A STUDY MATERIAL FOR SOME BASIC CONCEPTS OF ANALYSIS

(For Under Graduate Students)

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of

Bharathidasan University, Tiruchirappalli.

Prepared by

Mr. S. Riyasdeen, M.Sc., SET,

Assistant Professor

PG & Research Department of Mathematics,

Khadir Mohideen College,

Adirampattinam - 614 701.

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Chapter 1

Real Analysis

1.1 Introduction

1.1.1 Order Property of $\mathbb{R}$

1. Given x, $y \in \mathbb{R}$, with $x < y$ then

   (a) $\forall\ c\ \in \mathbb{R}(x + c\ <\ y + c)$

   (b) $\forall\ c\ \in \mathbb{R}, c > 0(x + c\ <\ y + c)$

2. Transitive: $\forall(x, y, z) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}$, with $x < y$ and $y < z$ then $x < z$

3. Law of Trichotomy: $\forall(x, y) \in \mathbb{R} \times \mathbb{R}$ exactly one of the following holds

   (a) $x < y$

   (b) $x = y$

   (c) $x > y$

Problem 1.1 Prove that $1 \neq 0$

Solution By law of Trichotomy, one of the following holds,$1 = 0$, $1 < 0$, $1 > 0$, Suppose $1 = 0 \implies \forall\alpha \in \mathbb{R}$, $\alpha.1 = \alpha.0 \implies \mathbb{R} = \{0\}$ which is a contradiction. Suppose $1 < 0 \implies -1 > 0 \implies -1. - 1 > 0. - 1 \implies 1 > 0$ contradiction to Law of Trichotomy. Therefore $1 > 0$.

Problem 1.2 Given $a \in \mathbb{R}$, $a^2 > 0$.

Solution If $a = 0$ then $a^2 = 0 \implies a^2 \geq 0$. If $a > 0$ then $a.a > 0.a$ (by order property) $\implies a^2 > 0 \implies a^2 \geq 0$. If $a < 0$ then $-a > 0 \implies -a.-a > 0.-a$ (by order property) $\implies a^2 > 0 \implies a^2 \geq 0$.

1.2 Least Upper Bound

1.2.1 Bounds

Definition 1.1 (Upper bound) Let $A \subseteq \mathbb{R}$. Given α is said to be an upper bound of A if $\forall\ a\ \in A$, $(a \leq \alpha)$.

Definition 1.2 (Lower bound) Let $A \subseteq \mathbb{R}$. Given β is said to be an lower bound of A if $\forall\ a\ \in A$, $(\beta \leq a)$.

Definition 1.3 (Bounded Above) Let $A \subseteq \mathbb{R}$, is said to be bounded above if $\exists \alpha \in \mathbb{R}$ such that α is an upper bound of A.

Definition 1.4 (Bounded Below) Let $A \subseteq \mathbb{R}$, is said to be bounded below if $\exists \beta \in \mathbb{R}$ such that β is a lower bound of A.

Definition 1.5 (Bounded) Let $A \subseteq \mathbb{R}$, is said to be bounded if $\exists \alpha$, $\beta \in \mathbb{R}$ such that α is an upper bound of A and β is a lower bound of A.

Definition 1.6 (Least upper bound) Let $A \subseteq \mathbb{R}$. Given α is said to be an least upper bound for A if

1. α is a upper bound for A.
2. Given any upper bound $\gamma \in \mathbb{R}$ for A then $\alpha \leq \gamma$ or given any real number $\gamma < \alpha$, then γ is not a upper bound for A.

Definition 1.7 (Greatest lower bound) Let $A \subseteq \mathbb{R}$. Given β is said to be a greatest lower bound for A if

1. β is a lower bound for A.

2. Given any lower bound $\gamma \in \mathbb{R}$ for A then $\beta \geq \gamma$ or given any real number $\gamma < \beta$, then γ is not a lower bound for A.

1.2.2 Least Upper Bound

Remark 1.1 (LUB Axiom) Let A be a non-empty subset of $\mathbb{R}$, then $\exists \alpha \in \mathbb{R}$ such that α is a LUB for A.

Theorem 1.1 (Archimedian Property) $\mathbb{N}$ is not bounded above.

Proof. Suppose $\mathbb{N}$ is bounded above. Since $1 \in \mathbb{N}$, $\mathbb{N}$ is non-empty and hence by remark (1.1), $\exists \alpha \in \mathbb{R}$ such that $\alpha = \text{lub}(\mathbb{N}) \implies \alpha - 1$ is not a $\text{lub}(\mathbb{N}) \implies \exists m \in \mathbb{N}$ such that $m > \alpha - 1 \implies m + 1 > \alpha$, Since $m + 1 \in \mathbb{N}$ contradiction to definition (1.6). Hence $\mathbb{N}$ is not bounded above. □

Theorem 1.2 (Avatars of Archimedian Property) The following are equivalent

i $\mathbb{N}$ is not bounded above

ii $\forall x > 0$ & $\forall y \in \mathbb{R}(\exists n \in \mathbb{N}(nx > y))$

iii $\forall a \in \mathbb{R}(\exists n \in \mathbb{N}(n > a))$

Proof. (i) $\implies$ (ii) Let $x > 0$ and $y \in \mathbb{R}$. Suppose for all $n \in \mathbb{N}$, $nx \leq y$ implies $n \leq \dfrac{y}{x}$ implies $\dfrac{y}{x}$ is a upper bound for $\mathbb{N}$, a contradiction to (i). Hence $\exists n \in \mathbb{N}$ such that $nx > y$.

(ii) $\implies$ (iii) Let $a \in \mathbb{R}$, let $x = 1 > 0$ therefore by (ii), $\exists n \in \mathbb{N}$ such that $n > a$.

(iii) $\implies$ (i) Suppose $\mathbb{N}$ is bounded, $\exists \alpha \in \mathbb{R}$ such that α is a lub for $\mathbb{N}$ implies that $\forall n \in \mathbb{N}$, $n \leq \alpha$ □

Exercise 1.3 $\mathbb{Z}$ is not a bounded set.

Theorem 1.4 ($\mathbb{Q}$ is dense in $\mathbb{R}$) Given $a, b \in \mathbb{R}$ with $a < b$ then there exists $x \in \mathbb{Q}$ such that $a < x < b$.

Proof. Given that $a < b$ implies $b - a > 0$. Therefore by (ii) of theorem (1.2)(Archimedian Property), there exists $q \in \mathbb{N}$ such that $q(b-a) > 1$. Let $S := \{m \in \mathbb{Z} : m \leq qa\}$. Suppose S is empty, $\forall m \in \mathbb{Z}$, $m > qa$ implies qa is a lower bound for $\mathbb{Z}$, a contradiction to exercise (1.3). Therefore S is non-empty. Clearly S is bounded above ($\forall m \in S$, $m \leq qa$ by definition of S). Therefore there exists $\alpha \in \mathbb{R}$ such that α is a lub for S(by remark (1.1)), this implies $\alpha - 1$ is not a upper bound for S, implies there exists $m \in S$ such that $m > \alpha - 1 \implies m + 1 > \alpha$. Choose $p = m + 1$. It is enough to prove that $qa < p$ and $p < qb$. Suppose $p \leq qa$, implies $p \in S$, a contradiction to the choice of p. Also suppose $p \geq qb$, consider $1 = m + 1 - m \geq qb - qa = q(b-a) > 1$ which is a contradiction to Law of trichotomy. Therefore $qa < p$ and $p < qb$, implies $qa < p < qb$, implies $a < \frac{p}{q} < b$.

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Theorem 1.5 (Existance of n^{th} root) Let α be a non-negative real number and $n \in \mathbb{N}$. Then there exists a unique non-negative $x \in \mathbb{R}$ such that $x^n = \alpha$.

Proof. If $\alpha = 0$ then $x = 0$ also suppose $y^n = 0$ implies $y = 0$. Let $\alpha > 0$, $S := \{t \in \mathbb{R} : t \geq 0 \& t^n \leq \alpha\}$. $0 \in S$ implies $S \neq \phi$. By Archimedian Property (theorem (1.1)), there exists $N \in \mathbb{N}$ such that $\mathbb{N} > \alpha$. Suppose N is not a upper bound for S, then there exists $x \in S$ such that $x > N \implies x^n > N^n > N > \alpha$, a contradiction to the definition of S. Therefore N is a upper bound for S. Therefore there exists $\alpha \in \mathbb{R}$ such that α is a lub for S(by remark (1.1)). It is enough to prove that $x^n = \alpha$. Suppose $x^n \neq \alpha$. Therefore by Law of Trichotomy $x^n < \alpha$ or $x^n > \alpha$.

case 1: $x^n < \alpha$.

$$\begin{aligned}(x+\frac{1}{k})^n &= x^n + \binom{n}{1}x^{n-1}\frac{1}{k} + \binom{n}{2}x^{n-2}\frac{1}{k^2} + \ldots + \binom{n}{n}\frac{1}{k^n} \\ &\leq x^n + \binom{n}{1}x^{n-1}\frac{1}{k} + \binom{n}{2}x^{n-2}\frac{1}{k} + \ldots + \binom{n}{n}\frac{1}{k} \qquad (\text{Since } \frac{1}{k} > \frac{1}{k^n}) \\ &= x^n + \sum_{j=1}^{n}\binom{n}{j}x^{n-j}\frac{1}{k}\end{aligned}$$

$$\therefore (x+\frac{1}{k})^n \leq x^n + \frac{c}{k}, \qquad \text{where } c = \sum_{j=1}^{n}\binom{n}{j}x^{n-j}$$

By Archimedian property $\exists k \in \mathbb{N}$ such that $k > \dfrac{c}{\alpha - x^n} \implies \dfrac{1}{k} < \dfrac{\alpha - x^n}{c}$ i.e., $x^n + \dfrac{c}{k} < \alpha \implies (x+\dfrac{1}{k})^n < \alpha \implies x + \dfrac{1}{k} \in S$, also $x < x + \dfrac{1}{k} \implies x$ is not upper bound of S, a contradiction to x is a lub of S. Therefore $x^n \not< \alpha$.

case 2: $x^n > \alpha$

$$\begin{aligned}(x-\frac{1}{k})^n &= x^n - \binom{n}{1}x^{n-1}\frac{1}{k} + \binom{n}{2}x^{n-2}\frac{1}{k^2} - \ldots + (-1)^n\binom{n}{n}\frac{1}{k^n} \\ &\geq x^n - \binom{n}{1}x^{n-1}\frac{1}{k} - \binom{n}{2}x^{n-2}\frac{1}{k} - \ldots - \binom{n}{n}\frac{1}{k} \qquad (\text{Since } \frac{-1}{k} < \frac{-1}{k^n}) \\ &= x^n + \sum_{j=1}^{n}\binom{n}{j}x^{n-j}\frac{1}{k}\end{aligned}$$

$$\therefore (x-\frac{1}{k})^n \geq x^n - \frac{c}{k}, \qquad \text{where } c = \sum_{j=1}^{n}\binom{n}{j}x^{n-j}$$

By Archimedian property $\exists k \in \mathbb{N}$ such that $k > \dfrac{c}{x^n - \alpha} \implies k(x^n - \alpha) > c$ i.e., $x^n - \dfrac{c}{k} > \alpha \implies (x - \dfrac{1}{k})^n > \alpha$

Claim: $x - \dfrac{1}{k}$ is a upper bound of S.

Suppose there exists $y \in S$ such that $y > x - \dfrac{1}{k}$ then $y^n > (x - dfrac1k)^n > \alpha$,

a contradiction to $y \in S$. Therefore $x - dfrac1k$ is a upper bound of S. This is a contradiction since $x - \dfrac{1}{k} < x$. Therefore $x^n \ngtr \alpha$.
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Corollary 1.1 Between any two real numbers there are infinitely many rationals.
Hint: Suppose finite.

Theorem 1.6 (Greatest Integer Function) Let $x \in \mathbb{R}$. Then there exists a unique m in $\mathbb{Z}$, such that $m \leq x < m+1$.

Proof. Consider the set $S := \{m \in \mathbb{Z} : m \leq \mathbb{Z}\}$. Suppose $S = \phi$, then for all $m \in \mathbb{Z}$, $m > x \implies x$ is a lower bound for $\mathbb{Z}$, contradiction to problem (1.3). By the definition of S, x is a upper bound for S, therefore S is bounded above. Hence by LUB axiom(remark (1.1)) $\exists \alpha \in \mathbb{R}$ such that α is lub of S.Since $\alpha - 1$ is not a upper bound for S, there exists $m \in S$ such that $m > \alpha - 1$. Since $m \in S$, $m \leq x$. Suppose $x \geq m+1 \implies m+1 \in S \implies m+1 \leq \alpha \implies m \leq \alpha - 1$, a contradiction. *Uniqueness:* Suppose there exists n such that $n \leq x < n+1$, W.L.G let $m < n \implies m+1 \leq n$, therefore $m \leq x < m+1 \leq n \implies x < n$, a contradiction to law of Trichotomy.

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1.3 Sequence

Definition 1.8 (Sequence) Let $X \subseteq \mathbb{R}$ be a non-empty set. A Sequence in X is a function $f : \mathbb{N} \to X$. We let $x_n := f(n)$ and call x_n as the n^{th} term of the sequence. The sequence is denoted by (x_n).

Definition 1.9 (Bounded Sequence) A sequence (x_n) is said to be bounded sequence if there exists $M > 0$, such that $\forall n \in \mathbb{N}$, $|x_n| \leq M$.

Definition 1.10 (Constant c-sequence) A sequence (x_n) is said to be constant c-sequence if $\forall n \in \mathbb{N}$, $x_n = c$.

Definition 1.11 (Constant sequence) A sequence (x_n) is said to be constant sequence if there exists $c \in X$ such that (x_n) is a constant c-sequence($\forall n \in \mathbb{N}$, $x_n = c$).

Definition 1.12 (Eventually constant sequence) A sequence (x_n) is said to be eventually constant sequence if there exists $c \in X$, $n_0 \in \mathbb{N}$ such that $\forall n \geq n_0$, $x_n = c$.

Definition 1.13 (Convergent Sequence) A sequence (x_n) is said to be convergent sequence if there exists $a \in \mathbb{R}$ such that the terms of the sequence gets arbitrarilly close to a. i.e., $\forall \epsilon > 0(\exists n_0 \in \mathbb{N}(\forall n \geq n_0(|x_n - a| < \epsilon)))$. We denote as $x_n \to a$

Note 1 We say a as a limit of the sequence (x_n).

Definition 1.14 (Cauchy Sequence) A sequence (x_n) is said to be cauchy sequence if the terms of the sequence gets arbitrarilly closer. i.e.,$\forall \epsilon > 0(\exists n_0 \in \mathbb{N}(\forall n,m \geq n_0(|x_n - x_m| < \epsilon)))$.

Exercise 1.7 Let $x,y \in \mathbb{R}$. If for all $\epsilon > 0$, $|x - y| < \epsilon$ then $x = y$.

Theorem 1.8 (Limit of a sequence is unique) Let (a_n) be a converging sequence. Then the limit of the sequence (a_n) is unique.

Proof. Let $a_n \to a$ and $a_n \to b$. Let $\epsilon > 0$, $a_n \to a \implies \exists n_1 \in \mathbb{N}$ such that $\forall n \geq n_1, |a_n - a| < \frac{\epsilon}{2}$ and $a_n \to b \implies \exists n_2 \in \mathbb{N}$ such that $\forall n \geq n_2, |a_n - b| < \frac{\epsilon}{2}$. Choose $N = \max\{n_1, n_2\}$, now $|a - b| = |a - a_N + a_N - b| \leq |a - a_N| + |a_N - b| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$. Therefore for all $\epsilon > 0$, $|a - b| < \epsilon \implies a = b$ by exercise (1.7).Hence limit of a sequence is unique. □

Theorem 1.9 Every convergent sequence is bounded.

Proof. Let (x_n) be a convergent sequence and x be its limit. Therefore for $\epsilon = 1(\exists n_0 \in \mathbb{N}(\forall n \geq n_0(|x_n - x| < 1)))$. Consider $|x_n| \leq |x_n - x| + |x| < 1 + |x|$(for $n \geq n_0$). Choose M =max $\{|x_1|,\ldots,|x_{n_0-1}|,1+|x|\}$, then for all $n \in \mathbb{N}$, $|x_n| < M$. Hence (x_n) is bounded. □

Theorem 1.10 (Sandwich Lemma) Let (x_n),(y_n) and (z_n) be sequences such that $\forall n \in \mathbb{N}$, $x_n \leq y_n \leq z_n$. If $x_n \to a$ and $z_n \to a$ then $y_n \to a$.

Proof. Let $\epsilon > 0$, $x_n \to a$ implies $\exists n_1 \in \mathbb{N}$ such that $\forall n \geq n_1$, $x_n \in (a - \epsilon, a + \epsilon)$ implies $\forall n \geq n_1$, $a - \epsilon < x_n$ and $z_n \to a$ implies $\exists n_2 \in \mathbb{N}$ such that $\forall n \geq n_2$, $z_n \in (a - \epsilon, a + \epsilon)$ implies $\forall n \geq n_2$, $z_n < a + \epsilon$. Choose $n_0 = \max\{n_1, n_2\}$, then $\forall n \geq n_0$, $a - \epsilon < x_n \leq y_n \leq z_n < a + \epsilon$. Therefore $y_n \to a$. □

1.3.1 Algebra of limits

Theorem 1.11 Let $x_n \to x$ and $y_n \to y$ then $x_n + y_n \to x + y$.

Proof. Let $\epsilon > 0$, $x_n \to x$ implies $\exists n_1 \in \mathbb{N}$ such that $\forall n \geq n_1$,

$$|x_n - x| < \frac{\epsilon}{2} \tag{1.1}$$

$y_n \to y$ implies $\exists n_2 \in \mathbb{N}$ such that $\forall n \geq n_2$,

$$|y_n - y| < \frac{\epsilon}{2} \tag{1.2}$$

Let $n_0 = \max\{n_1, n_2\}$, then $\forall n \geq n_0$,

$$\begin{aligned} |x_n + y_n - (x + y)| &\leq |x_n - x| + |y_n - y| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \qquad \text{by eqns ((1.1)) \& ((1.2)).} \end{aligned}$$

□

Theorem 1.12 Let $x_n \to x$ and $\alpha \in \mathbb{R}$ then $\alpha x_n \to \alpha x$.

Proof. Let $\epsilon > 0$, $x_n \to x$ implies $\exists n_1 \in \mathbb{N}$ such that $\forall n \geq n_1$,

$$|x_n - x| < \frac{\epsilon}{1+|\alpha|} \tag{1.3}$$

then $\forall n \geq n_1$,

$$\begin{aligned} |\alpha x_n - \alpha x| &= |\alpha||x_n - x| \\ &< (1+|\alpha|)|x_n - x| \\ &< (1+|\alpha|)\frac{\epsilon}{1+|\alpha|} = \epsilon \qquad \text{by equation ((1.3))} \end{aligned}$$

reason for we go for $1+|\alpha|$ is α may be equal to zero. □

Theorem 1.13 Let $x_n \to x$ and $y_n \to y$ then $x_n y_n \to xy$.

Proof. Consider

$$\begin{aligned} |x_n y_n - xy| &= |x_n y_n - x_n y + x_n y - xy| \\ &\leq |x_n||y_n - y| + |y||x_n - x| \\ &< |x_n||y_n - y| + (1+|y|)|x_n - x| \end{aligned} \tag{1.4}$$

Let $\epsilon > 0$, (x_n) is convergent implies it is bounded (by theorem (1.9)). Therefore there exists $M > 0$ such that $\forall n \in \mathbb{N}$, $|x_n| < M$. $y_n \to y$ implies $\exists n_1 \in \mathbb{N}$ such that $\forall n \geq n_1$,

$$|y_n - y| < \frac{\epsilon}{2M} \tag{1.5}$$

$x_n \to y$ implies $\exists n_2 \in \mathbb{N}$ such that $\forall n \geq n_2$,

$$|x_n - y| < \frac{\epsilon}{2(1+|y|)} \tag{1.6}$$

Let $n_0 = \max\{n_1, n_2\}$, then $\forall n \geq n_0$,

$$\begin{aligned} |x_n y_n - xy| &< |x_n||y_n - y| + (1+|y|)|x_n - x| \qquad \text{by equation ((1.4))} \\ &< M\frac{\epsilon}{2M} + (1+|y|)\frac{\epsilon}{2(1+|y|)} = \epsilon \qquad \text{by eqns ((1.5)) \& ((1.6)).} \end{aligned}$$

□

Theorem 1.14 Let $x_n \to x$ and $\forall n \in \mathbb{N}$, $x_n \neq 0 \& x \neq 0$ then $\dfrac{1}{x_n} \to \dfrac{1}{x}$.

Proof. Consider

$$\begin{aligned} |x| &= |x_n - x_n + x| \\ &\leq |x_n| + |x_n - x| \qquad (1.7) \\ \implies |x_n| &\geq |x| - |x_n - x| \end{aligned}$$

Let $\epsilon > 0$, $x_n \to x$ implies $\exists n_1 \in \mathbb{N}$ such that $\forall n \geq n_1$,

$$|x_n - x| < \frac{|x|}{2} \qquad (1.8)$$

$x_n \to x$ implies $\exists n_2 \in \mathbb{N}$ such that $\forall n \geq n_2$,

$$|x_n - x| < \frac{|x|^2 \epsilon}{2} \qquad (1.9)$$

Let $n_0 = \max\{n_1, n_2\}$, then $\forall n \geq n_0$,

$$\begin{aligned} \left|\frac{1}{x_n} - \frac{1}{x}\right| &= \frac{|x - x_n|}{|x||x_n|} \\ &< \frac{2|x - x_n|}{|x||x|} \qquad \text{by equations ((1.8)) \& ((1.7))} \\ &< \frac{2\frac{|x|^2\epsilon}{2}}{|x|^2} = \epsilon \qquad \text{by equation ((1.9)).} \end{aligned}$$

□

Exercise 1.15 Let $x_n \to x$. If for $\forall n \in \mathbb{N}$, $x_n \geq 0$ then prove that $x \geq 0$.

Theorem 1.16 If for $\forall n \in \mathbb{N}$, $x_n \geq 0$ then $\sqrt{x_n} \to \sqrt{x}$

Proof. Let $\epsilon > 0$. If $x = 0$ then $x_n \to 0$ implies $\exists n_1 \in \mathbb{N}$ such that $\forall n \geq n_1$, $|x_n| < \epsilon^2 \implies \sqrt{x_n} < \epsilon$. If $x \neq 0$ then $x_n \to 0$ implies $\exists n_2 \in \mathbb{N}$ such that $\forall n \geq n_2$,

$$\left|\sqrt{x_n} - \sqrt{x}\right| < \sqrt{x}\epsilon \qquad (1.10)$$

Let $n \geq n_2$,

$$\begin{aligned}
\left|\sqrt{x_n} - \sqrt{x}\right| &= \left|\frac{(\sqrt{x_n} - \sqrt{x})(\sqrt{x_n} + \sqrt{x})}{\sqrt{x_n} + \sqrt{x}}\right| \\
&= \frac{|x_n - x|}{\sqrt{x_n} + \sqrt{x}} \\
&\leq \frac{|x_n - x|}{\sqrt{x}} \qquad \text{since} \quad x_n \geq 0 \\
&< \frac{\sqrt{x}\epsilon}{\sqrt{x}} = \epsilon \qquad \text{by equation ((1.10)).}
\end{aligned}$$

□

1.3.2 Some Problems

Problem 1.3 Prove that $\frac{1}{n} \to 0$.

Solution Let $\epsilon > 0$, By Archimedian Property $\exists n_0 \in \mathbb{N}$ such that $n_0 > \frac{1}{\epsilon}$. Therefore $\forall n \geq n_0$, $\frac{1}{n} \leq \frac{1}{n_0} < \epsilon$.

Problem 1.4 Prove that $\frac{1}{n^k} \to 0$, for $k \in \mathbb{N}$.

Solution Let $\epsilon > 0$, By Archimedian Property $\exists n_0 \in \mathbb{N}$ such that $n_0 > \frac{1}{\epsilon}$. Therefore $\forall n \geq n_0$, $\frac{1}{n^k} \leq \frac{1}{n} \leq \frac{1}{n_0} < \epsilon$.

Problem 1.5 For $a \geq 0$, $a^{\frac{1}{n}} \to 1$.

Solution

Problem 1.6 For $0 \leq a \leq 1$, $a^n \to 0$.

Solution $a < 1 \implies \frac{1}{a} > 1 \implies \exists h > 0$ such that $\frac{1}{a} = 1 + h \implies \frac{1}{a^n} = (1 + h)^n > nh \implies a^n < \frac{1}{nh}$. Also since $0 \leq a^n$ and by Sandwich lemma (theorem (1.10)), $a^n \to 0$.