Unit-III / Geodesic....... / 04.08.2016

5m(c) Geodesic :- A line on a Surface b/w 2 points which works as the shortest distance b/w these 2 points.

Let A & B be two given points on the Surface S and let these points be joined by Curves lying on S. Then any curve possesing stationary length for small variation over Surface S is called "Geodesis."

The Geodesic on a Surface may be defined as a curve of shortest distance b/w two points on the Surface.

Differential Eqn- of Geodesic :-

Let two point A & B on the surfaces. $r = r(u,v)$ be joined by two arcs whose equations are of the type $u = u(t)$ & $v(t) = v$ where $u(t)$ & $v(t)$ are of class 2.

Let us further assume Without loss of Generality that for every one α, $t=0$ at A and $t=1$ at B. So that t is given by $0 \le t \le 1$.

Since the arc Length s is related to Parameter t by a relation.

$$\left(\frac{ds}{dt}\right)^2 = E\left(\frac{du}{dt}\right)^2 + 2F\left(\frac{du}{dt}\right)\left(\frac{dv}{dt}\right) + G\left(\frac{dv}{dt}\right)^2$$

$$\dot{s}^2 = E\dot{u}^2 + 2F\dot{u}\dot{v} + G\dot{v}^2$$

$$\Rightarrow \dot{s} = (E\dot{u}^2 + 2F\dot{u}\dot{v} + G\dot{v}^2)^{1/2}$$

∴ The length of arc α is given by.

$$S(\alpha) = \int_0^1 \dot{s}\, dt$$

$$= \int_0^1 (E\dot{u}^2 + 2F\dot{u}\dot{v} + G\dot{v}^2)^{1/2} dt \longrightarrow ①$$

Now let the arc α be formed slightly and the new arc to be α' as the end pt/- A & B are kept fixed when the eqn/- of α' will be of the form.

$$u'(t) = u(t) + g(t)$$
$$v'(t) = v(t) + t\,h(t)$$

In the above eqn/- t is small and g, h are arbitrary fun/- of class 2 in the interval $0 \le t \le 1$ such that $g = h = 0$ at $t = 0$ & $t = 1$.

(i) At the end points the length α is obtained from the relation of ① after replacing u & v by u' & v'.

$$\therefore S(\alpha') = \int_0^1 (E\dot{u}'^2 + 2F\dot{u}'\dot{v}' + G\dot{v}'^2)^{1/2} dt$$

3 Variation of the Length α :-

Variation of the Length α is given by $S(\alpha') - S(\alpha)$ and it is a magnitude of order ε^2. For small variation in α when $S(\alpha)$ is said to be stationary and α is Geodesic.

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Lemma:

If $g(t)$ is a conts/- fun/- for $0 < t < 1$ and if $\int_0^1 g(t) v(t)\,dt = 0$ for all admissable funs/- $v(t)$ as defined by above then $g(t) = 0$.

Pf:

Suppose $\int_0^1 v(t) g(t)\,dt = 0 \longrightarrow (1)$ for all admissable fun/- for $v(t)$ & $g(t) \neq 0$ Then $\exists$ t_0 such that $0 < t_0 < 1$ and $g(t_0) \neq 0 \Rightarrow g(t_0) > 0$.

$\therefore$ $g(t_0)$ is conts/- in $(0,1)$ $\exists$ a nghd/- (a,b) of t_0. Such that $g(t_0) > 0$ in (a,b) where $0 \leq a \leq t < b \leq 1$. ($\exists$ a t_0 b/w 0 & 1 $\ni$: $g(t_0) \neq 0$).

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Now let us define a fun/- $v(t)$ as follows.

$$v(t) = \begin{cases} (t-a)^3 (b-t)^3 & a \leq t \leq b \\ 0 & 0 \leq t \leq a \\ 0 & b \leq t \leq 1 \end{cases}$$

$v(t)$ is an admissable fun/- in $(0,1)$. so that eqn/- (1) can be written as,

$$\int_0^1 g(t) v(t)\,dt = \int_0^a g(t) v(t)\,dt + \int_a^b g(t) v(t)\,dt + \int_b^1 g(t) v(t)\,dt \longrightarrow (2)$$

Using $v(t)$ in $(0,1)$ in the above step,

we get,

$$\int_0^1 v(t)g(t)dt = \int_a^b (t-a)^3(b-t)^3 g(t)\,dt \longrightarrow (3)$$

$\because (t-a)^3(b-t)^3 > 0$ in (a,b) & $g(t) > 0$ for $a < t < b$, we get $\int_0^1 v(t)g(t)dt > 0$

This contradicts to the hypothesis.

$\int_0^1 v(t)g(t)dt = 0$ for all admissible fun/- $v(t)$.

$\therefore$ our assumption $g(t_0) \neq 0$ is false.

$\therefore g(t) = 0 \ \forall \ t$ in $(0,1)$.

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Theorem:

Necessary & Sufficient condition for a curve $u = u(t)$ & $v = v(t)$ on a surface $r = r(u,v)$ to be geodesic is that.

$$U \cdot \frac{\partial T}{\partial \dot{v}} - V \cdot \frac{\partial T}{\partial \dot{u}} = 0 \longrightarrow (1)$$

Where,

$$\left.\begin{aligned} U &= \frac{d}{dt}\left[\frac{\partial T}{\partial \dot{u}}\right] - \frac{\partial T}{\partial u} = \frac{1}{2T}\frac{dT}{dt} \cdot \frac{\partial T}{\partial \dot{u}} \\ V &= \frac{d}{dt}\left[\frac{\partial T}{\partial \dot{v}}\right] - \frac{\partial T}{\partial v} = \frac{1}{2T}\frac{dT}{dt} \cdot \frac{\partial T}{\partial \dot{v}} \end{aligned}\right\} \longrightarrow (2)$$

eqn/- (2) are called geodesic eqn/-.

Proof:-

let $f(u,v,\dot{u},\dot{v}) = \sqrt{2T}$

Where,

$$2T \cancel{(u,v,\dot{u},\dot{v})} = \dot{s}^2$$
$$= E\dot{u}^2 + 2F\dot{u}\dot{v} + G\dot{v}^2.$$

interms of F the arc length $s(\alpha)$ is

$$s(\alpha) = \int_0^1 \dot{s}\,dt = \int_0^1 \sqrt{2T}\,dt$$

$$= \int_0^1 f(u, v, \dot{u}, \dot{v})\, dt$$

after a slight deformation the arc length

$$S(\alpha') = \int_0^1 f(u+\varepsilon\lambda, v+\varepsilon\mu, \dot{u}+\varepsilon\dot{\lambda}, \dot{v}+\varepsilon\dot{\mu})\, dt$$

Hence the variation is,

$$S(\alpha') - S(\alpha) = \int_0^1 [f(u+\varepsilon\lambda, v+\varepsilon\mu, \dot{u}+\varepsilon\dot{\lambda}, \dot{v}+\varepsilon\dot{\mu}) - f(u, v, \dot{u}, \dot{v})]\, dt \longrightarrow (3)$$

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Using Taylor thm!- for several variables we've,

$$f(u+\varepsilon\lambda, v+\varepsilon\mu, \dot{u}+\varepsilon\dot{\lambda}, \dot{v}+\varepsilon\dot{\mu}) - f(u, v, \dot{u}, \dot{v})$$

$$= \varepsilon\lambda \frac{\partial f}{\partial u} + \varepsilon\mu \frac{\partial f}{\partial v} + \varepsilon\dot{\lambda} \frac{\partial f}{\partial \dot{u}} + \varepsilon\dot{\mu} \frac{\partial f}{\partial \dot{v}} + o(\varepsilon^2) \longrightarrow (4)$$

Sub (4) in (3) we get,

$$S(\alpha') - S(\alpha) = \int_0^1 \varepsilon\left(\lambda \frac{\partial f}{\partial u} + \mu \frac{\partial f}{\partial v} + \dot{\lambda} \frac{\partial f}{\partial \dot{u}} + \dot{\mu} \frac{\partial f}{\partial \dot{v}}\right) dt + o(\varepsilon^2) \longrightarrow (5)$$

Consider,

$$\int_0^1 \dot{\lambda}\left(\frac{\partial f}{\partial \dot{u}}\right) dt = \left(\frac{\partial f}{\partial \dot{u}}\lambda\right)_0^1 - \int_0^1 \lambda \frac{d}{dt}\left(\frac{\partial f}{\partial \dot{u}}\right) dt$$

$$\left(\because u = \frac{\partial f}{\partial \dot{u}} \Rightarrow du = \frac{d}{dt}\left(\frac{\partial f}{\partial \dot{u}}\right),\ dv = \dot{\lambda} \Rightarrow v = \lambda\right)$$

We've $\left(\lambda \frac{\partial f}{\partial \dot{u}}\right)_0^1 = 0$

$$\Rightarrow \int_0^1 \dot{\lambda} \frac{\partial f}{\partial \dot{u}}\, dt = -\int_0^1 \lambda \frac{d}{dt} \frac{\partial f}{\partial \dot{u}}\, dt \longrightarrow (6)$$

lll^ly we have,

$$\int_0^1 \dot{\mu} \frac{\partial f}{\partial \dot{v}}\, dt = -\int_0^1 \mu \frac{d}{dt}\left(\frac{\partial f}{\partial \dot{v}}\right) dt \longrightarrow (7)$$

Using (6), (7) in (5) we get,

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$$S(\alpha') - S(\alpha) = \int_0^1 \varepsilon \left\{ \lambda \left(\frac{\partial f}{\partial u} - \frac{d}{dt}\left(\frac{\partial f}{\partial \dot u}\right)\right) + \mu \left(\frac{\partial f}{\partial v} - \frac{d}{dt}\left(\frac{\partial f}{\partial \dot v}\right)\right)\right\} dt + o(\varepsilon^2)$$

$$S(\alpha') - S(\alpha) = \int_0^1 \varepsilon (\lambda L + \mu M)\, dt + o(\varepsilon^2) \longrightarrow (8)$$

Where,

$$L = \frac{\partial f}{\partial u} - \frac{d}{dt}\left(\frac{\partial f}{\partial \dot u}\right) \text{ and}$$

$$M = \frac{\partial f}{\partial v} - \frac{d}{dt}\left(\frac{\partial f}{\partial \dot v}\right)$$

For the arc length α to be geodesis on S. S(α) should be stationary.

It is stationary iff $S(\alpha') - S(\alpha)$ is atmost of order ε^2 $\forall$ small variation and since $\varepsilon > 0$ Then eqn (8) becomes.

$$\int_0^1 (\lambda L + \mu M)\, dt = 0 \longrightarrow (9)$$

For all admissable funs λ, μ in the interval $0 < t < 1$ and $\lambda = \mu = 0$ at $t = 0$ & $t = 1$.

$\therefore$ E, F, G are of class one and $\lambda(t)$, $\mu(t)$ are of class 2 the fun L & M are the cont funs satisfying the condition as that of g(t) of that previous Lemma.

We can apply previous lemma to eqn (9) we choosing λ, μ & g as follows.

Case : (i)

$\lambda = v(t)$, $\mu = 0$ & $g(t) = L$.

(9) $\Rightarrow \therefore \int_0^1 (\lambda L + \mu M)\,dt = \int_0^1 \lambda L\,dt = 0$

$\therefore L = 0$ by using previous Lemma.

($\because g(t) = L$ & $g(t) = 0$. By previous lemma).

Case : (ii)

Let $\mu = v(t)$ and $\lambda = 0$.

$g(t) = M$

$\int_0^1 (\mu M + \lambda L)\,dt = \int_0^1 \mu M\,dt = 0$

$\mu = 0$ by previous Lemma.

$\therefore L = M = 0$ are the differential eqn of $u(t)$ & $v(t)$

$\therefore$ These two eqn denote involve the two points A & B explicitly the eqn

$L = 0$, $M = 0$ are same for all geodesic on the Surface.

Let us rewrite $L = 0$, $M = 0$ interms of T.

W.K.T $F = \sqrt{2T}$

$$L = \frac{\partial F}{\partial u} - \frac{d}{dt}\left(\frac{\partial F}{\partial \dot{u}}\right)$$

$$= \frac{1}{\sqrt{2T}}\frac{\partial T}{\partial u} - \frac{d}{dt}\left(\frac{1}{\sqrt{2T}} \cdot \frac{\partial T}{\partial \dot{u}}\right)$$

$$= \frac{1}{\sqrt{2T}} \cdot \frac{\partial T}{\partial u} - \left[\frac{1}{\sqrt{2T}} \cdot \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{u}}\right) - \frac{1}{(2T)^{3/2}} \cdot \frac{dT}{dt} \cdot \frac{\partial T}{\partial \dot{u}}\right]$$

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$$L = \frac{1}{\sqrt{2T}}\left[\frac{\partial T}{\partial u} - \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{u}}\right) + \frac{1}{2T}\frac{dT}{dt}\left(\frac{\partial T}{\partial \dot{u}}\right)\right]$$

$$0 = \frac{1}{\sqrt{2T}}\left[\frac{\partial T}{\partial u} - \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{u}}\right) + \frac{1}{2T}\frac{dT}{dt}\left(\frac{\partial T}{\partial \dot{u}}\right)\right]$$

$$\Rightarrow \frac{\partial T}{\partial u} - \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{u}}\right) + \frac{1}{2T}\frac{dT}{dt}\cdot\frac{\partial T}{\partial \dot{u}} = 0$$

$$\Rightarrow \frac{1}{2T}\left(\frac{dT}{dt}\right)\frac{\partial T}{\partial \dot{u}} = \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{u}}\right) - \frac{\partial T}{\partial u} \longrightarrow (10)$$

lly

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{v}}\right) - \frac{\partial T}{\partial v} = \frac{1}{2T}\frac{dT}{dt}\cdot\frac{\partial T}{\partial \dot{v}} \longrightarrow (11)$$

eqn (10) & (11) gives the diff eqn of the geodesic & they are usually written as.

$$U = \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{u}}\right) - \frac{\partial T}{\partial u} = \frac{1}{2T}\cdot\frac{dT}{dt}\cdot\frac{\partial T}{\partial \dot{u}} \longrightarrow (12)$$

$$V = \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{v}}\right) - \frac{\partial T}{\partial v} = \frac{1}{2T}\cdot\frac{dT}{dt}\cdot\frac{\partial T}{\partial \dot{v}} \longrightarrow (13)$$

Where, $2T(u, v, \dot{u}, \dot{v}) = j^2$

This completes the proof of eqn (2)

To prove :-

eqn (1) as the necessary and sufficient condition for α to be geodesic on surface S.

To prove :

The necessary condition.

Let α be the geodesic on the surface S & that $u(t), v(t)$, satisfy eqn (2) from the 2nd expression of u & v in eqn (12) (13) we get,

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$$\frac{U}{V} = \frac{\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{u}}\right) - \frac{\partial T}{\partial u}}{\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{v}}\right) - \frac{\partial T}{\partial v}} = \frac{\frac{1}{2T}\frac{dT}{dt}\cdot\frac{\partial T}{\partial \dot{u}}}{\frac{1}{2T}\frac{dT}{dt}\cdot\frac{\partial T}{\partial \dot{v}}}$$

$$\frac{U}{V} = \frac{\partial T/\partial \dot{u}}{\partial T/\partial \dot{v}}$$

$$V \neq 0 \Rightarrow U\,\partial T/\partial \dot{v} = V\frac{\partial T}{\partial \dot{u}} \Rightarrow U\frac{\partial T}{\partial \dot{v}} - V\frac{\partial T}{\partial \dot{u}} = 0$$

Which proves the Necessary condition.

To prove:

Sufficient condition.

We need the following lemma which is true for any curve whether it is a Geodesic or not.

Continuous

Lemma:

If U & V are as in ① Then $\dot{u}U + \dot{v}V = \frac{dT}{dt} \rightarrow (14)$

Since each of U & V have two equal expression for it, we shall prove eqn. (14) by corresponding the following 2 cases.

Case: (i)

In this case we prove eqn. (14) by

Consider the 1st expression for U & V.

uniformity → after same level → degree

Since t is a homogeneous fun. of degree 2 in $\dot{u}$ & $\dot{v}$ we have by Euler's thm. $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu$

$$\dot{u}\frac{\partial T}{\partial \dot{u}} + \dot{v}\frac{\partial T}{\partial \dot{v}} = 2T \longrightarrow (15)$$

Since T is a fun. of $(u, v, \dot{u}, \dot{v})$

we get,

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$$\frac{dT}{dt} = \frac{\partial T}{\partial u}\dot{u} + \frac{\partial T}{\partial v}\dot{v} + \frac{\partial T}{\partial \dot{u}}\ddot{u} + \frac{\partial T}{\partial \dot{v}}\ddot{v} \longrightarrow (16)$$

Using (15) & (16) we prove eqn:- (14)

Substituting for U & V we have, from (12)

$$\dot{u}U + \dot{v}V = \dot{u}\left(\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{u}}\right) - \frac{\partial T}{\partial u}\right) + \dot{v}\left[\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{v}}\right) - \frac{\partial T}{\partial v}\right] \longrightarrow (17)$$

Consider,

$$\frac{d}{dt}\left(\dot{u}\left(\frac{\partial T}{\partial \dot{u}}\right)\right) = \dot{u}\,\frac{d}{dt}\frac{\partial T}{\partial \dot{u}} + \frac{\partial T}{\partial \dot{u}}\ddot{u}$$

$$\Rightarrow \dot{u}\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{u}}\right) = \frac{d}{dt}\left(\dot{u}\left(\frac{\partial T}{\partial \dot{u}}\right)\right) - \ddot{u}\frac{\partial T}{\partial \dot{u}} \longrightarrow (18)$$

$$\& \quad \frac{d}{dt}\left(\dot{v}\left(\frac{\partial T}{\partial \dot{v}}\right)\right) = \dot{v}\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{v}}\right) + \ddot{v}\left(\frac{\partial T}{\partial \dot{v}}\right)$$

$$\Rightarrow \dot{v}\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{v}}\right) = \frac{d}{dt}\left(\dot{v}\left(\frac{\partial T}{\partial \dot{v}}\right)\right) - \ddot{v}\frac{\partial T}{\partial \dot{v}} \longrightarrow (19)$$

Sub (18) & (19) in (17),

$$\dot{u}U + \dot{v}V = \frac{d}{dt}\left(\dot{u}\left(\frac{\partial T}{\partial \dot{u}}\right)\right) - \ddot{u}\frac{\partial T}{\partial \dot{u}} - \dot{u}\frac{\partial T}{\partial u} + \frac{d}{dt}\left(\dot{v}\frac{\partial T}{\partial \dot{v}}\right) - \ddot{v}\frac{\partial T}{\partial \dot{v}} - \dot{v}\frac{\partial T}{\partial v}$$

$$= \frac{d}{dt}\left[\dot{u}\left(\frac{\partial T}{\partial \dot{u}}\right) + \dot{v}\left(\frac{\partial T}{\partial \dot{v}}\right)\right] - \left[\ddot{u}\frac{\partial T}{\partial \dot{u}} + \ddot{v}\frac{\partial T}{\partial \dot{v}} + \dot{u}\frac{\partial T}{\partial u} + \dot{v}\frac{\partial T}{\partial v}\right]$$

$$= \frac{d}{dt}(2T) - \frac{dT}{dt} \quad \text{(by (15) \& (16))}$$

$$= 2\frac{dT}{dt} - \frac{dT}{dt}$$

$$= \frac{dT}{dt}$$

$$\therefore \dot{u}U + \dot{v}V = \frac{dT}{dt} \longrightarrow (20).$$