Linear Independence :

In $V_3(R)$ let $s = \{e_1, e_2, e_3\}$ we have seen that $L(S) = V_3 R$ Thus s is subset of $V_3(R)$ which spans the whole space $V_3(R)$

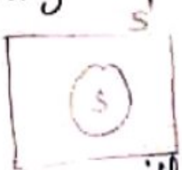
Definition :- finite dimensional

Let V be a vector space over a field F V is said to be finite dimensional if there exists a finite subset S of V such that $L(S) = V$.

Theorem :

Any subset of a linearly independent set in linearly independent.

proof :- Let V be a vector space over a field F, let $S = \{v_1, v_2, \ldots v_n\}$ be a linearly independent set



Let S' be a subset of S without loss of generality we takes $S' = \{v_1, v_2 \ldots v_K$ where $k \leq n$

Suppose S' is a linearly dependent

Then $\exists$ $\alpha_1, \alpha_2 \ldots \alpha_K$ in F not all zer

$$\alpha_1 v_1 + \alpha_2 . v_2 + \ldots + \alpha_K v_K = 0$$

Linear Independence :

In $V_3(R)$ let $S = \{e_1, e_2, e_3\}$ we have seen that $L(S) = V_2 R$ Thus S is subset of $V_3(R)$ which spans the whole space $V_3(R)$

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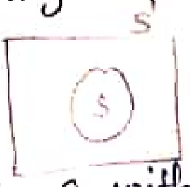

Let S' be a subset of S without loss of generality we takes $S' = \{v_1, v_2 \dots v_k$ where $k \le n$

Suppose S' is a linearly dependent

Then ∃ $\alpha_1, \alpha_2 \dots \alpha_k$ in F not all zer

$$\alpha_1 v_1 + \alpha_2 . v_2 + \dots + \alpha_k v_k = 0$$

hence $\alpha_1 v_1 + \alpha_2 v_2 + \cdots \alpha_K v_K + 0 v_{K+1} + \cdots + 0 v_n = 0$

is a non-trivial linear combination giving the zero vector.

Hence S is a linearly dependent set which is a contradiction.

Hence S' is linearly independent.

⊗ 2M **Definition** : linearly independent

Let V be vector space over a field F. A finite set of vectors $v_1 v_2 \ldots v_n$ in V is said to be linearly independent if

$$\alpha_1 v_1 + \alpha_2 v_2 + \cdots \alpha_n v_n = 0$$

$$\Rightarrow \alpha_1 = \alpha_2 = \cdots = \alpha_n = 0$$

if $v_1 v_2 \ldots v_n$ are not linearly independent then they are said to be linearly dependent.

Note :-

If $v_1 v_2 \ldots v_n$ are linearly dependent then $\exists$ scalars $\alpha_1 \alpha_2 \ldots \alpha_n$ not all zero, $\ni \alpha_1 v_1 + \cdots \alpha_n v_n = 0$

Theorem :-

⊗ · Any set containing a linearly dependent set is also linearly dependent

of:

Let V be a vector space

Let S be a linearly dependent set

Let $S' \supset S$

if S' is linearly independent S is also linearly independent (by theo 5.11) which is a contradiction.

Hence S' is linearly dependent.

Theorem :-

Let $S = \{v_1 v_2 \dots v_n\}$ be a linearly independent set of vector in a vector space V over a field F. Then every elements of $L(S)$ can be uniquely written in the from $\alpha_1 v_1 + \alpha_2 v_2 + \dots \alpha_n v_n$ where $\alpha_i \in F$

proof :- By definition every elements of $L(S)$ is of the form $\alpha_1 v_1 + \alpha_2 v_2 + \dots \alpha_n v_n$

now. let.

$$\alpha_1 v_1 + \alpha_2 v_2 + \dots \alpha_n v_n = \beta_1 v_1 + \beta_2 v_2 \dots \beta_n v_n$$

Hence $(\alpha_1 - \beta_1) v_1 + (\alpha_2 - \beta_2) v_2 + \dots (\alpha_n - \beta_n) v_n = 0$

Since S is a linearly independent set

$$\alpha_i - \beta_i = 0 \ \forall i$$

$\therefore \alpha_i = \beta_i \ \forall i$ Hence the theorem.

Theorem :-

$S = \{v_1, v_2 \dots v_n\}$ is a linearly depen

set of vectors in V iff $\exists$ a vector $v_k \in$

$\ni v_k$ is a linear combination of the proceed

$\alpha_3 = K$

These values of $\alpha_1, \alpha_2\ \alpha_3$ for any K satisfy eqn- (3) also

if K = 1 then

$\alpha_1 = -2,\ \alpha_2 = -3,\ \alpha_3 = 1$ as a non-trivial solution

Hence the three vectors are linearly dependent

$$\begin{vmatrix} 1 & -2 & -4 \\ 4 & 1 & 11 \\ -2 & 3 & 5 \end{vmatrix} = 1(5-33) + 2(20+22) - 4(12+2)$$

$$= -28 + 84 - 56 = 0$$

Hence the three vectors are linearly . depende

4) Let V be a vector space over a field F Then any subset S of V containing the zero vector is linearly dependent

proof: Let, $S = \{0, v_1, \ldots v_n\}$

clearly . $\alpha 0 + 0v_1 + \ldots + 0v_n = 0$ Where α is any element of F.

Hence for any $\alpha \neq 0$, we get a n trivial Linear Combination of vectors in S giving the zero vector.

Hence S is linearly dependent.

Theorem 5.15

1) Let V be vector space over F. Let $S = \{v_1, v_2 \ldots v_n\}$ and $L(S) = W$ then $\exists$ a linearly independent subset S' of $S \ni L(S) = W$

proof :- Let, $S = \{v_1, v_2 \ldots v_n\}$ $L(S') =$

If S is linearly independent there is nothing to prove.

If not, Let v_K be the first vector in S which is a linear combination of the preceding vectors.

Let, $S_1 = \{v_1, v_2 \ldots v_{K-1}, v_{K+1} \ldots v_n\}$

i,e S_1 is Obtained by deleting the vector v_K from S.

we claim that

$$L(S_1) = L(S) = W$$

Since $S_1 \subseteq S$. $L(S_1) \subseteq L(S)$ [refer theorem 5.10]

Now, Let $v \in L(S)$

Then,

$$v = \alpha_1 v_1 + \alpha_2 v_2 + \cdots \alpha_K v_K + \ldots + \alpha_n v_n \rightarrow (1)$$

Now, v_K is a linear combination of the preceding vector.

Let, $v_K = \beta_1 v_1 + \cdots + \beta_{K-1} v_{K-1}$

Hence,

$$v = \alpha_1 v_1 + \ldots + \alpha_{K-1} v_{K-1} + \alpha_K (\beta_1 v_1 + \ldots + \beta_{K-1} v_{K-1})$$

$$+ \alpha \quad v \quad \ldots + \alpha_n v_n$$

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$\therefore$ v can be expressed as a linear combination of the vectors of S_1. So that, $v \in L(S_1)$

Hence $L(S) \subseteq L(S_1)$

Thus $L(S) = L(S_1) = W$

Now, if S_1 is linearly independent the proof is complete. if not, we continue the above process of removing a vector ~~form~~ from S_1, which is linear combination of the preceding vectors untill we arrive at a linearly independent subset S' of $S \ni L(S') = W$.

Basis and dimension

Definition :- (basis)

A linearly independent subset S of a vector space over V which spans the whole space is called basis of the vector space.

Theorem :-

Any finite dimensional vector space V Contains a finite number of linearly independ vectors which span V (i,e..) A finite dimens-n vector space has a basis consisting of a finite number of vectors.

proof :-

Since V is finite dimensional ∃ a finite subset S of V ∋ $L(S) = V$ [by theorem 5.15]

this set S contains a linearly independent subset

$$S' = \{v_1 v_2 \dots v_n\}$$

$$\ni L(S') = L(S) = V$$

Hence S' is a basis for V.

2) Theorem :-

.17

Let V be a vector space over a field F. Then $S = \{v_1 v_2 \dots v_n\}$ is a basis for V iff every element of V can be uniquely expressed as a linear combination of element of S

proof : Let S be a basis for V then by definition of basis S is linearly independent and $L(S) = V$

Hence by theorem 5.13 every element of V can be uniquely expressed as a linear combination of element of S.

Conversely,

Suppose every element of V can be uniquely expressed as a linear combinatic of element of S

clearly $L(S) = V$

Now, $+ \dots + \alpha_n v_n = 0$

Also. $0v_1 + 0v_2 + \dots + 0v_n = 0$

Thus we have expressed 0 as a lin combination of vectors of S in two wa

$\therefore$ By hypothesis $\alpha_1 = \alpha_2 = \dots = \alpha_n = 0$

Hence S in linearly independent

Hence S is a basis

Theorem : 5·18

Let V be a vector space over F les $S = \{v_1 v_2 \dots v_n\}$ span V. Let $S = \{w_1 \dots w_n\}$ be a linearly independent set of vectors in V then $m \leq n$

proof :: $\therefore L(S) = V$, every vector in V and in particular w_1, is a linear combination of $v_1 v_2 \dots v_n$

Hence $S_1 = \{w_1, v_1, v_2 \dots v_n\}$ is a linearly independent set of vectors. Hence $\exists$ a vector $v_K \neq w_1$ in S_1 which is a linear combination of preceding vectors.

Let $S_2 = \{w_1, v_1 \dots v_{K-1}, v_{K+1} \dots v_n\}$

Clearly $L(S_1) = V$

Hence w_2 is a linear combination of the vectors in S_2

Hence $S_3 = \{w_2, w_1, v_1 \dots v_{K-1}, v_{K+1} \dots v_n\}$ is linearly dependent. Hence $\exists$ a vector in S_3

∵ The w_i's are linearly independent this vector cannot be w_2 are w_1 and Hence must be some v_j where $j \neq k$ (say with $j > k$). De of v_j from the set S_3 gives the set

$$S_4 = \{w_2, w_1, v_1, v_2 \ldots v_{k-1}, v_{k+1} \ldots v_{j-1}, v_{j+1} \ldots v_n)$$

of n vectors spanning V.

In this process at each step we insert one vector from $\{w_1, w_2 \ldots w_m\}$ and delete one vector from $\{v_1 v_2 \ldots v_n\}$

If $m > n$ after repeating this process n times we arrive at the set $\{w_n, w_{n-1} \ldots w_1\}$ which spans V.

Hence w_{n+1} is a linear combination of $w_1 w_2 \ldots w_n$ hence $\{w_1 w_2 \ldots w_n, w_{n+1} \ldots w_m\}$ is linearly dependent which is a contradiction

Hence $m \leq n$.

Theorem: 5.19

Any two basis of a finite dimensional vector space V have the same number of elements.

proof: ∵ V is finite dimensional it has a basis say $S = \{v_1 v_2 \ldots v_n\}$

Let $S' = \{w_1 w_2 \ldots w_m\}$ be any other basis for V

Now, $L(S) = V$ and S' is a set of m linearly independent vectors. Hence

W.K.T $m \leq n$.

Also, $\because L(S') = V$ and S is a set of n linearly independent vectors, $n \leq m$

Hence $m = n$.

Definition (Dimension) :-

Let V be a finite dimensional vector space over a field F. The number of elements in any basis of V is called the dimension of V and is denoted by dimension V

Theorem 5.20 :-

Let V be a vector space of dimension n. Then

i) any set of m vectors where $m > n$ is linearly dependent.

ii) Any set of m vectors where $m > n$ cannot span V.

proof :

i) Let $S = \{v_1, v_2 \dots v_n\}$ be a basis for V hence $L(S) = V$

Let S' be any set consisting of m vectors where $m > n$, suppose S' linearly Independent.

$\therefore$ S spans V by theo 5.18, $m < n$ which is a contradiction

Hence S' is linearly dependent

ii) Let S' be a set consisting of m vector where $m < n$. suppose $L(S') = V$

now,

$S = \{v_1, v_2 \dots v_n\}$ is a basis for V and hence linearly independent.

Hence by theorem 5.18, $n < m$ which is a contradiction.

Hence S' cannot span V.

Theorem : 5.21

Let V be finite dimensional vector space over a field F. Any linearly Independent set of vectors in V is part of a basis.

Proof :-

Let $S = \{v_1, v_2 \dots v_r\}$ be a linearly independent set of vectors

if $L(S) = V$ then S itself is a basis

if $L(S) \neq V$, choose an element $v_{r+1} \in V - L(S)$

now, Consider $S_1 = \{v_1, v_2 \dots v_r, v_{r+1}\}$

We shall P.T S_1, is linearly independent by showing that no vector in S_1 is a linearly combination of the preceeding vectors (the 5

$\therefore$ $\because$ $\{v_1 v_2 \dots v_r\}$ is linearly independent v_r where $1 \leq i \leq r$ is not a a linear Combination of the preceding vectors.

Also $v_{r+1} \notin L(S)$ and hence v_{r+1} is not a linear combination of $v_1 v_2 \ldots v_r$

Hence S_1 is linearly independent.

If $L(S_1) = V$, then S_1 is a basis for V.

If not we take an element $v_{r+2} \in V - L(S_1)$ and proceed as before the dimension of V is finite thus process must stop at a certain stage giving the required basis containing S.

1) $S = \{(1,0,0)\ (0,1,0)\ (0,0,1)\}$ is a basis for $V_3(R)$

Soln:- $v_1 = (1,0,0)$ $v_2 = (0,1,0)$ $v_3 = (0,0,1)$

$$\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 = 0$$

$$\alpha_1(1,0,0) + \alpha_2(0,1,0) + \alpha_3(0,0,1) = 0$$

$$\alpha_1 + 0 + 0\}\ 0 + \alpha_2 + 0\ ,\ 0 + 0 + \alpha_3 = 0$$

$$(\alpha_1, \alpha_2, \alpha_3) = 0$$

$$\alpha_1 = 0,\ \alpha_2 = 0,\ \alpha_3 = 0$$

$V_3(R)$ is a linearly independent.

Also, $(a,b,c) = a(1,0,0) + b(0,1,0) + c(0,0,1)$

$$= a + 0 + 0\ ,\ 0 + b + 0\ ,\ 0 + 0 + c$$

$$= (a, b, c)$$

Hence ~~S~~ S is a basis for $V_3(R)$

2) $S = \{(1,0,0)\ (0,1,0)\ (1,1,1)\}$ is a basi

Theorem : 5·23

Definition :-

Let V be a vector space and S=$\{v_1 \dots v_n\}$ be a set of independent vectors in V. Then S is a called a maximal linearly indepent set if for every $v \in V - S$, the set $\{v, v_1, v_2 \dots v_n\}$ is linearly dependent

Definition :-

Let $S=\{v_1 v_2 \dots v_n\}$ be a set of vectors in V and let $L(S)=V$ then S is called a minimal generating set if for any $v_1 \in S$ $L(S-\{v_1\}) \neq v_1$

Theorem : 5-23

Let V be a vector space over a field F. Let $S=\{v_1, v_2 \dots v_n\} \leq V$ Then the following are equivalent

i) S is a basis for

ii) S is a maximal linearly independent set

iii) S is a minimal generating set

Proof :- i) => ii)

Let $S=\{v_1 \dots v_n\}$ be a basis for V.

Then by theorem 5·20

Any n+1 vectors in V are linearly dependent and hence S is a maximal "

ii) → i)

Let $S = \{v_1, v_2 \ldots v_n\}$ be a maximum linear independent set. Now to p.t S is a basis for V we shall show that $L(S) = V$.

Obviously $L(S) \subseteq V$

Now, let $v \in V$

If $v \in S$, then $v \in L(S)$ $(\because S \subset L(S))$

If $v \notin S$, $S' = \{v_1, v_2 \ldots v_n, v\}$ is a linearly dependent set ($\because$ S is a maximal linearly independent set)

$\therefore$ ∃ a vector in S' which is a linear combination of the preceeding vector.

$\because$ $v_1 v_2 \ldots v_n$ are linearly independent this vector must be v. Thus v is a linear combination of $v_1 v_2 \ldots v_n$

Therefore $v \in L(S)$

Hence $V \subseteq L(S)$, Thus $V = L(S)$

i) ⇒ iii)

Let $S = \{v_1 v_2 \ldots v_n\}$ be a basis. Then $L(S) = V$. If S is not minimal, ∃ $v_i \in S$ ∋ $L(S - \{v_i\}) = V$.

$\because$ S is linearly independent, $S - \{v_i\}$ is also linearly independent.

Thus $S - \{v_i\}$ is a basis consisting of $n-1$ elemence which is a contradiction.

Hence S is a minimal generating set.

iii) => i)

Let $S = \{v_1 v_2 \cdots v_n\}$ be a minimal generating set. To P.T S is a bases we have to show that S is linearly independent. If S is linearly dependent, ∃ a vector v which is a linear

clearly $L(S - \{v_k\}) = V$ Contradicting the combination of the preceeding vector minimal of S.

Thus, S is a linearly Independent and

∵ $L(S) = V$, S is a basis for V.

Theorem: 5 : 24

Any vector space of dimension n. Let a field F is Isomorphic to $V_n(F)$.

Proof:

Let V be a vector space of dimension n, Let $\{v_1 v_2 \cdots v_n\}$ be a basis for V.

Then we know that if $v \in V$, v can be written uniquely as $v = \alpha_1 v_1 + \alpha_2 v_2 + \cdots \alpha_n v_n$ where $\alpha_i \in F$.

Now, Consider the map $f : V \to V_n(F)$ Given by.

$$f(\alpha_1 v_1 + \cdots + \alpha_n v_n) = (\alpha_1, \alpha_2 \cdots \alpha_n)$$

Clearly f is 1-1 and onto

Let $v, w \in V$

Then, $v = \alpha_1 v_1 + \cdots + \alpha_n v_n$ and $w = \beta_1 v_1 + \cdots + \beta_n v_n$

$f(v+w) = f[(\alpha_1+\beta_1)v_1 + \quad + (\alpha_n+\beta_n)v_n]$

$= ((\alpha_1+\beta_1)(\alpha_2+\beta_2) \cdots (\alpha_n+\beta_n))$

$= (\alpha_1 \alpha_2 \cdots \alpha_n) + (\beta_1 \beta_2 \cdots \beta_n)$

$= f(v) + f(w)$

Also, $f(\alpha v) = f(\alpha\alpha_1 v_1 + \cdots + \alpha\alpha_n v_n)$

$= (\alpha\alpha_1, \alpha\alpha_2 \cdots \alpha\alpha_n)$

$= \alpha(\alpha_1, \alpha_2 \cdots \alpha_n)$

$= \alpha f(v)$

Hence f is an isomorphism of V to $V_n(F)$

Corollary :

Any two vector spaces of the same dimension over a field F are isomorphic

are of dimension

$\alpha_1 = \alpha_2 = \cdots \alpha_n = 0$ (∵ T is 1-1)

(∵ $v_1 v_2 \dots v_n$ are linearly independent)

∴ $T(v_1), T(v_2) \dots T(v_n)$ are linearly independen

Now, let $w \in W$. Then T is onto, ∃ a vector $v \in V$ ∋ $T(v) = W$

let $v = \alpha_1 v_1 + \dots + \alpha_n v_n$

Then $w = T(v)$

$= T(\alpha_1 v_1 + \dots + \alpha_n v_n)$

$= \alpha_1 T(v_1 + \dots + \alpha_n T(v_n))$

Thus w is a linear combination of the vector. $T(v_1) \dots T(v_n)$

∴ $T(v_1) \dots T(v_n)$ span w and hence is a basis for w.

Corollary:

Two finite dimensional vector spaces V over a field F are isomorphic iff they have same dimension.

Theorem: 5.26

Let V and W be finite dimensional vector spaces over a field F. Let $\{v_1, v_2 \dots v_n\}$ be a basis for V and let $w_1, w_2 \dots w_n$ be any n vectors in W (not necessarily distinct) Then ∃ a unique linear transformation $T : V \to W$ ∋ $T(v_i) = w$, $i = 1, 2, 3 \dots n$

proof :

Let $v = \alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n$

we define $T(v) = \alpha_1 w_1 + \alpha_2 w_2 + \dots + \alpha_n w_n \rightarrow (+)$

now, let $x, y \in V$

Let $x = \alpha_1 v_1 + \dots + \alpha_n v_n$ and

$y = \beta_1 v_1 + \dots + \beta_n v_n$

$\therefore x + y = (\alpha_1 + \beta_1) v_1 + \dots + (\alpha_n + \beta_n) v_n$

$T(x+y) = (\alpha_1 + \beta_1) w_1 + \dots + (\alpha_n + \beta_n) w_n$

$= (\alpha_1 w_1 + \dots + \alpha_n w_n) + (\beta_1 w_1 + \dots \beta_n w_n)$

$= T(x) + T(y)$

$|||^{ly}$ $T(\alpha x) = \alpha T(x)$

Hence T is a linear transformation.

Also, $v_1 = 1 v_1 + 0 v_2 + \dots + 0 v_n$

Hence $T(v_1 = 1 w_1 + 0 w_2 + 0 w_n = w$

$|||^{ly}$ $T(v_i) = w_i \ \forall \ i = 1, 2 \dots n$

How, to prove the uniqueness, let $T' : V \to W$ be any other linear transformation ∋:

$T'(v_i) = w_i \rightarrow (*)$

Let $V = \alpha_1 v_1 + \dots + \alpha_n v_n \in V$

$T'(V) = \alpha_1 T'(v_1) + \dots + \alpha_n T'(v_n)$

$= \alpha_1 w_1 + \dots \alpha_n w_n \Rightarrow T(V)$

Hence $T = T'$

Remark :

the above theorem shows that a linear transformation is completely determination by its values on the elements of a bases.

Theorem : 5.27

Let V be a finite dimensional vectors let x space over a field F. Let W be a subspaces of V. Then.

i) dim W $\leq$ dim V, ii) dim V/W = dim V - dim W

proof: i) Let $S = \{w_1 w_2 \cdots w_m\}$ be a basis for W $\because$ W is a subspaces of V, S is a part of a basis for V.

Hence dim W $\leq$ dim V

ii) Let dim V = n and dim W = m

Let $S = \{w_1, w_2 \cdots w_m\}$ be a basis for W

Clearly S is a linearly independent set of vectors in V.

Hence S is a part of basis in V.

Let $\{w_1, w_2 \cdots w_m, v_1 v_2 \cdots v_r\}$ be a basis for V

Them $m + r = n$

now, we claim $S' = \{W+v_1, W+v_2 \cdots W+v_r\}$ is a bases for V/W.

$$\alpha_1(W+v_1) + \alpha_2(W+v_2) + \cdots + \alpha_r(W+v_r) = W+0$$

$$(W+\alpha_1 v_1) + (W+\alpha_2 v_2) + \cdots + (W+\alpha_r v_r) = W$$

$$W + \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_r v_2 = W$$

$$\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_r v_r \in W$$

Now, $\because \{w_1, w_2 \cdots w_m\}$ is a bases for W

$$\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_r v_r = \beta_1 w_1 + \cdots + \beta_m w_m$$

$\therefore \alpha_1 v_1 + \dots + \alpha_r v_r - \beta_1 w_1 \dots - \beta_{\infty m} w_m = 0.$

$\alpha_1 = \alpha_2 = \alpha_r = \beta_1 = \beta_2 \dots \beta_m = 0$

$\therefore$ S' is a linearly independent set

Now, let $w + v \in V/w$

Then $w + v = w + (\alpha_1 v_1 + \dots + \alpha_r v_r) + \beta_1 w_1 + \dots \beta_m w_m$

$= w + (\alpha_1 v_1 + \dots \alpha_r v_r)$ $[\because (\beta_1 w_1 + \dots \beta_m w_m \in w)$

$= (w + \alpha_1 v_1) + \dots + (w + \alpha_r v_r)$

$= \alpha_1 (w + v_1) + \dots + \alpha_r (w + v_r)$

Hence S' spans V/w so that S' is a basis for V/w,

$\therefore \dim V/w = r = n - m$

$= \dim V - \dim w$

$\dim V/w = \dim V - \dim w$

Theorem : 5 . 28

Let V be a finite dimensional vector space over a field F. Let A and B be subspaces of V. Then $\dim (A+B) = \dim A + \dim B - \dim (A \cap B)$

Proof $\because$ A and B are ~~dimension~~ subspaces of V, hence $A \cap B$ is subspaces of V.

let $\dim (A \cap B) = r$

Let $S = \{v_1 \dots v_r\}$ be a basis for $A \cap B$.

$\because$ $A \cap B$ is a subspace of A and B, S is a part of a basis for A and B

Let $\{v_1 v_2 \dots v_r, u_1, u_2 \dots u_s\}$ be a basis for A and $\{v_1 v_2 \cdots v_r\ w_1, w_2 \cdots w_t\}$ be a basis for B.

We shall prove that $S' = \{v_1 v_2 \cdots v_r, u_1, \cdots u_s, w_1 \cdots w_t\}$ is a basis for A+B.

Let $\alpha_1 v_1 + \cdots \alpha_r v_r + \beta_1 w_1 + \cdots \beta_s u_s + \gamma_1 w_1 + \cdots + \gamma_t w_t = 0$

Then, $\beta_1 w_1 + \beta_2 w_2 + \cdots \beta_s u_s = -(\gamma_1 w_1 + \cdots + \gamma_t w_t) - (\alpha_1 v_1 + \cdots + \alpha_r v_r) \in B$

Hence $\beta_1 u_1 + \cdots + \beta_s u_s \in A$

$\therefore \beta_1 u_1 + \cdots \beta_s u_s = \delta_1 v_1 + \cdots + \delta_r v_r$

$\therefore \beta_1 u_1 + \cdots + \beta_s u_s = \delta_1 v_1 - \cdots - \delta_r v_r = 0$

$\therefore \beta_1 = \cdots = \beta_s = \delta_1 = \cdots = \delta_r = 0$

($\because \{u_1 \cdots u_s, v_1 \cdots v_r\}$ is linearly-independent

III^{ly} we can prove $\gamma_1 = \gamma_2 = \gamma \cdots = \gamma_t = 0$

$\therefore \alpha_i = \beta_j = \gamma_k = 0$ for $1 \le i \le r$

$1 \le j \le s$; $1 \le k \le k$

Thus S' is a linearly independent set

clearly S' spans A+B

$\therefore S'$ is a basis for A+B

Hence dim $(A+B) = r+s+t$

Also, dim $A = r+s$; dim $B = r+t$ &

dim $(A \cap B) = r$

$\therefore \dim A + \dim B - \dim A \cap B = (r+s) + (r+t) - r$

$= r+s+t$

$= \dim(A+B)$

Rank and Nullity

Definition :-

Let $T : V \to W$ be a linear transformation Then the dimension of $T(V)$ is called the rank of T, the dimension of $\ker T$ is called the nullity of T.

Theorem : 5.29

Let $T : V \to W$ be a linear Transformation Then $\dim V = \text{rank } T + \text{nullity } T$

proof:

W.K.T

$$V/\ker T = T(V)$$

$$\because \dim V - \dim(\ker T) = \dim(T(V))$$

$$\dim V - \text{nullity } T = \text{rank } T$$

$$\therefore \dim V = \text{nullity } T + \text{rank } T.$$

Note :-

$\ker T$ is also called null space of T.

Definition:

A linear Transformation $T : V \to W$ is called non-singular if T is 1-1; Otherwise T is called singular.