VECTOR CALCULAS AND FOURIER SERIES

Subject code : 16SCCMM7

Unit : $\underline{\overline{V}}$

Half - Range Fourier Series

Problem for sine series $(0,\pi)$:-

) Find sine series for $f(x) = c$ in $0 < x < \pi$ deduce that

$$\frac{\pi}{4c} = 1 - 1/3 + 1/5 \cdots$$

Soln:-

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

$$b_n = 2/\pi \int_0^{\pi} f(x) \sin nx \, dx$$

$$= 2/\pi \int_0^{\pi} c \sin nx \, dx$$

$$= \frac{2c}{\pi} \left[\frac{-\cos nx}{n} \right]_0^{\pi}$$

$$= \frac{2c}{\pi} \left[\frac{-\cos n\pi}{n} + \frac{\cos 0}{n} \right]$$

$$= \frac{2c}{\pi} \left[\frac{-(-1)^n}{n} + 1/n \right]$$

$$\boxed{b_n = \frac{2c}{\pi} \left[\frac{1 - (-1)^n}{n} \right]}$$

$$f(x) = \sum_{n=1}^{\infty} \frac{2c}{\pi} \left[\frac{1 - (-1)^n}{n} \right] \sin nx$$

$$f(x) = \frac{2c}{\pi} \left[\frac{2 \sin x}{1} + \frac{2 \sin 3x}{3} + \frac{2 \sin 5x}{5} \cdots \right]$$

$$f(x) = \frac{4c}{\pi} \left[\sin x + \sin 3x/3 + \sin 5x/5 \cdots \right]$$

put $x = \pi/2$

$$= \frac{4c}{\pi}\left[\sin(\pi/2) + \frac{\sin 3(\pi/2)}{3} + \frac{\sin 5(\pi/2)}{5} \cdots\right]$$

$$= \frac{4c}{\pi}\left[1 - 1/3 + 1/5 - \cdots\right]$$

$$\boxed{\therefore \frac{\pi}{4c} = 1 - 1/3 + 1/5 - \cdots}$$

If $f(x) = x$ in $(0 < x < \pi/2)$, $\pi - x$ in $(\pi/2 < x < \pi)$ find sine series

(or) $f(x) = \begin{cases} x & 0 < x < \pi/2 \\ \pi - x & \pi/2 < x < \pi \end{cases}$

soln :-

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

$$b_n = 2/\pi \int_0^{\pi} f(x) \sin nx \, dx$$

$$= 2/\pi \int_0^{\pi/2} x \sin nx \, dx + \int_{\pi/2}^{\pi} (\pi - x) \sin nx \, dx$$

$$= 2/\pi \left\{ \left[x \cdot \left(-\frac{\cos nx}{n}\right) - 1\left(-\frac{\sin nx}{n^2}\right) \right\}_0^{\pi/2} + \right.$$

$$\left\{ (\pi - x)\left(-\frac{\cos nx}{n}\right) - (-1)\left(-\frac{\sin nx}{n^2}\right) \right\}_{\pi/2}^{\pi}$$

$$= 2/\pi \left[\frac{\sin n\pi/2}{n^2} - \frac{\sin n(0)}{n^2}\right] + \left[\frac{-\sin n\pi}{n^2} + \frac{\sin n\pi/2}{n^2}\right]$$

$$= 2/\pi \left[1/n^2 + 1/n^2\right]$$

$$\boxed{b_n = \frac{4}{\pi n^2}}$$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

$$= \sum_{n=1}^{\infty} \frac{4}{\pi n^2} \sin nx$$

$$\boxed{f(x) = \frac{4}{\pi}\left[\frac{\sin 1x}{1^2} + \frac{\sin 2x}{2^2} + \frac{\sin 3x}{3^2} \dots\right]}$$

Find sine series $f(x) = x$ in $0 < x < \pi$ (or) prove that

$$f(x) = 2\left[\frac{\sin x}{1} - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} \dots\right]$$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x) \sin nx \, dx$$

$$= \frac{2}{\pi}\left[x\left(-\frac{\cos nx}{n}\right) + 1\left(\frac{-\sin nx}{n^2}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[x\left(\frac{-\cos nx}{n}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\pi\left(\frac{-\cos n\pi}{n}\right)\right]$$

$$\boxed{b_n = 2\left[\frac{-(-1)^n}{n}\right]}$$

$$f(x) = \sum_{n=1}^{\infty} \frac{-2(-1)^n}{n} \sin nx$$

$$= -2\left[\frac{-\sin x}{1} + \frac{\sin 2x}{2} - \frac{\sin 3x}{3} \dots\right]$$

$$= 2\left[\sin x - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} \cdots\right]$$

$$\boxed{\therefore f(x) = 2\left[\sin x - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \cdots\right]}$$

Find sine series $f(x) = x$ in $0 < x < \pi/2$

soln :

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

$$b_n = \frac{2}{\pi}\int_0^{\pi/2} f(x) \sin nx \, dx$$

$$= \frac{2}{\pi}\int_0^{\pi/2} x \sin nx \, dx$$

$$= \frac{2}{\pi}\left[x\left(\frac{-\cos nx}{n}\right) - \left(\frac{-\sin nx}{n^2}\right)\right]_0^{\pi/2}$$

$$= \frac{2}{\pi}\left[\frac{\sin n\pi/2}{n^2}\right]$$

$$\boxed{b_n = \frac{2}{n^2\pi}}$$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

$$= \sum_{n=1}^{\infty} 2/n^2\pi \sin nx$$

$$= \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{\sin nx}{n^2}$$

$$\boxed{f(x) = \frac{2}{\pi}\left[\frac{\sin 1x}{1^2} + \frac{\sin 2x}{2^2} + \frac{\sin 3x}{3^2} \cdots\right]}$$

Find sine series

i) $f(x) = 0$ in $0 < x < \pi/2$; $f(x) = c$ in $\pi/2 < x < \pi$

ii) $f(x) = \pi x - x^2$ in $0 < x < \pi$

iii) $f(x) = \pi - x$ in $0 < x < \pi$

Soln:

i) $f(x) = \sum_{n=1}^{\infty} b_n \sin nx$

$$b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx \, dx$$

$$= \frac{2}{\pi} \left[\int_0^{\pi/2} 0 \, dx + \int_{\pi/2}^{\pi} c \sin nx \, dx \right]$$

$$= \frac{2c}{\pi} \left[\frac{-\cos nx}{n} \right]_{\pi/2}^{\pi}$$

$$= \frac{2c}{\pi} \left[-\frac{\cos n\pi}{n} + \frac{\cos n\pi/2}{n} \right]$$

$$\boxed{b_n = \frac{2c}{\pi} \left[\frac{\cos n\pi/2 - (-1)^n}{n} \right]}$$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

$$= \sum_{n=1}^{\infty} \frac{2c}{\pi} \left[\frac{\cos n\pi/2 - (-1)^n}{n} \right] \sin nx$$

$$= \frac{2c}{\pi} \sum_{n=1}^{\infty} \left(-\frac{(-1)^n}{n} \sin nx \right)$$

$$\boxed{f(x) = \frac{2c}{\pi} \left[\frac{\sin x}{1} - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \cdots \right]}$$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} (\pi x - x^2)\sin nx\,dx$$

$$= \frac{2}{\pi}\left\{\left[(\pi x - x^2)\left(-\frac{\cos nx}{n}\right) - (\pi - 2x)\left(-\frac{\sin nx}{n^2}\right) + (-2)\frac{\cos nx}{n^3}\right]\right\}_0^{\pi}$$

$$= \frac{2}{\pi}\left[(-2)\left(\frac{(-1)^n}{n^3} + \frac{2}{n^3}\right)\right]$$

$$= \frac{2}{\pi}(2)\left[\frac{1}{n^3} - \frac{(-1)^n}{n^3}\right]$$

$$\boxed{b_n = \frac{4}{\pi}\left[\frac{1-(-1)^n}{n^3}\right]}$$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

$$= \sum_{n=1}^{\infty} \frac{4}{\pi}\left[\frac{1-(-1)^n}{n^3}\right]\sin nx$$

$$= \frac{4}{\pi}\left[\frac{2\sin x}{1^3} + \frac{2\sin 3x}{3^3} \cdots\right]$$

$$\boxed{\therefore f(x) = \frac{8}{\pi}\left[\frac{\sin x}{1} + \frac{\sin 3x}{3^3} + \frac{\sin 5x}{5^3} \cdots\right]}$$

iii) $f(x) = \sum_{n=1}^{\infty} b_n \sin nx$

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx$$

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$$= \frac{2}{\pi}\int_0^{\pi}(\pi - x)\sin nx\,dx$$

$$= \frac{2}{\pi}\left[(\pi - x)\left(-\frac{\cos nx}{n}\right) - (-1)\left(-\frac{\sin nx}{n^2}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[-(\pi - x)\frac{\cos nx}{n} - \frac{\sin nx}{n^2}\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[-(\pi - \pi)\frac{\cos n\pi}{n} + (\pi - 0)\frac{\cos(0)}{n}\right]$$

$$= \frac{2}{\pi}\left[0 + \pi(1/n)\right]$$

$$= 2/\pi \times \pi/n$$

$$\boxed{b_n = 2/n}$$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx\,dx$$

$$= \sum_{n=1}^{\infty} 2/n \sin nx\,dx$$

$$\boxed{f(x) = 2\left[\frac{\sin x}{1} + \frac{\sin 2x}{2} + \frac{\sin 3x}{3} \cdots\right]}$$

Find cosine series $f(x) = x$ in $0 < x < \pi$

soln:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

$$a_0 = \frac{2}{\pi} \int_0^{\pi} f(x)\, dx$$

$$= \frac{2}{\pi} \int_0^{\pi} x\, dx$$

$$= \frac{2}{\pi} \left[\frac{x^2}{2} \right]_0^{\pi} = 2/\pi \left(\pi^2/2 \right)$$

$$a_0 = \pi$$

$$a_n = 2/\pi \int_0^{\pi} f(x) \cos nx\, dx$$

$$= 2/\pi \int_0^{\pi} x \cos nx\, dx$$

$$= 2/\pi \left[x \left(\frac{\sin nx}{n} \right) - \left(\frac{-\cos nx}{n^2} \right) \right]_0^{\pi}$$

$$= 2/\pi \left[(-1)^n / n^2 - \left(\frac{\cos 0}{n^2} \right) \right]$$

$$= 2/\pi \left[\frac{(-1)^n}{n^2} - \frac{1}{n^2} \right]$$

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$$a_n = \frac{2}{\pi}\left[\frac{(-1)^n - 1}{n^2}\right]$$

$$f(x) = \frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{2}{\pi} \frac{(-1)^n - 1}{n^2} \cos nx$$

$$= \frac{\pi}{2} + \frac{2}{\pi}\left[\sum_{n=1}^{\infty} \frac{(-1)^n - 1}{n^2} \cos nx\right]$$

$$= \frac{\pi}{2} + \frac{2}{\pi}\left[\frac{-2\cos x}{1^2} - \frac{2\cos 3x}{3^2} \cdots\right]$$

$$f(x) = \frac{\pi}{2} - \frac{4}{\pi}\left[\cos x + \cos 3x/3^2 \cdots\right]$$

Find cosine series $f(x) = \pi - x$ in $0 < x < \pi$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} (\pi - x)\,dx$$

$$= \frac{2}{\pi}\left[\pi x - \frac{x^2}{2}\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\pi^2 - \frac{\pi^2}{2}\right]$$

$$= \frac{2}{\pi}\left[\frac{\pi^2}{2}\right]$$

$$a_0 = \pi$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} (\pi - x) \cos nx\,dx$$

$$= \frac{2}{\pi}\left[(\pi - x)\frac{\sin nx}{n} - (-1)\left(-\frac{\cos nx}{n^2}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[-\frac{(-1)^n}{n^2} + \frac{1}{n^2}\right]$$

$$a_n = \frac{2}{\pi}\left[\frac{1 - (-1)^n}{n^2}\right]$$

$$f(x) = \pi/2 + \sum_{n=1}^{\infty} 2/\pi \left(\frac{1-(-1)^n}{n^2} \right) \cos nx$$

$$= \pi/2 + 2/\pi \left[\sum_{n=1}^{\infty} \frac{1-(-1)^n}{n^2} \right] \cos nx$$

$$= \pi/2 + 2/\pi \left[\frac{2\cos x}{1^2} + \frac{2\cos 3x}{3^2} \cdots \right]$$

$$f(x) = \pi/2 + 4/\pi \left[\cos x + \frac{\cos 3x}{3^2} \cdots \right]$$

) Find cosine series $f(x) = \begin{cases} 0 & 0 < x < \pi/2 \\ c & \pi/2 < x < \pi \end{cases}$ show that

$$f(x) = 2c/\pi + \pi/4 \left[-\cos x + \cos 3x/3 - \cos 5x/5 \cdots \right]$$

<u>soln</u>

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

$$a_0 = 2/\pi \int_0^{\pi} f(x)\, dx$$

$$= 2/\pi \int_0^{\pi/2} 0\, dx + \int_{\pi/2}^{\pi} c\, dx$$

$$= \frac{2c}{\pi} \left[x \right]_{\pi/2}^{\pi}$$

$$= \frac{2c}{\pi} \left[\pi - \pi/2 \right]$$

$$= 2c/\pi \times \pi/2$$

$$a_0 = c$$

$$a_n = 0 + 2/\pi \int_{\pi/2}^{\pi} c \cos nx\, dx$$

$$= 2c/\pi \left[\frac{\sin nx}{n} \right]_{\pi/2}^{\pi}$$

$$= \frac{2c}{\pi}\left[\frac{-\sin \pi/2}{n}\right]$$

$$= \frac{2c}{\pi}\left[-1/n\right]$$

$$a_n = \frac{-2c}{\pi}$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

$$= c/2 + \sum_{n=1}^{\infty} \frac{-2c}{\pi} \cos nx$$

$$f(x) = c/2 - \frac{2c}{\pi}\left[\frac{\cos x}{1} + \frac{\cos 2x}{2} + \cdots\right]$$

Find cosine series $f(x) = \sin x$ in $0 < x < \pi$ S.T

$$4/\pi\left[1/2 - \frac{\cos 2x}{3} - \frac{\cos 4x}{15} - \frac{\cos 6x}{35} \cdots \frac{\cos 2nx}{4n^2-1}\right]$$

Soln :-

Let the required fourier cosine series be

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \text{ where}$$

$$a_0 = 2/\pi \int_0^{\pi} f(x)\, dx$$

$$= 2/\pi \int_0^{\pi} \sin x\, dx$$

$$= 2/\pi \left[-\cos x\right]_0^{\pi} = \frac{-2}{\pi}\left[\cos x\right]_0^{\pi} = \frac{-2}{\pi}\left[-1-1\right] = \frac{4}{\pi}$$

$$a_0 = \frac{4}{\pi}$$

$$a_n = 2/\pi \int_0^{\pi} f(x) \cos nx\, dx = 2/\pi \int_0^{\pi} \sin x \cos nx\, dx$$

$$= 2/\pi \int_0^{\pi} \cos nx \sin x\, dx = \frac{2}{2\pi}\int_0^{\pi}$$

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$$= \frac{2}{\pi} \cdot \frac{1}{2} \int_0^{\pi} (\sin(1+n)x + \sin(1-n)x \, dx$$

$$= \frac{1}{\pi} \left[\frac{-\cos(1+n)x}{1+n} - \frac{\cos(1-n)x}{1-n} \right]_0^{\pi}$$

$$= \frac{-1}{\pi} \left[\frac{\cos(1+n)\pi}{1+n} + \frac{\cos(1-n)\pi}{(1-n)} - \frac{\cos(1+n)0}{1+n} - \frac{\cos(1-n)0}{1-n} \right]$$

$$= \frac{-1}{\pi} \left[\frac{1}{1+n} + \frac{1}{1-n} - \frac{1}{1+n} - \frac{1}{1-n} \right]$$

$a_n = 0$

when n is even $(1+n)$ and $(1-n)$ is odd

n is even

$\cos(1+n)\pi = -1$

$\cos(1-n)\pi = -1$

$$a_n = \frac{-1}{\pi} \left[\frac{-1}{1+n} - \frac{1}{1-n} - \frac{1}{1+n} - \frac{1}{1-n} \right]$$

$$= \frac{1}{\pi} \left[\frac{2}{1+n} + \frac{2}{1-n} \right]$$

$$= \frac{-2}{\pi} \left[\frac{1-n+1+n}{1-n^2} \right]$$

$$a_n = \frac{4}{\pi} \left[\frac{1}{1-n^2} \right]$$

$$f(x) = \frac{a_0}{2} + \sum_{n=2,4}^{\infty} a_n \cos nx \, dx$$

$$= \frac{4/\pi}{2} + \sum_{n=2,4}^{\infty} \frac{-4}{\pi} \left[\frac{1}{1-n^2} \right] \cos nx$$

$$f(x) = \frac{2}{\pi} - \frac{4}{\pi} \sum_{n=2}^{\infty} \frac{1}{1-n^2} \cos nx$$

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put in eqn ①

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

$$= \frac{2}{\pi}(e^{\pi}-1) \times 1/2 + \sum_{n=1}^{\infty} \frac{2}{\pi(1+n^2)}(e^{\pi}(-1)^n - 1) \cos nx$$

$$f(x) = \frac{e^{\pi}-1}{\pi} + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{e^{\pi}(-1)^n - 1}{1+n^2} \cos nx$$

Write the half range cosine series in the limit $(0, l)$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{l}$$

$$a_0 = \frac{2}{l} \int_0^l f(x)\,dx$$

$$a_n = \frac{2}{l} \int_0^l f(x) \cos \frac{n\pi x}{l}\,dx$$

Problem based for cosine series in the limit $0 < x < l$

Find a cosine series for $f(x) = x$ in $0 < x < 1$

Soln :-

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{l} \rightarrow ①$$

$$a_0 = 2/l \int_0^l f(x)\,dx$$

Here, $l = 1$

$$a_0 = 2\int_0^1 f(x)\,dx = 2\left[\frac{x^2}{2}\right]_0^1$$

$$a_0 = 1$$

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}\,dx$$

$l = 1$

$$a_n = 2\int_0^1 x\cos n\pi x\,dx$$

$$= 2\left[x\left(\frac{\sin n\pi x}{n\pi}\right) - (1)\left(-\frac{\cos n\pi x}{n^2\pi^2}\right)\right]_0^1$$

$$= 2\left[\frac{\sin n\pi}{n\pi} + \frac{\cos n\pi}{n^2\pi^2} - \frac{\cos(0)}{n^2\pi^2}\right]$$

$$= 2\left[\frac{\cos n\pi}{n^2\pi^2} - \frac{1}{n^2\pi^2}\right]$$

$$= 2\left[\frac{(-1)^n - 1}{n^2\pi^2}\right]$$

$$a_n = \frac{2}{n^2\pi^2}\left[(-1)^n - 1\right]$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

$$= \frac{1}{2} + \sum_{n=1\cdot 3\cdot 5}^{\infty} \frac{2}{n^2\pi^2}\left((-1)^n - 1\right)\cos nx$$

$$f(x) = \frac{1}{2} + \frac{2}{\pi^2}\sum_{n=1\cdot 3\cdot 5}^{\infty} \frac{(-1)^n - 1}{n^2}\cos nx$$

Find the cosine series $f(x) = x(l-x)$ in the interval $(0,l)$

soln:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} \rightarrow (1)$$

$$a_0 = \frac{2}{l}\int_0^l f(x)\,dx$$

$$= \frac{2}{l}\int_0^l (xl - x^2)\,dx$$

$$= \frac{2}{l}\left[\frac{x^2}{2}l - \frac{x^3}{3}\right]_0^l$$

$$= \frac{2}{l}\left[l\cdot\frac{l^2}{2} - \frac{l^3}{3}\right]$$

$$= \frac{2}{l}\left[\frac{3l^3 - 2l^3}{6}\right]$$

$$= \frac{2}{l}\left[\frac{l^3}{6}\right]$$

$$a_0 = \frac{l^2}{3}$$

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}\,dx$$

$$= \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}\,dx$$

$$= \frac{2}{l}\int_0^l (xl - x^2)\cos\frac{n\pi x}{l}\,dx$$

$$= \frac{2}{l}\left\{(lx - x^2)\frac{\sin\frac{n\pi x}{l}}{\frac{n\pi}{l}} - (l-2x)\frac{-\cos\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}} + (-2)\left(\frac{-\sin\frac{n\pi x}{l}}{\frac{n^3\pi^3}{l^3}}\right)\right]_0$$

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$$= \frac{2}{l}\left[\left\{ (l-2l)\frac{\cos n\pi}{\frac{n^2\pi^2}{l^2}} + 0 \right\} - \left\{ (l-0)\left(\frac{\cos 0}{\frac{n^2\pi^2}{l^2}}\right) + 0 \right\} \right]$$

$$= \frac{2}{l}\left[\frac{-l\cos n\pi}{\frac{n^2\pi^2}{l^2}} - \frac{l}{\frac{n^2\pi^2}{l^2}} \right]$$

$$= \frac{2}{l} \times l \left[\frac{-\cos n\pi}{\frac{n^2\pi^2}{l^2}} - \frac{1}{\frac{n^2\pi^2}{l^2}} \right] = \frac{-2l^2}{n^2\pi^2}\left[\cos n\pi + 1\right]$$

$$a_n = \frac{-2l^2}{n^2\pi^2}\left[(-1)^n + 1\right]$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}$$

$$= \frac{l^2}{2\times 3} + \sum_{n=2,4,6}^{\infty} \frac{-4l^2}{n^2\pi^2}\cos\frac{n\pi x}{l}$$

$$f(x) = \frac{l^2}{6} - \frac{4l^2}{\pi^2}\sum_{n=2,4\dots}^{\infty} \frac{1}{n^2}\cos\frac{n\pi x}{l}$$

Problems in the Interval $(0,2l)$

Note :

* Find fourier series in the interval $(0,4)$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

$$a_0 = \frac{1}{l}\int_0^{2l} f(x)\,dx$$

$$a_n = \frac{1}{l}\int_0^{2l} f(x)\cos\frac{n\pi x}{l}\,dx$$

$$b_n = \frac{1}{l}\int_0^{2l} f(x)\sin\frac{n\pi x}{l}\,dx$$

Here; $2l = 4$

$l = 4/2$

$l = 2$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{2} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{2}$$

$$a_0 = \frac{1}{2}\int_0^4 f(x)\,dx$$

$$a_n = \frac{1}{2}\int_0^4 f(x)\cos\frac{n\pi x}{2}\,dx$$

$$b_n = \frac{1}{2}\int_0^4 f(x)\sin\frac{n\pi x}{2}\,dx$$

Explain briefly change of Interval:-

In practice, we often require to find a fourier series for an interval which is not of length 2π

In many problems, the period of the function to be expanded is not 2π, but some other interval say $2l$.

Suppose $f(x)$ is defined in the interval $(-l, l)$

Let $z = \frac{\pi x}{l}$, Hence $x = \frac{l z}{\pi}$

Also, when $x = -l$ we have $z = -\pi$ and

when $x = l$ we have $z = \pi$ and

Hence, the function $F(z) = f\left(\frac{l z}{\pi}\right)$ is defined in the interval $(-\pi, \pi)$. The fourier series of $f(z)$ is given by

$$F(z) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nz + \sum_{n=1}^{\infty} b_n \sin nz$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(z)\,dz, \quad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(z) \cos nz\,dz$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(z) \sin nz\,dz$$

$$f\left(\frac{lz}{\pi}\right) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nz + \sum_{n=1}^{\infty} b_n \sin nz$$

where, $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f\left(\frac{lz}{\pi}\right) dz$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f\left(\frac{lz}{\pi}\right) \cos nz\,dz$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f\left(\frac{lz}{\pi}\right) \sin nz\,dz$$

Now $x = \frac{lz}{\pi}$, $dx = \frac{1}{\pi}dz$

Thus $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$

where $a_0 = \frac{1}{l}\int_{-l}^{l} f(x)\,dx,$

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x) \cos\left(\frac{n\pi x}{l}\right) dx$$

$$b_n = \frac{1}{l}\int_{-l}^{l} f(x) \sin\left(\frac{n\pi x}{l}\right) dx$$

Find the fourier series expansion of period $2l$ for the function $f(x) = (l-x)^2$ in the range $0 < x < 2l$.

soln

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} \rightarrow (1)$$

$$a_0 = \frac{1}{l}\int_0^{2l} f(x)\,dx$$

$$= \frac{1}{l}\int_0^{2l} (l-x)^2 dx = \frac{1}{l}\left[\frac{(l-x)^3}{-3}\right]_0^{2l}$$

$$= \frac{(l-2l)^3 - (l-0)^3}{-3l} = \frac{-l^3 - l^3}{-3l}$$

$$= \frac{2l^3}{3l}$$

$$a_0 = \frac{2}{3} l^2 \rightarrow (2)$$

$$a_n = \frac{1}{l}\int_0^{2l} f(x) \cos\frac{n\pi x}{l}\,dx$$

$$= \frac{1}{l}\int_0^{2l} (l-x)^2 \cos\frac{n\pi x}{l}\,dx$$

$$= \frac{1}{l}\left[\left\{ l-x^2\left(\frac{\sin\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right) - 2(l-x)(-1)\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right) + 2\left(\frac{-\sin\frac{n\pi x}{l}}{\frac{n^3\pi^3}{l^3}}\right)\right]\right.$$

$$= \frac{1}{l}\left\{(l-x^2)\left(\frac{\sin\frac{n\pi x}{l}}{n\pi/l}\right) - 2(l-x)\left(\frac{\cos\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right) - 2\left(\frac{\sin\frac{n\pi x}{l}}{\frac{n^3\pi^3}{l^3}}\right)\right\}_0^{2l}$$

$$= \frac{1}{l}\left\{(l-2l)^2(0) - 2(l-2l)\left(\frac{\cos\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right) - (0)\right\}$$

$$- \left\{0 - 2(l-0)\left(\frac{\cos(0)}{\frac{n^2\pi^2}{l^2}}\right) - 0\right\}$$

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$$= \frac{1}{l}\left[\frac{2l}{\frac{n^2\pi^2}{l^2}} + \frac{2l}{\frac{n^2\pi^2}{l^2}}\right] \qquad \because \sin 2n\pi = 0, \quad \cos 2n\pi = 1$$

$$= \frac{1}{l}\left[\frac{4l}{\frac{n^2\pi^2}{l^2}}\right]$$

$$a_n = \frac{4l^2}{n^2\pi^2} \rightarrow (3)$$

$$b_n = \frac{1}{l}\int_0^{2l} f(x) \sin\frac{n\pi x}{l}\, dx$$

$$= \frac{1}{l}\int_0^{2l} (l-x)^2 \sin\frac{n\pi x}{l}\, dx$$

$$= \frac{1}{l}\left[(l-x)^2\left\{\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right\} - 2(l-x)(-1)\left\{\frac{-\sin\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right\} + 2\left\{\frac{\cos\frac{n\pi x}{l}}{\frac{n^3\pi^3}{l^3}}\right\}\right]_0^{2l}$$

$$= \frac{1}{l}\left\{-(l-x)^2\left(\frac{\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right) - 2(l-x)\left(\frac{\sin\frac{n\pi x}{l}}{\frac{n^2\pi^2}{l^2}}\right) + 2\left(\frac{\cos\frac{n\pi x}{l}}{\frac{n^3\pi^3}{l^3}}\right)\right\}_0^{2l}$$

$$= \frac{1}{l}\left\{-(l-2l)^2\left(\frac{\cos 2n\pi}{\frac{n\pi}{l}}\right) - 2(l-2l)(0) + \frac{2\cos 2n\pi}{\frac{n^3\pi^3}{l^3}}\right\}$$

$$- \left\{-(l-0)^2\left(\frac{\cos(0)}{\frac{n\pi}{l}}\right) - 2(l-0)(0) + \left(\frac{2\cos 0}{\frac{n^3\pi^3}{l^3}}\right)\right\}$$

$$= \frac{1}{l}\left[\frac{-l^2\cos 2n\pi}{\frac{n\pi}{l}} + \frac{2\cos 2n\pi}{\frac{n^3\pi^3}{l^3}} + \frac{l^2}{\frac{n\pi}{l}} - \frac{2}{\frac{n^3\pi^3}{l^3}}\right]$$

$$= \frac{1}{l}\left[\frac{-l^2}{\frac{n\pi}{l}} + \frac{2}{\frac{n^3\pi^3}{l^3}} + \frac{l^2}{\frac{n\pi}{l}} - \frac{2}{\frac{n^3\pi^3}{l^3}}\right]$$

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$b_n = 0 \rightarrow (4)$

$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}$

$f(x) = \frac{l^2}{3} + \frac{4l^2}{\pi^2} \sum_{n=1}^{\infty} \frac{\cos n\pi x}{l n^2}$

obtain fourier series $f(x) = \begin{cases} l-x & 0 < x < l \\ 0 & l < x < 2l \end{cases}$

soln :

$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$

$a_0 = \frac{1}{l}\int_0^{2l} f(x)\,dx$

$= \frac{1}{l}\left[\int_0^{l} f(x)\,dx + \int_l^{2l} f(x)\,dx\right]$

$= \frac{1}{l}\left[\int_0^{l} (l-x)\,dx\right]$

$= \frac{1}{l}\left[\frac{(l-x)^2}{-2}\right]_0^{l}$

$= \frac{1}{l}\left[\frac{(l-l)^2}{-2} - \frac{(l-0)}{-2}\right]$

$= \frac{-1}{2l}\left[0 - l^2\right]$

$a_0 = \frac{l}{2} \rightarrow (2)$

$a_n = \frac{1}{l}\int_0^{2l} f(x) \cos\frac{n\pi x}{l}\,dx$

$= \frac{1}{l}\int^{l} f(x) \cos\frac{n\pi x}{l}\,dx + \int_l^{2l} f(x) \cos\frac{n\pi x}{l}\,dx$

$$bn = \frac{l}{n\pi} \rightarrow (4)$$

substituting (2), (3) & (4) in (1)

$$f(x) = \frac{1}{4} + \sum_{n=1\cdot3\cdot5}^{\infty} \frac{2l}{n^2\pi^2} \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} \frac{l}{n\pi} \sin\frac{n\pi x}{l}$$

$$f(x) = \frac{1}{4} + \frac{2l}{\pi^2} \sum_{n=1\cdot3}^{\infty} \frac{\cos n\pi x/l}{n^2} + \frac{l}{\pi} \sum_{n=1}^{\infty} \frac{\sin n\pi x/l}{n}$$

Find the fourier series expansion of $f(x) = e^x$ in $(0, 2l)$

Soln:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

$$a_0 = \frac{1}{l}\int_0^{2l} e^x\,dx = \frac{1}{l}\left[e^x\right]_0^{2l}$$

$$= \frac{1}{l}\left[e^{2l} - e^0\right] = \frac{1}{l}\left[e^{2l} - 1\right]$$

$$\therefore a_0 = \frac{1}{l}\left[e^{2l} - 1\right]$$

$$a_n = \frac{1}{l}\int_0^{2l} f(x)\cos\frac{n\pi x}{l}\,dx = \frac{1}{l}\int_0^{2l} e^x \cos\frac{n\pi x}{l}\,dx$$

$$= \frac{1}{l}\left[\frac{e^x}{1 + \frac{n^2\pi^2}{l^2}}\left(\cos\frac{n\pi x}{l} + \frac{n\pi}{l}\sin\frac{n\pi x}{l}\right)\right]_0^{2l}$$

$$= \frac{1}{l}\left[\frac{l^2 e^x}{l^2 + n^2\pi^2}\left(\cos\frac{n\pi x}{l} + \frac{n\pi}{l}\sin\frac{n\pi x}{l}\right)\right]_0^{2l}$$

$$= \frac{l}{l^2+n^2\pi^2}\left[e^{x}\left(\cos\frac{n\pi x}{l} + \frac{n\pi}{l}\sin\frac{m\pi x}{l}\right)\right]_0^{2l}$$

$$= \frac{l}{l^2+n^2\pi^2}\left[e^{2l}(1+0) - 1(1+0)\right]$$

$$\therefore a_n = \frac{l}{l^2+n^2\pi^2}\left[e^{2l}-1\right]$$

$$b_n = \frac{1}{l}\int_0^{2l} f(x)\sin\frac{n\pi x}{l}\,dx$$

$$= \frac{1}{l}\int_0^{2l} e^{x}\sin\frac{n\pi x}{l}\,dx$$

$$= \frac{1}{l}\left[\frac{e^{x}}{1+\frac{n^2\pi^2}{l^2}}\left(\sin\frac{n\pi x}{l} - \frac{n\pi}{l}\cos\frac{n\pi x}{l}\right)\right]_0^{2l}$$

$$= \frac{l}{l}\left[\frac{l^2 e^{x}}{l^2+n^2\pi^2}\left(\sin\frac{n\pi x}{l} - \frac{n\pi}{l}\cos\frac{m\pi x}{l}\right)\right]_0^{2l}$$

$$= \frac{l}{l^2+n^2\pi^2}\left[e^{x}\left(\sin\frac{n\pi x}{l} - \frac{n\pi}{l}\cos\frac{n\pi x}{l}\right)\right]_0^{2l}$$

$$= \frac{l}{l^2+n^2\pi^2}\left[e^{2l}\left(0-\frac{n\pi}{l}\right) - 1\left(0-\frac{n\pi}{l}\right)\right]$$

$$= \frac{l}{l^2+n^2\pi^2}\left[e^{2l}\left(-\frac{n\pi}{l}\right) + \left(\frac{n\pi}{l}\right)\right]$$

$$= \frac{l}{l^2+n^2\pi^2}\left(\frac{n\pi}{l}\right)\left[e^{-2l}+1\right]$$

$$= \frac{n\pi}{l^2+n^2\pi^2}\left[1-e^{2l}\right]$$

$$\therefore b_n = \frac{-n\pi}{l^2+n^2\pi^2}\left[e^{2l}-1\right]$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

$$= \frac{e^{2l}-1}{2} + \sum_{n=1}^{\infty} \frac{l}{l^2+n^2\pi^2}(e^{2l}-1)\cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} \frac{-n\pi}{l^2+n^2\pi^2}(e^{2l}-1)\sin\frac{n\pi x}{l}$$

$$= \frac{e^{2l}-1}{2} + l(e^{2l}-1)\sum_{n=1}^{\infty} \frac{1}{l^2+n^2\pi^2}\cos\frac{n\pi x}{l} - \pi(e^{2l}-1)\sum_{n=1}^{\infty} \frac{n}{l+n^2\pi^2}\sin\frac{n\pi x}{l}$$