(60) Example : 2.

Show that the surface $x^2+y^2+z^2 = cx^{2/3}$ co form an equipotential family of surfaces and find the general form of the potential fun/.

Soln :

$$\text{Given } f(x,y,z) = \frac{x^2+y^2+z^2}{x^{2/3}} = c$$

$$f = x^{-2/3}(x^2+y^2+z^2) \longrightarrow (1)$$

$$f = x^{-2/3}x^2 + x^{-2/3}(y^2+z^2)$$

$$f = x^{4/3} + x^{-2/3}(y^2+z^2) \longrightarrow (2)$$

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} \longrightarrow (3)$$

$$\nabla^2 f = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\left(x^{4/3} + x^{-2/3}(y^2+z^2)\right)$$

$$\nabla^2 f = \frac{\partial^2}{\partial x^2}\left(x^{4/3} + x^{-2/3}(y^2+z^2)\right) + \frac{\partial^2}{\partial y^2}\left(x^{4/3} + x^{-2/3}(y^2+z^2)\right) + \frac{\partial^2}{\partial z^2}\left(x^{4/3} + x^{-2/3}(y^2+z^2)\right)$$

Take

$$\frac{\partial^2}{\partial x^2}\left(x^{4/3} + x^{-2/3}(y^2+z^2)\right) = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}\left\{x^{4/3} + x^{-2/3}(y^2+z^2)\right\}\right)$$

$$= \frac{\partial}{\partial x}\left[4/3\, x^{1/3} - \frac{2}{3}x^{-5/3}(y^2+z^2)\right]$$

$$= \frac{4}{3} \times \frac{1}{3} x^{-2/3} + \frac{10}{9} x^{-8/3}(y^2+z^2)$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{4}{9}x^{-2/3} + \frac{10}{9}x^{-8/3}(y^2+z^2) \longrightarrow (4)$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial}{\partial y}\left\{x^{4/3} + x^{-2/3}(y^2+z^2)\right\}\right)$$

$$\frac{\partial^2 f}{\partial y^2} = 2x^{-2/3} \longrightarrow (5)$$

$$\frac{\partial^2 f}{\partial z^2} = \frac{\partial}{\partial z}\left[\frac{\partial}{\partial z}\left(x^{4/3} + x^{-2/3}(y^2+z^2)\right)\right]$$

$$= \frac{\partial}{\partial z}\left(x^{-2/3} \cdot 2z\right)$$

$$\frac{\partial^2 f}{\partial z^2} = 2x^{-2/3} \longrightarrow (6)$$

Sub/- equ/- (4), (5) and (6) in (3)

$$\nabla^2 f = \frac{4}{9}x^{-2/3} + \frac{10}{9}x^{-8/3}(y^2+z^2) + 2x^{-2/3} + 2x^{-2/3}$$

$$= \frac{4}{9}x^{-2/3} + \frac{10}{9}x^{-8/3}(y^2+z^2) + 4x^{-2/3}$$

$$= \frac{4}{9}x^{-2/3} + 36\frac{x^{-2/3}}{9} + \frac{10}{9}x^{-8/3}(y^2+z^2)$$

$$= \frac{40x^{-2/3}}{9} + \frac{10}{9}x^{-8/3}(y^2+z^2)$$

$$= \frac{40}{9}x^{-2/3-2+2} + \frac{10}{9}x^{-8/3}(y^2+z^2)$$

$$= \frac{40}{90}x^{-8/3} \cdot x^2 + \frac{10}{9}x^{-8/3}(y^2+z^2)$$

$$\nabla^2 f = \frac{10}{9}x^{-8/3}\left[4x^2 + y^2 + z^2\right] \longrightarrow (7)$$

$$\nabla f = \left(\frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}\right)F$$

$$\nabla f = \left(\frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}\right)\left(x^{4/3} + x^{-2/3}(y^2+z^2)\right)$$

$$\nabla f = \frac{\partial}{\partial x}\vec{i}\left[x^{4/3} + x^{-2/3}(y^2+z^2)\right] + \frac{\partial}{\partial y}\vec{j}\left[x^{4/3} + x^{-2/3}(y^2+z^2)\right]$$

$$+ \frac{\partial}{\partial z}\vec{k}\left[x^{4/3} + x^{-2/3}(y^2+z^2)\right] \longrightarrow (8)$$

$$\frac{\partial}{\partial x}\vec{i}\left[x^{4/3} + x^{-2/3}(y^2+z^2)\right] = \left\{\frac{4}{3}x^{1/3} - \frac{2}{3}x^{-5/3}(y^2+z^2)\right\}\vec{i} \longrightarrow (9)$$

$$\frac{\partial}{\partial y}\vec{j}\left[x^{4/3} + x^{-2/3}(y^2+z^2)\right] = x^{-2/3} \cdot 2y\vec{j}$$

$$= 2x^{-2/3}yj \longrightarrow (10)$$

$$\frac{\partial}{\partial z}\vec{k}\left[x^{4/3} + x^{-2/3}(y^2+z^2)\right] = x^{-2/3} \cdot 2z\vec{k}$$

$$= 2x^{-2/3}z\vec{k} \longrightarrow (11)$$

(62) Sub/- equ/- (9), (10), (11) in equ/- (8)

$$\nabla f = \left[\frac{4}{3}x^{1/3} - \frac{2}{3}x^{-5/3}(y^2+z^2)\right]\vec{i} + 2x^{-2/3}y\vec{j} + 2x^{-2/3}z\vec{k}$$

$$= \left[\frac{4}{3}x^{1/3} - \frac{2}{3}x^{-5/3}(y^2+z^2)\right]\vec{i} + \frac{6x^{-2/3}y\vec{j}}{3} + \frac{6x^{-2/3}z\vec{k}}{3}.$$

$$= \left[\frac{4}{3}x^{1/3-2+2} - \frac{2}{3}x^{-5/3}(y^2+z^2)\vec{i}\right] + \frac{6x^{-2/3+1-1}}{3}\vec{j} + \frac{6x^{-2/3-1+1}}{3}\vec{k}$$

$$= \left[\frac{4}{3}x^{-5/3}x^2 - \frac{2}{3}x^{-5/3}y^2 - \frac{2}{3}x^{-5/3}z^2\right]\vec{i} + \frac{6x^{-5/3}xy}{3}\vec{j} + \frac{6x^{-5/3}xz}{3}\vec{k}$$

$\frac{2\times 2}{3}$ $\frac{2\times 3}{3}$ $\frac{2\times 3}{3}$

$$= \frac{2}{3}x^{-5/3}\left\{(2x^2-y^2-z^2)\vec{i} + 3xy\vec{j} + 3xz\vec{k}\right\}$$

$$|\nabla f| = \sqrt{\left(\frac{2}{3}x^{-5/3}\right)^2\left\{(2x^2-y^2-z^2)^2 + 9x^2y^2 + 9x^2z^2\right\}}$$

$$= \sqrt{\frac{4}{9}x^{-10/3}\left\{(2x^2-y^2-z^2)^2 + 9x^2y^2 + 9x^2z^2\right\}}$$

$$= \sqrt{\frac{4}{9}x^{-10/3}\left\{4x^4+y^4+z^4-4x^2y^2+2y^2z^2-4x^2z^2 + 9x^2y^2 + 9x^2z^2\right\}}$$

$$= \sqrt{\frac{4}{9}x^{-10/3}\left\{4x^4+y^4+z^4+5x^2y^2+5x^2z^2+2y^2z^2\right\}}$$

$$= \sqrt{\frac{4}{9}x^{-10/3}\left\{(4x^2+y^2+z^2)(x^2+y^2+z^2)\right)}$$

$$|\nabla f|^2 = \frac{4}{9}x^{-10/3}(4x^2+y^2+z^2)(x^2+y^2+z^2) \longrightarrow (21)$$

$$\therefore \frac{\nabla^2 f}{|\nabla f|^2} = \frac{\frac{10}{9}x^{-8/3}(4x^2+y^2+z^2)}{\frac{4}{9}x^{-10/3}(4x^2+y^2+z^2)(x^2+y^2+z^2)}$$

$$= \frac{10}{9} x^{-8/3} \times \frac{9}{4} x^{10/3} \cdot \frac{1}{x^2+y^2+z^2}$$

$$= \frac{5}{2} x^{2/3} \times \frac{1}{x^2+y^2+z^2}$$

$$= \frac{5}{2} \times \frac{1}{x^{-2/3}(x^2+y^2+z^2)}$$

$$\frac{\nabla^2 f}{|\nabla f|^2} = \frac{5}{2f} = \chi(f) \longrightarrow (13)$$

The given set of surfaces therefore forms a family of equipotential surfaces.

The formula for the general form of Potential fun/- is

$$V = A \int e^{-\int \chi(f)\, df}\, df + B \longrightarrow (14)$$

Sub/- equ/- (13) in (14)

$$V = A \int e^{-\int \frac{5}{2f}\, df}\, df + B \longrightarrow (15)$$

Consider $\int \frac{5}{2f}\, df$

$$= \frac{5}{2} \int \frac{df}{f}$$

$$\therefore \int \frac{5}{2f}\, df = \frac{5}{2} \log f \longrightarrow (16)$$

Sub (16) in (15)

$$V = A \int e^{-5/2 \log f}\, df + B$$

$$V = A \int e^{\log f^{-5/2}}\, df + B$$

$$V = A \int f^{-5/2}\, df + B$$

$$V = A \left[\frac{f^{-5/2+1}}{-5/2+1} \right] + B$$

$$= A \left[\frac{f^{-3/2}}{-3/2} \right] + B$$

$$= -A\frac{2}{3}\left[f^{-3/2}\right] + B$$

$$v = -2\frac{A}{3}\left[f^{-3/2}\right] + B$$

The required soln is

$$v = \frac{-2}{3}A \cdot x^{2/3} (x^2+y^2+z^2)^{-3/2} + B.$$

Boundary Value Problems.

Laplace Equation:

The laplace equ. in two dimensions is
$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \longrightarrow (1)$$

The soln of equ. (1) is called two dimentional harmonic equation.

Boundary Value Problems:

Let S be the interior of a simple, closed, smooth curve V and F be a continuous function defined on the boundary V.

Boundary Value Problem for Laplace's equ.

There are two main types of boundary value problem for laplace's equation

i) Dirichlet
ii) Neumann

i) The First Boundary Value Problem the Dirchlet Problem:

A problem of finding a harmonic function $u(x,y)$ in D ∋ it coincides with F

F on the boundary B is called the Drichlet Problem.

ii) **The Second B.V.P the Neumann Problem:**

This involves finding a function (ie) $u(x,y) \ni$ it is harmonic in D and Satisfies $\frac{\partial u}{\partial n} = F(S)$ on B [where $\frac{\partial}{\partial n}$ is the directional derivative along the outwards normal] with the condition

$$\int_{B} F(S)\,ds = 0$$

Infact the vanishing of the integral is the necessary condition for the Soln to exist.

Interior Dirichlet Problem:

If f is a continuous function presribed on the boundary S of some finite region V, determine a function $\varphi(x,y,z)$ such that $\nabla^2\varphi = 0$ within v and $\varphi = f$ on S.

Exterior Dirichlet Problem:

If f is a continuous function prescribed on the boundary S of a finite simply connected region V, determine a fun/. $\varphi(x,y,z)$ which satisfies $\nabla^2\varphi = 0$ Outside V and is $\ni \varphi = f$ on S.

Interior Neumann Problem:

If f is a continuous fun/. which is defined uniquely at each point of the boundary S of a finite region V, determine a fun/. $\varphi(x,y,z) \ni \nabla^2\varphi = 0$ within v and its normal derivative $\frac{\partial \varphi}{\partial n}$ coincides with f at every point of S.

Exterior Neumann problem:

If f is a continuous funt prescribed at each point of the (smooth) boundary S of a bounded simply connected region V find a funt $\psi(x,y,z)$ satisfying $\nabla^2\psi = 0$ outside V and $\frac{\partial\psi}{\partial n} = f$ on S.

UNIQUENESS THEOREM:

The soln of the dirichlet problem if it exists is unique.

Proof:

Suppose $u_1(x,y)$ & $u_2(x,y)$ are two solnt of the dirichlet problem.

(ie) $\nabla^2 u_1 = 0$ in S and

$u_1 = f(s)$ on V

$\nabla^2 u_2 = 0$ in S

$u_2 = f(s)$ on V

u_1 & u_2 are harmonic

$u_1 - u_2$ also harmonic in S.

Clearly $u_1 - u_2 = 0$ on V.

By maximum & minimum Principle.

$\Rightarrow u_1 - u_2 = 0$ on S.

$\Rightarrow u_1 = u_2$

Hence the soln of the Dirichlet problem is unique.

State and Prove necessary condition for the Neumann Problem.

Let u be a soln of the Newmann Problem $\nabla^2 u = 0$ in S and $\frac{\partial u}{\partial n} = f(s)$ on V.

Then $\int_V f(s)\,ds = 0$.

Proof

Given, let u be a soln. of the Neumann Problem $\nabla^2 u = 0$ in S

$$\frac{\partial u}{\partial n} = f(s) \text{ on } V$$

To prove $\int_V f(s)\, ds = 0$

Consider the Green identity

$$\int_S (\psi \nabla^2 \phi - \phi \nabla^2 \psi)\, ds = \int_V \left(\psi \frac{\partial \phi}{\partial n} - \phi \frac{\partial \psi}{\partial n}\right) ds$$

Put $\psi = 1$, $\phi = u$ in the above equ.

$$\int_S [\nabla^2 u - u \nabla^2 (1)]\, ds = \int_V \left(1 . \frac{\partial u}{\partial n} - u \frac{\partial (1)}{\partial n}\right) ds$$

$$\int_S (0 - u.0)\, ds = \int_V (f(s) - 0)\, ds$$

$$0 = \int_V f(s)\, ds$$

$$\Rightarrow \int_V f(s)\, ds = 0$$

Hence the Proof.

Vanathi

THEOREM:

The soln of neumann Problem is unique upto the addition of a constant.

(or)

The solutions of a certain neumann problem can differ from one another by a constant only.

Proof:

Let u_1 & u_2 be a harmonic functions in S

by $\ni$ $\frac{\partial u_1}{\partial n} = \frac{\partial u_2}{\partial n} = f(s)$ on V $\longrightarrow$ ①

To Prove $u_1 - u_2$ is constant.

Consider

$$V = u_1 - u_2$$

$$\nabla^2 u_1 - \nabla^2 u_2 = 0$$

$$\nabla^2 (u_1 - u_2) = 0$$

$$\nabla^2 V = 0$$

Now $\frac{\partial V}{\partial n} = \frac{\partial}{\partial n}(u_1 - u_2)$

$$= \frac{\partial u_1}{\partial n} - \frac{\partial u_2}{\partial n}$$

$$= f(s) - f(s)$$

$$\frac{\partial V}{\partial n} = 0 \quad \text{on } V \quad \rightarrow (9)$$

Consider the Green identity

$$\int_S [\psi_x \phi_x + \psi \phi_{xx} + \phi_y \psi_y + \psi \phi_{yy}] \, ds = \int_V \psi \frac{\partial \phi}{\partial n} \, ds$$

Take $\psi = \phi = V$ in (3)

$$\int_S (V_x V_x + V V_{xx} + V_y V_y + V V_{yy}) \, ds = \int_B V \frac{\partial V}{\partial n} \, ds$$

$$\Rightarrow \int_D V_x^2 + V[V_{xx} + V_{yy}] + V_y^2 \, ds = \int_B V(0) \, ds$$

$$\Rightarrow \iint_D \{[V_x^2 + V_y^2] + V[V_{xx} + V_{yy}]\} \, ds = 0 \qquad [\because (9), (0)]$$

$$\Rightarrow \int_S (V^2 x + V^2 y) \, ds = 0$$

$$\Rightarrow \nabla u = 0$$

$\therefore$ V is constant.

Separation of Variables:

Find the soln of laplace equ in spherical co-ordinates by the method of separation of variables.

Soln:

Laplace equation in spherical