The general form of Cauchy's Theorem:

Chains & cycles:

Chain :-

Consider the equation

$$\int_{\gamma_1+\gamma_2+\cdots+\gamma_n} f\,dz = \int_{\gamma_1} f\,dz + \int_{\gamma_2} f\,dz + \cdots + \int_{\gamma_n} f\,dz \longrightarrow (1)$$

Which is Valid When $\gamma_1, \gamma_2 \cdots \gamma_n$ from a Subdivison of arc γ.

[An arbitrary formal Sum $\gamma_1+\gamma_2+\cdots+\gamma_n$ Which need not be an arc].

And we define the Corresponding integral by Mean of (1)

Such formal Sum of arcs are called chains. The following operations do not Change the identity of a Chain.

Example of chain:

(i) Permutation of two arcs.

(ii) Subdivison of an arc

(iii) Fusion of Subarc to a Single arc

(iv) Reparameterization of an arc.

(v) Cancellation of opposite arc.

Cycle:

A chain is a cycle if it can be represented as a sum of closed curve.

(i) A chain is a cycle iff in any representation the initial & end points of the individual are identical in pairs.

Note:

(i) The integral of an exact differentiable over any cycle is zero.

(ii) $n(\gamma_1+\gamma_2, a) = n(\gamma_1, a) + n(\gamma_2, a)$

Simple Connectivity: (Simple connected Region)

A region is simply connected if its complement w.r.to the extended plane is connected.

Thm: 1

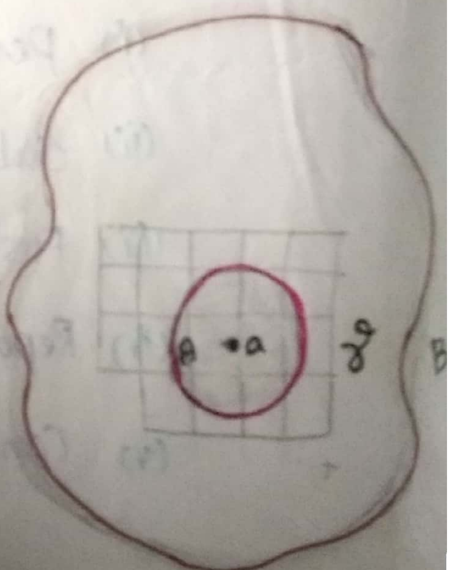
A region Ω is simply connected iff $n(\gamma, a) = 0$ $\forall$ cycles γ in Ω and all pts/- 'a' which $\notin$ to Ω.

Proof: (Necessary Part)

Let Ω be a simply region

Let γ be any simple closed curve in Ω

Let $a \notin \Omega$



Since Ω is simply connected region.

$\Rightarrow \Omega^c$ is connected

$\because a \notin \Omega \Rightarrow a \in \Omega^c$

$\Rightarrow$ 'a' belongs to unbounded region.

$$\therefore \frac{1}{2\pi i}\int_{\gamma}\frac{dz}{z-a} = 0$$

$$\Rightarrow n(\gamma, a) = 0$$

Sufficient part:

Let $n(\gamma, a) = 0$ for any cycle γ in Ω and $a \notin \Omega'$

To Prove: Ω is simple connected region.

Suppose,

Ω is not simply connected

By the defn/-

Ω^c is not simply connected.

Assume that

$$\Omega^c = A \cup B$$

Where A & B are non empty disjoint closed sets.

Here,

Either A or B should be unbounded.

Take,

B is unbounded region.

$$a \in A \cup B$$

$$a \in A \quad \text{or} \quad a \in B$$

Let $\delta > 0$ be the shortest b/w set A & B

Cover the whole plane with the net of square.

'Q' of side $< \delta/\sqrt{2}$

Choose $a \in A$ lies at the centre of square.

Considered the cycle $\gamma = \sum_j \partial Q_j$

Where ∂Q is denoted the boundary of the curve.

The sum ranges over all the squares Q_j in the net which have a common point w.r.to 'A'.

$$\therefore n(\gamma, a) = n\left(\sum_j \partial Q_j, a\right)$$

$$= \sum_j n(\partial Q_j, a) = 1$$

Since

a is a centre of one Q_j

Further γ does not meet B.

But the cancellation of carried out. It is clear that γ does not meet 'A'.

Thus, we prove that Ω is not simply connected.

Since, γ is contained in Ω.

$$\Rightarrow n(\gamma, a) \neq 0$$

Which is $\Rightarrow \Leftarrow$

$\therefore \Omega$ is simply connected region

Hence the proof.

Homology :-

Defn : Homologus :

The cycle 'γ' is an open set Ω is said to be Homologous to zero with respect to Ω if $n(\gamma, a) = 0$ for all pts/- 'a' in the complement of Ω'. $(a \in \Omega^c)$

Note :

In symbols we write $\gamma \sim 0 \pmod{\Omega}$.

$\gamma_1 \sim \gamma_2$ iff $\gamma_1 - \gamma_2 \equiv 0 \pmod{\Omega}$

2 R.Q.X. 10m.X. Sedj

General statement of cauchy's Theorem :

Cauchy's Theorem : (2) (General cauchy's thm) 10m

(statement only)

Statement :

If $f(z)$ is analytic in Ω Then $\int_\gamma f(z)\,dz = 0$ for every cycle γ which is homologies to 0 in Ω. (homologus)

Proof :

(i) First assume that

Case : (i)

Ω is bounded

But otherwise arbitrary.

Given $\delta > 0$, we cover the plane by a net of square of sides δ.

We denoted by Q_j, $j \in J$ the closed squares in the net which are contained in Ω

$\because \Omega$ is bounded, J is finite & if

'δ' is sufficiently small. It is not empty.

$$\Gamma_\delta = \sum_{j \in J} \partial Q_j$$

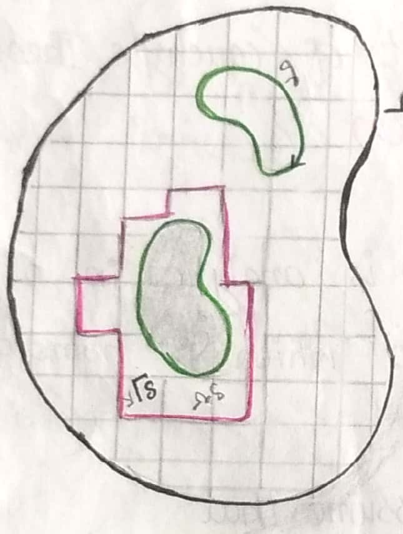
$\Rightarrow \Gamma_\delta$ is the sum of oriented line segments which are sides of exactly one Q_j

We denote $\Omega_\delta = \cup Q_j$ (Interior of Union Q_j)

Let $\gamma \equiv 0 \pmod{\Omega}$

Choose 'δ' so small that $\gamma \in \Omega_\delta$ $\quad \gamma \subset \Omega_\delta$

Consider one Q which is not Q_j



Consider a point $\xi \in \Omega - \Omega_\delta$. $\quad \xi \in \Omega$, $\xi \notin \Omega_\delta$

There is a point $\xi_0 \in Q$ which is not in Ω_δ. It is possible to join ξ & ξ_0 by a line segment which lies in Q. $\quad \xi_0 \in \Omega^c$

$\therefore$ It does not meet Ω_δ.

$$n(\gamma, \xi) = n(\gamma, \xi_0) = 0$$

[$\because \gamma$ ~~contained~~ considered as a pts/- set contained in Ω_δ]

In particular $n(\gamma, \xi) = 0 \ \forall$ pts/- ξ on Γ_δ. Suppose that $f(z)$ is analytic in Ω. If z lies in the interior of $Q_{j_0}^{\circ}$

Then

$$\frac{1}{2\pi i}\int_{\partial Q_j^{\circ}} \frac{f(\xi)}{(\xi - z)}\, d\xi = \begin{cases} f(z) & \text{if } j = J_0 \\ 0 & \text{if } j \neq J_0 \end{cases}$$

Both sides are conts/- funt- of z

We have

$$f(z) = \frac{1}{2\pi i}\int_{\partial Q_j^{\circ}} \frac{f(\xi)}{(\xi - z)}\, d\xi \quad \forall\ z \in \Omega_\delta$$

$$\int_\gamma f(z)\, d\xi = \int_\gamma \left[\frac{1}{2\pi i}\int_{\Gamma_\delta} \frac{f(\xi)}{(\xi - z)}\, d\xi\right] dz$$

By changing the order of $\int$ration we get,

$$\int_\gamma f(z)\, dz = \int_{\Gamma_\delta}\left[\frac{1}{2\pi i}\int_\gamma \frac{dz}{(\xi - z)}\right] f(\xi)\, d\xi$$

$$= \int_{\Gamma_\delta}\left[\frac{-1}{2\pi i}\int_\gamma \frac{dz}{z - \xi}\right] f(\xi)\, d\xi$$

$$= \int_{\Gamma_\delta}\left[-\underline{n(\gamma, \xi)}\, f(\xi)\, d\xi\right]$$

$$= 0$$

Hence the theorem is proved for bounded Region Ω.

Case: (ii)

Ω' is unbounded.

We replace it by its intersection of Ω' with a disc $|z| < R$ with large enough to contain γ.

Any point a in the complement of Ω' is either in the complement of Ω (or) lies outside the disc.

$$\therefore n(\gamma, a) = 0$$

$$\Rightarrow \gamma \sim 0 \pmod{\Omega}$$

Hence the thm/- is proved the arbitrary "Ω".

UQ

2000 Study

Locally Exact Differentiable:

A differentiable $p\,dx + q\,dy$ is said to be locally exact in Ω if its is exact in some nghd/- of each point in Ω.

10 m Study

State and prove Local Exact:

Thm: 3

If $p\,dx + q\,dy$ is said to locally exact in Ω. Then $\int_\gamma p\,dx + q\,dy = 0$ for every cycle $\gamma \sim 0$ in Ω'.

Proof:

Let γ be defined as $z = z(t)$ $a \leq t \leq b$. Then the fun/-

$z(t)$ is Uniformly Conts/- in $[a,b]$

Let δ be the least distance of γ from Ω^c $\exists$ $\delta > 0$ $\ni$:

$$|t - t'| < \delta$$

$$\Rightarrow |z(t) - z'(t)| < \rho \quad [\because z(t) \text{ is uniformly conts}]$$

Now.

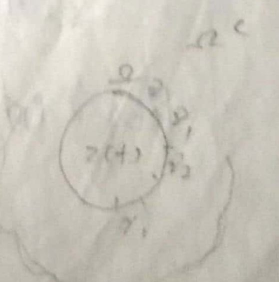
Divide $[a,b]$ into Sub intervals of length less than δ then the corresponding Sub arc γ_i of γ is such that it is contained in a disc of radius "ρ" which lies inside Ω.

Joins the end points of γ_i can be joint with in the disc by a polygon σ_i

Consisting of a horizontal & vertical Segment

Since the differential $pdx + qdy$ is exact in the disc.

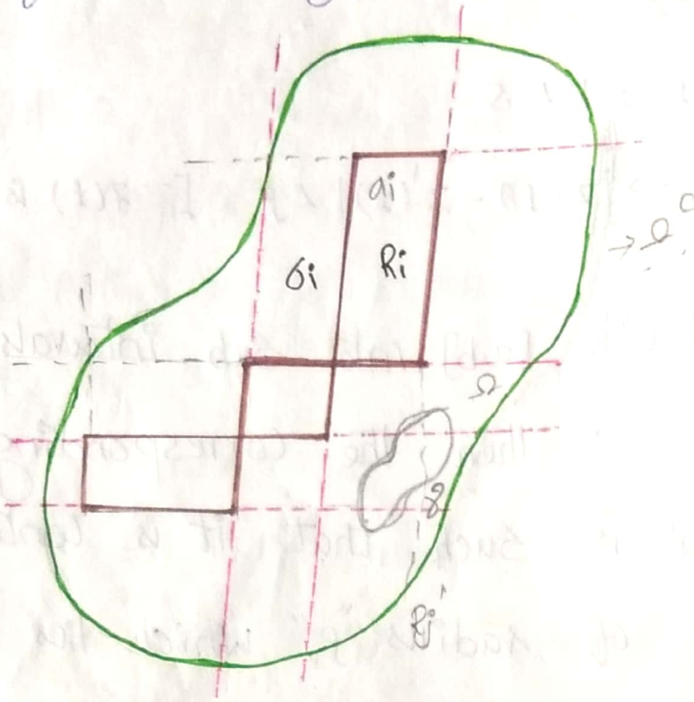
$$\text{(i.e)} \int_{\sigma_i} pdx + qdy = \int_{\gamma_i} pdx + qdy$$



If $\sum \sigma_i = \sigma$ then

$$\int_{\sigma} pdx + qdy = \int_{\gamma} pdx + qdy$$

Extended all line segment which makes σ to infinite line. They divide the plane into some finite rectangles R_i & unbounded regions R_j'.



Choose a point 'a_i' from the interior of each R_i.

And consider the cycle :

$$\sigma_0 = \sum_i n(\sigma, a_i)\, \partial R_i \longrightarrow (1)$$

Where the sum ranges over all finite rectangles.

Also.

$n(\sigma, a_i)$ is well defined for a_i lies in σ

$$n(\partial R_i, a_k) = \begin{cases} 0 & \text{if } k \neq i \\ 1 & \text{if } k = i \end{cases}$$

$|||^{ly}$

If a_j' is an interior point R_j'.

Then $n(\partial R_i, a_j') = 0 \ \forall j$

It follows that

$$n(\sigma, a_i) = n(\sigma_0, a_i) \ \forall i$$

$$n(\sigma_0, a_j') = 0 \ \forall j$$

Also

$$n(\sigma, a_j') = 0 \ \forall j$$

[$\because a_j'$ lies in the unbounded region]

$$\Rightarrow n(\sigma - \sigma_0, a) = 0 \ \forall \ a = a_i ; a = a_j'$$

From this property of $\sigma - \sigma_0$ we wish to conclude that σ_0 is identical with σ upto segments that cancel agains each other.

Let σ_{ik} be the common sides of two adjacent sides of Rectangle R_i, R_k.

We choose the orientation so that lies to the left of σ_{ik}.

Suppose that the reduced expression of $\sigma - \sigma_0$ contains the multiple $c\sigma_{ik}$.

Then the cycle $\sigma - \sigma_0 - c\partial R_i$ does not contain σ_{ik} and it follows that a_i & a_k must have the same index w.r.to the cycle.

$\therefore$ we conclude $c = 0$

$|||^{ly}$

σ_{ij} is the common side of a finite rectangle R_i and an infinite rectangle R_j'

Thus every side of a finite rectangle occurs with co-efficient zero in $\sigma - \sigma_0$ proves that.

$$\sigma = \sum_i n(\sigma, a_i)\, \partial R_i$$

We claim that.

All the R_i is different from zero($\neq 0$) are actually contained in Ω. Suppose that a pt/- a' in the closed rectangle R_i were not in Ω, then $n(\sigma, a) = 0$.

Where
$$\sigma \sim 0 \pmod{\Omega}$$

Again the line segment a & a_i does not intersect σ and hence $n(\sigma, a_i) = n(\sigma, a) = 0$

Now

we conclude by the local exact of $pdx + qdy$ that $\int_{\partial R_i} pdx + qdy = 0$

$$\Rightarrow \int_\sigma pdx + qdy = 0.$$

$$\int_{\sigma} p\,dx + q\,dy = \int_{\sum n(\sigma,a_i)\partial R_i} p\,dx + q\,dy$$

$$= \sum n(\sigma,a_i) \int_{\partial R_i} p\,dx + q\,dy.$$

$$= 0$$

$$\int_{\sigma} p\,dx + q\,dy = \int_{\gamma} p\,dx + q\,dy = 0$$

$$\int_{\gamma} p\,dx + q\,dy = 0.$$

H.T.P.

Note:

Let $\Omega' = \Omega - \{a_1, a_2 \cdots a_n\}$

Then $f(z)$ is analytic in Ω.

Since 'a_j' are the isolated singularities

to each a_j. these exist $\delta_j > 0 \ni$

$$0 < |z - a_j| < \delta_j < \Omega'$$

considered

A circle c_j about a_j of radius $< \delta_j$.

Let $P_j = \int_{c_j} f(z)\,dz$

Let $R_j = \dfrac{P_j}{2\pi i}$

$$\int_{c_j} \left(f(z) - \frac{R_j}{z-a_j} \right) dz = \int_{c_j} f(z)\,dz - R_j \int \frac{dz}{z-a_j}$$

$$= P_j - R_j\, 2\pi i$$

$$= P_j - \frac{P_j}{2\pi i} \cdot 2\pi i$$

$$= 0$$

$\Rightarrow f(z) - \frac{R_j}{z - a_j}$ is the derivative of a single valued analytic fun/- in $0 < |z - a_j| < \delta_j$

$$R_j = \underset{z=a_j}{\text{Res}} f(z) = \frac{1}{2\pi i} \int_{C_j} f(z)\, dz.$$

Calculus of Residues:

Let $f(z)$ be a function which is Analytic in a region Ω except for isolated Singularity.

Defn:

The Residues of $f(z)$ of an isolated Singularity at $z = a$ is the unique Complex no/- R which makes $f(z) - \frac{R}{z-a}$. The derivative of a single valued analytic function in an annulus $0 < |z-a| < \delta$. It is written as

$$R = \underset{z=a}{\text{Res}} f(z).$$

Theorem : 4

Cauchy's Residue Theorem

Let $f(z)$ be analytic except for isolated singularity a_j in a region Ω.

$$\frac{1}{2\pi i}\int_{\gamma} f(z)\,dz = \sum_j n(\gamma, a_j)\ \underset{z=a_j}{\text{Res}}\ f(z)$$ for any

cycle γ which is homologues to zero in Ω & does not pass through any of the pts a_j.

Pf:

Given $f(z)$ be analytic except for isolated singularities.

Case : (i)

Assume that there are only a finite no. of isolated singularities say $a_1, a_2 \dots a_n$

Claim:

$$\gamma \sim \sum_j n(\gamma, a_j)\, c_j$$

(γ be a cycle in Ω which is homologus to zero w.r.to Ω)

Let

$$\Gamma = \gamma - \sum_j n(\gamma, a_j)\, c_j$$

For a lies outside Ω

$$n(\Gamma, a) = n(\gamma, a) - n\left[\sum_j n(\gamma, a_j) \cdot c_j, a\right]$$

$$= n(\gamma, a) - \sum_j n(\gamma, a_j)\, n(c_j, a) \quad (1)$$

$$= 0$$

Suppose $a = a_i$ for some 'i'.

$1 \le i \le n$

$$n(\Gamma, a_i) = n(\gamma, a_i) - \sum n(\gamma, a_j)\, n(C_j, a_i) \quad \rightarrow (1)$$

$$= n(\gamma, a_i) - n(\gamma, a_i) \qquad |z - a_i| < \delta_i$$

$$n(\Gamma, a_i) = 0 \qquad \because n(C_j, a) = \begin{cases} 0 & \text{for } j \ne i \\ 1 & \text{for } j = i \end{cases}$$

$\Gamma \sim 0$

(e) $\gamma - \sum\limits_j n(\gamma, a_j)\, C_j \sim 0$

$$\int_\gamma f(z)\,dz = \int_{\sum\limits_j n(\gamma, a_j) C_j} f(z)\,dz$$

$$= \sum_j n(\gamma, a_j) \int_{C_j} f(z)\,dz$$

$$= n(\gamma, a_1) \int_{C_1} f(z)\,dz + n(\gamma, a_2) \int_{C_2} f(z)\,dz + \cdots + n(\gamma, a_n) \int_{C_n} f(z)\,dz$$

$$\frac{1}{2\pi i} \int_\gamma f(z)\,dz = \frac{1}{2\pi i} \left[n(\gamma, a_1) \int_{C_1} f(z)\,dz + n(\gamma, a_2) \int_{C_2} f(z) + \cdots + n(\gamma, a_n) \int_{C_n} f(z)\,dz \right]$$

$$\Rightarrow \frac{1}{2\pi i} \int_\gamma f(z)\,dz = \frac{1}{2\pi i} \sum_j n(\gamma, a_j)\, P_j$$

$$= \sum_j n(\gamma, a_j)\, R_j \qquad \left(\because R_j = \frac{P_j}{2\pi i}\right)$$

$$\therefore \frac{1}{2\pi i} \int_\gamma f(z)\,dz = \sum_j n(\gamma, a_j) \operatorname*{Res}_{z = a_j} f(z).$$

Claim :-

Case : (ii)

Suppose $f(z)$ has infinite no. of singularities

$$A = \left\{ z \,/\, n(\gamma, z) = 0 \right\}$$

Claim :

A is open. Let $a \in A$

$\therefore n(\gamma, a) = 0$

$$\frac{1}{2\pi i} \int_{\gamma} \frac{dz}{z-a} = 0$$

$$\therefore \int \frac{dz}{z-a} = 0$$

$\frac{1}{z-a}$ is the derivative of an analytic fun. in $0 < |z-a| < \varepsilon$

Consider, $|b-a| < \varepsilon$.

$N_\varepsilon(a)$ then for $b \in N_\varepsilon(a)$

$n(\gamma, b) = 0$

$\therefore \frac{1}{z-b}$ is also the derivative of an analytic in $N_\varepsilon(a)$

i.e.) $N_\varepsilon(a) \subset A$

$\therefore$ A is open

A^c is closed

Claim :

A^c is bounded.

Now $n(\gamma, z) \neq 0$

$\Rightarrow z \in$ to the bdd. region of γ. $n(\gamma, z) \neq 0$

This is true for any $z \in A^c$.

i.e.) A^c is bdd. region

$\therefore A^c$ is compact.

Claim:

A^c cannot have infinite no. of isolated singularities a_j.

Suppose, there are infinite no. of isolated singularities $a_j \in A^c$.

Since, A^c is compact, A^c has an accumulation point. (Unit-I)

Hence the accumulation pt. cannot be an isolated singularities.

Which is $\Rightarrow\Leftarrow$

$\therefore \exists$ only a finite no. of isolated singularities in A^c.

By case (i)

$$\frac{1}{2\pi i} \int_{\gamma} f(z)\,dz = \sum_j n(\gamma, a_j) \operatorname{Res}_{z=a_j} f(z)$$

Hence the Pf.

Defn:

The cycle γ is said to be bounded the region Ω iff $n(\gamma, a)$ is defined & equal to one $\forall a \in \Omega$ & equal to zero for all $a \notin \Omega$. either undefined or equal to zero for all $a \notin \Omega$.

R.08 X. 5m -X. Study

Argument Principle: (Thm: 5)

If $f(z)$ is meromorphic in 'Ω' with the zero's a_j & the poles b_k then

$$\frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)} dz = \sum_j n(\gamma, a_j) - \sum_k n(\gamma, b_k).$$

for every cycle γ which is homologues to zero in 'Ω' & does not pass through any of the zeros or poles.

Proof:

Let a_j be the zero's of $f(z)$ with order m_j & b_k be a poles of $f(z)$ with order m_k.

Enclose the zero by circle C_k & poles such that γ satisfies the homologys

$$\gamma \sim \sum_j n(\gamma, a_j) c_j + \sum_k n(\gamma, b_k) C_k$$

Now, a_j be the zero of order m_j

$f(z) = (z-a_j)^{m_j} g(z)$, where $g(z)$ is analytic.

$$g(a_j) \neq 0$$

$$f'(z) = (z-a_j)^{m_j} g'(z) + g(z)\, m_j (z-a_j)^{m_j - 1}$$

Divide by $f(z)$ $\left(\div \text{by } (z-a_j)^{m_j} f(z)\right)$

$$\frac{f'(z)}{f(z)} = \frac{m_j (z-a_j)^{\cancel{m_j-1}} \cancel{g(z)}}{(z-a_j)^{m_j} \cancel{g(z)}} + \frac{g'(z)(z-a_j)^{m_j}}{(z-a_j)^{m_j} g(z)}$$

$$\frac{f'(z)}{f(z)} = \frac{m_j}{z-a_j} + \frac{g'(z)}{g(z)}$$

(coeff of $\frac{1}{z-a_j}$ — m_j) $(z-z_0)$

$$\operatorname*{Res}_{z=a_j} \frac{f'(z)}{f(z)} = m_j$$

Let $f(z) = \dfrac{\phi(z)}{(z-b_k)^{m_k}}$

$$f'(z) = \frac{(z-b_k)^{m_k} \phi'(z) - \phi(z)\, m_k (z-b_k)^{m_k - 1}}{\left[(z-b_k)^{m_k}\right]^2}$$

$$\frac{f'(z)}{f(z)} = \frac{(z-b_k)^{m_k} \phi'(z)}{\left((z-b_k)^{m_k}\right)^2} - \frac{\phi(z)\, m_k (z-b_k)^{m_k}}{\left((z-b_k)^{m_k}\right)^2}$$

$$= \frac{\phi'(z)}{(z-b_k)^{m_k}} - \frac{\phi(z)\, m_k}{(z-b_k)^{m_k+1}}$$

$$\frac{f'(z)}{f(z)} = \frac{\phi'(z)}{\phi(z)} - \frac{m_k}{(z-b_k)}$$

$\dfrac{\phi'(z)(z-b_k)^{m_k}}{\phi(z)(z-b_k)^{m_k}}$

$$\operatorname*{Res}_{z \to b_k} \frac{f'(z)}{f(z)} = -m_k.$$

$$\frac{1}{2\pi i}\int_{\gamma}\frac{f'(z)}{f(z)}dz = m_j - m_k.$$

$$\frac{1}{2\pi i}\int_{\gamma}\frac{f'(z)}{f(z)}dz = \sum_j n(\gamma,a_j) - \sum_k n(\gamma,b_k),$$

H.T.P.

Corollary:

If $f(z)$ is analytic in Ω' & it has no poles [above thm]

$$\frac{1}{2\pi i}\int_{\gamma}\frac{f'(z)}{f(z)} = m_j - 0$$
$$= \sum_j n(\gamma,a_j)$$

Note:

If 'N' is the no/- of zeros of $f(z)$ inside γ then $N = \frac{1}{2\pi}\Delta\gamma\,[\arg f(z)]$ denote the change of arg $f(z)$ as z varies over γ.

proof:

$$N = \frac{1}{2\pi i}\int_{\gamma}\frac{f'(z)}{f(z)}dz \quad \text{By Corollary.}$$

Put $f(z) = Re^{i\theta}$

$|f(z)| = R$ & $\arg f(z) = \theta$

$$f'(z) = d(Re^{i\theta})$$
$$= Re^{i\theta} i\,d\theta + e^{i\theta}dR$$
$$= e^{i\theta}\{Ri\,d\theta + dR\}$$
$$= \frac{f(z)}{R}[Ri\,d\theta + dR]$$

$\because f(z) = Re^{i\theta}$

$\frac{f(z)}{R} = e^{i\theta}$

$$f'(z) = f(z)\left[i\,d\theta + \frac{dR}{R}\right]$$

$$\frac{f'(z)}{f(z)} = \frac{dR}{R} + i\,d\theta$$

Now,

$$N = \frac{1}{2\pi i}\int_{\gamma}\left[\frac{dR}{R} + i\,d\theta\right] dz$$

$$= \frac{1}{2\pi i}\left[\int_{\gamma}\left(\frac{dR}{R} + i\,d\theta\right)\right]$$

$$= \frac{1}{2\pi i}\int_{\gamma}\frac{dR}{R} + \frac{i}{2\pi i}\int_{\gamma} d\theta$$

$$= \frac{1}{2\pi i}\left[\log R\right]_{\gamma} + \frac{1}{2\pi}\left[\theta\right]_{\gamma}$$

$$= 0 + \frac{1}{2\pi}\left[\theta\right]_{\gamma}$$

$$= 0 + \frac{1}{2\pi}\Delta\gamma\left[\arg f(z)\right]$$

$$N = \frac{1}{2\pi}\Delta\gamma\left[\arg f(z)\right]$$

2m Study R.U.Q. 5m Study

Rouche's Theorem: 6

Let γ be homologues to zero in Ω & such that $n(\gamma, z)$ is either 0 or 1. For every pt/- z not on γ. Suppose that $f(z)$ & $g(z)$ analytic in Ω' and satisfies the inequality

$$|f(z) - g(z)| < |f(z)| \text{ on } \gamma.$$

Then $f(z)$ & $g(z)$ have the same no/- of zero's enclosed by γ.

Proof:

Let $\gamma \sim 0 \pmod{\Omega}$

Given $n(\gamma, z) = 0$ or 1 $\forall z \in \Omega - \{\gamma\}$

$f(z)$ & $g(z)$ are analytic in Ω such that

$$|f(z) - g(z)| < |f(z)| \text{ on } \gamma \longrightarrow (1)$$

If $f(z) = 0$ for some $z \in \gamma$

Then by eqn. (1) $|g(z)| < 0$

Which is a $\Rightarrow\Leftarrow$

$\therefore f(z) \neq 0 \ \forall z \in \gamma$

If $g(z) = 0$ for some $z \in \gamma$

Then by eqn (1)

$$|f(z)| < |f(z)|$$

Which is $\Rightarrow\Leftarrow$

$\therefore g(z) \neq 0 \ \forall z \in \gamma$

$\therefore$ The fun. $f(z)$ & $g(z)$ does not have zero's on γ.

Consider

$$\frac{1}{2\pi i} \int_{\gamma} \frac{f'(z)}{f(z)} \, dz = \sum_{j} n(\gamma, a_j)$$

Where a_j's is the zero's of $f(z)$.

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\,dz = \text{No/- of zero's of } f(z) \text{ inside } \gamma$$

Similarly,

$$\frac{1}{2\pi i}\int_\gamma \frac{g'(z)}{g(z)}\,dz = \text{No/- of zeros of } g(z) \text{ inside } \gamma.$$

To prove that

$f(z)$ & $g(z)$ have the same no/- of zero's inside γ.

i.e.) $$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\,dz = \frac{1}{2\pi i}\int_\gamma \frac{g'(z)}{g(z)}\,dz$$

Given

$(1) \Rightarrow |f(z)-g(z)| < |f(z)|$ on γ.

i.e.) $\left|1-\frac{g(z)}{f(z)}\right| < 1$ on γ.

$\Rightarrow |F(z)-1| < 1$ where $F(z) = \frac{g(z)}{f(z)}$

Γ be the image of γ under the transformation

$w = F(z)$

Then γ is contained in a circle with centre 1 radius 1.

$\therefore n(\Gamma, 0) = 0$

ii) $\frac{1}{2\pi i}\int_\gamma \frac{F'(z)}{F(z)}\,dz = 0$ $\quad [\because w = f(z)$
$dw = f'(z)dz]$

$$\int_\gamma \frac{F'(z)}{F(z)}\,dz = 0 \longrightarrow (2)$$

$\frac{1}{2\pi i}\int_\Gamma \frac{dw}{w-w_0}$

W.K.T

$$F(z) = \frac{g(z)}{f(z)}$$

Taking log on both sides.

$$\log F(z) = \log g(z) - \log f(z)$$

Diffl.-

$$\frac{F'(z)}{F(z)} = \frac{g'(z)}{g(z)} - \frac{f'(z)}{f(z)}$$

$$\frac{1}{2\pi i}\int_\gamma \frac{F'(z)}{F(z)}\,dz = \frac{1}{2\pi i}\int_\gamma \frac{g'(z)}{g(z)}\,dz - \frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\,dz$$

$$0 = \frac{1}{2\pi i}\int_\gamma \frac{g'(z)}{g(z)}\,dz - \frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\,dz$$

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\,dz = \frac{1}{2\pi i}\int_\gamma \frac{g'(z)}{g(z)}\,dz$$

Thm: 7

Fundamental Theorem of algebra:

If $p(z)$ is the polynomial of n^{th} degree with complex co-eff- has n zero's in C.

proof:

$$p(z) = a_0 + a_1 z + \cdots + a_n z^n$$ be a

Polynomial of degree n

Take $f(z) = a_n z^n$

$$g(z) = a_0 + a_1 z + \cdots + a_{n-1} z^{n-1}$$

Let 'c' be a circle $|z| = R$ $[R > 1]$

$$\frac{g(z)}{f(z)} = \frac{a_0 + a_1 z + \cdots + a_{n-1} z^{n-1}}{a_n z^n}$$

$$\left|\frac{g(z)}{f(z)}\right| = \frac{|a_0 + a_1 z + \cdots + a_{n-1} z^{n-1}|}{|a_n z^n|}$$

$$\leq \frac{|a_0| + |a_1||z| + \cdots + |a_{n-1}||z^{n-1}|}{|a_n||z^n|} \quad (|z| \leq R)$$

$$= \frac{|a_0| + |a_1| R + \cdots + |a_{n-1}| R^{n-1}}{|a_n| R^n}$$

Choose R is large enough such that R.H.S is < 1.

$$\left|\frac{g(z)}{f(z)}\right| < 1$$

$$\Rightarrow |g(z)| < |f(z)|$$

By Rouche's Thm:-

$g(z)$, $f(z)+g(z)$ will have same no:- of zero's inside c but.

$f(z) = a_n z^n$ has exactly n zero's inside c.

$p(z)$ has exactly n roots.

Thm: 8

If a single valued $f(z)$ has no singularity other than poles in the finite part of the plane or at ∞ then P.T $f(z)$ is rational functional.

(or)

P.T any fun:- which is meromorphic in the extended plane is rational.

Pf:

$f(z)$ has poles at $z = z_1, z_2 \cdots\cdots z_K$ of order $m_1, m_2 \cdots\cdots m_K$ in the finite part of the z- plane.

$$(z-z_1)^{m_1}(z-z_2)^{m_2}\cdots(z-z_k)^{m_k}$$

Where $p(z)$ is analytic for all finite values of z.

$$p(z)=(z-z_1)^{m_1}(z-z_2)^{m_2}\cdots(z-z_k)^{m_k}f(z)$$

Since $p(z)$ is analytic

By Taylor Thm/-

$$p(z)=\sum_{n=0}^{\infty}a_n z^n \longrightarrow (1)$$

Put $z=1/w$

$$p(1/w)=\sum_{n=0}^{\infty}a_n(1/w)^n$$

$$=\sum_{n=0}^{\infty}a_n w^{-n} \longrightarrow (2)$$

$p(z)$ at $z=\infty$ is the same as that of $p(1/w)$ at $w=0$

Now, $z=\infty$ is the pole of $p(z)$.

$\Rightarrow w=0$ is the poles of $p(1/w)$.

$\Rightarrow$ The no. of term in (2) should be finite

$\Rightarrow$ The no.of term in (1) should be finite.

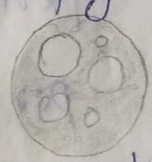
$$\therefore f(z) = \frac{P(z)}{(z-z_1)^{m_1}(z-z_2)^{m_2}\cdots(z-z_k)^{m_k}}$$

$\therefore f(z)$ is rational

Hence proved.

Multiple connected Region:

A region 'Ω' is said to be multiple connected if it is not simply connected.



'Ω' is said to have the finite connectivity 'n' if the complement of 'Ω' in the extended plane.

Modulus of periodicity:

Let $f(z)$ be an analytic in 'Ω' and γ be any cycle in γ which is homologues to a linear combination of $\gamma_1, \gamma_2 \ldots \gamma_{n-1}$

(i) $\gamma \sim c_1\gamma_1 + c_2\gamma_2 + \cdots + c_{n-1}\gamma_{n-1}$.

$$\gamma \sim \sum_i c_i \gamma_i$$

Then,

$$\int_{\gamma} f(z)\,dz = \int_{\sum_i c_i \gamma_i} f(z)\,dz$$

$$= \sum_i c_i \gamma_i \int_{\gamma_i} f(z)\,dz.$$

The number

$$p_i = \gamma \int_{\gamma_i} f(z)\,dz$$ depends only on

the funl- and not on γ.

They are called modulus of periodicity of differential $f(z)dz$ (or) The periods of indefinite integral.

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