\sin 7t \cos 3t)$

$\sin A \cos B = \frac{1}{2}\left[\sin(A+B) + \sin(A-B)\right)$