UNIT - III

1) $T : V_2(R) \to V_2(R)$ given by $T(a,b) = (a, a+b)$

i) standard basis

ii) $\{e_1, e_2\}$ & as a basis for the domain

$\{(1,1)\ (1,-1)$ as basis for range.

soln:- i) $T(e_1) = T(1,0) = (1,1) = e_1 + e_2$

$T(e_2) = T(0,1) = (0,1) = e_2$

Hence the matrix T is $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

ii) let $w_1 = (1,1)$, $w_2 = (1,-1)$

$T(e_1) = T(1,0) = (1,1) = W_1$

$T(e_2) = T(0,1) = (0,1) = \frac{1}{2}(1,1) - \frac{1}{2}(1,-1)$

$= \frac{1}{2} W_1 - \frac{1}{2} w_2$

Hence the matrix representation

T is $= \begin{pmatrix} 1 & 0 \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix}$

2) obtain the matrix representing linear transformation $T : V_3(R) \to V_3(R)$ gn- by $T(a,b,c) = (3a,\ a-b,\ 2a+b+c)$ with respect to standard basis.

soln:- $T(e_1) = T(1,0,0) = (3,1,2) = 3e_1 + e_2 + 2e_3$

$T(e_2) = T(0,1,0) = (0,-1,1) = -e_2 + e_3$

$T(e_3) = T(0,0,1) = (0,0,1) = e_3$

Hence the matrix representation T is $= \begin{pmatrix} 3 & 1 & 2 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$

3) Find the linear transformation $T : V_3(R) \to V_3(R)$ determine by the matrix $\begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ -1 & 3 & 4 \end{pmatrix}$ with respect to standard basis

soln: $T(e_1) = e_1 + 2e_2 + e_3 = (1,2,1)$

$T(e_2) = 0e_1 + e_2 + e_3 = (0,1,1)$

$T(e_3) = -e_1 + 3e_2 + 4e_3 = (-1,3,4)$

Now,

$(a,b,c) = a(1,0,0) + b(0,1,0) + c(0,0,1)$

$= ae_1 + be_2 + ce_3$

$T(a,b,c) = T(ae_1 + be_2 + ce_3)$

$= aT(e_1) + bT(e_2) + cT(e_3)$

$= a(1,2,1) + b(0,1,1) + c(-1,3,4)$

$T(a,b,c) = (a-c,\ 2a+b+3c,\ a+b+4c)$

1) Obtain the matrix for the following linear transformation a) $T: V_2(R) \to V_2(R)$ gvl- by $T(a,b) = (-b,a)$ w.r.t

i) standard basis

ii) The basis $(1,2)\ (1,-1)$ for both domain & range.

soln: i) $T(e_1) = T(1,0) = (-0,1) = (0,1) = 0e_1 + e_2$

$T(e_2) = T(0,1) = (-1,0) = -e_1$

Hence the matrix T is $= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

ii) Let $W_1 = (1,2)$, $W_2 = (1,-1)$

$T(w_1) = T(1,2) = (-2,1)$

$= -\frac{1}{3} W_1 - \frac{5}{3} W_2$

$= -\frac{1}{3}(1,2) - \frac{5}{3}(1,-1)$

$= -\frac{1}{3} W_1 - \frac{5}{3} W_2$

$T(W_2) = T(1,-1) = (1,1) = \frac{2}{3}(1,2) + \frac{1}{3}(1,-1)$

$= \frac{2}{3} w_1 + \frac{1}{3} w_2$

Hence the matrix T is $= \begin{pmatrix} -\frac{1}{3} & -\frac{5}{3} \\ \frac{2}{3} & \frac{1}{3} \end{pmatrix}$

b) $T : V_3(R) \to V_2(R)$ ...

W.r.t i) standard basis

ii) $(1,0,-1)$ $(1,1,1)$ $(1,0,0)$ as a basis for $V_3(R)$ and $\{(0,1),(1,0)\}$ for $V_2(R)$

soln: $T(e_1) = T(1,0,-1) = (1,-2) = 1(0,1) - 2(1,0)$

$= -2(w_1) + 1(w_2)$

$T(e_2) = T(1,1,1) = (2,1)$

$= 2(0,1) + 1(1,0)$

$= 1(w_1) + 2(w_2)$

$T(e_3) = T(1,0,0) = (1,-1)$

$= 1(0,1) + (-1)(1,0)$

$= -w_1 + w_2$

Hence the matrix T is $\begin{pmatrix} -2 & 1 \\ 1 & 2 \\ -1 & 1 \end{pmatrix}$

i) $T(e_1) = (1,0,0) = 1, -1$

$T(e_2) = (0,1,0) = 1, 0$

$T(e_3) = (0,0,1) = 0, 2$

Hence the matrix representation T is $\begin{pmatrix} 1 & -1 \\ 1 & 0 \\ 0 & 2 \end{pmatrix}$

c) $T_3 : V_3(R) \to V_3(R)$ gn by $T(a,b,c) = (3a+c, -2a ...$ $a+2b+4c)$ with respect to

i) standard basis

ii) The basis $(1,0,1)$ $(-1,2,1)$ $(2,1,1)$ for both domain and range.

soln: i) $T(e_1) = (1,0,0) = (3,-2,1)$

$T(e_2) = (0,1,0) = (0,1,2)$

$T(e_3) = (0,0,1) = (1,0,4)$

Hence the matrix representation T is

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$$\begin{bmatrix} & 1 & 2 \\ 1 & 0 & 4 \end{bmatrix}$$

ii) $T(w_1) = T(1,0,1) = (4,-2,5)$

$= \frac{17}{4}(1,0,1) - \frac{3}{4}(-1,2,1) - \frac{1}{2}(2,1,1)$

$= \frac{17}{4}(W_1) - \frac{3}{4}(W_2) - \frac{1}{3}(W_3)$

$T(w_2) = T(-1,2,1) = (-2,4,7)$

$= \frac{35}{4}(1,0,1) + \frac{15}{4}(-1,2,1) - \frac{7}{2}(2,1,1)$

$= \frac{35}{4} W_1 + \frac{15}{4} W_2 - \frac{7}{2} W_3$

$T(w_3) = T(2,1,1) = (7,-3,8)$

$= \frac{17}{2}(1,0,1) + \frac{3}{2}(-1,2,1) + 0(2,1,1)$

$= \frac{17}{2} W_1 + \frac{3}{2} W_2 + 0 W_3$

Hence the matrix representative T is

$$= \begin{bmatrix} 17/4 & -3/4 & -1/3 \\ 35/4 & 15/4 & -7/2 \\ 17/2 & 3/2 & 0 \end{bmatrix}$$

) Obtain the linear transformation determined by the following matrices.

a) $T: V_2(R) \to V_2(R)$ given by $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ with respect to the standard basis.

soln:- $T(e_1) = \cos\theta e_1 - \sin\theta e_2 = (\cos\theta, -\sin\theta)$

$T(e_2) = \sin\theta e_1 + \cos\theta e_2 = (\sin\theta, \cos\theta)$

Now, $(a,b,c) = a(1,0) + b(0,1)$

$= ae_1 + be_2$

$T(a,b) = T(ae_1 + be_2)$

$= aT(e_1) + bT(e_2)$

$= a(\cos\theta, -\sin\theta) + b(\sin\theta, \cos\theta)$

$= (a\cos\theta + b\sin\theta,\ -a\sin\theta + b\cos\theta)$

b) $T: V_3(R) \to V_3(R)$ given by $\begin{bmatrix} a & b & c \\ b & c & a \\ c & a & b \end{bmatrix}$ standard basis.

Soln: $T(e_1) = ae_1 + be_2 + ce_3 = (a, b, c)$

$T(e_2) = be_1 + ce_2 + ae_3 = (b, c, a)$

$T(e_3) = ce_1 + ae_2 + be_3 = (c, a, b)$

Now, $(a, b, c) = a(1,0,0) + b(0,1,0) + c(0,0,1)$

$= ae_1 + be_2 + ce_3$

$T(a,b,c) = aT(e_1) + bT(e_2) + cT(e_3)$

$= a(a,b,c) + b(b,c,a) + c(c,a,b)$

$T(a,b,c) = a^2+b^2+c^2,\ ab+bc+ca,\ ac+ab+bc$

$(x,y,z) = x(1,0,0) + y(0,1,0) + z(0,0,1)$

$= xe_1 + ye_2 + ze_3$

$T(x,y,z) = xT(e_1) + yT(e_2) + zT(e_3)$

$= x(a,b,c) + y(b,c,a) + z(c,a,b)$

$= (ax+by+cz),\ bx+cy+az,\ cx+ay+bz$

c) $T: V_2(R) \to V_2(R)$ gn by $\begin{bmatrix} 2 & 1 & -1 \\ 1 & 1 & -1 \end{bmatrix}$ which respect to the standard basis.

soln: $T(e_1) = 2e_1 + e_2 - e_3 = (2, 1, -1)$

$T(e_2) = e_1 + e_2 - e_3 = (1, 1, -1)$

Now, $(a,b,c) = a(1,0,0) + b(0,1,0)$

$= ae_1 + be_2$

$T(a,b,c) = aT(e_1) + bT(e_2)$

$= a(2,1,-1) + b(1,1,-1)$

$= (2a+b,\ a+b,\ -a-b).$

Inner product spaces.

Ex :

3) Let V be the set of all continuous complex valued functions defined on the closed interval [0,1]. S.T V is a complex inner product spaces with inner product defined by

$$(f,g) = \int_0^1 f(t)\,\overline{g(t)}\,dt.$$

soln: $f, g, h \in V$ & $\alpha \in R$.

i) $(f+g, h) = \int_0^1 f(t)+g(t)\,\overline{h(t)}\,dt$

$= \int_0^1 f(t)\,\overline{h(t)}\,dt + \int_0^1 g(t)\,\overline{h(t)}\,dt$

$= (f, h) + (g, h)$

ii) $(\alpha f, g) = \int_0^1 \alpha f(t)\,\overline{g(t)}\,dt$

$= \alpha \int_0^1 f(t)\,\overline{g(t)}\,dt$

$= \alpha\,(f, g)$

iii) $\overline{(g, f)} = \int_0^1 \overline{f(t)\,\overline{g(t)}}\,dt$

$= \int_0^1 f(t)\,\overline{g(t)}\,dt = \langle f, g\rangle$

iv) $\langle f, f\rangle = \int_0^1 |f(t)|^2 \geq 0$

$\langle f, f\rangle = 0$, iff $f = 0$

1) S.T $V_2(R)$ is an inner product space with inner product defined by.

$\langle x, y\rangle = x_1y_1 + x_2y_1 - x_1y_2 + 4x_2y_2$

where $x = (x_1, x_2)$ & $y = y_1 + y_2$

soln: Let $x, y, z \in V$ & $\alpha \in R$.

i) $\langle x+y, z\rangle = (x_1+y_1)z_1 + (x_2+y_2)z_1 - (x_1+y_1)z_2 + 4(x_2+y_2)z_2$

$= x_1z_1 + y_1z_1 + x_2z_1 + y_2z_1 - x_1z_2 + y_1z_2 + 4x_2z_2 + 4y_2z_2$

$= \langle x, z\rangle + \langle y, z\rangle$

ii) $\langle \alpha x, y\rangle = (\alpha x_1y_1 + \alpha x_2y_1 - \alpha x_1y_2 + 4\alpha x_2y_2)$

$= \alpha(x_1y_1 + x_2y_1 - x_1y_2 + 4x_2y_2)$

$= \alpha\langle x, y\rangle$

iii) $\langle x,y\rangle = \langle x_1y_1 + x_2y_1 - x_1y_2 + 4x_2y_2\rangle$

$= y_1x_1 + y_1x_2 - y_2x_1 + 4y_2x_2$

$= \langle y,x\rangle$

iv) $\langle x,x\rangle = x_1^2 + x_2x_1 - x_1x_2 + 4x_2^2$

$= x_1^2 - 4x_2^2 \geq 0$

$\langle x,x\rangle = 0$ iff $x=0$

2) S.T $V_2(C)$ is an inner product space with inner product defined by $\langle x,y\rangle = 2x_1\bar{y}_1 + x_1\bar{y}_2 + y_2\bar{y}_1 + x_2\bar{y}_2$ where. $x = (x_1,x_2)$ & $y = (y_1,y_2)$

soln: let $x,y \in, z \in C$ & $\alpha \in C$

i) $\langle (x+y), z\rangle = 2(x_1+y_1)\bar{z}_1 + (x_1+y_1)\bar{z}_2 + (z_2)(\bar{z}_1) + (x_2+y_2)\bar{z}_2$

$= 2x_1\bar{z} + 2y_1\bar{z}_1 + x_2\bar{z}_2 + y_1\bar{z}_2 + z_2\bar{z}_1 + x_2\bar{z}_2 + y_2\bar{z}_2$

$= 2x_1\bar{z} + x_1\bar{z}_2 + x_2\bar{z}_2$

3) which of the following are Inner products on $V_2(R)$ $x = (x_1,x_2)$ & $y = (y_1,y_2)$

a) $\langle x,y\rangle = x_1y_1 + 2x_1y_2 + 2x_2y_1 + 5x_2y_1$

soln: Let $x,y,z \in V$ & $\alpha \in R$

i) $\langle (x+y), z\rangle = (x_1 y_1)z_1 + 2(x_1+y_1)z_2 + 2(x_2+y_2) + 5(x_2+y_2)z_2$.

$= x_1z_1 + y_1z_1 + 2x_1z_2 + 2y_1z_2 + 2x_2z_1 + 2y_2z_2 + 5x_2z_2 + 5y_2$

$= x_1z_1 + 2x_1z_2 + 2x_2z_1 + 5x_2z_2 + y_1z_1 + 2y_1z_2 + 2y_2z_1 + 5y_2z_2$

$\langle (x+y), z\rangle = \langle x,y\rangle + \langle y,x\rangle$

ii) $\langle \alpha x, y\rangle = (\alpha x_1 y_1 + 3\alpha x_1 y_2 + 3\alpha x_2 y_1 + 5\alpha x_2 y_2)$

$= \alpha(x_1 y_1 + 3x_1 y_2 + 3x_2 y_1 + 5x_2 y_2)$

$= \alpha \langle x, y\rangle$

iii) $\langle x, y\rangle = x_1 y_1 + 3x_1 y_2 + 3x_2 y_1 + 5x_2 y_2$

$= y_1 x_1 + 3y_2 x_1 + 3y_1 x_2 + 5y_2 x_2$

$= \langle y, x\rangle$

iv) $\langle x, x\rangle = x_1^2 + 3x_1 x_2 + 3x_2 x_1 + 5x_2^2$

$= x_1^2 + 6x_1 x_2 + 5x_2^2 \geq 0$

$\langle x, x\rangle = 0$ iff $x = 0$

b) $\langle x, y\rangle = x_1^2 - 2x_1 y_2 - 2x_2 y_1 + y_1^2$

soln: let $x, y, z \in V$ & $\alpha \in \mathbb{R}$

i) $\langle (x+y), z\rangle = (x_1 + y_1)^2 - 2(x_1 + y_1)(z_2) - 2(x_2 + y_2)$
$+ (z_1)^2$

$= (x_1^2 + y_1^2 + 2x_1 y_1 - 2x_1 z_2 - 2y_1 z_2 - 2x_2 z_1$
$- 2y_2 z_1 + z_1^2$

$\langle (x+y) + z\rangle \neq x_1^2 + 2x_1 y_1 - 2x_1 z_2 - 2x_2 z_1 + y_1^2$
$- 2y_1 z_2 - 2y_2 z_1 + z_1^2$

ii) $\langle \alpha x, y\rangle = (\alpha x_1)^2 - 2(\alpha x_1) y_2 - 2(\alpha x_2) y_1 + y_1^2$

$= \alpha^2 x^2 - 2\alpha x_1 y_2 - 2\alpha x_2 y_1 + y_1^2$

iii) $\langle x, y\rangle = x_1^2 - 2x_1 y_2 - 2x_2 y_1 + y_1^2$

iv) $\langle x, x\rangle = x_1^2 - 2x_1^2 - 2x_2 x_1 + x_1^2$

$V_2(\mathbb{R})$ is not inner product

C) $\langle x, y\rangle = 6x_1 y_1 + 7x_2 y_2$

soln: let $x, y, z \in V$ & $\alpha \in \mathbb{R}$

i) $\langle (x+y), z\rangle = 6(x_1 + y_1) z_1 + 7(x_2 + y_2) z_2$
$= 6x_1 z_1 + 6y_1 z_1 + 7x_2 z_2 + 7y_2 z_2$

$= 6x_1z_1 + 7x_2z_2 + 6y_1z_1 + 7y_2z_2$

$= \langle x, z \rangle + \langle y, z \rangle$

ii) $\langle \alpha x, y \rangle = (6\alpha x_1 y_1 + 7\alpha x_2 y_2)$

$= \alpha(6x_1y_1 + 7x_2y_2)$

$= \alpha \langle x, y \rangle$

iii) $\langle x, y \rangle = (6x_1y_1 + 7x_2y_2)$

$= 6y_1x_1 + 7y_2x_2$

$= \langle y, x \rangle$

iv) $\langle x, x \rangle = 6x_1^2 + 7x_2^2 \geq 0$ &

$\langle x, x \rangle = 0$ iff $x = 0$

4) $\langle x, y \rangle = x_1y_1 - 2x_2y_1 - 2x_1y_2 + 4x_2y_2$

soln: Let $x, y, z \in V$ & $\alpha \in R$

i) $\langle x+y, z \rangle = (x_1+y_1)z_1 - 2(x_1+y_1)z_2 + 4(x_2+y_2)z_2$

$= x_1z_1 + y_1z_1 - 2x_2z_1 - 2y_2z_1 - 2x_1z_2 - 2y_1z_2 + 4x_2z_2 + 4y_2z_2$

$= x_1z_1 - 2x_2z_1 - 2x_1z_2 + 4x_2z_2 + y_1z_1 - 2y_2z_1 - 2y_1z_2 + 4y_2z_2$

$= \langle x, z \rangle + \langle y, z \rangle$

ii) $\langle \alpha x, y \rangle = (\alpha x_1 y_1 - 2\alpha x_2 y_1 - 2\alpha x_1 y_2 + 4\alpha x_2 y_2)$

$= \alpha(x_1y_1 - 2x_2y_1 - 2x_1y_2 + 4x_2y_2)$

$= \alpha \langle x, y \rangle$

iii) $\langle x, y \rangle = x_1y_1 - 2x_2y_1 - 2x_1y_2 + 4x_2y_2$

$= y_1x_1 - 2y_1x_2 - 2y_2x_1 + 4y_2x_2$

$= \langle y, x \rangle$

iv) $\langle x, x \rangle = x_1^2 - 2x_2x_1 - 2x_1x_2 + 4x_2x_2$

$= x_1^2 - 4x_1x_2 + 4x_2^2$

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1) S.T is an inner product space

i) $\langle \alpha u+\beta v, w\rangle = \alpha\langle u,w\rangle + \beta\langle v,w\rangle$

Soln: For, $\langle \alpha u+\beta v, w\rangle = \overline{\langle w, \alpha u+\beta v\rangle}$

$= \overline{\langle w, \alpha u\rangle} + \overline{\langle w, \beta v\rangle}$

$= \overline{\langle w, \alpha u\rangle} + \overline{\langle w, \beta v\rangle}$

$= \langle \alpha u, w\rangle + \beta\langle \beta v, w\rangle$

$\langle \alpha u+\beta v, w\rangle = \alpha\langle u,w\rangle + \beta\langle v,w\rangle$

ii) $\langle u, \alpha v+\beta w\rangle = \bar{\alpha}\langle u,v\rangle + \bar{\beta}\langle u,w\rangle$

Soln: For, $\langle u, \alpha u+\beta w\rangle = \overline{\langle \alpha u+\beta w, u\rangle}$

$= \overline{\langle \alpha v, u\rangle} + \overline{\langle \beta w, u\rangle}$

$= \bar{\alpha}\langle \overline{v,u}\rangle + \bar{\beta}\langle \overline{w,u}\rangle$

$= \bar{\alpha}\langle u,v\rangle + \bar{\beta}\langle u,w\rangle$

$\langle u, \alpha u+\beta w\rangle = \bar{\alpha}\langle u,v\rangle + \bar{\beta}\langle u,w\rangle$

iii) $\langle \alpha u+\beta v, \gamma w+\delta z\rangle = \alpha\bar{\gamma}\langle u,w\rangle + \alpha\bar{\delta}\langle u,z\rangle + \beta\bar{\gamma}\langle v,w\rangle + \beta\bar{\delta}\langle v,z\rangle$ where $\alpha,\beta,\gamma,\delta \in F$ & $u,v,w,z \in V$

Soln: $\langle \alpha u+\beta v, \gamma w+\delta z\rangle = \langle \alpha u, \gamma w\rangle + \langle \alpha u, \delta z\rangle + \langle \beta v, \gamma w\rangle + \langle \beta v, \delta z\rangle$

$= \alpha\langle u, \gamma w\rangle + \alpha\langle u, \gamma z\rangle + \beta\langle v, \gamma w\rangle + \beta\langle v, \delta z\rangle$

$\langle \alpha u+\beta v, \gamma w+\delta z\rangle = \alpha\bar{\gamma}\langle u,w\rangle + \alpha\bar{\delta}\langle u,z\rangle + \beta\bar{\gamma}\langle v,w\rangle + \beta\bar{\delta}\langle v,z\rangle$

P-1

Let V be the vectors space of polynomials with inner product given by $\langle f,g\rangle = \int_0^1 f(t)g(t)\,dt$ let $f(t) = t+2$ and $g(t) = t^2-2t-3$

ii) $\|g\|$

Soln: $\|g\|^2 = \langle g \cdot g \rangle$

$= \int_0^1 [g(t)]^2 \, dt = \int_0^1 (t^2 - 2t - 3)^2 dt$

$= \int_0^1 t^4 + 4t^2 + 9 - 2t^2(-2t) + 2t^2(-3) - 2t^2(-3) dt$

$= \int_0^1 t^4 + 4t^2 + 9 + 4t^3 - 6t^2 + 6t^2 \, dt$

$= \int_0^1 (t^4 + 4t^3 + 4t^2 + 9) \, dt$

$= \left[t^5/5 + 4t^4/4 + 4t^3/3 + 9t \right]_0^1$

$= 1/5 + 1 + 4/3 + 9 = 1/5 + 4/3 + 10$

$= \frac{3+20}{15} + 10 = 173/5$

1) find the form of the following vectors in the $V_3(R)$ with standard inner product

a) $(1,1,1)$ b) $(1,2,3)$ c) $(3,-4,0)$

Soln: a) $\langle 1,1,1 \rangle = \sqrt{1+1+1} = \sqrt{3}$

b) $(1,2,3)$

$\langle 1,2,3 \rangle = \sqrt{1+4+9} = \sqrt{14}$

c) $(3,-4,0)$

$\langle 3,-4,0 \rangle = \sqrt{9+16+0} = \sqrt{25} = 5$

d) $4x+5y$

$\langle x, x, x \rangle = \langle 1,-1,0 \rangle$

$= \sqrt{1+1+0} = \sqrt{2}$

$\langle y, y, y \rangle = \langle 1,2,3 \rangle$

$= \sqrt{1+4+9} = \sqrt{14}$

$2x+5y = 2\sqrt{2} + 5\sqrt{14}$

Theorem 6.1 :-

The norm defined in an inner product space V has the following properties.

i) $\|x\| \geq 0$ and $\|x\| = 0$ if $x = 0$

ii) $\|\alpha x\| = |\alpha| \|x\|$

iii) $|\langle x, y \rangle| \leq \|x\| \|y\|$ (Schwarz inequality)

iv) $\|x+y\| \leq \|x\| + \|y\|$ (Schwarz inequality)

Proof:

i) $\|x\| = \sqrt{\langle x, x \rangle} \geq 0$ and $\|x\| = 0$

if $x = 0$

ii) $\|\alpha x\|^2 = \langle \alpha x, \alpha x \rangle$

$= \alpha \langle x, \alpha x \rangle$

$= \alpha \bar{\alpha} \langle x, x \rangle$

$= \alpha \bar{\alpha} \|x\|^2$

$\|\alpha x\|^2 = |\alpha|^2 \|x\|^2$

Hence $\|\alpha x\| = |\alpha| \|x\|$

iii) The inequality is trivially true when $x = 0$ (or) $y = 0$ hence let $x \neq 0$ and $y \neq 0$

Consider $z = y - \frac{\langle y, x \rangle}{\|x\|^2} x$

Then $0 \leq \langle z, z \rangle$

$= \langle y - \frac{\langle y, x \rangle}{\|x\|^2} x, y - \frac{\langle y, x \rangle}{\|x\|^2} x \rangle$

$= \langle y, y - \frac{\langle y, x \rangle}{\|x\|^2} x \rangle + \langle - \frac{\langle y, x \rangle}{\|x\|^2} x,$

$y - \frac{\langle y, x \rangle}{\|x\|^2} x \rangle$

$= \langle y, y \rangle - \langle y, \frac{\langle y, x \rangle}{\|x\|^2} x \rangle +$

$\langle - \frac{\langle y, x \rangle}{\|x\|^2} x \rangle$

$= \langle y, y \rangle - \frac{\langle \overline{y, x} \rangle}{\|x\|^2} \langle y, x \rangle - 1 \frac{\langle y, x \rangle}{\|x\|^2} \langle x, y \rangle$

$+ \frac{\langle y, x \rangle}{\|x\|^2} x \rangle$

$= \|y^2\| - \frac{\langle y, x \rangle \langle y, x \rangle}{\|x\|^2} - \frac{\langle y, x \rangle \langle x, y \rangle}{\|x\|^2}$

$+ \frac{\langle y, x \rangle \langle y, x \rangle}{\|x\|^2}$

$= \|y\|^2 - \frac{\langle \overline{x, y} \rangle \langle x, y \rangle}{\|x\|^2}$

$0 \leq \|y\|^2 \|x\|^2 - |\langle x, y \rangle|^2$

$\therefore |\langle x, y \rangle|^2 \leq \|y\|^2 \|x\|^2$

$|\langle x, y \rangle| \leq \|x\| \|y\|$

iv) $\|x + y\|^2 = \langle x + y, x + y \rangle$

$= \langle x, x \rangle + \langle x, y \rangle + \langle y, x \rangle + \langle y, y \rangle$

$= \|x\|^2 + \langle x, y \rangle + \langle \overline{x, y} \rangle + \langle y, \ldots$

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$= \|x\|^2 + 2Re \langle x, y \rangle + \|y\|^2$

$\leq \|x\|^2 + 2|\langle x, y\rangle| + \|y\|^2$

$\leq \|x\|^2 + 2\|x\|\|y\| + \|y\|^2$

$\|x+y\|^2 \leq (\|x\| + \|y\|)^2$

$\|x+y\| \leq \|x\| + \|y\|$

2) S.T in any inner product space V

$\|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2)$

soln: $\|x+y\|^2 = \langle x+y, x+y \rangle$

$= \langle x, x\rangle + \langle x, y\rangle + \langle y, x\rangle + \langle y, y\rangle$

$= \|x\|^2 + \langle x, y\rangle + \langle y, x\rangle + \|y\|^2$

$\|x-y\|^2 = \langle x-y, x-y\rangle$

$= \langle x, x\rangle - \langle x, y\rangle - \langle y, x\rangle + \langle y, y\rangle$

$= \|x\|^2 - \langle x, y\rangle - \langle y, x\rangle + \|y\|^2$

$\|x+y\|^2 + \|x-y\|^2 = \|x\|^2 + \langle x, y\rangle + \langle y, x\rangle + \|y\|^2 +$

$\|x\|^2 - \langle x, y\rangle - \langle y, x\rangle + \|y\|^2$

$= 2\|x\|^2 + 2\|y\|^2$

$= 2(\|x\|^2 + \|y\|^2)$

Hence

$\|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2)$

3) S.T in any inner product space $\|\alpha x + \beta y\|^2 =$

$|\alpha|^2\|x\|^2 + \alpha\bar{\beta}\langle x, y\rangle + \bar{\alpha}\beta\langle y, x\rangle + |\beta|^2\|y\|^2$

soln: $\|\alpha x + \beta y\|^2 = \langle \alpha x + \beta y, \alpha x + \beta y \rangle$

$= \langle \alpha x, \alpha x\rangle + \langle \alpha x, \beta y\rangle + \langle \beta y, \alpha x\rangle +$

$\langle \beta y, \beta y\rangle$

$$\|x+y\|^2 = \|x\|^2 + \cancel{0+0} + \|y\|^2 = \|x\|^2 + \|y\|^2$$

$$= \alpha\bar{\alpha} \langle x, x \rangle + \alpha\bar{\beta} \langle x, y \rangle + \beta\bar{\alpha} \langle y, x \rangle + \beta\bar{\beta} \langle y, y \rangle$$

$$\|\alpha x + \beta y\|^2 = |\alpha|^2 \|x\|^2 + \alpha\bar{\beta} \langle x, y \rangle + \bar{\alpha}\beta \langle y, x \rangle + |\beta|^2 \|y\|^2$$

5\.

a) S.T in a real inner product space if $\langle x, y \rangle = 0$ Then $\|x+y\|^2 = \|x\|^2 + \|y\|^2$

soln:

$$\|x+y\|^2 = \langle x+y, x+y \rangle$$

$$= \langle x, x \rangle + \langle x, y \rangle + \langle y, x \rangle + \langle y, y \rangle$$

$$= \|x\|^2 + 0 + 0 + \|y\|^2$$

$$\|x+y\|^2 = \|x\|^2 + \|y\|^2$$

b) S.T in a real inner product space if then ~~then~~ $\|x\|^2 + \|y\|^2 = \|x+y\|^2$ then $\langle x, y \rangle = 0$

soln:

$$\|x+y\|^2 = \langle x+y, x+y \rangle$$

$$= \langle x, x \rangle + \langle x, y \rangle + \langle y, x \rangle + \langle y, y \rangle$$

$$= \|x\|^2 + \langle x, y \rangle + \langle y, x \rangle + \|y\|^2$$

$$\|x\|^2 + \|y\|^2 = \|x\|^2 + 2\langle x, y \rangle + \|y\|^2$$

$$\|x\|^2 + \|y\|^2 - \|x\|^2 - \|y\|^2 = 2\langle x, y \rangle$$

$$0 = 2\langle x, y \rangle \Rightarrow 0 = \langle x, y \rangle =$$

$\langle x, y$

b) In an inner product space we define t distance b/w any two vectors x & y by

$d\langle x, y \rangle = \|x - y\|$, S.T a) $d\langle x, y \rangle \geq 0$ & d if $x = y$ b) $d\langle x, y \rangle = d\langle y, x \rangle$

c) $d\langle x, y \rangle \leq d\langle x, z \rangle + d\langle y, z \rangle$

Soln: a) $d<x,y> = \|x-y\|$

≥ 0

$d<x,y> \geq 0$

$d<x,y> = 0$ iff $x-y=0$

$x=y$

c) $d<x,y> = \|x-y\|$

$= \|x-y+z-z\|$

$\leq \|x-z\| + \|z-y\|$

b) $d<x,y> = \|x-y\|$

$= \|y-x\|$

$d<x,y> = d<y,x>$

Problem 1.

Apply Gram-Schidt process to construct an orthonormal basis for $V_3(R)$ with the standard inner product for the basis $\{v_1, v_2, v_3\}$ where $v_1=(1,0,1)$ $v_2=(1,3,1)$ & $v_3=(3,2,1)$

Soln:: Take $w_1 = v_1$

$w_1 = (1,0,1)$

Then $\|w_1\|^2 = <w_1, w_1> = 1^2+0^2+1^2 \Rightarrow \|w_1\| = \sqrt{2}$

$<v_2, w_1> = (1,0,1) = 1+0+1 = 2$

Put. $w_2 = v_2 - \frac{<v_2, w_1>}{\|w_1\|^2} w_1$

$= (1,3,1) - \frac{2}{2}(1,0,1)$

$w_2 = (0,3,0)$

$w_3 = V_3 - \frac{<V_3, w_1>}{\|w_1\|^2} w_1$

$= V_3 - \frac{<V_3, w_1>}{\|w_1\|^2} w_1 - \frac{<V_3, w_2>}{\|w_2\|^2} w_2$

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$<v_3, w_1> = (3,0,1) = 3+0+1 = 4$

$<v_3, w_2> = (0,6,0) = 0+6+0 = 6$

$\|w_2\|^2 = <0,3,0> = 0^2+3^2+0^2 = 9 = \sqrt{3}$

$w_3 = (3,2,1) = 4/2\,(1,0,1) - 6/9\,(0,3,0)$

$= (3,2,1) - 2(1,0,1) - 2/3\,(0,3,0)$

$\|w_3\|^2 = (1,0,-1) \Rightarrow 1^2+0^2+(-1)^2 = 2 \Rightarrow \sqrt{2}$

The orthonormal basis is

$$\frac{w_i}{\|w_i\|} = \frac{w_1}{\|w_1\|}, \frac{w_2}{\|w_2\|}, \frac{w_3}{\|w_3\|}$$

$$= \left(\frac{(1,0,1)}{\sqrt{2}}, \frac{(0,3,0)}{3}, \frac{(1,0,-1)}{\sqrt{2}}\right)$$

$$= (1/\sqrt{2}, 0, 1/\sqrt{2})\ (0,1,0)\ (1/\sqrt{2}, 0 - 1/\sqrt{2}$$

The - 6.4 :-

Every finite dimensional inner product space has an orthonormal basis.

proof :- Let V be a finite dimensional inner product space. Let $\{v_1, v_2 \dots v_n\}$ be a basis for V.

From this basis we shall construct an ~~orthogon~~ orthonormal basis $\{w_1, w_2 \dots w_n\}$ by means of a construction known as gram-Schmidt orthogonalisation process

First we take $w_1 = v_1$

let $w_2 = v_2 - \frac{<v_2, w_1>}{\|w_1\|^2} w_1$

we claim that $w_2 \neq 0$

For if $w_2 = 0$ then v_2 is a scalar multiple of w_1 & hence for v_1 which is a contradiction.

$\because v_1 v_2$ are linearly independent.

$$w_2 = v_2 - \frac{\langle v_2, v_1 \rangle}{\|w_1\|^2} w_1$$

Also, $\langle w_2, w_1 \rangle = \langle v_2 - \frac{\langle v_2, w_1 \rangle}{\|w_1\|^2} w_1, w_1 \rangle$

$$= \langle v_2 - \frac{\langle v_2, v_1 \rangle}{\|v_1\|^2} v_1, v_1 \rangle$$

$(\because w_1 = v_1)$

$$= \langle v_2, v_1 \rangle - \frac{\langle v_2, v_1 \rangle}{\|v_1\|^2} \langle v_1, v_1 \rangle$$

$$= \langle v_2, v_1 \rangle - \frac{\langle v_2, v_1 \rangle}{\|v_1\|^2} \|v_1\|^2 =$$

$$\langle v_2, v_1 \rangle - \langle v_2, v_1 \rangle$$

Now, suppose that we have constructed non-zero ~~ob~~ orthogonal vectors $w_1 w_2 \dots w_k$

Then, put $w_{k+1} = v_{k+1} - \sum_{j=1}^{k} \frac{\langle v_{k+1}, w_j \rangle}{\|w_j\|^2} w_j$