MAXIMA AND MINIMA OF A FUNCTION OF TWO VARIABLES

Definitions :

Maximum value. $f(a, b)$ *is a maximum value of* $f(x, y)$, if there exists some neighbourhood of the point (a, b) such that for every point $(a+h, b+k)$ of the neighbourhood

$$f(a, b) > f(a+h, b+k)$$

Minimum value. $f(a, b)$ is a minimum value of $f(x, y)$, if there exists some neighbourhood of the point (a, b) such that for every point $(a+h, b+k)$ of the neighbourhood

$$f(a, b) < f(a+h, b+k)$$

Extremum value. $f(a, b)$ is said to be an extremum value of $f(x, y)$ if it is either a maximum or a minimum.

Necessary conditions for a maximum or a minimum

The necessary conditions for $f(a, b)$ to be an *extremum of* $f(x, y)$ *are that*

$$\frac{\partial f}{\partial x}(a, b) = 0 \text{ and } \frac{\partial f}{\partial y}(a, b) = 0$$

If $f(a, b)$ is an extremum of $f(x, y)$ of two variables, then it is also an extremum function of one variable $f(x, b)$.

Hence $\frac{\partial f}{\partial x}$ at $x = a$ must necessarily be zero

Theat is, $\frac{\partial f}{\partial x}(a,b) = 0$

Similarly, we can prove, $\frac{\partial f}{\partial y}(a,b) = 0$

Hence $f_x(a, b) = 0$ and $f_y(a, b) = 0$

Notations :

$$\frac{\partial f}{\partial x} = f_x, \quad \frac{\partial f}{\partial y} = f_y, \quad \frac{\partial^2 f}{\partial x^2} = f_{xx}$$

$$\frac{\partial^2 f}{\partial x \partial y} = f_{yx}, \quad \frac{\partial^2 f}{\partial y^2} = f_{yy}$$

Sufficient conditions

If $\quad f_x(a, b) = 0, \; f_y(a, b) = 0$

and $\qquad f_{xx}(a, b) = A, f_{xy}(a, b) = B, f_{yy}(a, b) = C$ then

(*i*) $f(a, b)$ is a maximum value if $AC - B^2 > 0$ and $A < 0$ (or $B < 0$)

(*ii*) $f(a, b)$ is a minimum value if $AC - B^2 > 0$ and $A > 0$ (0r $B > 0$)

(*iii*) $f(a, b)$ is not an extremum if $AC - B^2 < 0$ and

(*iv*) if $AC - B^2 = 0$, further considerations are necessary to decide. This case is doubtful. (proof is not expected of you)

Note : 1. If $AC - B^2 > 0$, then $A \neq 0, C \neq 0$

2. Stationary value. A function $f(x, y)$ is said to be stationary at (a, b) or $f(a, b)$ is said to be a stationary value of $f(x,y)$ if

$$f_x(a, b) = 0 \text{ and } f_y(a, b) = 0$$

3. Every extremum value is a stationary value but a stationary value need not be an extremum value.

Ex. 1. *Find maxima and minima of*

$$f(x, y) = x^3 + y^3 - 3axy.$$

Sol.
$$f(x, y) = x^3 + y^3 - 3axy$$

$$f_x = 3x^2 - 3ay \; ; f_y = 3y^2 - 3ax$$

For extremum, $\quad f_x = 0$ and $f_y = 0$

$$\therefore \quad 3x^2 - 3ay = 0 \text{ and } 3y^2 - 3ax = 0$$

i.e., $\quad x^2 = ay$ and $y^2 = ax$

Eliminating y, $\left(\dfrac{x^2}{a}\right)^2 = ax$

$$x^4 = a^3x$$

$$x(x^3 - a^3) = 0 \qquad \therefore \qquad x = 0 \text{ or } x = a$$

when $x = 0, y = 0$ and when $x = a, y = a$

Therefore, the stationary points are $(0, 0)$ and (a, a)

$$f_{xx} = 6x, f_{xy} = -3a, f_{yy} = 6y$$

$$\therefore \quad AC - B^2 = 36xy - 9a^2$$

At $(0, 0)$, $\quad AC - B^2 = -9a^2 < 0$

$\therefore$ There is neither a maximum nor a minimum at $(0, 0)$.

At (a, a), $AC - B^2 = 36a^2 - 9a^2 =$ positive value and $A(a, a) = 6a$

$\therefore$ If $a > 0$, then $A > 0$ and hence $f(a, a)$ is a minimum value at (a, a)

If $a < 0$, then $A < 0$ and hence f has a maximum at (a, a) and the maximum value is $f(a, a)$. Hence, the maximum value or minimum value at (a, a) is $f(a, a) = -a^2$.

Ex. 2. *Find the maximum and minimum of*

$$x^2 + y^2 + 6x + 12$$

$$\frac{\partial f}{\partial x} = 2x+6\,;\ \frac{\partial f}{\partial y} = 2y\,;\ \frac{\partial^2 f}{\partial x^2} = 2,\ \frac{\partial^2 f}{\partial x \partial y} = 0$$

and $$\frac{\partial^2 f}{\partial y^2} = 2,$$

For extremum, $\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0$

$\therefore$ Solving $2x+6=0$ and $2y=0$, we get the point $(-3, 0)$.

$$AC - B^2 = 4 - 0 = 4$$

At $(-3, 0)$, $AC - B^2 = 4 > 0$ and $A(-3, 0) = 2 > 0$

$\therefore$ $f(x, y)$ is minimum at $(-3, 0)$

The minimum value $= f(-3, 0) = 9 - 18 + 12 = 3$

Ex. 3. *Find the maxima or minima of*

$$f(x, y) = x^3 + y^3 - 3xy$$

Sol. This is a particular case of example 1 where $a = 1 > 0$.

$\therefore$ $(1, 1)$ is a minimum point and the minimum value is -1.

Ex. 4. *Show that the function*

$$f(x, y) = x^3 + y^3 - 63(x+y) + 12xy$$

has a maximum at $(-7, -7)$ and a minimum at $(3, 3)$. ***(AMIE, 1991)***

Sol. $$f(x, y) = x^3 + y^3 - 63(x+y) + 12xy$$

$$f_x = 3x^2 - 63 + 12y;\ f_y = 3y^2 - 63 + 12x$$

$$\frac{\partial^2 f}{\partial x^2} = 6x,\ \frac{\partial^2 f}{\partial x \partial y} = 12,\ \frac{\partial^2 f}{\partial y^2} = 6y$$

Solve $\frac{\partial f}{dx} = 0$ and $\frac{\partial f}{\partial y} = 0$

$\therefore$ $x^2 + 4y = 21$ and $y^2 + 4x = 21$

Subtracting, $x^2 - y^2 + 4(y - x) = 0$

i.e., $(x - y)(x + y - 4) = 0$

$\therefore$ $x - y = 0$ or $x + y = 4$

Taking $x = y$ and using $x^2 + 4y = 21$, we get,

$$x^2 + 4x - 21 = 0$$

i.e., $(x + 7)(x - 3) = 0$ $\quad \therefore x = 3, -7$

When $x = 3,\ y = 3$ (since $x = y$)

When $x = -7, y = -7$

Taking $x + y = 4$ and $x^2 + 4y = 21$, we get,

$$x^2 + 4(4 - x) = 21$$

i.e., $x^2 - 4x - 5 = 21$

$\therefore \quad (x-5)(x+1) = 0 \qquad \therefore x = 5$ or -1

When $x = 5,\ y = 4 - x = -1$

When $x = -1, y = 4 - x = 5$

$\therefore$ The stationary points are (3, 3), (−7, −7), (5, −1) and (−1, 5)

$$AC - B^2 = 36xy - 144$$

At (3, 3), $A = 18 > 0$ and $AC - B^2 = 324 - 144 > 0$

$\therefore\ f(x, y)$ is minimum at (3, 3)

At (−7, −7), $AC - B^2$ = positive value and $A < 0$

$\therefore\ f(x, y)$ is maximum at (−7, −7),

At (5, −1) and (−1, 5), $AC - B^2$ = negative value

Therefore, there is no extremum at (5, −1) and (−1, 5).

Ex. 5. *Find the maxima and minima of $xy(a - x - y)$.*

Sol. Let $\quad f(x, y) = xy(a - x - y) = axy - x^2y - xy^2$

$$f_x = ay - 2xy - y^2$$

$$f_y = ax - x^2 - 2xy$$

$$A = \frac{\partial^2 f}{\partial x^2} = -2y,\ B = a - 2x - 2y \text{ and } C = -2x$$

$$AC - B^2 = 4xy - (a - 2x - 2y)^2$$

Solving $ay - 2xy - y^2 = 0$ and $ax - x^2 - 2xy = 0$

We have $y(a - 2x - y) = 0$ and $x(a - x - 2y) = 0$

$\therefore \quad y = 0, a - 2x - y = 0, x = 0$ or $a - x - 2y = 0$

Solving, stationary points are (0,0), (a, 0), (0, a) and $\left(\frac{1}{3}a, \frac{1}{3}a\right)$

At (0, 0), $AC - B^2$ = negative value

At (a, 0), $AC - B^2$ = negative value

At (0, a,) $AC - B^2$ = negative value.

$\therefore\ f(x, y)$ does not have an extremum at these three points.

At $\left(\frac{1}{3}a, \frac{1}{3}a\right)$, $AC - B^2$ = positive and $A < 0$ if $a > 0$

$\therefore\ f(x, y)$ is maximum at $\left(\frac{1}{3}a, \frac{1}{3}a\right)$ if $a > 0$

$f(x, y)$ is minimum at this point if $a < 0$.

Maximum or minimum value

$$= f\left(\frac{1}{3}a, \frac{1}{3}a\right) = \frac{a^3}{27}$$

Ex. 6. *Find the points on $z^2 = xy + 1$ which is nearest to the origin.*

Sol. If the required point on the surface is (x, y, z) then its distance from the origin must be minimum.

Hence, we want to find the minimum of

$$(x-0)^2+(y-0)^2+(z-0)^2 = x^2+y^2+z^2$$

Let $f(x, y, z) = x^2+y^2+z^2$

using $z^2 = xy+1$,

$$f = x^2+y^2+xy+1$$

$$f_x = 2x+y, f_y = 2y+x$$

$$A = 2, B = 1, C = 2$$

$$AC-B^2 = 4-1 = 3 > 0 \text{ at all points}$$

Solving $2x+y = 0$ and $2y+x = 0$ we get $x = 0$ and $y = 0$

Putting $x = 0$ and $y = 0$ in $z^2 = xy+1$ we get $z = \pm 1$

Stationary points are (0, 0, 1) and (0, 0, –1)

Since $AC-B^2 = 3 > 0$ and $A > 0$, f is minimum at (0, 0, 1) and (0, 0, –1)

∴ The two nearest points are (0, 0, 1) and (0, 0, –1)

Ex. 7. *A rectangular box, open at the top, is to have a given quantity of 32 c.c. Find the dimensions of the box which requires least material for its construction.*

(*AMIE, 1992*)

Sol. Let x, y, z be the length, breadth and height of the box. Let S, V be the surface area and volume of the box.

$$xyz = V = 32 \text{ c. c.} \qquad \therefore \quad z = \frac{32}{xy}$$

and

$$S = 2(x+y)z + xy$$

$$= 2(x+y).\frac{32}{xy} + xy, \text{ using } z = \frac{32}{xy}$$

$$= 64\left(\frac{1}{x}+\frac{1}{y}\right) + xy$$

$$\frac{\partial S}{\partial x} = \frac{-64}{x^2}+y; \frac{\partial S}{\partial y} = \frac{-64}{y^2}+x$$

$$\frac{\partial^2 S}{\partial x^2} = \frac{128}{x^3}; \frac{\partial^2 S}{\partial x \partial y} = 1; \frac{\partial^2 S}{\partial y^2} = \frac{128}{y^3}$$

Solve $\frac{\partial S}{\partial x} = 0$ and $\frac{\partial S}{\partial y} = 0$ for stationary values.

$$y = \frac{64}{x^2} \text{ and } x = \frac{64}{y^2}$$

Dividing,

$$\frac{y}{x} = \frac{y^2}{x^2}$$

$$\frac{y}{x}\left(\frac{y}{x}-1\right) = 0 \quad \therefore \quad x = y$$

Put $y = x$ in $y = \dfrac{64}{x^2}$ $\therefore x^3 = 64$ $\therefore x = 4$, and $y = 4$

Since $xyz = 32, z = 2$

Stationary point is (4, 4, 2)

$$A = \frac{\partial^2 S}{\partial x^2} \text{ at } (4, 4, 2) = \frac{128}{64} = 2$$

$$B = 1, \; C = 2 \; \therefore AC - B^2 = 4 - 1 = 3 > 0 \text{ and } A > 0$$

$\therefore$ S is minimum (cost minimum) when $x = y = 4; z = 2$

(square base of side 4 c.m. and height 2 c.m.)

Ex. 8. *In a plane triangle ABC, find the maximum value of*

cos A cos B cos C.

Sol. In triangle ABC, $A + B + C = 180°$

Let
$$f = \cos A \cos B \cos C$$
$$= \cos A \cos B \cos [180° - (A + B)]$$
$$= -\cos A \cos B \cos (A + B)$$

$$\frac{\partial f}{\partial A} = -\cos B [-\sin A \cos (A + B) - \cos A \sin (A + B)]$$
$$= \cos B \sin (2A + B)$$

Similarly,
$$\frac{\partial f}{\partial B} = \cos A \sin (2B + A)$$

$$\frac{\partial f}{\partial A} = 0 \text{ and } \frac{\partial f}{\partial B} = 0 \text{ implies.}$$

$$\cos B \sin (2A + B) = 0 \text{ and } \cos A \sin (2B + A) = 0$$

$\therefore$ $B = 90°$ or $2A + B = 180°$ and $A = 90°$ or $2B + A = 180°$

The cases $A = 90°$ and $B = 90°$ are impossible (argue why ?)

$\therefore$ Only possibility is $2A + B = 180°$ and $2B + A = 180°$

Solving $A = B = 60° \quad \therefore \quad C = 60°$

$$\frac{\partial^2 f}{\partial A^2} = 2 \cos B \cos (2A + B)$$

$$\frac{\partial^2 f}{\partial B^2} = 2 \cos A \cos (2B + A)$$

$$\frac{\partial^2 f}{\partial A \partial B} = \cos B \cos (2A + B) - \sin B \sin (2A + B)$$
$$= \cos (2A + 2B)$$

At $A = B = C = 60°$

$$\frac{\partial^2 f}{\partial A^2} = 2\cos 60^\circ \cos 180^\circ = -1$$

$$\frac{\partial^2 f}{\partial B^2} = -1$$

$$\frac{\partial^2 f}{\partial A \partial B} = -\frac{1}{2}$$

$$\text{“}AC - B^2\text{”} = 1 - \frac{1}{4} = +\text{ ve and ‘}A\text{’} = -\text{ ve}$$

$\therefore$ f is maximum and maximum value $= \frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2} = \frac{1}{8}$

Ex. 9. *Find the minimum value of $x^2 + y^2 + z^2$ where*

$$ax + by + cz = p$$

Sol.

$$z = (p - ax - by)/c$$

$\therefore$

$$f = x^2 + y^2 + \frac{1}{c^2}(p - ax - by)^2 \qquad ...(1)$$

$$\frac{\partial f}{\partial x} = 2x - \frac{2a}{c^2}(p - ax - by)$$

$$\frac{\partial f}{\partial y} = 2y - \frac{2b}{c^2}(p - ax - by)$$

$$\frac{\partial^2 f}{\partial x^2} = 2 + \frac{2a^2}{c^2};\ \frac{\partial^2 f}{\partial y^2} = 2 + \frac{2b^2}{c^2};\ \frac{\partial^2 f}{\partial x \partial y} = \frac{2ab}{c^2}$$

$$AC - B^2 = 4\left(1 + \frac{a^2}{c^2}\right)\left(1 + \frac{b^2}{c^2}\right) - \frac{4a^2b^2}{c^4}$$

$$= 4\left(1 + \frac{a^2}{c^2} + \frac{b^2}{c^2}\right) = +\text{ ve}$$

and

$$A = 2\left(1 + \frac{a^2}{c^2}\right) = +\text{ve}$$

$$\frac{\partial f}{\partial x} = 0 \text{ and } \frac{\partial f}{\partial y} = 0 \text{ imply}$$

$$x = \frac{a}{c^2}(p - ax - by)$$

$$y = \frac{b}{c^2}(p - ax - by) \qquad ...(2)$$

Dividing,

$$\frac{x}{y} = \frac{a}{b} \qquad \therefore\ y = \frac{bx}{a} \qquad ...(3)$$

Putting in (1), $x = \frac{a}{c^2}\left(p - ax - \frac{b^2x}{a}\right)$

$\therefore \quad x\left(1+\frac{a^2+b^2}{c^2}\right) = \frac{ap}{c^2}$

$\therefore \quad x = \frac{ap}{a^2+b^2+c^2}$

$\therefore \quad y = \frac{bx}{a} = \frac{bp}{a^2+b^2+c^2}$

Substituting these values in $ax + by + cz = p$

we get $\quad z = \frac{cp}{a^2+b^2+c^2}$

f is minimum at $\left(\frac{ap}{\Sigma a^2}, \frac{bp}{\Sigma a^2}, \frac{cp}{\Sigma a^2}\right)$

Minimum value of $f = \frac{p^2}{a^2+b^2+c^2}$

Ex. 10. *Find the maxima or minima of*

$$f(x, y) = 2(x-y)^2 - x^4 - y^4$$

Sol. $\quad f(x, y) = 2(x-y)^2 - x^4 - y^4$

$$\frac{\partial f}{\partial x} = 4(x-y) - 4x^3$$

$$\frac{\partial f}{\partial y} = -4(x-y) - 4y^3$$

$$\frac{\partial^2 f}{\partial x^2} = 4 - 12x^2$$

$$\frac{\partial^2 f}{\partial x \partial y} = -4$$

$$\frac{\partial^2 f}{\partial y^2} = 4 - 12y^2$$

$\frac{\partial f}{\partial x} = 0$ implies $(x-y) - x^3 = 0$ ...(1)

and $\frac{\partial f}{\partial y} = 0$ implies $-(x-y) - y^3 = 0$ ...(2)

Adding (1) and (2), $x^3 + y^3 = 0$ *i.e.*, $(x+y)(x^2 - xy + y^2) = 0$

This implies $\quad x = -y$

Putting in (1), $2x - x^3 = 0 \quad \therefore x = 0$ or $x = \pm\sqrt{2}$

Using $x = -y$, the corresponding values of y are 0 and $\mp\sqrt{2}$

Hence the solutions of (1) and (2) are $(0, 0)$, $(\sqrt{2}, -\sqrt{2})$ and $(-\sqrt{2}, \sqrt{2})$

Let us discuss the extremum.

(*i*) At $(0, 0)$, $A = \frac{\partial^2 f}{\partial x^2} = 4$, $B = \frac{\partial^2 f}{\partial x \partial y} = -4$, $C = \frac{\partial^2 f}{\partial y^2} = 4$

Hence, $AC - B^2 = 16 - 16 = 0$; This case is doubtful.

(*ii*) At $(\sqrt{2}, -\sqrt{2})$ and at $(-\sqrt{2}, \sqrt{2})$ (both cases)

$$A = -20, B = -4, C = -20$$

$\therefore$ $AC - B^2 = 400 - 16 = 384 =$ positive and $A < 0$

$\therefore$ Hence, $f(x, y)$ is maximum at both points.

Max value is $f(\sqrt{2}, -\sqrt{2}) = 8$.

Ex. 11. *Discuss the maxima and minima of $x^3 y^2 (1 - x - y)$.*

Sol. Let $f(x, y) = x^3 y^2 (1 - x - y)$

$$\frac{\partial f}{\partial x} = (1 - x - y)\, 3x^2 y^2 - x^3 y^2$$

$$\frac{\partial f}{\partial y} = 2x^3 y (1 - x - y) - x^3 y^2$$

$$\frac{\partial^2 f}{\partial x^2} = 6(1 - x - y)\, xy^2 - 6x^2 y^2$$

$$\frac{\partial^2 f}{\partial y^2} = 2x^3 (1 - x - y) - 4x^3 y$$

$$\frac{\partial^2 f}{\partial x \partial y} = 6x^2 y (1 - x - y) - 2x^3 y - 3x^2 y^2$$

$$\frac{\partial f}{\partial x} = 0 \Rightarrow x^2 y^2 [3(1 - x - y) - x] = 0 \Rightarrow 4x + 3y = 3 \quad ...(1)$$

$$\frac{\partial f}{\partial y} = 0 \Rightarrow x^3 y [2(1 - x - y) - y] = 0 \Rightarrow 2x + 3y = 2 \quad ...(2)$$

Solving (1) and (2), we get $x = \frac{1}{2}, y = \frac{1}{3}$

Obtaining A, B, C at $\left(\frac{1}{2}, \frac{1}{3}\right)$, we get $A = -\frac{1}{9}$, $B = -\frac{1}{12}$, $C = -\frac{1}{8}$

$\therefore$ $AC - B^2 = \frac{1}{72} - \frac{1}{144} =$ positive and $A < 0$

$\therefore$ $f(x, y)$ is maximum at $\left(\frac{1}{2}, \frac{1}{3}\right)$.

$$\text{Max. value} = \left(\frac{1}{2}\right)^3\left(\frac{1}{3}\right)^2\left(1-\frac{1}{2}-\frac{1}{3}\right)=\frac{1}{432}$$

Ex. 12. *Find the maxima and minima of*

$$Z(x, y) = \sin x \sin y \sin (x+y) \text{ if } 0 < x, y < \pi$$

Sol. $\frac{\partial z}{\partial x} = \sin y\,[\sin x \cos (x+y) + \cos x \sin (x+y)]$

$= \sin y \sin (2x+y)$

Similarly, $\frac{\partial z}{\partial y} = \sin x \sin (x+2y)$

$$\frac{\partial^2 z}{\partial x^2} = 2 \sin y \cos (2x+y)$$

$$\frac{\partial^2 z}{\partial x \partial y} = \cos x \sin (x+2y) + \sin x \cos (x+2y) = \sin (2x+2y)$$

$$\frac{\partial^2 z}{\partial y^2} = 2 \sin x \cos (x+2y)$$

$$\frac{\partial z}{\partial x} = 0 \Rightarrow \sin y \sin (2x+y) = 0$$

Since $y \neq 0$ and $y \neq \pi$, $\sin y \neq 0$. Hence $\sin (2x+y) = 0$

$$\frac{\partial z}{\partial y} = 0 \Rightarrow \sin (x+2y) = 0$$

Solving $\sin (x+2y) = 0$ and $\sin (2x+y) = 0$

$x + 2y = \pi$ and $2x + y = \pi$ $\quad (\because x \neq 0, y \neq 0)$

$\therefore \quad x = y = \pi/3$

At $\left(\frac{\pi}{3}, \frac{\pi}{3}\right)$, $A = \frac{\partial^2 z}{\partial x^2} = -\sqrt{3}$; $B = -\frac{\sqrt{3}}{2}$, $C = -\sqrt{3}$

$AC - B^2 = 3 - \frac{3}{4}$ = positive and $A < 0$

$\therefore$ $z(x, y)$ is maximum at $\left(\frac{\pi}{3}, \frac{\pi}{3}\right)$

$$\text{Max. value} = z\left(\frac{\pi}{3}, \frac{\pi}{3}\right) = \left(\frac{\sqrt{3}}{2}\right)^3 = \frac{3\sqrt{3}}{8}$$

Lagrange's Method of Undetermined Multipliers

Suppose $f(x, y, z)$ be a function of three variables connected by the constraint

$$\phi(x, y, z) = 0$$

The values of x, y, z that give the stationary values of $f(x, y, z)$ are given by the equations