unit - IV

Renewal processes

Renewal processes and Theory Renewal processes define:

Renewal processes in discrete time

Consider sequence of repeated trails with possible out comes E_j, $j = 1, 2 \ldots$

The trails need not be independent we assume that the trials can be repeated infinitely.

Suppose that we are interested in a system outcome in a trial r a outcomes in a number of outcomes in denotes wherever E^* occurence we say that a Renewal has occured.

Renewal period:

The Interval between occurence of two successive renewals is called a renewal period of the process.

$f_n = Pr\{E^* \text{ occurs for the first at the } n^{th} \text{ trial}\}$

$P_n = Pr\{E^* \text{ occurs at the } n^{th} \text{ trial}\}$

Defn:

Renewal Interval

Let $f_0 = 0$, $P_0 = 1$

$$F(S) = \sum_{n=0}^{\infty} f_n S^n, \quad P(S) = \sum_{n=0}^{\infty} P_n S^n$$

Now

$f^* = \sum f_n$ is the probability that the renewal E^* occurs some trial in a long sequence of trial

we have $f^* \leq 1$ when $f^* = 1$

Then $\{f_n\}$ is proper probability distribution representing the distribution of the length of a Renewal period T

ie) $P\{T = n\} = f_n$

However $\{P_n\}$ is not a probability distribution. The Renewal event is terms presistant when $f^* = 1$ & transient when $f^* < 1$

Relation between F(S) & P(S): (8) m

The event that E^* occurs at the n^{th} trial is compound event such that E^* occurs for the 1st time at the r^{th} trial & again at the later ~~trial~~ trial number n i.e in subsequence $(n-r)$ trials.

And thus.

$$P_n = \sum_{r=1}^{n} f_r P_{n-r}, \quad n \ge 1 \longrightarrow (1)$$

The R.H.S in a convolution relation $\{f_n\}$ & $\{P_n\}$ multiply by s^n, $n=1,2,\dots$

$$P_n s^n = \sum_{r=1}^{n} f_r P_{n-r} s^n$$

$$P_n s^n = (f_1 s^1)(P_{n-1} s^{n-1}) + \dots + (f_n s^n) P_0$$

and adding $$\sum_{n=1}^{\infty} P_n s^n = (f_1 s)\sum_{n=1}^{\infty} P_{n-1} s^{n-1} + (f_2 s^2)\sum_{n=2}^{\infty} P_{n-2} s^{n-2}$$

$$P(S) - 1 = P(S)\left[\sum_{n=1}^{\infty} f_n s^n\right] = P(S)F(S) \longrightarrow (2)$$

Thus $$\boxed{P(S) = \frac{1}{1-F(S)}} \longrightarrow (*)$$

(A) → $$F(S) = \frac{P(S)-1}{P(S)} \longrightarrow (2)$$

Now from equation (*) it follows that

$$\sum P_n = P(1) = \frac{1}{1-F(1)}$$

is convergent iff $F(1) < 1$ f^* is the transistent. In other words for E^* is transistent iff

$$\sum P_n = P(1) \text{ is finite}$$

The probability that E^* ever occuring by

$$f^* = f(1) = \frac{\sum P_n - 1}{\sum P_n}$$

E^* is persistent iff $\sum P_n$ is divergent.

Define: periodic & aperiodic

(8) 2m/NR The Renewal event E^* is said to be periodic if there exist an Integer $m (>1)$ such that E^* can

occur only at trials numbered m, 2m, ...

The greatest m with this property is said to be the period of E^*. It is said to be aperiodic if no such m exist. The sequence $\{a_n\}$ is said to be periodic with period $m (>1)$ if $a_n = 0$ if $n \neq km$, $k = 1, 2, ...$ and m is the greatest Integer

Mean Recurrence time:

For a persistent and aperiodic renewal event $F'(1) = \Sigma n f_n = E(T)$ is the mean recurrence time [i.e] mean time between two consecutive renewals (or) mean waiting time between the two consecutive renewals.

$F'(1)$ may be finite (or) Infinite

Renewal Interval:

The Renewal Interval T has the p.m.f

$Pr\{T = n\} = f_n$

T is the proper renewal value when

$$\Sigma f_n = F(1) = 1.$$

with mean recurrence time

$$\Sigma n f_n = F'(1)$$

T is also called the waiting time for the occurrence of the renewal E^*

The generating function of T is

$$F(s) = \Sigma f_n s^n$$

The probability $f_n^{(2)}$ that E^* occurs for the second time at n^{th} trail is given by

$$f_n^{(2)} = \sum_{k=1}^{n-1} f_k f_{n-k}$$

$^{|||ly}$ The probability $f_n^{(r)}$ that E^* occurs for the time at n^{th} trial is given by

$$f_n^{(r)} = \sum_{k=1}^{n-1} f_k f_{n-k}$$

Thus $\{f_n^{(r)}\}$ gives the probability distribution

$T^{(r)} = T_1 + \dots + T_r$

Where T_i are identically independent distribution random variable as T.

The Generating function of $\{f_n^{(r)}\}$ is given by

$$F^{(r)}(s) = \sum_n f_n^{(r)} s^n$$

$$= [F(s)]^r$$

putting $s=1$, we get

$$\sum_n f_n^{(r)} = F^{(r)}_{(1)} = [F(1)]^r = (f^*)^r$$

$\sum_n f_n^{(r)}$ is the probability that f^* occurs atleast r time of the processes is continue indefintely if follows that

$Pr\{E^*$ occurs exactly are times if the process is continued Indefinitely$\}$

$$= (f^*)^r - (f^*)^{r+1}$$

$$= (f^*)^r [1 - f^*]$$

Generalized form:- ✱ 2m Ⓡ

Define:- Delayed Recurrent Event:-

The first occurrence of E^* is then called a delayed recurrent event yield the subsequence occurren are ordinary reccurrence equation event.

Notation:-

Denote $V_n = pr\{E^*$ occurs at n^{th} trail$\}$ —→ ①

Suppose that the first occurence of E^* happen at trial number K & Then renewal occurence of E^* occurs at the subsequence $(n-K)$ trial.

Thus we get

$$V_n = B_n + B_{n-1} \cdot P_1 + B_{n-2} \cdot P_2 + \dots + B_1 P_{n-1} + B_0 P_n$$

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That is $v_n = \{B_n\} * \{p_n\}$ → (2)

Denoting $V(s) = \Sigma v_n s^n$, $B(s) = \Sigma B_n s^n$

we can write

$$V(s) = B(s) \cdot P(s) \rightarrow (3)$$

$P(s) = \frac{1}{1-F(s)}$

$$= \frac{B(s)}{1-F(s)} \rightarrow (4)$$

Theorem: (2) 5m (R)

(2) 5m

If $p_n \to \alpha$ Then $v_n \to \alpha b \rightarrow (5)$ where $b = B[1]$ $= \Sigma b_n$. If Σb_n Converges Then $\Sigma v_n \to b\beta \rightarrow (6)$

10m (X) 10m

proof:

Denote $r_k = P_k\{1^{st}$ Renewal period $> k\}$

$$= b_{k+1} + b_{k+2} + \ldots \ldots$$

we can choose k sufficiently large such that $r_k < \epsilon$.

since $p_m \leq 1$ we get from (1)

$$b_0 p_n + b_1 p_{n-1} + \ldots + b_k p_{n-k} \leq v_n$$

$$b_0 p_n + b_1 p_{n-1} + \ldots \ldots + b_{n-1}(p_1 + b_n)\}$$

$$= b_0 p_n + \ldots + b_k p_{n-k} + \{b_{k+1} p_{n-(k+1)} + \ldots$$

$$= b_0 p_n + \ldots + b_k p_{n-k} + \{b_{k+1} + \ldots + b_n\}$$

$$\leq b_0 p_n + \ldots + b_k p_{n-k} + r_k \rightarrow (7)$$

$$\leq b_0 p_n + \ldots + b_k p_{n-k} + r_k$$

As $p_n \to \alpha$, $b_0 p_n + b_1 p_{n-1} + \ldots + b_k p_{n-k}$

$$v_n \Rightarrow (b_0 + b_1 + \ldots + b_k)\alpha$$

$$= (b - r_k)\alpha$$

$$> b\alpha - 2\epsilon$$

Since $p_n < 1$ Implies that $\alpha \leq 1 \leq 2$ & $b_0 p_n + \ldots + b_k p_{n-k} + r_k$

$$\to (b - r_k)\alpha + r_k$$

$$= b\alpha + r_k(1-\alpha)$$

$$< b\alpha + (1-\alpha)\epsilon$$

$$< b\alpha + 2\epsilon$$

So that from (7) we get

$$b\alpha - 2\epsilon < v_n < b\alpha + 2\epsilon$$

making ϵ sufficiently small we get

$$\lim_{n\to\infty} V_n \to b\alpha$$

which gives ⑤

From ③ we get

$$\Sigma V_n = V(1) = B(1)\,P(1)$$

$$\to b_\beta \left(\text{as } P(1) = \Sigma P_n \to \beta\right)$$

Corollary:-

If F^* is persistent then $V_n \to b/\mu$

proof:

$$V_n = \lim_{s\to 1-0} (1-s)\, V(s)$$

$$= \lim_{s\to 1-0} B(s)\, \frac{1-s}{1-F(s)}$$

$$\to B(1)\, \frac{1}{F'(1)} = b/\mu$$

Renewal Theory in Discrete time!

we shall now consider a results of greater generality. Suppose that $\{f_n,\ n=1,2\ldots\}$ $\{b_n,\ n=0,1,2\ldots\}$ are two sequence of Real number such that $f_n \geq 0$, $f = \Sigma f_n < \infty \to ①$

and $b_n \geq 0$, $b = \Sigma b_n < \infty \to ②$.

Define a new sequence $\{V_n,\ n=0,1,2\ldots\ldots\}$.

By thm

Convolution Relation

$$V_n = b_n + V_{n-1} f_1 + V_{n-2} f_2 + \ldots\ldots + V_0 f_n$$

$$= b_n + \sum_{r=1}^{n} f_r V_{n-r} \to ③$$

The above define V_n uniquely interms of sequence $\{b_n\}$ & sequence $\{f_n\}$ In-terms of the generating function we get

$$V(S) = B(S) + F(S)\, V(S)$$

$$V(S)(1 - F(S)) = B(S)$$

$$V(S) = \frac{B(S)}{1-F(S)} \longrightarrow (4)$$

$F(S)$ & $B(S)$ Converge atleast for $0 \le S < 1$ & if $F(S) < 1$ Then $V(S)$ is a power series in S. The sequence $\{f_n\}$ is periodic

If there exist an integer $m \ni$; $f_n = 0$ except for $n = k.m$

Theorem: Renewal theorem:

Suppose that the Relation $b_n \ge 0$ $b = \Sigma b_n < \infty$ hold and $\{f_n\}$ is not periodic

a) If $b < 1$, Then $V_n \to 0$ & $\Sigma V_n = \frac{b}{1-f}$

b) If $b = 1$, Then $V_n \to b/\mu$.

proof:

we get, when $f < 1$ & $S = 1$

we know that

$$V(S) = \frac{B(S)}{1-F(S)}$$

$$V(1) = \frac{B(1)}{1-F(1)}$$

$$\Sigma V_n = \frac{b}{1-f}$$

ΣV_n is Convergent $\Rightarrow$ $V_n \to 0$

when $f = 1$, $\Sigma V_n \to \infty$

we can see that, $V_n \to b/\mu$

$$\mu = F'(1)$$

$$= \Sigma n f_n$$

Renewal Processes. Continuous time

let $\{X_n, n = 1, 2 \ldots\}$ be a sequence of non-ve Independent random variables

Assume that $pr\{X_n = 0\} < 1$ and that the random variables are Identically distribution & are Continuous.

with a distribution function $F(\cdot)$

Since X_n is non negative it follows that $E\{X_n\}$ exist & let us denote

$$E(X_n) = \int_0^\infty x \, dF(x)$$
$$= \int_0^\infty x \, F'(x) \, dx$$
$$= \int_0^\infty x \, f_n(x) \, dx = \mu$$

where μ may be infinite whenever $\mu = \infty$, $1/\mu$ shall Interpreted as zero

Let $S_0 = 0$, $S_n = X_1 + X_2 + \ldots\ldots + X_n$, $n \geq 1$ &

Let $F_n(x) = pr\{S_n \leq x\}$ be the distribution functions of $\{S_n, n \geq 1\}$

$$F_0(x) = 1 \text{ if } x \geq 0 \text{ \& } F_0(x) = 0 \text{ if } x < 0$$

Define:- Renewal process

Define the Random variable.

$$N(t) = \sup\{n : S_n \leq t\}$$

The process $\{N(t), t \geq 0\}$ is called a renewal is said to occur at t. S_n gives the time of the n^{th} renewal & is called n^{th} Renewal epoch

$$N(t) = \sup\{n : S_n = t\}$$

Define:- palm flow of events

The Renewals where X_i are Identically random variable are called palm flow of events.

where X_i are Identically exponential the renewa are called ordinary (or) poisson flow of events

Renewal function (?) 2m

The function $\mu(t) = E\{N(t)\}$ is called the Renewal function of the process with distribution

$\{N(t) \geq n\} \Leftrightarrow \{S_n \leq t\}$

$\{N(t) < n\} \Leftrightarrow \{S_n > t\}$

Theorem:-

The distribution of $N(t)$ is given by $P_n(t) = pr\{N(t) = n\} = F_n(t) - F_{n+1}(t)$ and the expected number of Renewals by

$$M(t) = \sum_{n=1}^{\infty} F_n(t)$$

proof:

we have $pr\{N(t) = n\} = pr\{N(t) \geq n\} - pr\{N(t) \geq n+1\}$

$$= pr\{S_n \leq t\} - pr\{S_{n+1} \leq t\}$$

$$P_n(t) = F_n(t) - F_{n+1}(t).$$

Again

$$M(t) = E\{N(t)\} = \sum_{n=0}^{\infty} n P_n(t)$$

$$= \sum_{n=0}^{\infty} n\{F_n(t) - F_{n+1}(t)\}$$

$$= F_1(t) - F_2(t) + 2F_2(t) - 2F_3(t) + 3F_3(t) - 3F_4(t) \ldots$$

$$= F_1(t) + F_2(t) + F_3(t) + \ldots$$

$$M(t) = \sum_{n=1}^{\infty} F_n(t) = \sum_{n=1}^{\infty} P_r\{S_n \leq t\}$$

Renewal density

The derivative $m(t)$ of $M(t)$ The derivative $m(t)$ of $M(t)$

ie) $M'(t) = m(t)$ is called the Renewal Density

we have.

$$m(t) = \lim_{\Delta t \to 0} \frac{pr\{\text{one (or) more renewal in } (t, t+\Delta t)\}}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \frac{pr\{n^{th} \text{ renewal occurs in } (t, t+\Delta t)\}}{\Delta t}$$

$$= \sum_{n=1}^{\infty} \lim_{\Delta t \to 0} \frac{P_n(t)\,\Delta t + o(\Delta t)}{\Delta t}$$ [$\because$ assuming that $f(x)$ is absolutely continuous & $F_n'(t) = f_n(t)$]

$$= \sum_{n=1}^{\infty} f_n(t)$$

$$= \sum_{n=1}^{\infty} F_n'(t) = M'(t)$$

The function $m(t)$ specifies the mean number of Renewals to be expected in a narrow Interval near t.

Eg:
Let x_n have gamma distribution having density

$$f(x) = \frac{a^k x^{k-1} e^{-ax}}{(k-1)!} \quad x \geq 0$$
$$= 0 \quad \text{else where}$$

i) find $f^*(s)$ & $M^*(s)$

ii) find the distribution of $S_n = x_1 \ldots x_n$ and Hence find $P_n(t)$

(iii) If $k=1$, S.T The Renewal process becomes a poisson process.

proof:
Then $f^*(s) = \left(\frac{a}{s+a}\right)^k$ and the density $F_n'(x)$ of $S_n = x_1 + x_2 + \ldots + x_n$ has the Laplace transform $\left(\frac{a}{s+a}\right)^k$

Thus $$F_n'(x) = \frac{a^{nk} x^{nk-1} e^{-ax}}{(nk-1)!}$$ and

Hence.
$$F_n(x) = \int_0^x F_n'(y)\, dy = 1 - e^{-ax} \sum_{r=0}^{nk-1} \frac{(ax)^r}{r!}, \; n \geq 1$$

Thus,
$$P_n(t) = F_n(t) - F_{n+1}(t)$$
$$= e^{-at} \sum_{r=nk}^{(n+1)k-1} \frac{(ax)^r}{r!}$$

and by using these equation $m^*(s) = \frac{f^*(s)}{s[1 - f^*(s)]}$

$$M^*(s) = \frac{a^k}{s[(s+a)^k - a^k]} \longrightarrow (i)$$

(i) when $k=1$, x_n has -ve exponential distribution & The Renewal process then Reduces to a poisson process we have,

$$P_n(t) = e^{-at} \frac{(at)^n}{n!}$$ and

$$m^*(s) = \frac{a}{s^2}$$ so that $m(t) = at$

(ii) when $K = 2$ Then from equation (1)

$$m^*(S) = \frac{a^2}{S[(S+a)^2 - a^2]} = \frac{a}{2}\left[\frac{1}{S^2} - \frac{1}{S(S+2a)}\right]$$

$$= \frac{a}{2}\left[\frac{1}{S^2} - \frac{1}{2a}\left\{\frac{1}{S} - \frac{1}{S+2a}\right\}\right]$$

Inverting the laplace transform we get

$M(t) = \frac{a}{2}(t) - \frac{1}{4} + \frac{1}{4}e^{-2at}$ as the expected number of Renewal is an Interval of time t

Hyper - Exponential Distribution

no need

Let X_n have density

$f(t) = pae^{-at} + (1-p)be^{-bt}$, $0 \le p \le 1$, $a > b > 0 \rightarrow (1)$

As system has two kind of components 'a' proportion 'p' of components having a high failure rate 'a' and the Remaining proportion $(1-p)$ to components we have

$$f^*(S) = \frac{pa}{S+a} + \frac{(1-p)b}{S+b} \rightarrow (2)$$

So that

$$m^*(S) = \frac{ab + S[pa + (1-p)b]}{S^2[S + (1-p)b + pb]}$$

writing $A = pa + (1-p)b$ and

$B = (1-p)a + pb$

we get

$$m^*(S) = \frac{AS + ab}{S^2(S+B)} = \frac{A}{S(S+B)} + \frac{ab}{S^2(S+B)}$$

$$= \frac{A}{B}\left\{\frac{1}{S} - \frac{1}{S+B}\right\} + \frac{ab}{B}\left[\frac{1}{B} - \left\{\frac{1}{S} - \frac{1}{S+B}\right\}\frac{1}{B}\right]$$

Sing the laplace transform we get

$$M(t) = \frac{A}{B}\left(1 - e^{-Bt}\right) + \frac{ab}{B}\left[t - \frac{1 - e^{-Bt}}{B}\right]$$

$$= \frac{abt}{B} + c\left(1 - e^{-Bt}\right) \rightarrow (3)$$

where $C = \left(\frac{A}{B} - \frac{ab}{B^2}\right)$

$= \frac{P(1-P)(a-b)^2}{B^2} \geq 0$

when $p=1$ (or $p=0$) the distribution of X_n Reduces the -ve exponential & then $c=0$. The second term of equation (3) vanishes, so that we get

$$M(t) = \frac{ab\,t}{B} \quad ((\text{or})\ bt)$$

Renewal Equation:-

An Integral equation can be obtained for the Renewal function $M(t) = E\{N(t)\}$ which gives the expected number of renewals in $[0,t]$

10m

Theorem: 2m

State and prove Integral Equation of Renewal Equation

The Renewal function M satisfies the equation

$$M(t) = F(t) + \int_0^t M(t-x)\, dF(x).$$

Proof:

By conditioning the 1st Renewal x_1 we get

$$M(t) = E\{N(t)\} = \int_0^t E\{N(t)/x_1 = x\}\, dF(x) \longrightarrow (1)$$

<u>case (i)</u>

Consider $x > t$, given that $x_1 = x > t$ No Renewal occurs in $[0,t]$ so that

$$E\{N(t)/x_1 = x\} = 0.$$

<u>case (ii)</u>

Consider $0 \leq x \leq t$, given that the first Renewal occurs at $x (\leq t)$ then the process starts again at epoch x. and the expected number of Renewals in the Remaining interval of length $(t-x)$ is

$E\{N(t-x)\}$ so that

$$E\{N(t)/x_1 = x\} = 1 + \{E\, N(t-x)\} = 1 + m(t-x) \quad \text{--- (2)}$$

Sub equ (2) in (1) we get

$$M(t) = \int_0^t \{1 + m(t-x)\}\, dF(x)$$

$$= \int_0^t dF(x) + \int_0^t M(t-x)\, dF(x)$$

$$= F(t) + \int_0^t M(t-x)\, dF(x)$$

Integral equation of Renewal Theory:-

(8) imp. The argument used to derive it is known as "Renewal argument"

The Renewal equation is also expressed as

$M = F + M^* f$

Another method of Derivation of Integral equation

we know that

$$M(t) = \sum_{n=1}^{\infty} F_n(t)$$

$$= F_1(t) + \sum_{n=1}^{\infty} F_{n+1}(t)$$

$$= F(t) + \sum_{n=1}^{\infty} F_{n+1}(t-x)$$

$$= F(t) + \sum_{n=1}^{\infty} \left\{ \int_0^t F_{n+1}(t-x)\, dF(x) \right\}$$

F_{n+1} being the convolution of F_n & $F_1 = F$ the change of order of Integration and summation we get

$$M(t) = F(t) + \int_0^t \left\{ \sum_{n=1}^{\infty} F_{n+1}(t-x) \right\} dF(x)$$

$$M(t) = F(t) + \int_0^t M(t-x)\, dF(x)$$

Renewal Type Equation:

The Renewal equation can be generalised as follows.

$$V(t) = g(t) + \int_0^t V(t-x)\, dF(x), \quad t \geq 0$$

are g & f are known and V is an known the equation is called "Renewal equation".

Theorem:

If $v(t) = g(t) + \int_0^t v(t-x)\, dF(x),\ t \geq 0$

$$v(t) = g(t) + \int_0^t g(t-x)\, dm(x)$$

$$M(t) = \sum_{n=0}^{\infty} F_n(t)$$

proof:

given $v(t) = g(t) + \int_0^t v(t-x)\, dF(x)$

Taking laplace transform

$$\boxed{v^*(s) = g^*(s) + v^*(s) f^*(s)}$$

$$\Rightarrow v^*(s) - v^*(s) f^*(s) = g^*(s)$$

$$\Rightarrow v^*(s) = \frac{g^*(s)}{1-f^*(s)} \rightarrow g^*(s)\left[\frac{1}{1-f^*(s)}\right] = v^*(s)$$

Add and sub $f^*(s)$

$$v^*(s) \Rightarrow g^*(s)\left[\frac{1-f^*(s)+f^*(s)}{1-f^*(s)}\right]$$

$$= g^*(s)\left[\frac{1-f^*(s)}{1-f^*(s)} + \frac{f^*(s)}{1-f^*(s)}\right]$$

$$= g^*(s)\left[1 + \frac{f^*(s)}{1-f^*(s)}\right]$$

$$= g^*(s)\left[1 - sM^*(s)\right] = g^*(s) - s g^*(s) M^*(s)$$

Taking Inverse L.T we get

$$L^{-1}[v^*(s)] = L^{-1}[g^*(s)] + L^{-1}(s)\, L^{-1}[g^*(s)]\, L^{-1}[M^*(s)]$$

$$v(t) = g(t) + \int_0^t g(t-x)\, dm(x)$$ solution $v(t)$ is unique since a function is uniquely determined by L.T

Stopping time; Wald's Equation:-

Define:- Stopping time

An Integer valued random variable N is said to be a stopping time for the sequence $\{X_i\}$ If the event $\{N = n\}$ is independent of $X_{n+1}, X_{n+2}, \ldots$

for all $n = 1, 2, \ldots$ It can be shown that N is a proper random variable.

Eg:- Consider a coin tossing experiment. Let the outcome of the i^{th} toss be denoted by $X_i = 1$ (or) 0, depending on the result being head (or) tail respectively and let $Pr\{X_i = 1\} = p = 1 - Pr\{X_i = 0\}$ & Then $E(X_i) = p$

The sum $S_n = X_1 + X_2 + \ldots + X_n$ denote the cummulative number of heads in the first 'n' tosses. Suppose that m is a given +ve integer, then

$$N = \min\{n;\ S_n = m\}$$ is a stopping time.

Derive

Wald's Equation:- (5) sm (8) (8)

Let $\{X_i\}$ be a sequence of Independent random variable having the same expectation & let N be a stopping time for $\{X_i\}$ & $E(N) < \infty$ Then $E\left\{\sum_{i=1}^{n} X_i\right\} = E(X_i) \cdot E(N)$

proof:

Let $Z_i = \begin{cases} 0 & \text{if } N \geq i \\ 1 & \text{if } N < i \end{cases}$

So that $$\sum_{i=1}^{n} X_i = \sum_{i=1}^{\infty} X_i Z_i$$

and Thus,
$$E\left\{\sum_{i=1}^{n} X_i\right\} = E\left\{\sum_{i=1}^{\infty} X_i Z_i\right\}$$
$$= \sum_{i=1}^{\infty} E\{X_i Z_i\}$$
$$= \sum_{i=1}^{\infty} E(X_i)\, E(Z_i)$$

and assuming the validity of the change of expectation & summation

Now Z_i is determined by $\{N < i\}$ by $X_1, X_2, \ldots, X_{i-1}$ and is Independent of X_i

Thus,
$$E\left\{\sum_{i=1}^{n} X_i\right\} = \sum_{i=1}^{\infty} E(X_i)\, E(Z_i)$$
$$= E\{X_i\} \sum_{i=1}^{\infty} E\{Z_i\}$$
$$= E\{X_i\} \sum_{i=1}^{\infty} Pr\{N \geq i\} = E\{X_i\}\, E\{N\}$$

Corollary :-

Since $N(t)+1$ is a stopping time for the sequence $\{x_i\}$, we have

$$E\{S_N(t+1)\} = E\left\{\sum_{i=1}^{N(t)+1} x_i\right\}$$
$$= E\{x_i\}\ E\{N(t)+1\}$$
$$= E\{x_i\}\ \{M(t)+1\}$$

Remark :-

For wald's equation to holds the r.v of x_i need not be identically distributed but x_i's must be independent and have the same mean.

i.e) $E(X_i) = E(X)$ for all i.

→ If N is independent of $\{x_i\}$

Then $E\left\{\sum_{i=1}^{n} x_i/N\right\} = N\ E\{x_i\}$ and

$$E\{x_i\} = E\left[E\left\{\sum_{i=1}^{n} x_i/N\right\}\right] = E(N)\ E(x_i)$$

Eg:-

Consider a sequence of independent coin tosses the result of the n^{th} toss in denoted by x_n.

$x_n = 1$ if head occurs and $x_n = -1$ if trail occurs.

Let $pr\{x_n = 1\} = p$. $Pr\{x_n = -1\} = q$

Then $E\{x_n\} = p-q$ for all n

and $S_n = x_1 + \ldots\ldots + x_n$ gives. the number of heads and the number of tails in n tosses.

Let. $N = \min\{n : S_n = 1,\ n = 1, 2 \ldots\}$

N is the first toss in which the numbers of trails by exactly 1.

Here N is a stopping time & $S_N = 1$ for any value of N so that $E\{S_N\} = 1$

By wald's theorem

$$1 = E\{S_N\} = E\{x_i\}\ E\{N\}$$

Thus $E\{N\} = E\{x_i\} \; E\{N\}$

$E\{N\} = \infty$ when $p = q$ p

N is defective when $p < q$

Renewal Theorem:

poisson process (with parameter a) is a Renewal process having expectation inter arrival time X_n with mean $1/a$ we have

$$M(t) = at \quad (or)$$

$$\frac{M(t)}{t} = a = \frac{1}{E(X_n)} \quad \text{In case the Result}$$

$$\frac{M(t)}{t} \to \frac{1}{\mu} \;.\; \mu = E(X_n) \leq \infty \text{ as } t \to \infty$$

Theorem: (2) sm

With probability 1, $\frac{N(t)}{t} \to \frac{1}{\mu}$ as $t \to \infty$ where $\to$ (1)

$\mu = E(X_n) \leq \infty$

proof:

Consider an interval $[0, t]$ we have

$$S_{N(t)} \leq t < S_{N(t)+1} \to (2)$$

Now the strong law of large number hold's for the sequence $\{S_n\}$ show that as $n \to \infty$

$$\frac{S_n}{n} = \frac{X_1 + X_2 + \cdots + X_n}{n} \to E(X_n) = \mu \text{ with}$$

probability 1

Agains as $t \to \infty$, $N(t) \to \infty$ with probability 1

Thus, with probability 1

$$\frac{S_{N(t)}}{N(t)} \to \mu \text{ as } t \to \infty \to (3)$$

my with probability 1

$$\frac{S_{N(t)+1}}{N(t)} = \frac{S_{N(t)+1}}{N(t)+1} \cdot \frac{N(t)}{N(t)} \to \mu \text{ as } t \to \infty$$

$\to$ (4)

Thus from equ (3) relations we get that with probability 1

$$\frac{N(t)}{t} \to \frac{1}{\mu} \text{ as } t \to \infty$$

This theorem shows that for longest number of Renewals per unit time converges to $1/\mu$.

Elementary Renewal theorem: (5) 10m (R.R)

(5) 2m (6)m

We have $\frac{\mu(t)}{t} \to \frac{1}{\mu}$ as $t \to \infty$ → (1) where $\mu = E(X_n) < \infty$ The limit being interpreted as 0 when $\mu = \infty$

proof:

let $N = N(t)+1$ & $S_N = t + y(t)$.

where $y(t)$ is a residual life time of the unit in use at time t

Let $\{x_i^{(j)}, j = 1, 2, \ldots \infty\}$ be a sequence of Indepnd relations of the renewal process $\{X_i\}$ $\{S_n^j\}$ be the corresponding partial sums and $N(t)$ the corresponding number of Renewals in $[0, t]$

$$\text{let } Z = S_{N(t)}.\ M_K = N^1(t) + \ldots + N^x(t)$$

$$T_K = Z^{(1)} + \ldots + Z^{(K)}$$

Now. T_K is the sum of $(K+M_K)$ Identically random Variable X_i. Thus by the strong law of large numbers if $\mu = E(X_i) < \infty$ Then as $K \to \infty$

$$\frac{T_K}{K+M_K} = \frac{\sum_{i=1}^{K+M_K} X_i}{K+M_K} \to \mu \to (2)$$

with probability 1.

By the same law we get as $K \to \infty$ → (3)

$$\frac{M_K}{K} = \frac{\sum_{i=1}^{k} N^i(t)}{k} \to E\{N(t)\} = M(t) \text{ as } t \to \infty$$

$$\frac{T_K}{K} = \sum \frac{Z}{K} \to E(Z) = E\{t + y(t)\} \text{ as } \quad [\because Z = S_N]$$

$$\frac{T_K}{K} = t + E\{y(t)\} \to (4)$$

with probability 1

Combining equation (2) (3) (4) we get as $K \to \infty$

Now,

$$\frac{T_K}{K+M_K} = \frac{T_K}{K\left(1+\frac{M_K}{K}\right)}$$

$$= \frac{T_K}{K}\left[1+\frac{M_K}{K}\right]^{-1} \to \frac{t + E\{y(t)\}}{1+M(t)} \to \mu$$

Consider

$$\frac{t + E\{y(t)\}}{1+M(t)} \to \mu, \quad t + E[y(t)] \to \mu[1+M(t)]$$

$\div$ by t

i.e)
$$1 + \frac{E\{y(t)\}}{t} = \mu\left\{\frac{1}{t} + \frac{M(t)}{t}\right\}$$

and Now $y(t)$ is $+$ve $E\{y(t)\}$ is finite and Hence as $t \to \infty$

$$\lim_{t\to\infty} \inf \frac{M(t)}{t} = \frac{1}{\mu} \to (5)$$

To show that $\lim_{t\to\infty} \sup \frac{M(t)}{t} \leq \frac{1}{\mu}$

we define a new Renewal process as follows

Let A be a constant > 0 and for $n = 1, 2, \ldots$

and let $x_n^* = \begin{cases} x_n & \text{if } x_n \leq A \\ A & \text{if } x_n > A \end{cases}$

Let $S_n^* = \sum_{i=1}^{n} x_i^*$ and $N^*(t) = \sup\{n;\ S_n^* \leq t\}$

$$M^*(t) = E\{N^*(t)\}$$

Now,

$S_n^* \leq S_n$ & Hence $N^*(t) \geq N(t)$ and $M^*(t) \geq M(t)$.

Again $E(x_n^*) = \mu_A \leq \mu$ and $\mu_A \to \mu$ as $A \to \infty$

$$\frac{S^*_{N(t)+1}}{N(t)+1} \to \mu_A \Rightarrow E\left(S^*_{N(t)+1}\right) \to \mu_A\, E[N(t)+1]$$

$$\Rightarrow E[S^*_{N(t)+1}] = \mu_A[M^*(t)+1]$$

and Hence

$$\lim_{t\to 0} \sup \frac{M^*(t)}{t} \leq \frac{1}{\mu_A} \text{ and as } A \to \infty$$

Since $\frac{M^*(t)}{t} \geq \frac{M(t)}{t}$

$$\lim_{t\to\infty} \sup \frac{M(t)}{t} < \frac{1}{\mu} \text{ as } n\to\infty \to (6)$$

$$\frac{M(t)}{t} = \frac{1}{\mu} \text{ as } t\to\infty$$

(8) 2m Define lattice variable.

A non negative random variable x is said to be a lattice variable if it takes on values nd ($n = 0, 1, \ldots$ d...

ie) Integral multiples of a non-ve number d. The largest d is said to be the period of the distribution when $d=1$, a lattice variable becomes an integer valued variable

Define: Directly Riemann Integrable:

Let $f(x)$ be a function defined on $[0, \infty]$ for $h>0$, $n=1,2\ldots$

Let $\overline{m_n} = \max\{f(x); (n-1)h \leq x \leq nh\}$

$\underline{m_n} = \min\{f(x); (n-1)h \leq x \leq nh\}$

and let $\bar{\sigma} = h\Sigma \overline{m_n}$, $\underline{\sigma} = h\Sigma \underline{m_n}$

If both the series converge absolutely for every $h>0$ and if $\bar{\sigma} - \underline{\sigma} \to 0$ as $h \to 0$

Then $f(x)$ is said to be directly Rieman Integrable.

If $f(x)$ is non-ve and non increasing on $[0,\infty)$ $[f(\infty)=0]$ and in integrable there on in the ordinary sence.

Then $f(x)$ is directly Riemann Integrable

Theorem:

Let $H(t)$ be a non-ve, non increasing function of $t\geq 0$ such that $\int_0^\infty H(t)\,dt < \infty$ & let X_i be non-lattice Then as $t\to\infty$

$$\int_0^t H(t-x)\,dm(x) \to \frac{1}{\mu}\int_0^\infty H(t)\,dt \quad \text{---}(1)$$

The limit being interpreted as 0 when $\mu=\infty$

Proof:

We have

$$\int_0^t H(t-x)\,dm(x) = \int_0^{t/2} H(t-x)\,dm(x) + \int_{t/2}^t H(t-x)\,dm(x)$$

$$= I_1 + I_2 \quad \text{---}(2)$$

Since $H(t)$ is non-ve & non-increasing

$H(t-x) \le H(t/2)$ for $0 \le x \le t/2$ & so.

$$I_1 = \int_0^{t/2} H(t-x)\, dm(x) \le H(t/2) \int_0^{t/2} dm(x)$$

$$= H(t/2) \left[M(x) \right]_0^{t/2}$$

$$= H(t/2)\; M(t/2)$$

Now as $t \to \infty$

$$\frac{M(t/2)}{t/2} \to \frac{1}{\mu}$$

and $\left(\frac{t}{2}\right) H(t/2) \to 0$ Thus $I_1 \to 0$ as $t \to \infty \to (3)$

let $J = \int_0^{\infty} H(t)\, dt = \sum_{n=0}^{\infty} \int_{nh}^{nh+h} H(t)\, dt \to (4)$

Now, $H(nh+h) \le H(t) \le H(n h)$, for $nh \le t \le nh+h$

$$\sum_{n=0}^{\infty} \int_{nh}^{nh+h} H(nh+h)\, dt \le \sum_{n=0}^{\infty} \int_{nh}^{nh+h} H(t)\, dt \le \sum_{n=0}^{\infty} \int_{nh}^{nh+h} H(nh)\, dt$$

$$\Rightarrow h \sum_{n=0}^{\infty} H(nh+h) \le J \le h \sum_{n=0}^{\infty} H(nh)$$

$$\Rightarrow h \sum_{n=0}^{\infty} H(nh+h) \le J \le h\, H(0) + h \sum_{n=1}^{\infty} H(nh)$$

Thus, $h \sum_{n=0}^{\infty} H(nh+h) - h \sum_{n=1}^{\infty} H(nh) \le J - h \sum_{n=1}^{\infty} H(nh) \le h\, H(0)$

Thus,

$$0 \le J - h \sum_{n=1}^{\infty} H(nh) \le h\, H(0) < \epsilon$$

$$\Rightarrow 0 \le h\, H(0) < \epsilon \text{ if } 0 < h < \epsilon / H(0)$$

Again $I_2 = \int_{t/2}^{t} H(t-x)\, dm(x)$

put $y = t - x \Rightarrow x = t - y \Rightarrow dx = -dy$

$$I_2 = \int_{t/2}^{t} H(y) \left[-d\, m(t-y) \right]$$

$$= \sum_{n=0}^{N-1} \int^{nh+h} H(y) \left[dm(t-y) \right]$$

where N is the greatest integer contained in $t/_h$

i.e) $t/_{nh} = \mu$

It follows that

$$nh+h \le t-x \le nh.$$

$$\Rightarrow H(nh+h) \le H(t-x) \le H(nh).$$

$$\Rightarrow \sum_{n=0}^{N-1} \int_{nh}^{nh+h} H(nh+h)\, dM(x) \le \sum_{n=0}^{N-1} \int_{nh}^{nh+h} H(t-x)\, dM(x) \le \sum_{n=0}^{N-1} \int_{nh}^{nh+h} H(nh)\, dM(x)$$

$$\Rightarrow \sum_{n=0}^{N-1} H(nh+h) \left[M(x) \right]_{nh}^{nh+h} \le I_2 \le \sum_{n=0}^{N-1} H(nh) \left[M(x) \right]_{nh}^{nh+h}$$

$$\Rightarrow \sum_{n=0}^{N-1} H(nh+h) \left[M(t-y) \right]_{nh}^{nh+h} \le I_2 \le \sum_{n=0}^{N-1} H(nh) \left[M(t-y) \right]_{nh}^{nh+h}$$

$$\Rightarrow \sum_{n=0}^{N-1} H(nh+h) \left[M(t-nh) - M(t-nh-h) \right] \le I_2 \le \sum_{n=0}^{N-1} H(nh)$$

for largest we can make $\left[M(t-nh) - M(t-nh-h) \right]$

$$\left| \frac{M(t-nh) - M(t-nh-h)}{h} - \frac{1}{\mu} \right| < \epsilon$$

and $h \sum_{N+1}^{\infty} H(nh) < \epsilon$

we find that

$$\left(1/\mu - \epsilon \right) (J - \epsilon) < I_2 < \left(1/\mu + \epsilon \right) (J + \epsilon)$$

$I_2 \to J/\mu$ as $t \to \infty$ Hence $t \to \infty$

$$I = I_1 + I_2 \to 0 + J/\mu \to J/\mu \to \frac{1}{\mu} \int_0^{\infty} H(t)\, dt$$

$$I_1 + I_2 \to \frac{1}{\mu} \int_0^{\infty} H(t)\, dt$$

$$\therefore \int_0^{t} H(t-x)\, dM(x) \to \frac{1}{\mu} \int_0^{\infty} H(t)\, dt$$

Theorem: Central Limit Theorem for Renewals:

Let $\{ X_n, n = 1, 2, \ldots \}$ be Renewals process with distribution F, for which the mean $\mu = E(X_i)$ and

Variance $\sigma^2 = E\{(X_i - \mu)^2\}$ exist and are finite

Let $\{N(t), t \geq 0\}$ be the Renewal process generated by F.

Then $\lim_{t\to\infty} \left\{ \frac{N(t) - t/\mu}{\sqrt{t\sigma^2/\mu^3}} < x \right\} = \phi(x)$

where, $\phi(x) = \frac{1}{\sqrt{2\pi}} \int_0^x \exp(-\tfrac{1}{2}t^2)\, dt$ is the discrete freedom of the standard normal distribution.

proof:

From the central limit theorem.

$S_n = X_1 + X_2 + \dots + X_n$ we find that as $n \to \infty$

S_n is asymptotically Normal with mean $n\mu$ & variance $n\sigma^2$

$$\lim_{n\to\infty} Pr \left| \frac{X-\mu}{\sigma} \right| < \epsilon$$

$$\lim_{n\to\infty} pr \left\{ \frac{S_n - n\mu}{\sigma\sqrt{n}} < x \right\} = \phi(x)$$

Let x be fixed and let $n \to \infty$ and $t \to \infty$ such that

$$\lim_{n\to\infty} \frac{t - n\mu}{\sigma\sqrt{n}} \to -x$$

$$t \to (n\mu - x) \cdot \sigma\sqrt{n} \text{ as } n \to \infty,\ t \to \infty$$

we have, $\lim pr\{S_n > t\} = \lim \{1 - pr\{S_n \leq t\}\}$

$$= 1 - \lim_{n\to\infty} Pr\{S_n \leq t\}$$

$$= 1 - \lim_{n\to\infty} Pr \left\{ \frac{S_n - n\mu}{\sigma\sqrt{n}} \leq \frac{t - n\mu}{\sigma\sqrt{n}} \right\}$$

$$= 1 - \lim_{n\to\infty} Pr \left\{ \frac{S_n - n\mu}{\sigma\sqrt{n}} \leq -x \right\}$$

$$= 1 - \phi(-x) = \phi(x)$$

again $\{N(t) < n\} \Leftrightarrow \{S_n > t\}$, $\phi(x) = \lim_{n\to\infty} pr\{S_n > t\}$

$$\phi(x) = \lim_{\substack{n\to\infty \\ t\to\infty}} Pr\{N(t) < n\}$$

$$= \lim_{\substack{n\to\infty \\ t\to\infty}} Pr \left\{ \frac{N(t) - t/\mu}{\sqrt{(t\sigma^2/\mu^3)}} < \frac{n - t/\mu}{\sqrt{(1 - \sigma^2/\mu^3)}} \right\}$$

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$$= \lim_{t \to \infty} Pr\left\{ \frac{N(t) - t/\mu}{\sqrt{(t\sigma^2/\mu^3)}} < x \right\}$$

Then $N(t)$ is asymptotically normal with mean t/μ, Variance $t\sigma^2/\mu^3$

Theorem:

Black Well's Theorem

For X_i non - Lattice and for fixed $h \geq 0$ $M(t) - M(t-h)$ $\to h/\mu$ as $t \to \infty$ for a lattice X_i with period d

$$\lim_{n \to \infty} Pr\{\text{Renewal at } nd\} \to d/\mu$$

Smiths Theorem (or) key Renewal Theorem:

Let $H(t)$ be directly Riemann Integrable and $H(t) = 0$ for $t < 0$ If X_i is non Lattice Then

$$\int_0^t H(t-x)\, dM(x) \to 1/\mu \int_0^\infty H(t)\, dt \text{ as } t \to \infty$$

The limit being Interpreted as 0 when $\mu = \infty$

If X_i is lattice with period d. Then $H(c+nd) \to d/\mu \sum_{k=0}^{\infty} h(c+kd)$

Remark:

$$(Ht) = \begin{cases} 1/h & 0 \leq t \leq h \\ 0 & \text{, othewise.} \end{cases}$$

$$M(t) = M(t-h) \to 1/\mu \int_0^h dt = h/\mu$$

State and prove Gamma distribution

Let X_n have gamma distribution having density

$$f(x) = \left\{ \frac{a^k x^{k-1} e^{-ax}}{(k-1)!} \right., x \geq 0. \text{ elsewhere}$$

i) Find $f^*(s)$ & $M^*(s)$

ii) Find the distribution of $S_n = X_1 \ldots X_n$ and process becomes a poisson process.

proof:

$$L[f(x)] = L\left[\frac{a^k x^{k-1} e^{-ax}}{(k-1)!}\right]$$

$$= \frac{a^k}{(k-1)!} L\left[x^{k-1} e^{-ax}\right]$$

$$= \frac{a^k}{(k-1)!}\left[\frac{(k-1)!}{(s+a)^k}\right]$$

$$= \left(\frac{a}{s+a}\right)^k$$

$f^*(s) = \left(\frac{a}{s+a}\right)^k$ and the density function $S_n = x_1 + x_2 + \ldots\ldots + x_n$ has the laplace transform $\left(\frac{a}{s+a}\right)^{k}$

Thus,

$$f_n(x) = \frac{a^{nk} x^{nk-1} e^{-ax}}{(nk-1)!} \text{ and Hence}$$

$$F_n(x) = \int_0^x f_n(y)\,dy = \int_0^x \frac{a^{nk} y^{nk-1}}{nk-1} e^{-ay}\,dy$$

$$= \frac{a^{nk}}{(nk-1)!}\int_0^x e^{-ay} y^{nk-1}\,dy$$

$$= \frac{a^{nk}}{(nk-1)!}\left\{\left[y^{nk-1}\left(\frac{e^{-ay}}{-a}\right)\right]_0^x - \int_0^x e^{-ay}(nk-1)\, y^{nk-2}\,dy\right.$$

$$= \frac{a^{nk}}{(nk-1)!}\left[x^{nk-1}\left(\frac{e^{-ax}}{-a}\right) + \frac{nk-1}{a}\int_0^x e^{-ay} y^{nk-2}\,dy\right.$$

$$= \frac{a^{nk}}{(nk-1)!}\left[\frac{x^{nk-1} e^{-ax}}{-a} + \frac{nk-1}{a}\left\{\left[y^{nk-2}\frac{e^{-ay}}{-a}\right]_0^x + \int_0^x \frac{e^{-ay}}{a}(nk-2)\, y^{nk-3}\,dy\right\}\right]$$

$$= \frac{a^{nk}}{(nk-1)!}\left[\frac{x^{nk-1} e^{-ax}}{-a} - \frac{a^{nk-2}}{(nk-2)!} x^{nk-2} e^{-ax} \ldots\right]$$

$$= -e^{-ax}\left[\frac{a^{nk-1}}{(nk-1)!} x^{nk-1} + \frac{a^{nk-2}}{(nk-2)!} x^{nk-2} + \ldots\right]$$

$$= 1 - e^{-ax}\left[\sum_{r=0}^{nk-1} \frac{(ax)^r}{r!}\right] = \left[1 - e^{-at}\sum_{n=0}^{(n+1)k-1} \frac{at}{n!}\right]$$

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$$= 1 - e^{-ax}\left[\sum_{r=0}^{nk-1}\left(\frac{a^k}{r!}\right)\right] - \left[1 - e^{-at}\sum_{n=0}^{(n+1)k-1}\frac{at}{n!}\right]$$

$$P_n(t) = e^{-at}\sum_{r=nk}^{(n+1)k-1}\left(\frac{(ax)^r}{r!}\right)$$ and using

$$M^*(s) = \frac{f^*(s)}{s[1-f^*(s)]}$$ we get

$$= \frac{a^k}{(s+a)^k s} \times \frac{1}{1-(a/s+a)^k}$$

$$= \frac{a^k}{s(s+a)^k} \times \frac{1/(s+a)^k - a^k}{(s+a)^k}$$

$$m^*(s) = \frac{a^k}{s(s+a)^k} \times \frac{(s+a)^k}{(s+a)^k - a^k}$$

$$= \frac{a^k}{s(s+a)^k} \times \frac{(s+a)^k}{(s+a)^k - a^k}$$

$$m^*(s) = \frac{a^k}{s[(s+a)^k - a^k]}$$

put $k=1$

$$= \frac{a}{s[(s+a)-a]} = \frac{a}{s^2}$$

(i)

$k=1$ X_n has -ve exponential distribution and renewal process then reduces to poisson process. As It is expected we have then

$$P_n(t) = e^{-at}\sum_{r=kn}^{(n+1)(k-1)}\frac{(ax)^r}{r!}$$

$$= e^{-at}\left[\frac{(at)^n}{n!}\right]$$

$$= \frac{e^{-at}(at)^n}{n!}$$ and $m^*(s)\ a/s^2$

$$\therefore m^*(s) = \frac{a^k}{s[(s+a)^k - a^k]} = \frac{a}{s[s+a-a]} = a/s^2$$

$$m^*(s) = a/s^2$$

So that $M(t) = L^{-1}[M^*(s)] = L^{-1}(a/s^2)$

$= a\, L^{-1}(1/s^2)$

(ii) when $k=2$ gamma with slope parameter two then

$$M^*(s) = \frac{a^k}{s[(s+a)^k - a^k]} \quad \text{put } k=2 \text{ we get}$$

$$M^*(s) = \frac{a^2}{s[(s+a)^2 - a^2]} = \frac{a^2}{s[s^2+a^2+2as-a^2]} = \frac{a^2}{s[s^2+2as]}$$

$$= \frac{a^2}{s^2[s+2as]} = \frac{a^2}{s^3+2as^2} = \frac{a^2}{s^2[s+2a]}$$

using partial fraction we get

$$\frac{a^2}{s^2(s+2a)} = \frac{A}{s+2a} + \frac{B}{s} + \frac{C}{s^2}$$

$$a^2 = As^2 + B(s+2a)s + C(s+2a)$$

put $s=-2a$ $\Rightarrow a^2 = A(-2a^2) + B(0) + C(0)$

$a^2 = 4a^2 A \Rightarrow A = 1/4$

put $s=0$ $\Rightarrow a^2 = 2Ca \Rightarrow C = a/2$

Equating the co-eff of s^2 we get

$A+B=0 \Rightarrow B + 1/4 = 0 \Rightarrow B = -1/4$

$$\frac{a^2}{s^2(s+a)} = \frac{1}{4(s+2a)} - \frac{1}{4s} + \frac{a}{2s^2}$$

$$= \frac{a}{2}\left[\frac{1}{s^2} - \frac{1}{2a}\left(\frac{1}{s} - \frac{1}{s+2a}\right)\right]$$

Inversting Laplace transform we get

$$L^{-1}[M^*(s)] = L^{-1}\left[\frac{a}{2}\left(\frac{1}{s^2} - \frac{1}{2a}\left(\frac{1}{s} - \frac{1}{s+2a}\right)\right)\right]$$

$$M(t) = \frac{a}{2}\left[L^{-1}\left(\frac{1}{s^2}\right) - \frac{1}{2a} L^{-1}\left(\frac{1}{s}\right) + \frac{1}{2a} L^{-1}\left(\frac{1}{s+2a}\right)\right]$$

$$= \frac{a}{2}\left[t - \frac{1}{2a}(1) + \frac{1}{2a}\left(e^{-2at}\right)\right] = \frac{at}{2} - \frac{a}{4a} + \frac{a}{4a\, e^{2at}}$$

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$$= \frac{a}{2}t - \frac{1}{4} + \frac{1}{4}e^{-2at}$$

As the expand number of renewals in the interval of time t

$$L^{-1}\frac{(a^n \sqrt{n+1})}{(s+a)^{n+1}} = (at)^n e^{-at}$$

$\Rightarrow n = 0.$

$$L^{-1}\left[\frac{1}{s+a}\right] = e^{-at} \Rightarrow L^{-1}\left(\frac{1}{s+a}\right) = e^{-2at}$$

Hyper - Exponential distribution:

Let X_n have density $f(t) = pa\,e^{-at} + (1-p)b\,e^{-bt}$

$0 \le p \le 1$, $a > b > 0$ such a model may be used to describe a system which has 2 kinds of components a proportional 'p' of components having say a high failure rate 'a' and the remaining proportion $(1-p)$ of components having a different say a lower failure rate b we have.

$$L[f(t)] = pa\,L(e^{-at}) + (1-p)\,b\,L(e^{-bt})$$

$$f^*(s) = \frac{pa}{s+a} + \frac{(1-p)b}{s+b}$$

So that

$$m^*(s) = \left(\frac{f^*(s)}{s[1-f^*(s)]}\right)$$

$$= \frac{\left(pa/s+a + \frac{(1-p)b}{s+b}\right)}{s\left(1 - \left[pa/s+a + \left(\frac{(1-p)b}{s+b}\right)\right]\right)}$$

$$m^*(s) = \frac{(s+b)pa + (s+a)(b-pb)\,/\,(s+a)(s+b)}{s\left[(s+a)(s+b) - pa(s+b) - (1-p)b(s+a)\,/\,(s+a)(s+b)\right]}$$

$$= \frac{pas + abp + sb - spb + ab - abp}{s[s^2 + sb + as + ab - pas - pab - bs - ab + psb + pab]}$$

$$= \frac{ab + s(ap + b - pb)}{s^2[s + a - pa + pb]} = \frac{ab + s[ap + b(1-p)]}{s^2[s + a(1-p) + pb]}$$

writing $A = pa + (1-p)b$ and $B = (1-p)a + pb$, we get

$$M^*(s) = \frac{As+ab}{s^2(s+B)} = \frac{As}{s^2(s+B)} + \frac{ab}{s^2(s+B)}$$

$$= \frac{A}{s(s+B)} + \frac{ab}{s^2(s+B)} \rightarrow (1)$$

$$\frac{A}{s(s+B)} = \frac{C}{s} + \frac{D}{s+B} \rightarrow (2)$$

$$A = C(s+B) + Ds$$

put $s=0 \Rightarrow A = BC \Rightarrow C = A/B$

$s=-B$, $A = C(0) + D(-B) \Rightarrow A = -DB \Rightarrow D = -A/B$

$$\frac{ab}{s^2(s+B)} = \frac{E}{s+B} + \frac{F}{s} + \frac{G}{s^2} \rightarrow (3)$$

$$ab = E(s^2) + Fs(s+B) + G(s+B)$$

put $s=-B \Rightarrow B^2E = ab \Rightarrow E = ab/B^2$

put $s=0 \Rightarrow ab = G(B) \Rightarrow G = ab/B$

Equating the co-eff of s^2 we get

$$E+F=0 \Rightarrow F = \frac{-ab}{B^2}$$

Sub (2) & (3) in (1)

$$M^*(s) = C/s + D/_{s+B} + E/_{s+B} + F/s + G/s^2$$

$$= \frac{A}{Bs} - \frac{A}{B(s+B)} + \frac{ab}{B^2(s+B)} - \frac{ab}{B^2s} + \frac{ab}{Bs^2}$$

$$= \frac{A}{B}\left[1/s - 1/_{s+B}\right] + \frac{ab}{B}\left[\frac{1}{B(s+B)} - \frac{1}{Bs} + \frac{1}{s^2}\right]$$

Taking Inverse Linear transform we get

$$M(t) = \frac{A}{B}\left(1-e^{-Bt}\right) + \frac{ab}{B}\left[t - \frac{1}{B} + \frac{e^{-Bt}}{B}\right]$$

$$= ab/B\, t + \left(1-e^{-Bt}\right)\left(A/B - ab/B^2\right)$$

put $C = A/B - ab/B^2$

$$M(t) = \frac{ab}{B} + \left(1 - e^{-Bt}\right) c \longrightarrow (4)$$

$$C = A/_B - \frac{ab}{B^2} = \frac{AB - ab}{B^2}$$

$$C = [pa + (1-p) b]\,[(1-p) a + pb] - ab/B^2$$

$$= [(1-p)\, a^2 p + ab p^2 + (1-p)^2 ab + p(1-p) b^2 - ab]/B^2$$

$$= p(1-p)(a^2+b^2) + ab\left(p^2 + (1-p)^2 - 1\right)/B^2$$

$$= p(1-p)(a^2+b^2) + ab\left(p^2 + 1 + p^2 - 2p - 1\right)/B^2$$

$$= p(1-p)(a^2+b^2) + ab\left(2p^2 - 2p\right)/B^2$$

$$= \frac{p(1-p)(a^2+b^2) + 2ab\, p(p-1)}{B^2} = \frac{p(1-p)\left[a^2+b^2-2ab\right]}{B^2}$$

$$= \frac{p(1-p)\left[a^2+b^2-2ab\right]}{B^2} = \frac{p(1-p)(a-b)^2}{B^2}$$

$$M^*(s) = \frac{p(1-p)(a-b)^2}{B^2} \geq 0.$$

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