UNIT - V

Stochastic process In Queueing systems.

Stochastic process in queueing systems and Reliability Queueing System:- Explain waiting time ⊗ 5m

Define:- Queble (or) Waiting Time:- process

A queue or waiting time is formed when units needing some kind of service arrive at a service channel that offers such facility the basic features characteristic a system are

(i) Input
(ii) The service machanism
iii) The queue discipline
iv) The number of service channels.

Inter arrival time:-

The Interval between two consecutive arrivals is called the Inter arrival time or Interval

Service time:-

A service machanism describes the manner In which service In renewerd. A unit may be served either singly or In a batch the time required for serving a unit is the service time

Queue Discipline:-

The Queue Discipline Indicates the way In which the units form a queue and are served the usual discipline is first served or first out through Some times other rules such as last come first served (or) random ordering before are also adopted.

No. of service channel:-

The system may have a single channel (or) a number of parallel channels for service.

Mean Arrival rate!

The mean Arrival rate usually denoted by λ is the mean number of arrivals per unit time, its reciprocal is the mean of the time Distribution.

Traffic Intensity!

The mean service rate usually denoted by μ is the mean number of units served per unit time. The reciprocal being the mean service time In a single channel system the ratio $\rho = \lambda/\mu$ is called the traffic Intensity.

Queueing processes!

The Number $N(t)$ in the system of time, The number at time t waiting in the queue Including those being served.

The busy period which means the duration of the Interval from the moment the service commences with arrival at an unit at an empty counter to the moment the server becomes free for the first time.

The waiting time in the queue the duration of time a unit has to spend in the queue also the waiting time w_n of the n^{th} arrival

The virtual waiting time $w(t)$. The Interval of time as unit would have to wait in the queue It to arrive at the Instant t.

Notation!

It consists of a three-part describes A/B/C where the first & second symbols denoted the Inter arrival and service time distribution respectively and the 3rd denotes the number of channels (or) serveres A & B usually take one of the following symbols.

M: For Exponential (markovian) distribution
Ek: For Erlang - k distribution
G: For arbitrary (General) distribution
D: For (Deterministic) Interval.

Thus by an M/G/1 ~~he is meant~~ the same system with having a single channel queueing system having exponential interarrival time distribution By M/G/1/k is meant the same system with the fourth descriptor k denoting that the system has a limited holding capacity k

Steady state Distribution:

N(t) the number in the system at time t and its probability distribution denoted by $P_n(t) = Pr\{N(t) = n / N(0) = 0\}$ are both time dependent

To determine $P_n = \lim P_n(t)$ as $t \to \infty$ provided the limit exists when the limit exists It is called the system has reached equilibrium (or) steady state

Some General Relationships in queueing Theory:

The most Important one $L = \lambda W$ where λ is the arrival rate. L is the expected Number of units in the system and W is the expected waiting time In the steady in state.

Denote the expected number In the queue and the expected waiting time in the queue in steady state by LQ and WQ respectively.

$$LQ = \lambda WQ \rightarrow (1)$$

A relation which holds for G/G/1 queue in steady state is $\lambda = (1 - P_0) M$ (or) $P_0 = 1 - \rho \rightarrow (2)$

This follows from the principle of customer conservation which states that in equilibrium. The rate of arrival equals the rate of departure In a GI/G/k queue the expected number of Idle servers is $k(1-\rho)$

Little's formula :- $L = \lambda W$

Since W is the average waiting time of a unit average rate of departure per unit is $1/W$. The average number of units in the system being L, the average rate of departure is L/W now since the system is in equilibrium the average departure rate must be equal to the average arrival rate in which is
Thus $\lambda = L/W$

The Queueing model m/m/1 : Steady state m/m/1 Behaviour :-

The single server model envisages poisson input and exponential service time with FCFS queue discipline.

The arrivals occurs in accordance with the poisson process with intensity λ (say)

i.e) The probability that an in arrival occurs in an infinitesimal interval of length h is $\lambda h + o(h)$ while that of more than one arrival is $o(h)$.

This is equivalent to the statement that the distribution of the inter arrival times is exponential having p.d.f $a(t) = \lambda e^{-\lambda t}$

The distribution of the service times is exponential with parameter μ (say) having p.d.f $b(t) = \mu e^{-\mu t}$

In otherwords the probability that one service is completed in an interval of infinitesimal length h is $\mu h + o(h)$ while the probability of more than one completion of service is $o(h)$ times as well as service times are stochastic independent. we shall call such a queue and m/m/1 queue with parameters (λ, μ) the mean inter arrival time is $1/\lambda$ and the mean service time is $1/\mu$ the ratio $\rho = \lambda/\mu$ is the Lattice Intensity.

[also equal to good factor (or) utilization factor]

Let $N(t)$ be the number in the system There in the Queue including the one being served if any at Instand $t (\geq 0)$; Then $\{N(t); t \geq 0\}$ is a markov chain process in continuous time with denumerable number of states $\{0, 1, 2 \ldots\}$ There Transitions place only to two neighbouring states this is a type of birth and death process the arrivals here can be thought of as "Birth" and the service competetions as "death" the birth & death process with $\lambda_n = \lambda$, $n \geq 0$ and

$\mu_n = \mu$, $n \geq 1$, $\mu = 0$ is also known as Emmigration

- Immigration process.

Let $pr\{N(t) = n / N[0]\} = P_n(t)$, $n \geq 0$ using the same proper arguments,

$$P'(t) = -\lambda P_n(t) + \mu P_1(t)$$

$$P_n'(t) = -(\lambda + \mu) P_n(t) \; \lambda P_n(-t) + \mu P_{n+1}(t), \; n \geq 1$$

First left us assume the existence if a steady state Then as t tends to Infinity $P_n(t)$ tends to a limit P_n Independent t, The event of steady state probabilities can be obtained by putting $P_n'(t) = 0$ and $P_r(N=n) = P_n$ replacing $P_n(t)$ by P_n

$$0 = -\lambda P_n + \mu P_1$$

$$0 = -(\lambda + \mu) P_n + \lambda P_{n-1} + \mu P_{n+1} \quad n = 1, 2 \ldots$$

The above equation are called balance (or) equilibrium (or) conservation flow] The same equation for an markovian queueing system in steady state can also be obtained from the principle.

Busy period zero avoiding state probabilities:-

A busy period is defined as the Interval time commencing at the Instant zero when a unit arrive at an empty counter and determination at the Instant when the server becomes free from the first.

Let the length of the Interval which is the random variable t & $N^*(t)$ be the stochastic process denote the no. of unit present at the Instant 't' during this busy period

Let $q_n(t) = P_r\{N^*(t) = n/N_0 = 1\}$ → (1)

be the zero avoiding state probability $q_1(0) = 1$ $q_n(0) = 0$ for $n \geq 2$. $q_n(t)$ will satisfies the same differential equation as $P_n(t)$

ie) The equation

$$P_n'(t) = -(\lambda+\mu) P_n(t) + \lambda P_{n-1}(t) + \mu P_{n+1}(t)$$ will

holds also for equation $q_n(t)$ for $n = 2$

$$q_n'(t) = -(\lambda+\mu) q_n(t) + \lambda q_{n-1}(t) + \mu q_{n+1}(t) \rightarrow (2)$$

as the term $q_0(t)$ will not occur the same equation will be as follows.

Corresponds to $n=1$ & $n=0$

$$q_0'(t) = -\mu q_1(t)$$

$$q_1'(t) = -(\lambda+\mu) q_1(t) + \mu q_2(t)$$

Let.

$g_n(s)$ be the Laplace transform of $q_n(t)$.

$$g_n(s) = L[q_n(t)]$$

In general $L[f'(t)] = sL[f(t)] - f(0)$

$$L[q_n'(t)] = s\,L[q_n(t)] - q_n(0)] = s\,g_n(s)$$

Taking Laplace transform on b/s (2) for $n \geq 2$ we get

$L[q_n'(t)] = L[-(\lambda+\mu) q_n(t) + \lambda q_{n-1}(t) + \mu q_{n+1}(t)]$

$S g_n(s) = -(\lambda+\mu) L[q_n(t)] + \lambda L[q_{n-1}(t) + \mu L(q_{n+1}(t))]$

$= -(\lambda+\mu) g_n(s) + \lambda g_{n-1}(s) + \mu g_{n+1}(s)$

$= -(\lambda+\mu) g_n(s) + \lambda g_{n-1}(s) + \mu g_{n+1}(s)$

$g_n(s)[s+\lambda+\mu] - \lambda g_{n-1}(s) - \mu g_{n+1}(s) = 0$

The characteristentic equation is.

$g_n(s) = A\alpha^n + B\beta^n, \quad n \geq 2 \quad |\alpha|>1, |\beta|>1$

$x = \dfrac{-b \pm \sqrt{b^2 4ac}}{2a} = \dfrac{(s+\lambda+\mu) \pm \sqrt{(s+\lambda+\mu)^2 - 4\lambda\mu}}{2\mu}$

$= \dfrac{s+\lambda+\mu \pm k}{2\mu}, \quad \alpha = \dfrac{s+\lambda+\mu+k}{2\mu}, \quad \beta = \dfrac{s+\lambda+\mu-k}{2\mu}$

$k = \dfrac{2\mu}{s+\lambda+\mu}$

$k(s) = k = \sqrt{b^2 4ac} = \sqrt{(s+\mu+\lambda)^2 - 4\lambda\mu}$

$\sum_{n=0}^{\infty} g_n(s)$ must converge so that $A=0$

$g_n(s) = B\beta^n$

put $n=1$ $\quad g_1(s) = \cancel{\beta} B\beta \Rightarrow B = \dfrac{g_1(s)}{\beta}$

$g_n(s) = \dfrac{g_1(s)}{\beta}\beta^n \Rightarrow \dfrac{g_1(s)\beta^{n-1}}{1}$

$(s+\lambda+\mu) g_n(s) = \mu g_{n-1}(s) + \mu g_{n+1}(s)$

put $n=1$

$(s+\lambda+\mu) g_n(s) = 1 + \mu g_2(s)$

$\therefore g_n(s) = g_1(s)\beta^{n-1}$

put $n=2$

$g_2(s) = g_1(s)\beta^{2-1}$

$\therefore (s+\lambda+\mu) g_1(s) = 1 + \mu g_1(s)\beta$

$(s+\lambda+\mu-\mu\beta) g_1(s) = 1$

$$g_1(s) = \frac{1}{s+\lambda+\mu-\mu\beta}$$

$$= \frac{1}{s+\lambda+\mu - \frac{\mu(s+\lambda+\mu-k)}{2\mu}}$$

$$= \frac{2}{s+\lambda+\mu+k} = \frac{2\mu}{\mu(s+\lambda+\mu+k)} =$$

$$g_1(s) = \frac{1}{\mu\alpha}$$

The products of the roots $\alpha\beta = \lambda/\mu$

$$\Rightarrow \alpha = \lambda/\mu\beta \Rightarrow \beta = \lambda/\alpha\mu$$

$$g_1(s) = \frac{1}{\mu \cdot \lambda/\mu\beta} = \beta/\lambda$$

$$g_n(s) = \frac{\beta}{\lambda}\beta^{n-1} = \beta^n/\lambda = \beta^n\lambda^{-1}$$

$$g_n(s) = (\lambda/\alpha\mu)^n \lambda^{-1} = (\lambda/\mu)^n \frac{1}{\alpha^n\lambda}$$

$$= \frac{\rho^n}{\lambda\alpha^n}$$

Inverting the Laplace transform we get for $n=1,2\ldots$

$$L^{-1}[g_n(s)] = L^{-1}(\rho^n/\lambda\alpha^n)$$

$$g_n(t) = 1/\lambda \, L^{-1}\left(\frac{\rho^n}{\alpha^n}\right) = 1/\lambda \, L^{-1}\left[\frac{\lambda^n}{\mu^n}\frac{1}{\alpha^n}\right]$$

$$= 1/\lambda \, \frac{\lambda^n}{\mu^n} L^{-1}(1/\alpha^n)$$

$$= 1/\lambda \, \lambda^n/\mu^n \, L^{-1}\left[\frac{1}{\frac{[(s+\lambda+\mu)+(s+\lambda+\mu)^2-4\lambda\mu]^{1/2}}{2\mu}}\right]$$

$$= 1/\lambda \, L^{-1}\left\{\frac{1}{\mu^{n/2}}\left[\frac{2^n\lambda^n\mu^n}{(s+\lambda+\mu)+(s+\lambda+\mu)^2-4\lambda\mu)^{1/2}}\right]\right.$$

$$= \frac{1}{\lambda}\left(\frac{\lambda}{\mu}\right)^{n/2} e^{-(\lambda+\mu)t}\, \frac{n}{t}\, I_n\left(2t\sqrt{\lambda\mu}\right)$$

$$\therefore q_n(t) = \frac{n}{\lambda t}\, \rho^{n/2}\, e^{-(\lambda+\mu)t}\, I_n\left(2t\sqrt{\lambda\mu}\right)$$

Let $b(t)$ of the probability distribution fn of the busy period t Then for small δt

$b(t)\,\delta t = P\{t \le J \le t+\delta t\}$ equals the probability that the busy period terminals is $[t, t+\delta t]$

Thus, $b(t)\,\delta_t = pr\{N^*(t) = j / N^*(0) = 1\} \times pr\{J$ unit complete serve in $(t, t+\delta t)\}$

$= pr\{N^*(t) = 1 / N^*(0) = 1\}\ pr\{1$ unit complete service in $(t, t+\delta t)\} +$

$+ \sum_{j=2}^{\infty} pr\{N^*(t) = j / N^*(0) = 1\}\ pr\{j$ unit complete service in $(t+\delta t)\}$

$$\Rightarrow b(t)\,\delta(t) = q_1(t)\,\mu\,\delta t + \sum_{j=1}^{\infty} q_j(t)\, o(\delta t)$$

$\div$ by δt we get

$$b(t) = q_1(t)\,\mu + \sum_{j=1}^{\infty} q_j(t)\,\frac{o(\delta t)}{\delta t}$$

Taking limit as $\delta t \to 0$ we have

$$b(t) = q_1(t)\,\mu$$

Taking Laplace transform on both sides we get

$$L[b(t)] = \mu\, L[q_1(t)]$$

$$b^*(s) = \mu\, q_1(s) = \mu\left(\beta/\lambda\right)$$

$$b^*(s) = \beta/\frac{\lambda}{\mu} = \beta/\rho$$

$$\lim_{b\to 0} \beta(s) = \begin{cases} \rho & \text{when } \rho<1 \\ 1 & \text{when } \beta \ge 1 \end{cases}$$

$$pr\{T<\alpha\} = \lim_{s\to 0} b^*(s) = \lim_{s\to 0} \frac{\beta(s)}{\rho}$$

$$= 1/\rho \lim_{s\to 0} \beta(s) = \begin{cases} \rho \times 1/\rho & \text{when } \rho<1 \\ 1\times 1/\rho & \text{when } \rho \ge 1 \end{cases}$$

$$= \begin{cases} 1 & \text{when } \rho < 1 \\ 1/\rho & \text{when } \rho \geq 1 \end{cases}$$

To find the moments of T for $\rho < 1$

$\Rightarrow (\lambda/\mu) < 1 \Rightarrow \lambda < \mu$

Consider $E(T) = -\frac{d}{ds} \; b^*(s)/s = 0$

$$\frac{d}{ds}[\beta(s)] = \frac{d}{ds}\left\{ \frac{(s+\lambda+\mu) - \left[(s+\lambda+\mu)^2 - 4\lambda\mu\right]^{1/2}}{2\mu} \right\}$$

$$\left\{ \because K = \left[(s+\lambda+\mu)^2 - 4\lambda\mu\right]^{1/2} \right\}$$

$$= \frac{1}{2\mu}\frac{d}{ds}(s+\lambda+\mu) - \frac{1}{2\mu}\frac{d}{ds}\left[(s+\lambda+\mu)^2 - 4\lambda\mu\right]^{1/2}$$

$$= \frac{1}{2\mu}(1) - \frac{1}{2\mu}\,\frac{1}{2}\left[(s+\lambda+\mu)^2 - 4\lambda\mu\right]^{-1/2} 2(s+\lambda+\mu)$$

$$= \frac{1}{2\mu} - \frac{1}{2\mu}\,\frac{s+\lambda+\mu}{K} = 1/2\mu\left[1 - \frac{(s+\lambda+\mu)}{K}\right]$$

$$= \frac{1}{2\mu}\left[\frac{K - (s+\lambda+\mu)}{K}\right] = \frac{-1}{2\mu}\left[\frac{(s+\lambda+\mu) - K}{K}\right]$$

$$= \frac{-1}{K}(\beta)$$

$$\frac{d^2}{ds^2}[\beta(s)] = \frac{d}{ds}\left(\frac{d}{ds}[\beta(s)]\right) = \frac{d}{ds}\left\{\frac{1}{2\mu}(s+\lambda+\mu)\left[(s+\lambda+\mu)^2 - 4\lambda\mu\right]^{-1/2}\right\}$$

$$= \frac{-1}{2\mu}\left\{\frac{d}{ds}(s+\lambda+\mu)\left[(s+\lambda+\mu)^2 - 4\lambda\mu\right]^{-1/2}\right\}$$

$$= \frac{-1}{2\mu}\left\{\left[(s+\lambda+\mu)^2 - 4\lambda\mu\right]^{-1/2} - 1/2\left[(s+\lambda+\mu)^2 - 4\lambda\mu\right]^{-3/2} 2(s+\lambda+\mu)^2\right\}$$

$$= \frac{-1}{2\mu}\left[1/K - \frac{(s+\lambda+\mu)^2}{K^3}\right] = -1/2\mu\left[\frac{K^2 - (s+\lambda+\mu)^2}{K^3}\right]$$

$$= \frac{1}{2\mu}\left[\frac{(s+\lambda+\mu)^2 - K^2}{K^3}\right]$$

$$= \frac{1}{2\mu K^3}\left[(s+\lambda+\mu)^2 - (s+\lambda+\mu)^2 + 4\lambda\mu\right]$$

$$= \frac{1}{2\mu K^3}(4\lambda\mu) = \frac{2\lambda}{K^3}$$

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$$E(T) = \frac{-d}{ds}[b^*(s)] = \frac{d}{ds}(\beta/\rho)$$

$$= -1/\rho \frac{d}{ds}[\beta(s)] = -1/\rho\left(-\beta/k\right)$$

$$= 1/\rho\,(\beta/k) = 1/\rho\left[\frac{\frac{[(s+\lambda+\mu) - (s+\lambda+\mu)^2 - 4\lambda\mu]^{1/2}}{2\mu}}{[(s+\lambda+\mu)^2 - 4\lambda\mu]^{1/2}}\right.$$

$$= \frac{1}{\rho}\left[\frac{\frac{(\lambda+\mu) - [(\lambda+\mu)^2 - 4\mu\lambda]^{1/2}}{2\mu}}{[(\lambda+\mu)^2 - 4\lambda\mu]^{1/2}}\right]$$

$$= \frac{1}{\rho}\left[\frac{\frac{(\lambda+\mu) - [(-\lambda+\mu)^2]^{1/2}}{2\mu}}{[(-\lambda+\mu)^2]^{1/2}}\right]$$

$$= \frac{1}{\rho}\left[\frac{(\lambda+\mu+\lambda-\mu)/2\mu}{\mu-\lambda}\right] = \frac{1}{\rho}\left[\frac{2\lambda/2\mu}{\mu-\lambda}\right]$$

$$= \frac{1}{\rho}\left[\frac{\rho}{\mu-\lambda}\right]$$

$$E(t) = \frac{1}{\mu-\lambda} = \frac{1}{\mu[1-\lambda/\mu]} = \frac{1}{\mu[1-\rho]}$$

$$E(T^2) = \frac{d^2}{ds^2}[b^*(s)] = \frac{1}{\rho}\left(\frac{2\lambda}{k^3}\right)\Big/ s=0$$

$$= \frac{1}{\rho}\left\{\frac{2\lambda}{[(\lambda+\mu)^2 - 4\lambda\mu]^{3/2}}\right\} = 1/\rho\left[\frac{2\lambda}{[(\mu-\lambda)^2]^{3/2}}\right]$$

$$= \frac{1}{\rho}\left[\frac{2\lambda}{(\mu-\lambda)^3}\right] = \frac{1}{\rho}\left[\frac{2\lambda}{\mu^3(1-\lambda/\mu)^3}\right]$$

$$= \frac{1}{\rho}\left[\frac{2\rho}{\mu^2(1-\rho)^3}\right] = \frac{2}{\mu^2(1-\rho)^3}$$

$$Var(T) = E(T^2) - [E(T)]^2$$

$$= \frac{2}{\mu^2(1-\rho)^3} - \left[\frac{1}{\mu(1-\rho)}\right]^2$$

$$= \frac{2}{\mu^2(1-\rho)^3} - \frac{1}{\mu^2(1-\rho)^2}$$

$$V(T) = \frac{1}{\mu^2}\left[\frac{2-1+\rho}{(1-\rho)^3}\right] = \frac{1}{\mu^2}\left[\frac{1+\rho}{(1-\rho)^3}\right]$$

The co-eff of variance $= \left[\frac{Var(T)}{E(T)}\right]^{1/2} \times 100$

$$= \frac{\frac{1}{\mu}\left[\frac{1+\rho}{(1-\rho)^3}\right]^{1/2}}{1/\mu\,(1-\rho)} \times 100$$

$$= \frac{(1+\rho)^{1/2}}{(1-\rho)(1-\rho)^{1/2}} \Big/ \frac{1}{1-\rho} \times 100 = \frac{(1+\rho)^{1/2}}{(1-\rho)^{1/2}} \times 100$$

$$V = \left[\frac{1+\rho}{1-\rho}\right]^{1/2}$$

The following table gives $E(T)$ & $Var(T)$ for some values of λ/μ ($\lambda=3, \mu=1$)

λ	$E(T)$	$Var(T)$
0.2	1.25	2.344
0.4	1.667	6.481
0.5	2	12
0.8	5	225
0.9	10	1900
0.95	20	15600

Waiting time Distribution:

Waiting time in the Queue:

(3) 10m We assume that the queue dupline is FCFS.
Let the random variable W_Q denote the time spent in waiting in the occuring of a test unit now $W_Q = 0$ if at the instant of arrival of the test unit there is no unit in the system the probability $P_0 = 1-\rho$ the system $W_Q = S_n$ where $S_n = v_1' + v_2 + \dots + v_n$, v_1' being the residual service time of the customer ~~being~~

being served and $v_2 \ldots v_n$ being the service times of the units weighting in Queueing at the instant of arrival of the test unit. Now v_1 the residual of an exponential distribution with mean $1/\mu$ is again an exponential distribution with the same mean.

Then S_n is the sum of identically and independently distributed exponential distribution and to the variable with gamma distribution having probability density function.

$$\frac{\mu^n x^{n-1} e^{-\mu x}}{\Gamma n}, \quad x \geq 0$$

Hence we have for $x > 0$

$$W_q(x)\, dx = pr\{x \leq W_Q \leq x + dx\}$$

$$= \sum_{n=1}^{\infty} pr\{x \leq W_Q \leq x + dx / \text{The test unit finds } n \text{ in the system}\} \cdot pr\{\text{The test unit finds } n \text{ in the system}\}$$

$$= \sum_{n=1}^{\infty} \frac{\mu^n x^{n-1}}{\Gamma n} e^{-\mu x} P_n = \sum_{n=1}^{\infty} \frac{\mu^n x^{n-1} e^{-\mu x}}{\Gamma n} (1-\rho)\rho^n$$

$$= e^{-\mu x}(1-\rho)\mu\rho \sum_{n=1}^{\infty} \frac{(x\mu\rho)^{n-1}}{(n-1)!}$$

$$= e^{-\mu x}(1-\rho)\mu\rho \left[1 + \frac{x\mu\rho}{1!} + \frac{(x\mu\rho)^2}{2!} + \cdots\right]$$

$$= e^{-\mu x}(1-\rho)\mu\rho\, e^{x\mu\rho}$$

$$= (1-\rho)\mu\rho\, e^{x\mu(\rho-1)}$$

$$\therefore W_q(x) = \begin{cases} P_0 = 1-\rho & x = 0 \\ (1-\rho)\mu\rho e^{x\mu(\rho-1)} & x > 0 \end{cases}$$

$L[W_q(x)] = W_q^*(\alpha)$ is the Laplace transform of $W_q(x)$ is denoted by $W_q^*(\alpha)$.

$$W_q^*(\alpha) = \int_{-\infty}^{\infty} e^{-\alpha x} W_q(x)\, dx$$

$$= (1-\rho) + \int_0^{\infty} e^{-\alpha x} \mu\rho(1-\rho) e^{-\mu(1-\rho)x}\, dx$$

$$= (1-\rho) + \mu\rho(1-\rho)\int_0^{\infty} e^{-(\alpha+\mu-\mu\rho)x}\, dx$$

$$= (1-\rho) + \mu\rho(1-\rho)\left(\frac{e^{-(\alpha+\mu-\mu\rho)x}}{-(\alpha+\mu-\mu\rho)}\right)_0^\infty$$

$$= (1-\rho)\left[1+\mu\rho\left(\frac{1}{\alpha+\mu-\mu\rho}\right)\right]$$

$$= (1-\rho)\left(\frac{\alpha+\mu-\mu\rho+\mu\rho}{\alpha+\mu-\mu\rho}\right)$$

$$= \frac{(1-\rho)(\alpha+\mu)}{\alpha+\mu-\mu\rho} \rightarrow (7)$$

The distribution function $F(x)$ of W_q is given by

$F(x) = 1-\rho, \quad x=0.$

$$F(x) = \int_0^x W_q(t)\,dt = 1 - \int_x^\infty W_q(t)\,dt$$

$$= 1 - \int_x^\infty \mu(1-\rho)\, e^{-\mu(1-\rho)t}\,dt$$

$$= 1 - \mu\rho(1-\rho)\int_x^\infty e^{-\mu(1-\rho)t}\,dt$$

$$= 1 - \mu\rho(1-\rho)\left[\frac{e^{-\mu(1-\rho)t}}{-\mu(1-\rho)}\right]_x^\infty$$

$$= 1 - \mu\rho(1-\rho)\left[\frac{e^{-\mu(1-\rho)x}}{-\mu(1-\rho)}\right]$$

$$= 1 - \rho\, e^{-\mu(1-\rho)x}, \quad x \geq 0$$

The above gives the distribution of the waiting time of a test the unit has not to wait is $(1-\rho)$ and it has to wait is ρ.

$$E[W_Q] = \frac{-d}{d\alpha}\left[W_q^*(\alpha)\right]_{\alpha=0} = \frac{-d}{d\alpha}\left[\frac{(1-\rho)(\alpha+\mu)}{\alpha-\mu\rho+\mu}\right]$$

$$= \frac{-d}{d\alpha}\left[\frac{\alpha+\mu-\rho\alpha-\rho\mu}{\alpha-\mu\rho+\mu}\right]_{\alpha=0}$$

$$= \left[\frac{(1-\rho)(\alpha-\mu\rho+\mu)-(\alpha+\mu-\rho\alpha-\rho\mu)'}{(\alpha-\mu\rho+\mu)^2}\right]$$

$$= -\left[\frac{\alpha-\mu\rho+\mu-\rho\alpha+\mu\rho^2-\rho\mu-\alpha-\mu+\rho\alpha+}{(\alpha-\mu\rho+\mu)^2}\right]$$

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$$= -\left[\frac{-\mu\rho+\mu\rho^2}{(\alpha+\mu-\mu\rho)^2}\right] = \frac{\mu\rho(1-\rho)}{(\alpha+\mu-\mu\rho)^2}\Big/_{\alpha=0}$$

$$= \frac{\mu\rho(1-\rho)}{(\mu-\mu\rho)^2} = \frac{\mu\rho(1-\rho)}{\mu^2(1-\rho)^2} = \frac{\rho}{\mu(1-\rho)} \rightarrow (2)$$

$$E[W_Q^2] = \frac{d^2}{d\alpha^2}\left(W_q^*(\alpha)\right)\Big/_{\alpha=0}$$

$$= \frac{-d}{d\alpha}\left[\frac{d}{d\alpha}\left(W_q^*(\alpha)\right)\Big/_{\alpha=0}\right]$$

$$= \frac{-d}{d\alpha}\left[\frac{\mu\rho(1-\rho)}{(\alpha+\mu-\mu\rho)^2}\right]\Big/_{\alpha=0} \quad [\text{From } (2)]$$

$$= -\mu\rho(1-\rho)\frac{d}{d\alpha}(\alpha+\mu-\mu\rho)^{-2}\Big/_{\alpha=0}$$

$$= -\mu\rho(1-\rho)\left[-2(\alpha+\mu-\mu\rho)^{-3}\right]\Big/_{\alpha=0}$$

$$= \frac{2\mu\rho(1-\rho)}{(\alpha+\mu-\mu\rho)^3}\Big/_{\alpha=0} = \frac{2\mu\rho(1-\rho)}{(\mu-\mu\rho)^3}$$

$$= \frac{2\mu\rho(1-\rho)}{\mu^3(1-\rho)^3} = \frac{2\rho}{\mu^2(1-\rho)^2}$$

$$\mathrm{Var}[E(W_Q)] = E[W_Q^2] - [E(W_Q)]^2$$

$$= \frac{2\rho}{\mu^2(1-\rho)^2} - \frac{\rho^2}{\mu^2(1-\rho)^2} = \frac{2\rho-\rho^2}{\mu^2(1-\rho)^2}$$

$$= \frac{\rho(2-\rho)}{\mu^2(1-\rho)^2}$$

Waiting time in the Queue system (or) Response time (or) Sojourn Time:

The Random variable time spent in the system by a unit Includes the service time of unit &e is Queing time this Variable is given by $W_s = W_Q + S$.

Where W_Q is the Queueing time and S is the service time. Now the test unit has to wait in the system even if the system is empty the time

Spent being equal to his service time

$$N_s = S_{n+1}$$

where

$$S_{n+1} = V_1 + V_2 + \dots + V_{n+1}$$

where V_{n+1} his own service time

The p.d.f of W_s of x is denoted by

$$W_s(t) = P\{x \le W_s \le x+dx\}$$

$$= \sum_{n=0}^{\infty} \frac{\mu^{n+1} \cdot x^n e^{-\mu x}}{\Gamma_{n+1}} \cdot P_n$$

$$= \sum_{n=0}^{\infty} \frac{\mu^{n+1} x^n \cdot e^{-\mu x}}{\Gamma_{n+1}} \rho^n (1-\rho)$$

$$= e^{-\mu x} \mu (1-\rho) \sum_{n=0}^{\infty} \frac{(\mu \rho)^n}{n!}$$

$$= e^{-\mu x} \mu (1-\rho) e^{\mu x \rho}$$

$$= \mu (1-\rho) e^{-\mu(1-\rho)x}$$

To find the distribution function $F(x)$ of the response time

$$F(x) = \int_0^x W_s(x)\, dx = 1 - \int_x^{\infty} W_s(x)\, dx$$

$$= 1 - \int_x^{\infty} \mu (1-\rho) e^{-\mu(1-\rho)} dx$$

$$= 1 - \mu (1-\rho) \left[\frac{e^{-\mu(1-\rho)}}{\mu(1-\rho)} \right]_x^{\infty}$$

$$= 1 - \mu(1-\rho) \left[\frac{e^{-\mu(1-\rho)x}}{\mu(1-\rho)} \right]$$

$$= 1 - e^{-\mu(1-\rho)x} \qquad x \ge 0$$

M/G/1

8 M Method of Generating function : [Derive pallaczck khinchine

~~Starts with~~ Let us assume that $N(0) = i$. The process starts ~~with~~ with i units at time $t=0$

So that $Pr\{N(t)=n / N(0)=i\} = P_{in}(t)$

$$P_{ij}(0) = \delta_{ij} = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i \ne j \end{cases}$$

$$P_{i0}'(t) = -\lambda P_{i0}(t) + \mu P_{i1}(t) \longrightarrow (1)$$

and $P'_{in}(t) = -(\lambda+\mu) P_{in}(t) + \lambda P_{i(n-1)}(t) + \mu P_{i(n+1)}(t) \longrightarrow (2)$

$$P(z,t) = \sum_{n=0}^{\infty} P_{in}(t) z^n$$

be the Generating function of $\{P_{in}(t)\}$

Then $P(z,0) = z^i$

~~be the~~ ~~Gene~~ Let Laplace Transformation of $P_{in}(t)$

$$L[P_{in}(t)] = \bar{P}_{in}(t)$$

and $F(z,s) = L[P(z,t)]$

multiply (2) by z^n and adding for $n=1,2\ldots$. The resulting equation with one we get

$$\sum_{n=1}^{\infty} P'_{in}(t) z^n = -(\lambda+\mu)\sum_{n=1}^{\infty} P_{in}(t) z^n + \lambda \sum_{n=1}^{\infty} P_{i(n-1)}(t) z^n + \mu \sum_{n=1}^{\infty} P_{i(n+1)}(t) z^n$$

$$\sum_{n=1}^{\infty} P'_{in}(t) z^n + P'_{i0}(t) = -(\lambda+\mu)\sum_{n=1}^{\infty} P_{in}(t) z^n - \lambda P_{i0}(t) + \mu P_{i1}(t) + \lambda \sum_{n=1}^{\infty} P_{i(n-1)}(t) z^n + \mu \sum_{n=1}^{\infty} P_{i(n+1)}(t) z^n$$

$$\sum_{n=1}^{\infty} P'_{in}(t) z^n = -(\lambda+\mu)\sum_{n=1}^{\infty} P_{in}(t) z^n + \mu P_{i0}(t) - \mu P_{i0}(t) - \lambda P_{i0}(t) + \mu P_{i1}(t) + \lambda \sum_{n=1}^{\infty} P_{i(n-1)}(t) z^n + \mu \sum_{n=1}^{\infty} P_{i(n+1)}(t) z^n$$

$$= -(\lambda+\mu)\sum_{n=0}^{\infty} P_{in}(t) z^n + \mu P_{i0}(t) + \lambda \sum_{n=1}^{\infty} P_{i(n-1)}(t) z^n + \mu \sum_{n=0}^{\infty} P_{i(n+1)}(t) z^n$$

$$\sum_{n=0}^{\infty} P'_{in}(t) z^n = -(\lambda+\mu) P(z,t) + \mu P_{i0}(t) + \lambda z P(z,t) + \mu/z [P(z,t) - P_{i0}(t)]$$

$$\frac{\partial P(z,t)}{\partial t} = -(\lambda+\mu) P(z,t) + \mu P_{i0}(t) + \lambda z P(z,t) + \mu/z [P(z,t) - P_{i0}(t)]$$

x'y by z we get

$$z \, \partial/\partial t \, P(z,t) = -(\lambda+\mu) z P(z,t) + \mu z P_{i0}(t) + z^2 \lambda P(z,t) + \mu [P(z,t) - P_{i0}(t)]$$

$$= [\lambda z^2 - (\lambda+\mu) z + \mu] P(z,t) - \mu (1-z) P_{i0}(t)$$

Now,

$$\partial/\partial t \, P(z,t) = SF(z,s) - P(z,0)$$

$$= SF(z,s) - z^i]$$

Taking Laplace Transform on both sides.

$\therefore z\left(S_F(z,s) - z^i\right) = \left[\lambda z^2 - (\lambda+\mu)(z+\mu) F(z,s) - F_{10}(s)\right]$

$$F(z,s)\left[\lambda z^2 - (\lambda+\mu)(z+\mu) - zs\right] = \mu(1-z) F_{10}(z) - z^{i+1}$$

$$F(z,s) = \frac{\mu(1-z) F_{10}(s) - z^{i+1}}{\lambda z^2 - (\lambda+\mu+s)^{z+\mu}}$$

This involves $F_{10}(s)$ which can be evaluated as follows the denominator of the R.H.S has two zeros say E and η

$$E = (S+\lambda+\mu) z + \mu$$

$$\eta = z\lambda, \quad k = k(s)$$

Further $|E| < 1$ and $|\eta| > 1$ The roots of the equation $\lambda z^2 - (S+\lambda+\mu) z + \mu = 0$ are the reciprocal of the ... of the equation Thus ξ and η are the Reciprocal of α, β

$$\xi = 1/\alpha, \quad \eta = 1/\beta \text{ and this } |\xi| < 1 \text{ and } |\eta| > 1$$

It can be shown that

$$\lambda z^2 - (S+\lambda+\mu) z + \mu = 0$$ has only one zero $z = E$ in the circle since $F(z,s)$ converges in the region $|z| = 1$ The zeros in the unit circle of the ~~number~~ numerator and denominator of R.H.S of $F(z,s)$ must coincide so that $z = \xi$ must a zero of the ~~number~~ numerator also

$$\therefore F(z,s) = \frac{\mu(1-z) f_{10}(s) - z^{i+1}}{\lambda z^2 - (S+\lambda+\mu)^{z+\mu}}$$

$$z = \xi$$

$$\Rightarrow 0 = \mu(1-\xi) f_{10}(s) - \xi^{i+1} \Rightarrow f_{10}(s) = \frac{\xi^{i+1}}{\mu(1-\xi)}$$

$$\therefore F(z,s) = \frac{\mu(1-z) + \frac{\xi^{i+1}}{\mu(1-\xi)} - z^{i+1}}{\lambda(z-\xi)(z-\eta)}$$

$$= \frac{(1-z)\xi^{i+1} - z^{i+1}(1-\xi)}{\lambda(1-\xi)(z-\xi)(z-\eta)} \rightarrow (4)$$

Expanding the R.H.S of equation (4) in power ... can be obtained as the coeff...

particular case:

when $T=0$ Then $V(0)=0$

$$F(z,s) = \frac{(1-z)\xi - z(1-\xi)}{\lambda(1-\xi)(z-\xi)(z-\eta)}$$

$$= \frac{\xi - z\xi - z + z\xi}{\lambda(1-\xi)(z-\xi)(z-\eta)} = \frac{-(z-\xi)}{\lambda(1-\xi)(z-\xi)(z-\eta)}$$

$$F(z,s) = \frac{1}{\lambda(1-\xi)(\eta - z)}$$

put $\xi = 1/\alpha$ & $\eta = 1/\beta$

$$F(z,s) = \frac{1}{\lambda(1-1/\alpha)(1/\beta - z)} = \frac{\alpha\beta}{\lambda(\alpha-1)(1-z\beta)}$$

$$= \frac{\alpha\beta}{\lambda(\alpha-1)}(1-z\beta)^{-1}$$

$$= \frac{\alpha\beta}{\lambda(\alpha-1)}\left[1+z\beta+(z\beta)^2+\dots\right]$$

$$= \frac{\alpha\beta}{\lambda(\alpha-1)}\sum_{n=0}^{\infty} z^n\beta^n$$

$$= \frac{\lambda}{\mu\lambda(\alpha-1)}\sum_{n=0}^{\infty} z^n\beta^n$$

Let $-s = \mu(\alpha-1)(\beta-1)$

$$\Rightarrow \mu(\alpha-1) = \frac{-s}{\beta-1} \Rightarrow \mu(\alpha-1) = \frac{s}{1-\beta}$$

$$\therefore F(z,s) = \frac{1-\beta}{s}\sum_{n=0}^{\infty} z^n\beta^n$$

Let $f_n(s) = f_{0n}(s) =$ co-eff of z^n in $F(z,s)$

$$f_n(s) = 1-\beta/s \cdot \beta^n$$

Transistent Behaviour of M/M/I model!

we have now to solve the equation

$$P_0'(t) = -\lambda P_0(t) + \mu P_1(t) \longrightarrow (1)$$

$$P_n'(t) = -(\lambda+\mu)P_n(t) + \lambda P_{n-1}(t) + \mu P_{n+1}(t),\ n\geq 0 \longrightarrow (2)$$

(i) Difference equation Technique:

we assume that the no. of units at the Instant $t=0$ is zero.

Let $P_n(t) = P\{N(t)=n \,/\, N(0) = \}$

$P_n(0) = 0, \quad P_0(0) = 1$

Let $f_n(s)$ be the Laplace Transform of $P_n(t)$

$$L[P_n(t)] = f_n(s)$$

$$L[P_n'(t)] = SL[P_n(t)] - P_n(0) \qquad [\because P_n(0) =$$

$$= SL[P_n(t)] = S f_n(S)$$

Taking Laplace Transform of both sides of equation 2 we get

$$L[P_n'(t)] = -(\lambda+\mu) L[P_n(t)] + \lambda L[P_{n-1}(t)] + \mu L[P_{n+1}(t)$$

$$S f_n(S) = -(\lambda+\mu) f_n(s) + \lambda f_{n-1}(s) + \mu f_{n+1}(S)$$

$$(\lambda+\mu+S) f_n(S) = \lambda f_{n-1}(S) + \mu f_{n+1}(s) \longrightarrow (3)$$

Taking Laplace Transform of both sides of equa (1) we get.

$$L[P_0'(t)] = -\lambda L[P_0(t)] + \mu L[P_1(t)]$$

$$SL[P_0(t)] - P_0(0) = -\lambda f_0(s) + \mu f_1(s)$$

$$S f_0(S) - 1 = -\lambda f_0(s) + \mu f_1(S) \longrightarrow (4)$$

The characteristic equation of (3) is.

$$\mu x^2 - (\lambda+\mu+s)x + \lambda = 0 \longrightarrow (5)$$

If α, β are roots of equation (5) Then the solution of equation (3) is.

$$f_n(S) = A\alpha^n + B\beta^n, \; n \geq 1$$

where A & B are fn of x

To find value of A & B

Consider The equation (5) $\mu x^2 - (\lambda+\mu+s)x + \lambda = 0$

Let $K(S) = K = \sqrt{b^2-4ac} = \sqrt{(S+\mu+\lambda)^2 - 4\mu\lambda}$

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a} = \frac{S+\lambda+\mu \pm \sqrt{(S+\lambda+\mu)^2 - 4\mu\lambda}}{2\mu}$$

$$= \frac{(S+\lambda+\mu) \pm K}{2\mu}$$

$$\alpha = \frac{(S+\lambda+\mu) + K}{2\mu}, \quad \beta = \frac{(S+\lambda+\mu) - K}{2\mu}$$

Here $\alpha = \alpha(s)$ and $\beta = \beta(s)$ It can be seen that for all real $s > 0$, μ, λ are real we have $|\alpha| > 1$, $|\beta| < 1$

Now, $\sum_{n=0}^{\infty} P_n(t) = 1 \longrightarrow (*)$

But $L[P_n(t)] = f_n(s)$ Taking Laplace on b/s $(*)$

$$\sum_{n=0}^{\infty} L[P_n(t)] = L(1)$$

$$\sum_{n=0}^{\infty} f_n(s) = 1/s \longrightarrow (7)$$

$$\therefore f_n(s) = A\alpha^n + B\beta^n$$

$$\sum_{n=0}^{\infty} f_n(s) = \sum_{n=0}^{\infty} A\alpha^n + \sum_{n=0}^{\infty} B\beta^n \text{ by (7)}$$

$$1/s = \sum_{n=0}^{\infty} (A\alpha^n + B\beta^n)$$

$$1 = s \sum_{n=0}^{\infty} (A\alpha^n + B\beta^n)$$

$\because |\alpha| > 1$ the convergence of

$$\sum_{n=0}^{\infty} f_n(s) = \sum_{n=0}^{\infty} (A\alpha^n + B\beta^n)$$

$$A = 0$$

So that $f_n(s) = B\beta^n, \ n \geq 1$

put $n = 0$, $f_0(s) = B$

$$\therefore f_n(s) = f_0(s)\beta^n \longrightarrow (**)$$

$$\sum_{n=0}^{\infty} f_n(s) = \sum_{n=0}^{\infty} f_0(s)\beta^n \quad [\because \text{by (7)}]$$

$$1/s = f_0(s) \sum_{n=0}^{\infty} \beta^n = f_0(s)(1 + \beta + \beta^2 + \cdots)$$

$$1/s = f_0(s)(1-\beta)^{-1}$$

$$\Rightarrow f_0(s) = 1/s \cdot \frac{1}{(1-\beta)^{-1}} = \frac{1-\beta}{s}$$

$$f_0(s) = \frac{1-\beta}{s}$$

$(**)$

$$\Rightarrow f_n(s) = \frac{1-\beta}{s}\beta^n \longrightarrow (6)$$

Now to find probability $P\{N \geq n\}$

$$f_n(s) = \frac{1-\beta}{s} \cdot \beta^n$$

$$\therefore \sum_{r=n}^{\infty} f_r(s) = \frac{1-\beta}{s} \sum_{r=n}^{\infty} \beta^r$$

$$= \frac{1-\beta}{s}(\beta^n + \beta^{n+1} + \dots)$$

$$= \frac{1-\beta}{s}\beta^n(1+\beta+\dots)$$

$$= \frac{1-\beta}{s}\beta^n \frac{1}{1-\beta} = \beta^n/s$$

By equation (5)

$$\mu x^2 - (s+\lambda+\mu)x + \lambda = 0$$

product of the roots $\alpha\beta = \lambda/\mu$

$$\beta = \lambda/\mu\alpha$$

$$\sum_{r=n}^{\infty} f_r(s) = \beta^n/s = \left(\lambda/\mu\alpha\right)^n \cdot 1/s$$

$$= \left(\lambda/\mu\right)^n \cdot \alpha^{-n}/s$$

$$P_n(t) = Pr\{N(t) \geq n\} = P_n(t) + P_{n+1}(t) + \dots$$

$$L[P_n(t)] = L[P_n(t)] + L[P_{n+1}(t)] + \dots$$

$$= f_n(s) + f_{n+1}(s) + \dots$$

$$L[P_n(t)] = \sum_{r=n}^{\infty} f_r(s) = \left(\lambda/\mu\right)^n \cdot \alpha^{-n}/s$$

Taking Inverse Laplace transform on

$$P_n(t) = \left(\lambda/\mu\right)^n \alpha^{-n}(1) \longrightarrow 7$$

from the table of Laplace transform we find the

$$n/t \; I_n(at) = \left\{\frac{a}{s+(s^2-a^2)^{1/2}}\right\}^n$$

$$a_n(t) = \left(\lambda/\mu\right)^{n/2} e^{-(\lambda+\mu)t} \cdot n/t \; I_n\left[2t\sqrt{\lambda\mu}\right]$$

$$L[a_n(t)] = \left(\lambda/\mu\right)^{n/2} L\left\{e^{-(\lambda+\mu)t}\left[n/t \; I_n\left(2t\sqrt{\lambda\mu}\right)\right]\right\}$$

$$= \left(\lambda/\mu\right)^{n/2}\left\{\frac{\left(2\sqrt{\lambda\mu}\right)^n}{\left[(s+\lambda+\mu) + \left[(s+\lambda+\mu)^2 - (4\lambda\mu)\right]^{1/2}\right]^n}\right\}$$

$$= \frac{\lambda^{n/2}}{\mu^{n/2}} \frac{2^n \lambda^{n/2} \mu^{n/2}}{\left[(s+\lambda+\mu) + \left[(s+\lambda+\mu)^2 - 4\lambda\mu\right]^{1/2}\right]^n}$$

$$= \frac{2^n \lambda^n}{\left\{(s+\lambda+\mu) + \left[(s+\lambda+\mu)^2 - 4\lambda\mu\right]^{1/2}\right\}^n}$$

M^{ly} ÷ by μ^n we get

$$= \left(\lambda/\mu\right)^n \frac{2^n \mu^n}{\left\{(s+\lambda+\mu) + \left[(s+\lambda+\mu)^2 - 4\lambda\mu\right]^{1/2}\right\}^n}$$

$$= \left(\lambda/\mu\right)^n \frac{1}{\left\{\frac{(s+\lambda+\mu) + \left[(s+\lambda+\mu)^2 - 4\lambda\mu\right]^{1/2}}{2\mu}\right\}^n}$$

$$= \left(\lambda/\mu\right)^n \alpha^{-n}$$

Then using the Laplace transform of Integration

$$L\left(\int_0^t F(x)\,dx\right) = F(s)/s$$

$$L\left[\int_0^t a_n(u)\,du\right] = \left(\lambda/\mu\right)^n \alpha^{-n}/s$$

Now,

$$\beta(s) = \frac{(s+\lambda+\mu) - \left[(s+\lambda+\mu)^2 - 4\lambda\mu\right]^{1/2}}{2\mu}$$

$$\lim_{s\to 0} \beta(s) = \frac{(\lambda+\mu) - \left[(\lambda+\mu)^2 - 4\lambda\mu\right]^{1/2}}{2\mu}$$

$$= \frac{\left[(\lambda+\mu) - (\lambda+\mu)^2\right]^{1/2}}{2\mu}$$

$$\lim_{s\to 0} \beta(s) = \begin{cases} \frac{\lambda+\mu-\mu-\lambda}{2\mu}, & \text{when } \lambda < \mu \\ \frac{\lambda+\mu-\lambda-\mu}{2\mu}, & \text{when } \lambda \geq \mu \end{cases}$$

$$\lim_{s\to 0} \beta(s) = \begin{cases} \lambda/\mu & \text{when } \lambda < \mu \\ 1 & \text{when } \lambda \geq \mu \end{cases}$$

$$\therefore \beta = \lim_{s\to 0} \beta(s) = \begin{cases} \rho, & \lambda < \mu \\ 1, & \lambda \geq \mu \end{cases}$$

$$P_n = \lim_{s\to 0} P_n(t) = \lim_{t\to\infty} s\, f_n(s)$$

$$= \lim_{t\to\infty} s \frac{(1-\rho)}{s} \rho^n$$

$$= (1-\rho)\cdot \rho^n$$

$$P_n = \begin{cases} (1-\rho)\rho^n, & \text{when } \lambda/\mu < 1 \\ 0, & \text{when } \lambda/\mu \geq 1 \end{cases}$$

∴ The must $P_n = 0$ $\rho \geq 1$ may be Interpreted by saying when the traffic Idensity is greater than ~~are~~ equal to unique.

Queues with poisson Input model $M/G/1$

(7) 5m Assume that the Input process is poisson with Intensity λ and that the service times are i.i.d random variable having an arbitrary distribution with mean $1/\mu$.

Denote the service time by v.t.z d.f by $B(t)$ its p.d.f when it exists by $b(t) = B'(t)$ and its L.T by,

$$B^*(s) = \int_0^\infty e^{-st}\, d[B(t)] \longrightarrow (*)$$

Let t_n, $n = 1, 2, \ldots$ $(t_0 = 0)$ be the n^{th} departure epoch i.e) The Instant at which the n^{th} unit completes his service and leaves the system.

The point t_n are the regeneration points of process $\{N(t)\}$. The sequence of pt $\{t_n\}$ form a Renewal process, $N(t_n+0)$ The number in the system Immediately after the n^{th} departure has a denumerable state space $\{0, 1, 2, \ldots\}$ write a ~~denumerable state~~ $N(t_n\, t_0) = X_n$, $n = 0, 1, \ldots$

$$X_{n+1} = X_{n-1} + A_{n+1} \quad \text{if } x_n \geq 1$$
$$= A_{n+1} \quad \text{if } x_n = 0$$

Now the service times of all the units have the same distribution so that

$A_n = A$ for $n = 1, 2 \ldots\ldots$ we have

$$pr\{A = r \text{ service time of a units is } t\}$$
$$= \frac{e^{-\lambda t}(\lambda t)^r}{r!}$$

$$p_r = pr\{A = r\} = \int_0^\infty \frac{e^{-\lambda t}(\lambda t)^r}{r!} \, d\, B(t)$$

$r = 0, 1, 2 \ldots\ldots$

Gives the distribution of A the number of arrivals During the service time of a unit the probability.

$$P_{ij} = pr\{x_{n+1} = j / x_n = i\}$$

Are given by

$$P_{ij} = K_{j-i+1}, \quad i \geq 1, \; j \geq i-1$$
$$i \geq 1, \; j < i-1$$

$$P_{0j} = P_{ij} = K_j \quad j \geq 0$$

The Relation clearly Indicate time that $\{x_n, n \geq 0\}$ is a morkov chain having t.p.m.

$$P = P_{ij} = \begin{pmatrix} K_0 & K_1 & K_2 & \cdots \\ K_0 & K_1 & K_2 & \cdots \\ 0 & K_0 & K_1 & \cdots \end{pmatrix}$$

As every state can be reached from every other state the morkov chain $\{x_n\}$ is Irreducible

Again as $P_{i1} \geq 0$ The chain is aperiodic It can also be shown that when the traffic Intensity

$P = \lambda/\mu < 1$

The chain is persistent non-null and of

M.C.

Pollaczek - Khinchine formula (P-K-F)

The limiting probability

$$v_j = \lim_{n\to\infty} p_{ij}^{(n)}, \quad j = 0,1,2 \ldots$$

Exist and are Independent of the Initial St

The probability

$$v = v_0, v_1, \ldots \leq v_j = 1$$ are given a

unique solution of

$$v = vp$$

Let $K(s) = \sum k_j s^j$ denote the p.g.f of the

Distribution of $\{k_j\}$ and $\{v_j\}$ respectively.

we have

$$K(s) = \sum_{j=0}^{\infty} k_j s^j = \sum_{j=0}^{\infty} s^j$$

$$= \left\{ \int_0^{\infty} \frac{e^{-\lambda t}(\lambda t)^j}{j!} \, dB(t) \right\}$$

$$= \int_0^{\infty} e^{-(+\lambda - \lambda s)t} \, dB(t)$$

$$= B^*(\lambda - \lambda s).$$

$$E(A) = K'(1) - \lambda B^{*}(1)(0)$$

$$= \lambda/\mu = P$$

Hence now $v = vp$ gives an Infinite system equation multiply the $(k+1)$ st equation by s^k, $k=$ and adding over k, we get on simplication for $0<$

$$V(s) = \frac{\{1 - K'(1)\}(1-s)\, K(s)}{K(s) - s}$$

putting $K'(1) = P$ we get

$$V(s) = \frac{(1-P)(1-s)\, K(s)}{K(s) - s}$$

$$V(s) = \frac{(1-P)(1-s)\, B^*(\lambda-\lambda s)}{B^*(\lambda-\lambda s) - s}$$

This is known as pollaczek - khinchine (P.K) formula.