SEMESTER - IV

UNIT - III

VECTOR CALCULUS
AND
FOURIER SERIES

(16SCCMM7)

## GAUSS DIVERGENCE THEOREM:

The Surface interval of the normal components of a Vector point function $\vec{F}$ over a closed Surface 'S' is equal to Volume integral of divergence of $\vec{F}$ taken Over the Volume V enclosed by a Surface 'S'.

(i.e) $$\iint_S \vec{F} \cdot \hat{n}\, ds = \iiint_V \nabla \cdot \vec{F}\, dv$$

where $\hat{n}$ is the Unit Vector along the Outward drawn normal at any point p on the Surface 'S'.

$\vec{F} = 2(x^2-yz)\vec{i} + (y^2-zx)\vec{j} + (z^2-xy)\vec{k}$, $0 \le x \le a$, $0 \le y \le b$, $0 \le z \le c$

Soln: using Gauss divergence thm

$$\iint_S \vec{F}\cdot\hat{n}\, ds = \iiint_V \nabla\cdot\vec{F}\, dv$$

Consider, R.H.S $= \iiint_V \nabla\cdot\vec{F}\, dv$

$$\nabla\cdot\vec{F} = \left(\partial/\partial x\,\vec{i} + \partial/\partial y\,\vec{j} + \partial/\partial z\,\vec{k}\right)\left[2(x^2-yz)\vec{i} + (y^2-zx)\vec{j} + (z^2-xy)\vec{k}\right]$$

$$= 2(x^2-yz)\,\partial/\partial x + (y^2-zx)\,\partial/\partial y + (z^2-xy)\,\partial/\partial z$$

$$= 4x + 2y + 2z$$

$$= 2(2x+y+z)$$

$$\iiint_V \nabla\cdot\vec{F}\, dv = 2\int_0^a\int_0^b\int_0^c (2x+y+z)\, dz\, dy\, dx$$

$$= 2\int_0^a\int_0^b \left(2xz + yz + z^2/2\right)_0^c\, dy\, dx$$

$$= 2\int_0^a\int_0^b \left(2xc + yc + c^2/2\right) dy\, dx$$

$$= 2\int_0^a \left(2xyc + y^2c/2 + c^2/2\, y\right)_0^b\, dx$$

$$= 2\int_0^a \left(2xbc + \frac{b^2c}{2} + \frac{c^2b}{2}\right) dx$$

$$= 2\left[\frac{2x^2bc}{2} + \frac{xb^2c}{2} + \frac{xc^2b}{2}\right]_0^a$$

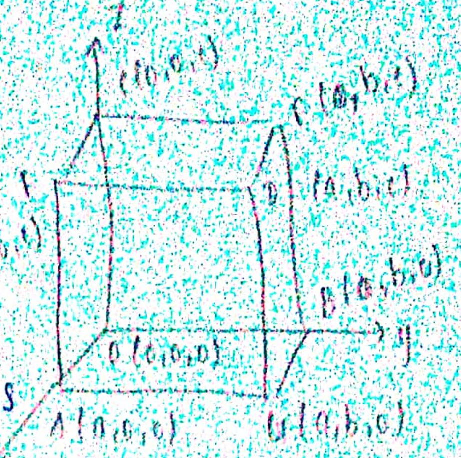
$$= 2\left[a^2bc + \frac{ab^2c}{2} + \frac{abc^2}{2}\right]$$

$$= 2a^2bc + \frac{2ab^2c}{2} + \frac{2abc^2}{2}$$

$= 2a^2bc + ab^2c + abc^2$

$= abc(2a+b+c)$

Consider, L.H.S $= \iint_S \vec{F}\cdot\hat{n}\, ds$




$$\iint_S \vec{F}\cdot\hat{n}\, ds = \iint_{AGDF} \vec{F}\cdot\hat{n}\, ds + \iint_{OBEC} \vec{F}\cdot\hat{n}\, ds$$

$$+ \iint_{BGDE} \vec{F}\cdot\hat{n}\, ds + \iint_{OAFC} \vec{F}\cdot\hat{n}\, ds + \iint_{CFDE} \vec{F}\cdot\hat{n}\, ds$$

$$+ \iint_{OAGB} \vec{F}\cdot\hat{n}\, ds$$

Consider, $\iint_{AGDF} \vec{F}\cdot\hat{n}\, ds$

$\hat{n} = \vec{i}$, $ds = dy\,dz$, $x = a$

$y$ varies from $0$ to $b$, $z$ varies from $0$ to $c$.

$$= 2\int_0^c\int_0^b (x^2 - yz)\, dy\, dz = 2\int_0^c\int_0^b (a^2 - yz)\, dy\, dz$$

$$= 2\int_0^c \left[a^2y - \frac{y^2z}{2}\right]_0^b dz = 2\int_0^c \left(a^2b - \frac{b^2z}{2}\right) dz$$

$$= 2\left[a^2bz - \frac{b^2z^2}{4}\right]_0^c$$

$$= 2\left[a^2bc - \frac{b^2c^2}{4}\right]$$

$$= 2a^2bc - \frac{b^2c^2}{2}$$

Consider, $\iint_{OBEC} \vec{F}\cdot\hat{n}\, ds$

$\hat{n} = -\vec{i}$, $ds = dy\,dz$, $x = 0$

$$= 2\int_0^c\int_0^b yz\, dy\, dz = 2\int_0^c \left[\frac{y^2z}{2}\right]_0^b dz$$

$$= 2\int_0^c \frac{b^2 z}{2}\, dz$$

$$= 2\left(\frac{b^2 z^2}{4}\right)_0^c = \frac{b^2 c^2}{2}$$

Consider, $\iint_{BGDE} \vec{F}\cdot\hat{n}\, ds$

$\hat{n} = \vec{j}$ $\quad ds = dx\,dz \quad y = b$

$x$ varies from $0$ to $a$, $z$ varies from $0$ to $c$

$$= \int_0^c\int_0^a (b^2 - zx)\, dx\, dz$$

$$= \int_0^c \left(b^2 x - \frac{zx^2}{2}\right)_0^a dz$$

$$= \int_0^c b^2 a - \frac{za^2}{2}\, dz$$

$$= \left[b^2 a z - \frac{z^2 a^2}{4}\right]_0^c$$

$$= ab^2 c - a^2c^2/4$$

Consider $\iint_{OAFC} \vec{F}\cdot\hat{n}\, ds$

$\hat{n} = -\vec{j}$ $\quad ds = dx\,dz \quad y = 0$

$$= -\int_0^c\int_0^a -zx\, dx\, dz = \int_0^c \left(\frac{zx^2}{2}\right)_0^a dz$$

$$= \int_0^c \frac{za^2}{2}\, dz$$

$$= \left[\frac{z^2a^2}{4}\right]_0^c$$

$$= \frac{c^2 a^2}{4}$$

Consider, $\iint\limits_{CFDE} \vec{F}\cdot\hat{n}\, ds$

$\hat{n} = \vec{K} \quad ds = dx\,dy \quad z = c$

$x$ and $y$ varies from 0 to $a$ and 0 to $b$

$$= \int_0^b\int_0^a (c^2 - xy)\, dx\, dy$$

$$= \int_0^b \left[c^2x - \frac{x^2y}{2}\right]_0^a dy$$

$$= \int_0^b \left(ac^2 - \frac{a^2y}{2}\right) dy = \left[ac^2y - \frac{a^2y^2}{4}\right]_0^b$$

$$= abc^2 - a^2b^2/4$$

Consider, $\iint\limits_{OAGB} \vec{F}\cdot\hat{n}\, ds$

$\hat{n} = -\vec{K} \quad ds = dx\,dy \quad z = 0$

$$= \int_0^b\int_0^a xy\, dx\, dz = \int_0^b \left(\frac{x^2y}{2}\right)_0^a dy = \int_0^b \left(\frac{a^2y}{2}\right) dy$$

$$= \left(\frac{a^2y^2}{4}\right)_0^b = \frac{a^2b^2}{4}$$

$$\iint\limits_S \vec{F}\cdot\hat{n}\, ds = 2a^2bc - \frac{b^2c^2}{2} + \frac{b^2c^2}{2} + ab^2c - \frac{c^2a^2}{4} + \frac{c^2a^2}{4} + abc^2 - \frac{a^2b^2}{4} + \frac{a^2b^2}{4}$$

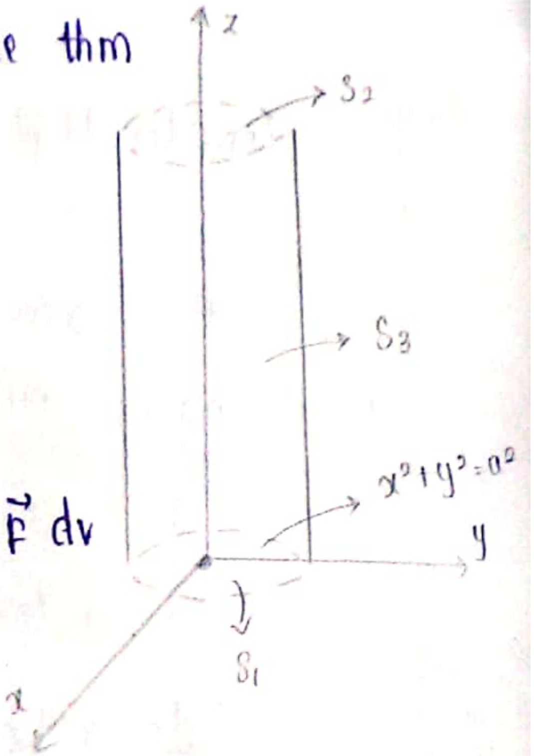
$$= 2a^2bc + ab^2c + abc^2$$

$$= abc\,(2a+b+c)$$

$\vec{F} = x\vec{i} + y\vec{j} + (z^2-1)\vec{k}$, $x^2+y^2=a^2$ where $z=0$ to $1$.

Soln: By using Gauss divergence thm

$$\iint_S \vec{F}\cdot\hat{n}\, ds = \iiint_V \nabla\cdot\vec{F}\, dv$$




To find Volume integral

$$\vec{F} = x\vec{i} + y\vec{j} + (z^2-1)\vec{k}$$

Now, we consider R.H.S $\iiint_V \nabla\cdot\vec{F}\, dv$

$$\nabla\cdot\vec{F} = 1+1+2z = 2+2z = 2(1+z)$$

$z$ Varies from 0 to 1

Gn: $x^2+y^2=a^2$

$$y^2 = a^2-x^2$$

$$y = \pm\sqrt{a^2-x^2}$$

$$x^2 = a^2 \Rightarrow x = \pm a$$

$y$ Varies from $-\sqrt{a^2-x^2}$ to $\sqrt{a^2-x^2}$

$x$ Varies from $-a$ to $a$

$$\iiint_V \nabla\cdot\vec{F}\, dv = \int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}}\int_0^1 2(1+z)\, dz\, dy\, dx$$

$$= 2\int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \left(z+\frac{z^2}{2}\right)_0^1 dy\, dx$$

$$= 2\int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} (1+\tfrac{1}{2})\, dy\, dx$$

$$= 2\times \tfrac{3}{2}\int_{-a}^{a} (y)_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}}\, dx$$

$$= 3\int_{-a}^{a} (\sqrt{a^2-x^2}+\sqrt{a^2-x^2})\, dx$$

$$= 6\times 2\int_{0}^{a} \sqrt{a^2-x^2}\, dx = 12\int_{0}^{a}\sqrt{a^2-x^2}\, dx$$

using the formula,

$$\int\sqrt{a^2-x^2}\, dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)$$

$$= 12\left(\tfrac{x}{2}\sqrt{a^2-x^2} + \tfrac{a^2}{2}\sin^{-1}\left(\tfrac{x}{a}\right)\right]_0^a$$

$$= 12\left[\frac{a}{2}\sqrt{a^2-a^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{a}{a}\right)\right]$$

$$= 12\left[\frac{a^2}{a}\sin^{-1}(1)\right] = 12\left[\frac{a^2}{2}\,\frac{\pi}{2}\right]$$

$$= 3a^2\pi$$

To find surface integral

$$\iint_S \vec{F}\cdot\hat{n}\, ds = \iint_{S_1}\vec{F}\cdot\hat{n}\, ds + \iint_{S_2}\vec{F}\cdot\hat{n}\, ds + \iint_{S_3}\vec{F}\cdot\hat{n}\, ds$$

Consider, $\iint_{S_1}\vec{F}\cdot\hat{n}\, ds$

$\hat{n} = -\vec{k}$ $\quad ds = dx\,dy \quad z=0$

$$= \iint_{S_1} (x\vec{i}+y\vec{j}+(z^2-1)\vec{k})\,(-\vec{k})\, dx\, dy$$

$$= -\iint_{S_1}(z^2-1)\, dx\, dy = -\iint_{S_1}(-1)\, dx\, dy = \iint_{S_1 \, -a}^{a} dx\, dy$$

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$= \pi a^2$

Consider, $\iint_{S_2} \vec{F}.\hat{n}\, ds$

$\hat{n} = \vec{K}, \quad z = 1, \quad ds = dx dy$

$\iint_{S_2} \vec{F}.\hat{n}\, ds = \iint_{S_2} (z^2-1)\, ds = 0$

Consider, $\iint_{S_3} \vec{F}.\hat{n}\, ds$

S is the curved surface and we have to consider the polar co-ordinates, put

$x = r \cos\theta, \quad y = r \sin\theta$

$x = a \cos\theta, \quad y = a \sin\theta$

$ds = r\, dz\, d\theta \Rightarrow ds = a\, dz\, d\theta$

z varies from 0 to 1.

θ varies from 0 to 2π

$\phi = x^2+y^2-a^2$

$\nabla\phi = 2x\vec{i} + 2y\vec{j}$

$|\nabla\phi| = 2\sqrt{x^2+y^2} = 2\sqrt{a^2} = 2a$

$\hat{n} = \frac{2(x\vec{i}+y\vec{j})}{2a} = \frac{x\vec{i}+y\vec{j}}{a}$

$\vec{F}.\hat{n} = \frac{x^2+y^2}{a} = \frac{a^2}{a} = a$

$\iint_{S_3} \vec{F}.\hat{n}\, ds = a \int_0^{2\pi}\int_0^1 a\, (dz\, d\theta) = a^2 \int_0^{2\pi} [z]_0^1\, d\theta$

$= a^2 \int_0^{2\pi} d\theta = a^2 [\theta]_0^{2\pi} = a^2 2\pi$

$\iint_S \vec{F}.\hat{n}\, ds = \pi a^2 + 0 + 2\pi a^2 = 3\pi a^2.$

Using Gauss divergence thm. evaluate $\iint_S \vec{F}\cdot\hat{n}\, ds$, where $\vec{F} = x^3\vec{i} + y^3\vec{j} + z^3\vec{k}$ and $S$ is the surface of the sphere $x^2+y^2+z^2=a^2$.

Soln: By using Gauss divergence thm

$$\iint_S \vec{F}\cdot\hat{n}\, ds = \iiint_V \nabla\cdot\vec{F}\, dv$$

$$\nabla\cdot\vec{F} = \left(\frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}\right)\cdot\left(x^3\vec{i} + y^3\vec{j} + z^3\vec{k}\right)$$

$$= 3x^2 + 3y^2 + 3z^2.$$

$$= 3(x^2+y^2+z^2)$$

$$\iint_S \vec{F}\cdot\hat{n}\, ds = \iiint_V \nabla\cdot\vec{F}\, dv$$

$$= \iiint_V 3(x^2+y^2+z^2)\, dv.$$

To evaluate the volume integral we have to change cartesian to spherical polar co-ordinates.

$$x = r\sin\theta\cos\phi \quad y = r\sin\theta\sin\phi \quad z = r\cos\theta$$

$$dv = r^2\sin\theta\, dr\, d\theta\, d\phi$$

$r$ varies from $0$ to $a$

$\theta$ varies from $0$ to $\pi$

$\phi$ varies from $0$ to $2\pi$

$$\iint_S \vec{F}\cdot\hat{n}\, ds = \int_0^a\int_0^\pi\int_0^{2\pi} 3r^2\left(r^2\sin\theta\, d\phi\, d\theta\, dr\right)$$

$$= 3\int_0^a\int_0^\pi\int_0^{2\pi} r^4\sin\theta\, d\phi\, d\theta\, dr$$

$$= 3\int_0^a\int_0^\pi r^4\sin\theta\, [\phi]_0^{2\pi}\, d\theta\, dr$$

$$= 6\pi\int_0^a\int_0^\pi r^4\sin\theta\, d\theta\, dr.$$

$$= 6\pi \int_0^a r^4 \left(-\cos\theta\right)_0^\pi \, dr$$

$$= 6\pi \int_0^a r^4 \left(-(-1-1)\right) dr \, .$$

$$= 12\pi \int_0^a r^4 \, dr = 12\pi \left[\frac{r^5}{5}\right]_0^a = \frac{12\pi a^5}{5}$$

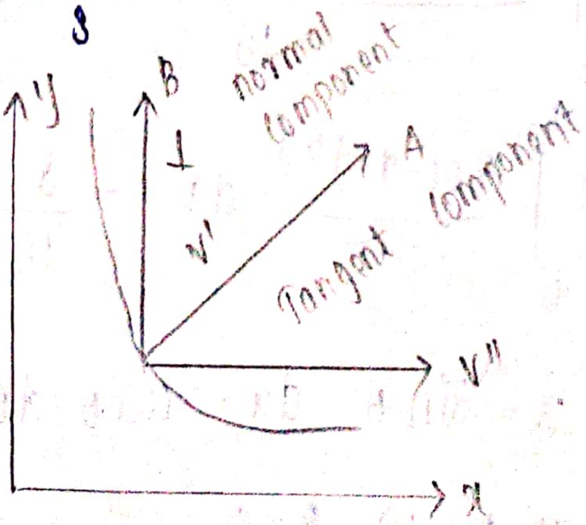
$$\iint_S \vec{F} \cdot \hat{n} \, ds = \frac{12 a^5 \pi}{5}$$

STROKE'S THM:

The line integral of the tangential component of a vector function $\vec{F}$ [finite & differentiable], around a simple closed curve 'c' is equal to the surface integral of the normal component of curl $\vec{F}$ over any surface 's' having as it's boundary, symbollicaly.

$$\int_c \vec{F}\cdot d\vec{r} = \iint_S \text{curl}\ \vec{F}\cdot \hat{n}\ ds \quad \text{(or)}$$

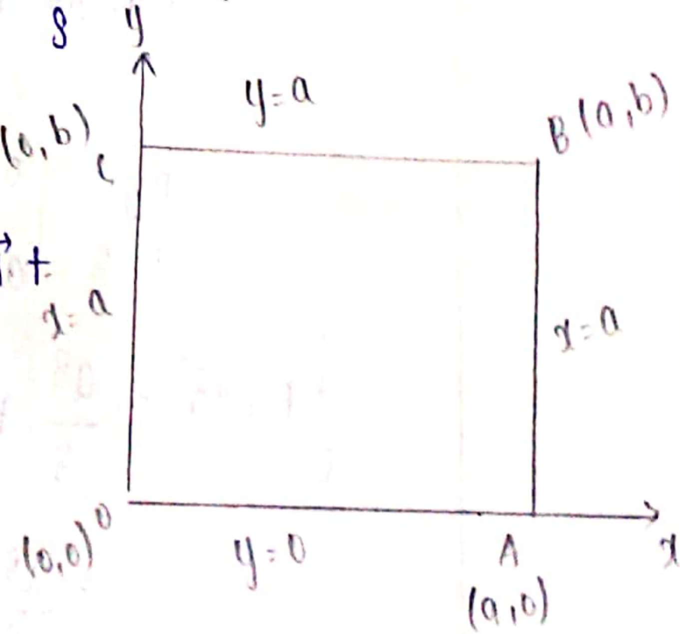
$$\int_c \vec{F}\cdot d\vec{r} = \iint_S (\nabla\times\vec{F})\cdot \hat{n}\ ds$$



Verify Stokes thm for $\vec{F} = x^2\vec{i} + xy\vec{j}$ taken around the Square in the $xy$ plane the two Sides are $x=0$, $x=a$, $y=0$, $y=b$.

**Soln:** By Stoke's thm $\int_C \vec{F}.d\vec{r} = \iint_S (\nabla \times \vec{F})\,\hat{n}\,ds$

Consider L.H.S $\int_C \vec{F}.d\vec{r}$




$$\int_C \vec{F}.d\vec{r} = \int_{OA} \vec{F}.d\vec{r} + \int_{AB} \vec{F}.d\vec{r} + \int_{BC} \vec{F}.d\vec{r} + \int_{OC} \vec{F}.d\vec{r}$$

Consider, $\int_{OA} \vec{F}.d\vec{r}$

$\vec{F} = x^2\vec{i} + xy\vec{j}$, $d\vec{r} = dx\vec{i} + dy\vec{j}$

$x$ Varies from 0 to a

$y=0$, $dy=0$

$$= \int_0^a x^2\,dx + xy\,dy = \int_0^a x^2\,dx = \left[\frac{x^3}{3}\right]_0^a = \frac{a^3}{3}$$

Consider, $\int_{AB} \vec{F}.d\vec{r}$

$y$ Varies from 0 to a.

$x = a,\ dx = 0$

$$= \int_0^a x^2\,dx + xy\,dy = \int_0^a a\left(\frac{y^2}{2}\right)_0^a = \frac{a^3}{2}$$

Consider $\int_{CA} \vec{F}\cdot d\vec{r}$

$y$ Varies from $a$ to $0$

$x = 0,\ dx = 0$.

$$= \int_a^0 x^2\,dx + xy\,dy = 0$$

Consider, $\int_{BC} \vec{F}\cdot d\vec{r}$

$x$ Varies from $a$ to $0$, $y = b$, $dy = 0$

$$= \int_a^0 x^2\,dx + xy\,dy = \int_a^0 x^2\,dx + 0$$

$$= \left[\frac{x^3}{3}\right]_a^0 = \frac{-a^3}{3}$$

$$\int_C \vec{F}\cdot d\vec{r} = \frac{a^3}{3} + \frac{a^3}{2} - \frac{a^3}{3}$$

$$= \frac{a^3}{2}$$

Consider, R.H.S $\iint_S \nabla\times\vec{F}\cdot\hat{n}\,ds$

$ds = dx\,dy$, $\hat{n} = \vec{k}$

$x$ Varies from $0$ to $a$, $y$ Varies from $0$ to $a$.

$$\nabla\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^2 & xy & 0 \end{vmatrix}$$

$$= \vec{i}\left(\partial/\partial y(0) - \partial/\partial z(xy)\right) - \vec{j}\left(\partial/\partial x(0) - \partial/\partial z(x^2)\right) + \vec{k}\left(\partial/\partial x(xy) - \frac{\partial}{\partial y}(x^2)\right)$$

$$= 0\vec{i} - \vec{j}(0) + \vec{k}(y)$$

$$= y\vec{k}$$

$$\iint_S (\nabla \times \vec{F})\, \hat{n}\, ds = \int_0^a \int_0^a y \, dx\, dy \quad = \int_0^a (xy)_0^a \, dy$$

$$= a\int_0^a y\, dy \quad = a\left[\frac{y^2}{2}\right]_0^a = \frac{a^3}{2}$$

$$\int_C \vec{F} \cdot d\vec{r} = \iint_S (\nabla \times \vec{F})\, \hat{n}\, ds \quad = a^3/2.$$

Evaluate by Shoke's thm $\int_C yz\,dx + zx\,dy + xy\,dz$ where $C$ is the Curve $x^2+y^2=1$; $y^2=x$

Soln: By Shoke's thm

$$\int_C \vec{F}\cdot d\vec{r} = \iint_S (\nabla\times\vec{F})\,\hat{n}\,ds$$

$$\vec{F} = yz\vec{i} + zx\vec{j} + xy\vec{K}$$

Consider $\iint_S (\nabla\times\vec{F})\,\hat{n}\,ds$

$$\nabla\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{K} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ yz & zx & xy \end{vmatrix}$$

$$= \vec{i}\left[\partial/\partial y\,(xy) - \partial/\partial z\,(xz)\right] - \vec{j}\left[\partial/\partial x\,(xy) - \partial/\partial z\,(yz)\right] + \vec{K}\left[\partial/\partial x\,(xz) - \partial/\partial y\,(yz)\right].$$

$$= \vec{i}\,(x-x) - \vec{j}\,(y-y) + \vec{K}\,(z-z)$$

$$= 0$$

$$\iint_S (\nabla\times\vec{F})\,\hat{n}\,ds = 0$$

$$\int_C \vec{F}\cdot d\vec{r} = 0$$

$$\int_C yz\,dx + zx\,dy + xy\,dz = 0.$$

Verify Stoke's thm $\vec{F} = y\vec{i} + z\vec{j} + x\vec{k}$ where S is the upper half Surface $x^2+y^2+z^2=1$ and C is the boundary.

Soln: By Stoke's thm

$$\int_C \vec{F}\cdot d\vec{r} = \iint_S (\nabla\times\vec{F})\,\hat{n}\,ds$$

evaluation of $\int_C \vec{F}\cdot d\vec{r}$

$$\vec{F} = y\vec{i} + z\vec{j} + x\vec{k}$$

$$d\vec{r} = dx\vec{i} + dy\vec{j} + dz\vec{k}$$

Hence C is the boundary of the upper half of the given sphere which is clearly a circle $x^2+y^2=1$

$\because$ C lies on the xy plane, we have in the plane $z=0$

$$\vec{F} = y\vec{i} + z\vec{j}$$

$$d\vec{r} = dx\vec{i} + dy\vec{j}$$

$$\vec{F}\cdot d\vec{r} = y\,dx + z\,dy$$

$$\because z = 0$$

$$\int \vec{F}\cdot d\vec{r} = \int y\,dx$$

The parametric representation of the circle $x^2+y^2=1$ is

$$x = \cos\theta\,,\ y = \sin\theta$$

$dx = -\sin\theta \, d\theta$ , $dy = \cos\theta \, d\theta$ .

$\theta$ Varies from $0$ to $2\pi$

$\therefore \int \vec{F} \cdot d\vec{r} = \int_C y \, dx$

$$= \int_0^{2\pi} \sin\theta(-\sin\theta) \, d\theta$$

$$= -\int_0^{2\pi} \sin^2\theta \, d\theta$$

$$= -\int_0^{2\pi} \left(\frac{1-\cos 2\theta}{2}\right) d\theta$$

$$= -\left[\frac{1}{2}\theta - \frac{\sin 2\theta}{4}\right]_0^{2\pi}$$

$$= -\frac{1}{2}(2\pi) - \frac{\sin 2(2\pi)}{4}$$

$$\int_C \vec{F} \cdot d\vec{r} = -\pi$$

Evaluation of $\iint_S (\nabla \times \vec{F}) \hat{n} \, ds$

$$\vec{F} = y\vec{i} + z\vec{j} + x\vec{k}$$

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y & z & x \end{vmatrix}$$

$$= \vec{i}\left(\partial/\partial y (x) - \partial/\partial z (z)\right) - \vec{j}\left(\partial/\partial x (x) - \partial/\partial z (y)\right) + \vec{k}\left(\partial/\partial x (z) - \partial/\partial y (y)\right)$$

$$= -\vec{i} - \vec{j} - \vec{k}$$

The plane region $S$ of the $xy$ plane bounded $x^2 + y^2 = 1$

$$\therefore \hat{n} = \vec{k}$$

$$\therefore (\nabla \times \vec{F}) \cdot \hat{n} = (-\vec{i}-\vec{j}-\vec{k}) \cdot (\vec{k})$$

$$= -1$$

$$\iint_S (\nabla \times \vec{F}) \cdot \hat{n} \, ds = -\iint_S ds$$

$$= \iint (-1) \qquad \text{(area of the circle)}$$

$$= (-1)(\pi)$$

$$\iint_S (\nabla \times \vec{F}) \cdot \hat{n} \, ds = -\pi$$

$$\int_C \vec{F} \cdot d\vec{r} = \iint_S (\nabla \times \vec{F}) \cdot \hat{n} \, ds.$$

GREEN'S THM:

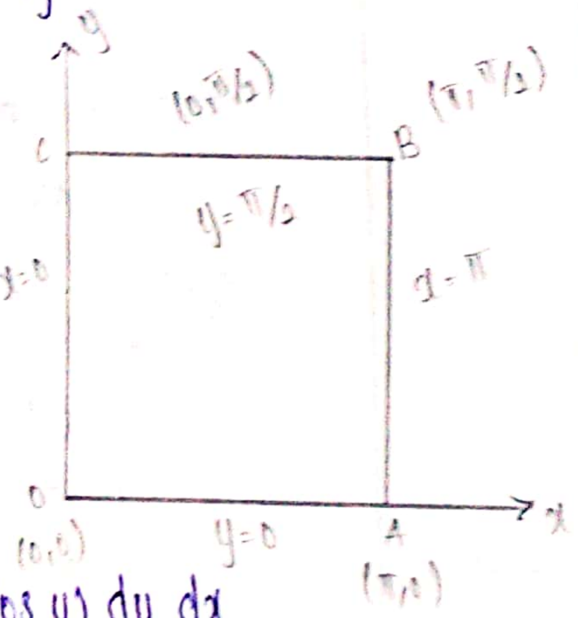
Let R be a closed curve in C. Let P and Q be continuous fn of x and y having continuous partial derivatives in R.

Then, $$\int_C P\,dx + Q\,dy = \iint_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy.$$

Verify green's thm $\int_c e^{-x} \sin y \, dx + e^{-x} \cos y \, dy$ where c is a rectangle which Vertices $(0,0), (\pi,0), (\pi, \pi/2), (0, \pi/2)$.

Soln: By Stroke's thm

$$\int_c p\,dx + Q\,dy = \iint_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$




$$\vec{F} = e^{-x} \sin y \,\vec{i} + e^{-x} \cos y \,\vec{j}$$

$$P = e^{-x} \sin y \,, \quad Q = e^{-x} \cos y$$

$$\frac{\partial P}{\partial y} = -e^{-x} \cos y \,, \quad \frac{\partial Q}{\partial x} = -e^{-x} \cos y$$

Consider $R.H.S = \int_0^{\pi}\int_0^{\pi/2} (-e^{-x} \cos y - e^{-x} \cos y)\, dy\, dx$

$$= \int_0^{\pi}\int_0^{\pi/2} (-2e^{-x} \cos y)\, dy\, dx$$

$$= -2 \int_0^{\pi}\int_0^{\pi/2} e^{-x} \cos y \, dy\, dx$$

$$= -2 \int_0^{\pi} e^{-x} (\sin y)_0^{\pi/2} \, dx$$

$$= -2 \int_0^{\pi} e^{-x} (1)\, dx$$

$$= -2 \left[-e^{-x}\right]_0^{\pi} = 2e^{-\pi} - 2e^{0} = 2e^{-\pi} - 2$$

$$\iint_S \left(\frac{\partial \phi}{\partial x} - \frac{\partial P}{\partial y}\right) dy\, dx = a(e^{\pi}-1)$$

To find line integral

$$\int_C pdx + Qdy = \int_{OA} pdx + Qdy + \int_{AB} pdx + Qdy + \int_{BC} pdx + Qdy + \int_{CD} pdx + Qdy.$$

Consider $\int_{OA} pdx + Qdy$

$y=0,\ dy=0$, $x$ Varies from $0$ to $\pi$

$$= \int_0^{\pi} -e^{-x} \sin y\, dx + e^{-x} \cos y\, dy = 0$$

Consider, $\int_{AB} pdx + Qdy$

$x=\pi,\ dx=0$, $y$ Varies from $0$ to $\pi/2$

$$= \int_0^{\pi/2} e^{-\pi} \cos y\, dy$$

$$= e^{-\pi} \int_0^{\pi/2} \cos y\, dy$$

$$= e^{-\pi} \left[\sin y\right]_0^{\pi/2}$$

$$= e^{-\pi} \left[\sin \pi/2\right]$$

$$= e^{-\pi}$$

Consider $\int_{BC} pdx + Qdy$

$y = \pi/2,\ dy=0$, $x$ Varies from $\pi$ to $0$.

$$= \int_{\pi}^{0} e^{-x} \sin y\, dx$$

$$= \int_{\pi}^{0} e^{-x} \sin \pi/2\, dx$$

$$= \int_{\pi}^{0} e^{-x}\, dx = \left[-e^{-x}\right]_{\pi}^{0} = -e^{-0} - (-e^{-\pi}) = -1 + e^{-\pi}$$

$$= e^{-\pi} - 1$$

Consider, $\int_{CA} p\,dx + Q\,dy$

$x = 0$, $dx = 0$, $y$ varies from $\pi/2$ to $0$.

$$= \int_{\pi/2}^{0} e^{-x} \cos y \, dy$$

$$= \int_{\pi/2}^{0} e^{-0} \cos y \, dy = \int_{\pi/2}^{0} \cos y \, dy$$

$$= \left[\sin y\right]_{\pi/2}^{0} = \sin 0 - \sin \pi/2 = 0 - 1 = -1$$

$$\int_{C} p\,dx + Q\,dy = 0 + e^{-\pi} + e^{-\pi} - 1 - 1$$

$$= 2e^{-\pi} - 2 = 2(e^{-\pi} - 1)$$

$$\therefore \int p\,dx + Q\,dy = \iint_{S} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dy\, dx$$

$$= 2(e^{-\pi} - 1)$$

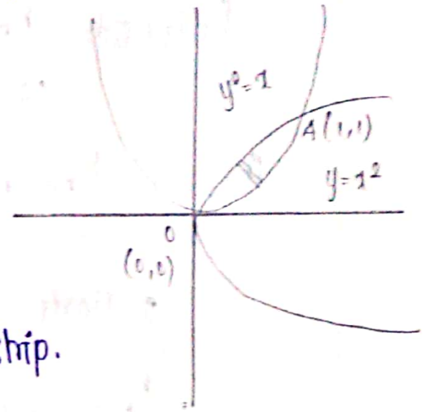
Hence green's thm is verified.

$$\oint_C p\,dx + Q\,dy = 1 + 8/3 - 2 = 8/3 - 1 = 5/3$$

$\therefore$ Green thm is Verified.

Verify Green's thm $\oint_C (3x^2 - 8y^2)\,dx + (4y - 6xy)\,dy$ where $C$ is the boundary of the point $y = x^2$ and $y^2 = x$ are the parabola.

Soln: $$\oint_C p\,dx + Q\,dy = \iint_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$




To find Surface integral

$P = 3x^2 - 8y^2$, $Q = 4y - 6xy$

$\frac{\partial P}{\partial y} = -16y$, $\frac{\partial Q}{\partial x} = -6y$

First we consider the horizontal strip.

$x$ Varies from $y^2$ to $\sqrt{y}$

Then we move the strip vertically

$y$ Varies from 0 to 1

$$= \int_0^1 \int_{y^2}^{\sqrt{y}} -6y - (-16)y \; dx\,dy$$

$$= \int_0^1 \int_{y^2}^{\sqrt{y}} 10y \; dx\,dy$$

$$= 10 \int_0^1 [yx]_{y^2}^{\sqrt{y}} \, dy$$

$$= 10 \int_0^1 [y\sqrt{y} - y^3] \, dy$$

$$= 10 \int_0^1 (y^{3/2} - y^3) \, dy$$

$$= 10 \left[\frac{y^{5/2}}{5/2} - \frac{y^4}{4}\right]_0^1$$

$$= 10 \left[\frac{2y^{5/2}}{5} - \frac{y^4}{4}\right]_0^1$$

$= 10\left[\frac{2}{5} - \frac{1}{4}\right]$

$= 10\left[\frac{3}{20}\right]$

$= \frac{3}{2}$

To find line integral

$$\int_C pdx + Qdy = \int_{OA} pdx + Qdy + \int_{AO} pdx + Qdy$$

Consider, $\int_{OA} pdx + Qdy$, $y = x^2$, $dy = 2x\,dx$

$x$ varies from 0 to 1

$$= \int_0^1 (3x^2 - 8y^2)\,dx + (4y - 6xy)\,dy$$

$$= \int_0^1 (3x^2 - 8x^4)\,dx + (4x^2 - 6x^3)\,2x\,dx$$

$$= \int_0^1 [3x^2 - 8x^4 + 8x^3 - 12x^4]\,dx$$

$$= \int_0^1 (3x^2 + 8x^3 - 20x^4)\,dx$$

$$= \left[\frac{3x^3}{3} + \frac{8x^4}{4} - \frac{20x^5}{5}\right]_0^1$$

$= \frac{3}{3} + \frac{8}{4} - \frac{20}{5}$

$= 1 + 2 - 4$

$= -1$

Consider, $\int_{AO} pdx + Qdy$, $x = y^2$, $dx = 2y\,dy$

$y$ varies from 1 to 0.

$$= \int_1^0 (3x^2 - 8y^2)\,dx + (4y - 6xy)\,dy$$

$$= \int_1^0 (3y^4 - 8y^2)\, 2y\,dy + (4y - 6y^3)\,dy$$

$$= \int_1^0 (6y^5 - 16y^3 + 4y - 6y^3)\,dy$$

$$= \int_1^0 (6y^5 - 22y^3 + 4y)\,dy$$

$$= \left[ \frac{6y^6}{6} - \frac{22y^4}{4} + \frac{4y^2}{2} \right]_1^0$$

$$= -\left[ 6/6 - 22/4 + 4/2 \right]$$

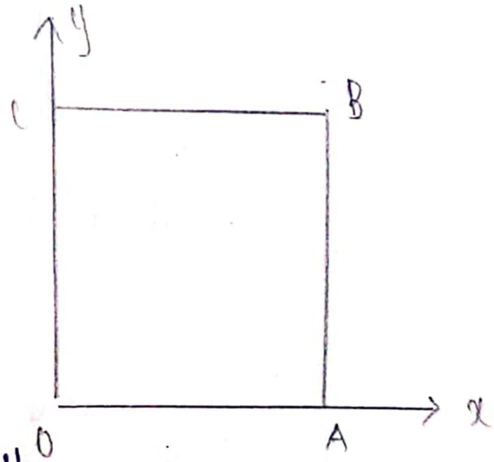
$$= -1 + 11/2 - 2$$

$$= 5/2$$

$$\int_C p\,dx + Q\,dy = -1 + 5/2 = 3/2$$

$\therefore$ Green thm is verified.

Verify green's thm $\int_c (x^2-y^2)\,dx + 2xy\,dy$ where c is the boundary of the rectangular in the xy plane bounded by the lies $x=0$, $x=a$, $y=0$, $y=b$.




Soln: By green's thm

$$\int_c P\,dx + Q\,dy = \iint_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

Consider, R.H.S $= \iint_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$

Gn: $\vec{F} = (x^2-y^2)\,\vec{i} + 2xy\,\vec{j}$

$P = x^2 - y^2$ , $Q = 2xy$

$\frac{\partial P}{\partial y} = -2y$ $\quad \frac{\partial Q}{\partial x} = 2y$

$= \int_0^b \int_0^a (2y + 2y)\, dx\, dy$

$= \int_0^b \int_0^a 4y\, dx\, dy$

$= 4 \int_0^b [x]_0^a\, y\, dy$

$= 4a \int_0^b y\, dy = 4a \left[ \frac{y^2}{2} \right]_0^b = \frac{4ab^2}{2}$

$= 2ab^2$

To find line integral

$\int_C p\,dx + Q\,dy = \int_{OA} p\,dx + Q\,dy + \int_{AB} p\,dx + Q\,dy + \int_{BC} p\,dx + Q\,dy +$

$\int_{CO} p\,dx + Q\,dy$.

Consider, $\int_{OA} p\,dx + Q\,dy$

$y = 0$, $dy = 0$, $x$ Varies from $0$ to $a$

$= \int_0^a (x^2 - y^2)\, dx + 2xy\, dy$

$= \int_0^a x^2\, dx = \left[ \frac{x^3}{3} \right]_0^a = \frac{a^3}{3}$

Consider, $\int_{AB} p\,dx + Q\,dy$

$x = a$, $dx = 0$, $y$ Varies from $0$ to $b$.

$= \int_0^b 2xy\, dy$

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$= 2a\int_0^b y\,dy = 2a\left[\frac{y^2}{2}\right]_0^b = \frac{2ab^2}{2}$

$= ab^2$

Consider $\int_{BC} p\,dx + Q\,dy$

$y = b,\ dy = 0$ $x$ varies from $a$ to $0$.

$= \int_a^0 (x^2 - y^2)\,dx = \int_a^0 (x^2 - b^2)\,dx$

$= \left(\frac{x^3}{3} - b^2 x\right)_a^0 = -\left(\frac{a^3}{3} - ab^2\right)$

$= ab^2 - \frac{a^3}{3}$

Consider, $\int_{CO} p\,dx + Q\,dy$

$x = 0,\ dx = 0$ $y$ varies from $b$ to $0$.

$= \int (x^2 - y^2)\,dx + 2xy\,dy$

$= 0$

$\therefore \int_C p\,dx + Q\,dy = \frac{a^3}{3} + ab^2 - \frac{a^3}{3} + ab^2$

$= 2ab^2$