General Linear Programming Problems – Simplex Methods

3.1.1 General Linear Programming Problem

The linear programming involving more than two variables may be expressed as follows :

Maximize (or) Minimize $Z = c_1x_1 + c_2x_2 + c_3x_3 + \ldots\ldots c_nx_n$

subject to the constraints

$$a_{11}x_1 + a_{12}x_2 + \ldots\ldots\ldots\ldots\ldots + a_{1n}x_n \le \text{ or } = \text{ or } \ge b_1$$
$$a_{21}x_1 + a_{22}x_2 + \ldots\ldots\ldots\ldots\ldots + a_{2n}x_n \le \text{ or } = \text{ or } \le b_2$$
$$a_{31}x_1 + a_{32}x_2 + \ldots\ldots\ldots\ldots\ldots + a_{3n}x_n \le \text{ or } = \text{ or } \le b_3$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$a_{m1}x_1 + a_{m2}x_2 + \ldots\ldots\ldots\ldots\ldots + a_{mn}x_n \le \text{ or } = \text{ or } \ge b_m$$

and the non-negativity restrictions

$$x_1, x_2, x_3, \ldots\ldots\ldots\ldots\ldots, x_n \ge 0.$$

Note : Some of the constraints may be equalities, some others may be inequalities of ($\le$) type and remaining ones inequalities of ($\ge$) type or all of them are of same type.

Definition (1) : A set of values $x_1, x_2 \ldots\ldots x_n$ which satisfies the constraints of the LPP is called its ***solution.***

Definition (2) : Any solution to a LPP which satisfies the non-negativity restrictions of the LPP is called its ***feasible solution.***

Definition (3) : Any feasible solution which optimizes (maximizes or minimizes) the objective function of the LPP is called its ***optimum solution*** or ***optimal solution.***

Definition (4) : If the constraints of a general LPP be

$$\sum_{j=1}^{n} a_{ij} x_j \le b_i \quad (i = 1, 2, 3, \ldots k) \qquad \ldots(1)$$

then the non-negative variables s_i which are introduced to convert the inequalities (1) to the equalities

$$\sum_{j=1}^{n} a_{ij} x_j + s_i = b_i \qquad (i = 1, 2, 3, \ldots k)$$

are called ***slack variables.*** The value of these variables can be interpreted as the amount of unused resource.

Definition (5) : If the constraints of a general LPP be

$$\sum_{j=1}^{n} a_{ij} x_j \geq b_i \qquad (i = k, k+1, \ldots) \qquad \ldots(1)$$

then the non-negative variables s_i which are introduced to convert the inequalities (1) to the equalities

$$\sum_{j=1}^{n} a_{ij} x_j - s_i = b_i \qquad (i = k, k+1, \ldots)$$

are called ***surplus variables.*** The value of these variables can be interpreted as the amount over and above the required level.

3.1.2 Canonical and Standard forms of LPP :

After the formulation of LPP, the next step is to obtain its solution. But before any method is used to find its solution, the problem must be presented in a suitable from. Two forms are dealt with here, the canonical form and the standard form.

The canonical form : The general linear programming problem can always be expressed in the following form :

Maximize $Z = c_1x_1 + c_2x_2 + c_3x_3 + \ldots\ldots c_nx_n$

subject to the constraints

$$a_{11}x_1 + a_{12}x_2 + \ldots\ldots\ldots\ldots + a_{1n}x_n \leq b_1$$

$$a_{21}x_1 + a_{22}x_2 + \ldots\ldots\ldots\ldots + a_{2n}x_n \leq b_2$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$a_{m1}x_1 + a_{m2}x_2 + \ldots\ldots\ldots\ldots + a_{mn}x_n \leq b_m$$

and the non-negativity restrictions

$$x_1, x_2, \ldots\ldots\ldots\ldots, x_n \geq 0.$$

This form of LPP is called the ***canonical form*** of the LPP.

In matrix notation the canonical form of LPP can be expressed as :

Maximize Z = CX (objective function)

subject to AX $\leq$ b (constraints)

and X $\geq$ 0 (non-negativity restrictions)

where C = $(c_1\ c_2 \ldots\ldots c_n)$,

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots\cdots\cdots & a_{1n} \\ a_{12} & a_{22} & \cdots\cdots\cdots & a_{2n} \\ \cdots & \cdots\cdots\cdots\cdots & & \cdots \\ \cdots & \cdots\cdots\cdots\cdots & & \cdots \\ a_{m1} & a_{m2} & \cdots\cdots\cdots & a_{mn} \end{pmatrix}, \quad X = \begin{pmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{pmatrix}, \quad b = \begin{pmatrix} b_1 \\ b_2 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ b_m \end{pmatrix}$$

Characteristics of the Canonical form :

(i) The objective function is of maximization type.

(ii) All constraints are of ($\leq$) type.

(iii) All variables x_i are non-negative.

The Standard Form :

The general linear programming problem in the form

Maximize $Z = c_1x_1 + c_2x_2 + \ldots\ldots + c_nx_n$

subject to the constraints

$$a_{11}x_1 + a_{12}x_2 + \ldots\ldots\ldots\ldots\ldots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + \ldots\ldots\ldots\ldots\ldots + a_{2n}x_n = b_2$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$a_{m1}x_1 + a_{m2}x_2 + \ldots\ldots\ldots\ldots\ldots + a_{mn}x_n = b_m$$

and $x_1, x_2, \ldots\ldots\ldots\ldots\ldots, x_n \geq 0$ is known as ***standard form.***

In matrix notation the standard form of LPP can be expressed as :

Maximize $Z = CX$ (objective function)

subject to constraints

$$AX = b \text{ and } X \geq 0$$

Characteristics of the standard form :

(i) The objective function is of maximization type.

(ii) All constraints are expressed as equations.

(iii) Right hand side of each constraint is non-negative.

(iv) All variables are non-negative.

Note :

(1) : The minimization of a function $f(x)$ is equivalent to the maximization of the negative expression of this function.

i.e., $\text{Min } f(x) = -\text{Max } \{-f(x)\}$

i.e., $\text{Min } Z = -\text{Max } (-Z)$

e.g : $\text{Min } Z = c_1x_1 + c_2x_2$ is equivalent to

$\text{Max } (-Z) = -c_1x_1 - c_2x_2$

(2) : An inequality in one direction can be converted into an inequality in the opposite direction by multiplying both sides by (–1).

e.g :
$$ax_1 + bx_2 \geq c$$
$$\Rightarrow -ax_1 - bx_2 \leq -c$$

(3) : An equality constraint can be expressed as two inequalities.

$$e.g: ax_1 + bx_2 = c \Rightarrow \left.\begin{array}{l} ax_1 + bx_2 \leq c \\ ax_1 + bx_2 \geq c \end{array}\right\} \Rightarrow \begin{array}{l} ax_1 + bx_2 \leq c \\ -ax_1 - bx_2 \leq -c \end{array}$$

(4) : An inequality constraint with its left hand side in the absolute form can be expressed as two inequalities.

$$e.g.: |ax_1 + bx_2| \leq c \Rightarrow \begin{array}{l} ax_1 + bx_2 \leq c \\ ax_1 + bx_2 \geq -c \end{array}$$

(5) : If a variable is unconstrained or unrestricted (without specifying its sign), it can always be expressed as the difference of two non-negative variables.

e.g. : If x_2 is unrestricted, then

$$x_2 = x_2' - x_2'' \text{ where } x_2', x_2'' \geq 0.$$

(6) : Whenever slack / surplus variables are introduced in the constraints, they should also appear in the objective function with zero coefficients.

Example 1 : Express the following LPP in the canonical form.

$$\text{Maximize } Z = 2x_1 + 3x_2 + x_3$$

subject to the constraints
$$4x_1 - 3x_2 + x_3 \leq 6$$
$$x_1 + 5x_2 - 7x_3 \geq -4$$
$$\text{and } x_1, x_3 \geq 0, \; x_2 \text{ is unrestricted}$$

Solution : As x_2 is unrestricted, $x_2 = x_2' - x_2''$ where $x_2', x_2'' \geq 0$. $\therefore$ The given LPP becomes

$$\text{Maximize } Z = 2x_1 + 3(x_2' - x_2'') + x_3$$

subject to
$$4x_1 - 3x_2' + 3x_2'' + x_3 \leq 6$$
$$x_1 + 5x_2' - 5x_2'' - 7x_3 \geq -4$$
$$x_1, x_2', x_2'', x_3 \geq 0.$$

Convert the second constraint $\leq$ type by multiplying both sides by –1. Now the LPP becomes

$$\text{Maximize } Z = 2x_1 + 3x_2' - 3x_2'' + x_3$$

subject to
$$4x_1 - 3x_2' + 3x_2'' + x_3 \leq 6$$
$$-x_1 - 5x_2' + 5x_2'' + 7x_3 \leq 4$$
$$\text{and } x_1, x_2', x_2'', x_3 \geq 0.$$

which is in the canonical form.

Example 2 : Express the following LPP in standard form

$$\text{Minmize } Z = 5x_1 + 7x_2$$

subject to the constraints

$$x_1 + x_2 \le 8$$
$$3x_1 + 4x_2 \ge 3$$
$$6x_1 + 7x_2 \ge 5 \text{ and } x_1, x_2 \ge 0.$$

Solution : Since Min Z = – Max (–Z) = – Max Z*,

The given LPP becomes $\text{Maximize } Z^* = -5x_1 - 7x_2$

subject to

$$x_1 + x_2 \le 8$$
$$3x_1 + 4x_2 \ge 3$$
$$6x_1 + 7x_2 \ge 5 \text{ and } x_1, x_2 \ge 0.$$

By introducing slack variable s_1 and surplus variables s_2, s_3 the standard form of the LPP is given by

$$\text{Maximize } Z^* = -5x_1 - 7x_2 + 0s_1 + 0s_2 + 0s_3$$

subject to

$$x_1 + x_2 + s_1 = 8$$
$$3x_1 + 4x_2 - s_2 = 3$$
$$6x_1 + 7x_2 - s_3 = 5$$
$$\text{and } x_1, x_2, s_1, s_2, s_3 \ge 0.$$

Example 3 : Express the following LPP in standard (Matrix) form

$$\text{Maximize } Z = 4x_1 + 2x_2 + 6x_3$$

subject to

$$2x_1 + 3x_2 + 2x_3 \ge 6$$
$$3x_1 + 4x_2 = 8$$
$$6x_1 - 4x_2 + x_3 \le 10 \text{ and } x_1, x_2, x_3 \ge 0.$$

Solution : By introducing the surplus variable s_1 and slack variable s_2, the standard form of the LPP becomes

$$\text{Maximize } Z = 4x_1 + 2x_2 + 6x_3 + 0s_1 + 0s_2$$

subject to

$$2x_1 + 3x_2 + 2x_3 - s_1 + 0s_2 = 6$$
$$3x_1 + 4x_2 + 0x_3 + 0s_1 + 0s_2 = 8$$
$$6x_1 - 4x_2 + x_3 + 0s_1 + s_2 = 10$$
$$\text{and } x_1, x_2, x_3, s_1, s_2 \ge 0.$$

Thus the given problem in matrix form is

$$\text{Maximize } Z = CX$$
$$\text{subject to } AX = b$$
$$X \ge 0$$
$$\text{where } C = (4, 2, 6, 0, 0)$$

$$A = \begin{pmatrix} 2 & 3 & 2 & -1 & 0 \\ 3 & 4 & 0 & 0 & 0 \\ 6 & -4 & 1 & 0 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} 6 \\ 8 \\ 10 \end{pmatrix}, \quad X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ s_1 \\ s_2 \end{pmatrix}$$

3.1.3 The Simplex Method

While solving a LPP graphically, the region of feasible solutions was found to be convex. The optimal solution if it exists, occured at some vertex. If the optimal solution was not unique, the optimal points were on an edge. These observations also hold for the general LPP. Essentially the problem is that of finding the particular vertex of the convex region which corresponds to the optimal solution. The most commonly used method for locating the optimal vertex is the ***simplex method or simplex technique or simplex algorithm*** which was developed by G. Dantzig in 1947.

The simplex method provides an algorithm which consists in moving from one vertex of the region of feasible solutions to another vertex in such a way that the value of the objective function at the succeeding vertex is more (or less, as the case may be) than at the preceding vertex. This procedure of jumping from one vertex to another is then repeated. Since the number of vertices is finite, the method leads to an optimal vertex in a finite number of steps or indicates the existence of an unbounded solution.

Definition (1): Given a system of m linear equations with n variables $(m < n)$. The solution obtained by setting $(n - m)$ variables equal to zero and solving for the remaining m variables is called a ***basic solution.***

The m variables are called ***basic variables*** and they form the basic solution. The $(n - m)$ variables which are put to zero are called as ***non-basic variables.***

Definition (2) : A basic solution is said to be a ***non–degenerate basic solution*** if none of the basic variables is zero.

Definition (3) : A basic solution is said to be a ***degenerate basic solution*** if one or more of the basic variables are zero.

Definition (4) : A feasible solution which is also basic is called a ***basic feasible solution.***

Example 1 : **Find all the basic solutions to the following problem :**

$$\text{Maximize } Z = x_1 + 3x_2 + 3x_3$$

$$\text{subject to } x_1 + 2x_2 + 3x_3 = 4$$

$$2x_1 + 3x_2 + 5x_3 = 7$$

Also find which of the basic solutions are

(i) **basic feasible**

(ii) **non–degenerate basic feasible**

(iii) **optimal basic feasible.**

Solution : Since there are $m = 2$ equations with $n = 3$ variables, the basic solutions are obtained by setting $(n - m) = (3 - 2) = 1$ variable equal to zero and solving for the remaining two variables. Since there are 3 variables with 2 equations. We shall have $3C_2 = 3$ different basic solutions, which are given in the following table.

S. No	*Basic variables*	*Non-basic variables*	*Values of the basic variables given by the constraint equations*	*Value of the objective function*	*Is the solution feasible ?*	*Is the solution non–degenerate ?*	*Is the solution feasible and optimal ?*
1	x_1, x_2	$x_3 = 0$	$x_1 + 2x_2 = 4$ $2x_1 + 3x_2 = 7$ $\Rightarrow x_1=2, x_2 = 1$	5	yes	yes	yes
2	x_1, x_3	$x_2 = 0$	$x_1 + 3x_3 = 4$ $2x_1 + 5x_3 = 7$ $\Rightarrow x_1=1, x_3 =1$	4	yes	yes	No
3	x_2, x_3	$x_1 = 0$	$2x_2 + 3x_3 = 4$ $3x_2 + 5x_3 = 7$ $\Rightarrow x_2=-1, x_3 =2$	3	No	yes	No

From the table, we see that, the first two solutions are non–degenerate basic feasible solutions and the third is non–degenerate and infeasible. The first solution is the optimal one.

∴ The optimal solution is

$$\text{Maximize } Z = 5, \; x_1 = 2, x_2 = 1, x_3 = 0.$$

Example 2 : Obtain all the basic solutions to the following system of linear equations :

$$2x_1 + 6x_2 + 2x_3 + x_4 = 3$$

$$6x_1 + 4x_2 + 4x_3 + 6x_4 = 2$$

Which of them are basic feasible solutions and which are non-degenerate basic solutions ? Is the non-degenerate solution feasible ?

Solution : Since there are 4 variables with 2 equations, we shall have $4C_2 = 6$ different basic solutions, which are given in the following table.

S.No	*Basic variables*	*Non-basic variables*	*Values of the basic variables given by the constraint equations*	*Is the solution feasible ?*	*Is the solution non-degenerate ?*	*Is the solution feasible and non-degenerate?*
1	x_1, x_2	$x_3 = x_4 = 0$	$\left.\begin{array}{l}2x_1 + 6x_2 = 3\\6x_1 + 4x_2 = 2\end{array}\right\}$ $\Rightarrow x_1=0,\ x_2 = 1/2$	yes	No	No
2	x_1, x_3	$x_2 = x_4 = 0$	$\left.\begin{array}{l}2x_1 + 2x_3 = 3\\6x_1 + 4x_3 = 2\end{array}\right\}$ $\Rightarrow x_1=-2, x_3 = 7/2$	No	yes	No
3	x_1, x_4	$x_2 = x_3 = 0$	$\left.\begin{array}{l}2x_1 + x_4 = 3\\6x_1 + 6x_4 = 2\end{array}\right\}$ $\Rightarrow x_1 = \frac{8}{3}, x_4 = \frac{-7}{3}$	No	yes	No
4	x_2, x_3	$x_1 = x_4 = 0$	$\left.\begin{array}{l}6x_2 + 2x_3 = 3\\4x_2 + 4x_3 = 2\end{array}\right\}$ $\Rightarrow x_2 = \frac{1}{2}, x_3 = 0$	yes	No	No
5	x_2, x_4	$x_1 = x_3 = 0$	$\left.\begin{array}{l}6x_2 + x_4 = 3\\4x_2 + 6x_4 = 2\end{array}\right\}$ $\Rightarrow x_2 = \frac{1}{2}, x_4 = 0$	yes	No	No
6	x_3, x_4	$x_1 = x_2 = 0$	$\left.\begin{array}{l}2x_3 + x_4 = 3\\4x_3 + 6x_4 = 2\end{array}\right\}$ $\Rightarrow x_3 = 2, x_4 = -1$	No	yes	No

Definition (5) : Let X_B be a basic feasible solution to the LPP :

$$\text{Maximize } Z = CX$$
$$\text{subject to } AX = b$$
$$\text{and } X \geq 0.$$

Then the vector $C_B = (C_{B1}, C_{B2}, \ldots C_{Bm})$, where C_{Bi} are components of C associated with the basic variables, is called the ***cost vector associated*** with the basic feasible solution X_B.

Remarks:

1 : If a LPP has a feasible solution, then it also has a basic feasible solution.

2 : There exists only finite number of basic feasible solutions to a LPP.

3 : Let a LPP have a feasible solution. If we drop one of the basic variables and introduce another variable in the basis set, then the new solution obtained is also a basic feasible solution.

Definition (6): Let $X_B = B^{-1} b$ be a basic feasible solution to the LPP: Maximize $Z = CX$, where $AX = b$ and $X \geq 0$,

Let C_B be the cost vector corresponding to X_B. For each column vector a_j in A, which is not a column vector of B, let

$$a_j = \sum_{i=1}^{m} a_{ij} b_i$$

Then the number $$Z_j = \sum_{i=1}^{m} C_{B_i} a_{ij}$$

is called the ***evaluation*** corresponding to a_j and the number $(Z_j - C_j)$ is called the ***net evaluation*** corresponding to a_j.

Remark 1 : If $(Z_j - C_j) = 0$ for atleast one j for which $a_{ij} > 0$, $i = 1,2, \ldots, m$; then another basic feasible solution is obtained which gives an unchanged value of the objective function.

Remark 2 : (Unbounded solution). Let there exist a basic feasible solution to a given LPP. If for atleast one j, for which $a_{ij} \leq 0$ $(i = 1,2,\ldots, m)$ and $(Z_j - C_j)$ is negative, then there does not exist any optimum solution to this LPP.

Remark 3 : A necessary and sufficient condition for a basic feasible solution to a LPP to be an optimum (maximum) is that $(Z_j - C_j) \geq 0$ for all j, for which $a_j \notin B$.

Remark 4 : The two fundamental conditions on which the simplex method is based are : ***(i) Feasibility condition :*** It ensures that if the initial (Starting) solution is basic feasible then during computation only basic feasible solutions will be obtained. ***(ii) Optimality condition:*** It guarantees that only improved solutions will be obtained.

3.1.4 The Simplex Algorithm

Assuming the existence of an initial basic feasible solution, an optimal solution to any LPP by simplex method is found as follows :

Step 1 : Check whether the objective function is to be maximized or minimized. If it is to be minimized, then convert it into a problem of maximization, by

$$\text{Minimize } Z = -\text{Maximize } (-Z)$$

Step 2 : Check whether all b_i's are positive. If any of the b_i's is negative, multiply both sides of that constraint by −1 so as to make its right hand side positive.

Step 3 : By introducing slack / surplus variables, convert the inequality constraints into equations and express the given LPP into its standard form.

Step 4 : Find an initial basic feasible solution and express the above information conveniently in the following simplex table.

C_j (C_1 C_2 C_3 ……… 0 0 0 ……)

C_B	Y_B	X_B	x_1	x_2	x_3 ………	s_1	s_2	s_3	………
C_{B_1}	s_1	b_1	a_{11}	a_{12}	a_{13} ………	1	0	0	………
C_{B_2}	s_2	b_2	a_{21}	a_{22}	a_{23} ………	0	1	0	………
C_{B_3}	s_3	b_3	a_{31}	a_{32}	a_{33} ………	0	0	1	………
⋮	⋮	⋮	⋮	⋮	⋮ ………				………
			body matrix				unit matrix		
$(Z_j - C_j)$		Z_0	$Z_1 - C_1$	$Z_2 - C_2$	… … …		………………		

(Where C_j - row denotes the coefficients of the variables in the objective function. C_B – column denotes the coefficients of the basic variables in the objective function. Y_B – column denotes the basic variables. X_B – column denotes the values of the basic variables. The coefficients of the non-basic variables in the constraint equations constitute the body matrix while the coefficients of the basic variables constitute the unit matrix. The row $(Z_j - C_j)$ denotes the net evaluations (or) index for each column).

Step 5 : Compute the net evaluations $(Z_j - C_j)$ (j =1, 2, … n) by using the relation $Z_j - C_j = C_B\, a_j - C_j$.

Examine the sign of $Z_j - C_j$

(a) If all $(Z_j - C_j) \geq 0$ then the current basic feasible solution X_B is optimal.

(b) If atleast one $(Z_j - C_j) < 0$, then the current basic feasible solution is not optimal, go to the next step.

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Step 6 : (To find the entering variable)

The entering variable is the non-basic variable corresponding to the most negative value of $(Z_j - C_j)$. Let it be x_r for some $j = r$. The entering variable column is known as the ***key column*** (or) ***pivot column*** which is shown marked with an arrow at the bottom. If more than one variable has the same most negative $(Z_j - C_j)$, any of these variables may be selected arbitrarily as the entering variable.

Step 7 : (To find the leaving variable)

Compute the ratio $\theta = \text{Min}\left\{\frac{X_{B_i}}{a_{ir}},\ a_{ir} > 0\right\}$ (*i.e.*, the ratio between the solution column and the entering variable column by considering only the positive denominators)

(a) If all $a_{ir} \leq 0$, then there is an unbounded solution to the given LPP.

(b) If atleast one $a_{ir} > 0$, then the leaving variable is the basic variable corresponding to the minimum ratio θ. If $\theta = \frac{X_{B_k}}{a_{kr}}$, then the basic variable x_k leaves the basis. The leaving variable row is called the ***key row*** or ***pivot row*** (or) ***pivot equation***, and the element at the intersection of the pivot column and pivot row is called the ***pivot element or key element (or) leading element.***

Step 8 : Drop the leaving variable and introduce the entering variable along with its associated value under C_B column. Convert the pivot element to unity by dividing the pivot equation by the pivot element and all other elements in its column to zero by making use of

(i) New pivot equation = old pivot equation ÷ pivot element

(ii) New equation (all other rows including $(Z_j - C_j)$ row)

$$= \text{Old equation} - \begin{pmatrix}\text{Corresponding}\\ \text{column}\\ \text{coefficient}\end{pmatrix} \times \begin{pmatrix}\text{New pivot}\\ \text{equation}\end{pmatrix}$$

Step 9 : Go to step (5) and repeat the procedure unitl either an optimum solution is obtained or there is an indication of an unbounded solution.

Note (1) : For maximization problems:

(i) If all $(Z_j - C_j) \geq 0$, then the current basic feasible solution is optimal.

(ii) If atleast one $(Z_j - C_j) < 0$, then the current basic feasible solution is not optimal.

(iii) The entering variable is the non-basic variable corresponding to the most negative value of $(Z_j - C_j)$.

Note (2) : For minimization problems :

(i) If all $(Z_j - C_j) \le 0$, then the current basic feasible solution is optimal.

(ii) If atleast one $(Z_j - C_j) > 0$, then the current basic feasible solution is not optimal.

(iii) The entering variable is the non-basic variable corresponding to the most positive value of $(Z_j - C_j)$.

Note (3) : For both maximization and minimization problems, the leaving variable is the basic variable corresponding to the minimum ratio θ.

Example 1 : Use simplex method to solve the LPP

$$\text{Maximize } Z = 4x_1 + 10x_2$$

subject to

$$2x_1 + x_2 \le 50$$

$$2x_1 + 5x_2 \le 100$$

$$2x_1 + 3x_2 \le 90 \text{ and } x_1, x_2 \ge 0.$$

[BRU BSc.86, MKU.B.Sc 94, MU BSc. 84, MU. BE. Nov 91, Nov 94]

Solution : By introducing the slack variables s_1, s_2 and s_3, the problem in standard form becomes

$$\text{Maximize } Z = 4x_1 + 10x_2 + 0s_1 + 0s_2 + 0s_3$$

subject to

$$2x_1 + x_2 + s_1 + 0s_2 + 0s_3 = 50$$

$$2x_1 + 5x_2 + 0s_1 + s_2 + 0s_3 = 100$$

$$2x_1 + 3x_2 + 0s_1 + 0s_2 + s_3 = 90$$

$$\text{and } x_1, x_2, s_1, s_2, s_3 \ge 0.$$

Since there are 3 equations with 5 variables, the initial basic feasible solution is obtained by equating $(5 - 3) = 2$ variables to zero.

$\therefore$ The initial basic feasible solution is $s_1 = 50$, $s_2 = 100$, $s_3 = 90$

$(x_1 = 0,\ x_2 = 0,$ non-basic$)$

The initial simplex table is given by

C_j (4 10 0 0 0)

C_B	Y_B	X_B	x_1	x_2	s_1	s_2	s_3	$\theta = \min \frac{X_{Bi}}{a_{ir}}$
0	s_1	50	2	1	1	0	0	50
0	s_2	100	2	(5)	0	1	0	20*
0	s_3	90	2	3	0	0	1	30
$Z_j - C_j$		0	–4	–10	0	0	0	

Here the net evaluations are calculated as $Z_j - C_j = C_B\, a_j - C_j$.

$Z_1 - C_1 = C_B\, a_1 - C_1 = (0\ 0\ 0)\ [2\ 2\ 2]^T - 4$

[where T denotes transpose]

$$= (0\ 0\ 0)\begin{pmatrix}2\\2\\2\end{pmatrix} - 4 = -4$$

$Z_2 - C_2 = C_B\, a_2 - C_2 = (0\ 0\ 0)\ [1\ 5\ 3]^T - 10 = -10$

$Z_3 - C_3 = C_B\, a_3 - C_3 = (0\ 0\ 0)\ [1\ 0\ 0]^T - 10 = 0$

$Z_4 - C_4 = C_B\, a_4 - C_4 = (0\ 0\ 0)\ [0\ 1\ 0]^T - 0 = 0$

$Z_5 - C_5 = C_B\, a_5 - C_5 = (0\ 0\ 0)\ [0\ 0\ 1]^T - 0 = 0$

Since there are some $(Z_j - C_j) < 0$, the current basic feasible solution is not optimal.

To find the entering variable :

Since $(Z_2 - C_2) = -10$ is the most negative, the corresponding non-basic variable x_2 enters the basis. The column corresponding to this x_2 is called the key column or pivot column.

To find the leaving variable :

Find the ratio $\theta = \min\left\{\frac{X_{Bi}}{a_{ir}},\ a_{ir} > 0\right\}$

$= \min\left\{\frac{X_{Bi}}{a_{i2}},\ a_{i2} > 0\right\}$

$\theta = \min\left\{\frac{50}{1}, \frac{100}{5}, \frac{90}{3}\right\}$

$= \min\{50, 20, 30\} = 20$, which corresponds to s_2

∴ The leaving variable is the basic variable s_2 which corresponds to the minimum ratio $\theta = 20$. The leaving variable row is called the pivot – row or key row or pivot equation and 5 is the pivot element. Now,

New pivot equation = old pivot equation ÷ pivot element.

$$= (100 \quad 2 \quad 5 \quad 0 \quad 1 \quad 0) \div 5$$

$$= 20 \quad \frac{2}{5} \quad 1 \quad 0 \quad \frac{1}{5} \quad 0$$

$$\text{New } s_1 \text{ equation} = \text{old } s_1 \text{ equation} - \begin{pmatrix}\text{corresponding} \\ \text{column} \\ \text{coefficient}\end{pmatrix} \times \begin{pmatrix}\text{New} \\ \text{pivot} \\ \text{equation}\end{pmatrix}$$

	=	50	2	1	1	0	0
(–)		20	$\frac{2}{5}$	1	0	$\frac{1}{5}$	0
	=	30	$\frac{8}{5}$	0	1	$-\frac{1}{5}$	0
New s_3 equation	=	90	2	3	0	0	1
(–)		60	$\frac{6}{5}$	3	0	$\frac{3}{5}$	0
	=	30	$\frac{4}{5}$	0	0	$\frac{-3}{5}$	1
New $(Z_j - C_j)$ eqn.	=	0	–4	–10	0	0	0
(–)		–200	$\frac{-20}{5}$	–10	0	$\frac{-10}{5}$	0
	=	200	0	0	0	2	0

∴ The improved basic feasible solution is given in the following simplex table.

First Iteration :

		C_j	4	10	0	0	0
C_B	Y_B	X_B	x_1	x_2	s_1	s_2	s_3
0	s_1	30	$8/5$	0	1	$-1/5$	0
10	x_2	20	$2/5$	1	0	$1/5$	0
0	s_3	30	$4/5$	0	0	$-3/5$	1
$Z_j - C_j$		200	0	0	0	2	0

Since all $Z_j - C_j \geq 0$ the current basic feasible solution is optimal.

∴ The optimal solution is Max $Z = 200, x_1 = 0, x_2 = 20$.

Example 2 : **Find the non-negative values of x_1, x_2 and x_3 which**

Maximize $Z = 3x_1 + 2x_2 + 5x_3$

subject to

$$x_1 + 4x_2 \leq 420$$

$$3x_1 + 2x_3 \leq 460$$

$$x_1 + 2x_2 + x_3 \leq 430$$

[MU. MBA. Nov 95]

Solution : Given the LPP

$$\text{Maximize } Z = 3x_1 + 2x_2 + 5x_3$$

subject to

$$x_1 + 4x_2 \leq 420$$

$$3x_1 + 2x_3 \leq 460$$

$$x_1 + 2x_2 + x_3 \leq 430$$

$$x_1, x_2, x_3 \geq 0.$$

By introducing non-negative slack variables s_1, s_2 and s_3, the standard form of the LPP becomes

$$\text{Maximize } Z = 3x_1 + 2x_2 + 5x_3 + 0s_1 + 0s_2 + 0s_3$$

subject to

$$x_1 + 4x_2 + 0x_3 + s_1 + 0s_2 + 0s_3 = 420$$

$$3x_1 + 0x_2 + 2x_3 + 0s_1 + s_2 + 0s_3 = 460$$

$$x_1 + 2x_2 + x_3 + 0s_1 + 0s_2 + s_3 = 430$$

$$\text{and } x_1, x_2, x_3, s_1, s_2, s_3 \geq 0.$$

Since there are 3 equations with 6 variables, the initial basic feasible solution is obtained by equating $(6 - 3) = 3$ variables to zero.

∴ The initial basic feasible solution is $s_1 = 420$, $s_2 = 460$, $s_3 = 430$ ($x_1 = x_2 = x_3 = 0$, non-basic)

The initial simplex table is given by

Initial iteration :

		C_j	(3	2	5	0	0	0)	
C_B	Y_B	X_B	x_1	x_2	x_3	s_1	s_2	s_3	θ
0	s_1	420	1	4	0	1	0	0	–
0	s_2	460	3	0	(2)	0	1	0	$\frac{460}{2} = 230^*$
0	s_3	430	1	2	1	0	0	1	$\frac{430}{1} = 430$
$Z_j - C_j$		0	–3	–2	–5	0	0	0	

Since there are some $(Z_j - C_j) < 0$, The current basic feasible solution is not optimal.

To find the entering variable : Since $(Z_3 - C_3) = -5$ is the most negative, the corresponding non-basic variable x_3 enters into the basis. The column corresponding to this x_3 is called the key column or pivot column.

To find the leaving variable :

Find the ratio $\theta = \min\left\{\frac{X_{Bi}}{a_{ir}}, a_{ir} > 0\right\} = \min\left\{\frac{X_{Bi}}{a_{i3}}, a_{i3} > 0\right\}$

$$\theta = \min\left\{\frac{460}{2}, \frac{430}{1}\right\} = \min\{230, 430\} = 230$$

∴ The leaving variable is the basic variable s_2 which corresponds to the minimum ratio $\theta = 230$. The leaving variable row is called the key row or pivot equation and 2 is the pivot element.

New pivot equation = old pivot equation ÷ pivot element

$= (460 \quad 3 \quad 0 \quad 2 \quad 0 \quad 1 \quad 0) \div 2$

$= 230 \quad \frac{3}{2} \quad 0 \quad 1 \quad 0 \quad \frac{1}{2} \quad 0$

New s_1 equation = old s_1 equation − (its entering column coefficient) × (New pivot equation)

=	420	1	4	0	1	0	0
(−)	0	0	0	0	0	0	0
=	420	1	4	0	1	0	0

New s_3 equation = old s_3 equation − (its entering column coefficient) × (New pivot equation)

=	430	1	2	1	0	0	1
(−)	230	$\frac{3}{2}$	0	1	0	$\frac{1}{2}$	0
=	200	$-\frac{1}{2}$	2	0	0	$-\frac{1}{2}$	1

New $(Z_j - C_j)$ eqn. = old $(Z_j - C_j)$ equation − (its entering column coefficient) × (New pivot equation)

=	0	−3	−2	−5	0	0	0
(−)	−1150	$\frac{-15}{2}$	0	−5	0	$\frac{-5}{2}$	0
=	1150	$\frac{9}{2}$	−2	0	0	$\frac{5}{2}$	0

The improved basic feasible solution is given in the following simplex table.

First iteration :

C_B	Y_B	C_j → X_B	x_1 (3)	x_2 (2)	x_3 (5)	s_1 (0)	s_2 (0)	s_3 (0)	θ
0	s_1	420	1	4	0	1	0	0	$\frac{420}{4} = 105$
5	x_3	230	$^3/_2$	0	1	0	$^1/_2$	0	–
0	s_3	200	$-^1/_2$	(2)	0	0	$-^1/_2$	1	$\frac{200}{2} = 100$*
$Z_j - C_j$		1150	$^9/_2$	–2	0	0	$^5/_2$	0	

Since there is an $(Z_2 - C_2) = -2$, the current basic feasible solution is not optimal.

∴ Here the non-basic variable x_2 enters into the basis and the basic variable s_3 leaves the basis.

Second iteration :

C_B	Y_B	C_j → X_B	x_1 (3)	x_2 (2)	x_3 (5)	s_1 (0)	s_2 (0)	s_3 (0)
0	s_1	20	2	0	0	1	1	–2
5	x_3	230	$^3/_2$	0	1	0	$^1/_2$	0
2	x_2	100	$-^1/_4$	1	0	0	$-^1/_4$	$^1/_2$
$Z_j - C_j$		1350	4	0	0	0	2	1

Since all $(Z_j - C_j) \geq 0$, the current basic feasible solution is optimal.

∴ The optimal solution is Max $Z = 1350, x_1 = 0, x_2 = 100, x_3 = 230$.

Example 3 : Solve the following :

Maximize : $15x_1 + 6x_2 + 9x_3 + 2x_4$

subject to

$$2x_1 + x_2 + 5x_3 + 6x_4 \leq 20$$

$$3x_1 + x_2 + 3x_3 + 25x_4 \leq 24$$

$$7x_1 + x_4 \leq 70$$

$$x_1, x_2, x_3, x_4 \geq 0.$$

[MU. MCA. May 95]

Solution : By introducing non-negative slack variables s_1, s_2 and s_3, the standard form of the LPP becomes.

Maximize $Z = 15x_1 + 6x_2 + 9x_3 + 2x_4 + 0s_1 + 0s_2 + 0s_3$

subject to
$$2x_1 + x_2 + 5x_3 + 6x_4 + s_1 + 0s_2 + 0s_3 = 20$$
$$3x_1 + x_2 + 3x_3 + 25x_4 + 0s_1 + s_2 + 0s_3 = 24$$
$$7x_1 + 0x_2 + 0x_3 + x_4 + 0s_1 + 0s_2 + s_3 = 70$$
$$x_1, x_2, x_3, x_4, s_1, s_2, s_3 \geq 0.$$

$\therefore$ The initial basic feasible solution is $s_1 = 20$, $s_2 = 24$, $s_3 = 70$ $(x_1 = x_2 = x_3 = x_4 = 0$, non-basic)

The initial simplex table is given by

Initial iteration :

C_B	Y_B	X_B	x_1	x_2	x_3	x_4	s_1	s_2	s_3	θ
	C_j		(15	6	9	2	0	0	0)	
0	s_1	20	2	1	5	6	1	0	0	$\frac{20}{2} = 10$
0	s_2	24	(3)	1	3	25	0	1	0	$\frac{24}{3} = 8^*$
0	s_3	70	7	0	0	1	0	0	1	$\frac{70}{7} = 10$
$Z_j - C_j$		0	–15	–6	–9	–2	0	0	0	

Since there are some $(Z_j - C_j) < 0$, the current basic feasible solution is not optimal.

The non-basic variable x_1 enters into the basis and the basic variable s_2 leaves the basis.

First iteration :

C_B	Y_B	X_B	x_1	x_2	x_3	x_4	s_1	s_2	s_3	θ
	C_j		(15	6	9	2	0	0	0)	
0	s_1	4	0	$(^1/_3)$	3	$-\frac{32}{3}$	1	$-^2/_3$	0	12*
15	x_1	8	1	$^1/_3$	1	$^{25}/_3$	0	$^1/_3$	0	24
0	s_3	14	0	$-^7/_3$	–7	$\frac{-172}{3}$	0	$-^7/_3$	1	–
$Z_j - C_j$		120	0	–1	6	123	0	5	0	

Since $(Z_2 - C_2) = -1 < 0$, the current basic feasible solution is not optimal.

The non-basic variable x_2 enters into the basis and the basic variable s_1 leaves the basis.

Second iteration :

C_j (15 6 9 2 0 0 0)

C_B	Y_B	X_B	x_1	x_2	x_3	x_4	s_1	s_2	s_3
6	x_2	12	0	1	9	−32	3	−2	0
15	x_1	4	1	0	−2	$\frac{57}{3}$	−1	1	0
0	s_3	42	0	0	14	−132	7	−7	1
$Z_j - C_j$		132	0	0	15	91	3	3	0

Since all $(Z_j - C_j) \geq 0$, the current basic feasible solution is optimal.

$\therefore$ The optimal solution is given by

Max $Z = 132$, $x_1 = 4$, $x_2 = 12$, $x_3 = 0$ and $x_4 = 0$.

Example 4 : **Solve the following LPP by simplex method :**

Minimize $Z = 8x_1 - 2x_2$.

$$\begin{aligned} \textbf{subject to} \quad -4x_1 + 2x_2 &\leq 1 \\ 5x_1 - 4x_2 &\leq 3 \\ \textbf{and } x_1, x_2 &\geq 0. \end{aligned}$$

[MKU. BE. 1989]

Solution : Since the given objective function is of minimization type, we shall convert it in to a maximization type as follows :

$$\text{Maximize } (-Z) = \text{Maximize } Z^* = -8x_1 + 2x_2$$

subject to

$$\begin{aligned} -4x_1 + 2x_2 &\leq 1 \\ 5x_1 - 4x_2 &\leq 3 \\ x_1, x_2 &\geq 0. \end{aligned}$$

By introducing non-negative slack variables s_1, s_2, the standard form of the LPP becomes

$$\text{Maximize } Z^* = -8x_1 + 2x_2 + 0s_1 + 0s_2$$

subject to the constraints

$$\begin{aligned} -4x_1 + 2x_2 + s_1 + 0s_2 &= 1 \\ 5x_1 - 4x_2 + 0s_1 + s_2 &= 3 \\ \text{and } x_1, x_2, s_1, s_2 &\geq 0. \end{aligned}$$

$\therefore$ The initial basic feasible solution is given by $s_1 = 1$, $s_2 = 3$, ($x_1 = x_2 = 0$, non-basic).

Initial iteration :

C_j (−8 2 0 0)

C_B	Y_B	X_B	x_1	x_2	s_1	s_2	θ
0	s_1	1	−4	(2)	1	0	$\frac{1}{2}$*
0	s_2	3	5	−4	0	1	–
$Z_j^* - C_j$		0	8	−2	0	0	

Since $(Z_2^* - C_2) = -2 < 0$, the current basic feasible solution is not optimal.

The non-basic variable x_2 enters the basis and the basic variable s_1 leaves the basis.

First iteration :

C_j (−8 2 0 0)

C_B	Y_B	X_B	x_1	x_2	s_1	s_2
2	x_2	$\frac{1}{2}$	−2	1	$\frac{1}{2}$	0
0	s_2	5	−3	0	2	1
$Z_j^* - C_j$		1	4	0	1	0

Since all $(Z_j^* - C_j) \geq 0$, the current basic feasible solution is optimal.

∴ The optimal solution is given by maximize $Z^* = 1, x_1 = 0, x_2 = {}^1/_2,$

But Minimize $Z = -$ Maximize $(-Z) = -$ Maximize $Z^* = -1$

$$\therefore \text{ Min } Z = -1;\ x_1 = 0,\ x_2 = \frac{1}{2}.$$

Aliter : The above problem can be solved without converting the objective function in to maximization type.

$$\text{Given Minimize } Z = 8x_1 - 2x_2$$

subject to the constraints

$$-4x_1 + 2x_2 \leq 1$$
$$5x_1 - 4x_2 \leq 3$$
$$x_1, x_2, \geq 0.$$

∴ By introducing the non-negative slack variable s_1, s_2 the LPP becomes

$$\text{Minimize } Z = 8x_1 - 2x_2 + 0s_1 + 0s_2$$

subject to the constraints

$$-4x_1 + 2x_2 + s_1 + 0s_2 = 1$$
$$5x_1 - 4x_2 + 0s_1 + s_2 = 3$$
$$x_1, x_2, s_1, s_2 \geq 0.$$

The initial basic feasible solution is given by
$s_1 = 1, s_2 = 3$ (basic) $(x_1 = x_2 = 0$, non-basic)

Initial iteration :

		C_j	(8	–2	0	0)	
C_B	Y_B	X_B	x_1	x_2	s_1	s_2	θ
0	s_1	1	–4	(2)	1	0	$\frac{1}{2}$*
0	s_2	3	5	–4	0	1	–
$Z_j - C_j$		0	–8	2	0	0	

Since $(Z_2 - C_2) = 2 > 0$, the current basic feasible solution is not optimal.

To find the entering variable :

Since $(Z_2 - C_2) = 2$ is most positive, the corresponding non-basic variable x_2 enters into the basis.

To find the leaving variable :

The leaving variable is the basic variable s_1 corresponding to the minimum ratio $\theta = \frac{1}{2}$.

First iteration :

		C_j	(8	–2	0	0)
C_B	Y_B	X_B	x_1	x_2	s_1	s_2
–2	x_2	$\frac{1}{2}$	–2	1	$\frac{1}{2}$	0
0	s_2	5	–3	0	2	1
$Z_j - C_j$		–1	–4	0	–1	0

Since all $(Z_j - C_j) \le 0$, the current basic feasible solution is optimal.

∴ The optimal solution is Min $Z = -1, x_1 = 0, x_2 = \frac{1}{2}$

Example 5 : Use simplex method to

$$\text{Min } Z = x_2 - 3x_3 + 2x_5$$

subject to

$$3x_2 - x_3 + 2x_5 \le 7$$
$$-2x_2 + 4x_3 \le 12$$
$$-4x_2 + 3x_3 + 8x_5 \le 10$$
$$\text{and } x_2, x_3, x_5 \ge 0$$

[BRU. MSc. 1986 MKU. BSc 1992, MU. BE. Nov 93, Nov 95, MU. B.Tech. Leather Oct 96]

Solution : Since the given objective function is of minimization type, we shall convert it in to a maximization type as follows :

$$\text{Maximize } (-Z) = \text{Maximize } Z^* = -x_2 + 3x_3 - 2x_5$$

subject to $3x_2 - x_3 + 2x_5 \leq 7$

$-2x_2 + 4x_3 \leq 12$

$-4x_2 + 3x_3 + 8x_5 \leq 10$

$x_2, x_3, x_5 \geq 0.$

By introducing non-negative slack variables s_1, s_2 and s_3, the standard form of the LPP becomes

Maximize $Z^* = -x_2 + 3x_3 - 2x_5 + 0s_1 + 0s_2 + 0s_3$

subject to the constraints

$3x_2 - x_3 + 2x_5 + s_1 + 0s_2 + 0s_3 = 7$

$-2x_2 + 4x_3 + 0x_5 + 0s_1 + s_2 + 0s_3 = 12$

$-4x_2 + 3x_3 + 8x_5 + 0s_1 + 0s_2 + s_3 = 10$

and $x_2, x_3, x_5, s_1, s_2, s_3 \geq 0.$

∴ The initial basic feasible solution is given by $s_1 = 7$, $s_2 = 12$, $s_3 = 10$ $(x_2 = x_3 = x_5 = 0$, non-basic)

Initial iteration :

		C_j	(−1	3	−2	0	0	0)	
C_B	Y_B	X_B	x_2	x_3	x_5	s_1	s_2	s_3	θ
0	s_1	7	3	−1	2	1	0	0	−
0	s_2	12	−2	(4)	0	0	1	0	$\frac{12}{4} = 3^*$
0	s_3	10	−4	3	8	0	0	1	$\frac{10}{3} = 3.33$
$Z_j^* - C_j$		0	1	−3	2	0	0	0	

Since $(Z_2^* - C_2) < 0$, the current basic feasible solution is not optimal.

The non-basic variable x_3 enters into the basis and the basic variable s_2 leaves the basis.

First iteration :

		c_j	(−1	3	−2	0	0	0)	
C_B	Y_B	X_B	x_2	x_3	x_5	s_1	s_2	s_3	θ
0	s_1	10	$\left(\frac{5}{2}\right)$	0	2	1	$\frac{1}{4}$	0	$\frac{20}{5} = 4^*$
3	x_3	3	$\frac{-1}{2}$	1	0	0	$\frac{1}{4}$	0	−
0	s_3	1	$\frac{-5}{2}$	0	8	0	$\frac{-3}{4}$	1	−
$Z_j^* - C_j$		9	$\frac{-1}{2}$	0	2	0	$\frac{3}{4}$	0	

Since $(Z_1^* - C_1) < 0$, the current basic feasible solution is not optimal. The non-basic variable x_2 enters into the basis and the basic variable s_1 leaves the basis.

Second iteration :

C_j (−1 3 2 0 0 0)

C_B	Y_B	X_B	x_2	x_3	x_5	s_1	s_2	s_3
−1	x_2	4	1	0	$\frac{4}{5}$	$\frac{2}{5}$	$\frac{1}{10}$	0
3	x_3	5	0	1	$\frac{2}{5}$	$\frac{1}{5}$	$\frac{3}{10}$	0
0	s_3	11	0	0	10	1	$\frac{-1}{2}$	1
$(Z_j^* - C_j)$		11	0	0	$\frac{12}{5}$	$\frac{1}{5}$	$\frac{4}{5}$	0

Since all $(Z_j^* - C_j) \geq 0$, the current basic feasible solution is optimal.

∴ The optimal solution is given by

$$\text{maximize } Z^* = 11, \quad x_2 = 4, \quad x_3 = 5, \quad x_5 = 0$$

$$\text{But Minimize } Z = -\text{Maximize } Z^* = -11$$

$$\therefore \text{Min } Z = -11, \; x_2 = 4, \; x_3 = 5, \; x_5 = 0.$$

3.2 Artificial Variables Techniques

To solve a LPP by simplex method, we have to start with the initial basic feasible solution and construct the initial simplex table. In the previous problems, we see that the slack variables readily provided the initial basic feasible solution. However, in some problems, the slack variables can not provide the initial basic feasible solution. In these problems atleast one of the contraints is of $=$ or $\geq$ type. To solve such linear programming problems, there are two (closely related) methods available.

(i) The ***"Big M–method"*** or ***"M–technique"*** or the ***"Method of penalties"*** due to A. Charnes.

(ii) The ***"Two phase"*** method due to Dantzig, Orden and Wolfe.

3.2.1 The Big M – method :

Step (1) : Express the linear programming problem in the standard form by introducing slack and/or surplus variables, if necessary.

Step (2) : Introduce the non-negative artificial variables R_1, R_2 ... to the left hand side of all the constraints of $\geq$ or $=$ type. The purpose of introducing artificial variables is just to obtain an initial basic feasible solution. However, addition of these artificial variables causes violation of the corresponding constraints. Therefore we would like to get rid of these variables and would not allow them to appear in the final solution. To achieve this we assign a very large penalty (–M for maximization problems and +M for minimization problems) as the coefficients of the artificial variables in the objective function.

Step (3) : Solve the modified linear programming problem by simplex method.

While making iterations, using simplex method, one of the following three cases may arise :

(i) If no artificial variable remains in the basis and the optimality condition is satisfied, then the current solution is an optimal basic feasible solution.

(ii) If atleast one artificial variable appears in the basis at zero level (with zero value of X_B column) and the optimality condition is satisfied, then the current solution is an optimal basic feasible (though degenerated) solution .

(iii) If atleast one artificial variable appears in the basis at non–zero level (with positive value in X_B column) and the optimality condition is satisfied, then the original problem has no feasible solution. The solution satisfies the constraints but does not optimize the objective function since it contains a very large penalty M and is called ***pseudo–optimal solution.***

Note : While applying simplex method, whenever an artificial variable happens to leave the basis, we drop that artificial variable and omit all the entries corresponding to its column from the simplex table.

Example 1 : **Solve the following LPP by simplex method :**

$$\text{Maximize } Z = 3x_1 + 2x_2$$

subject to

$$2x_1 + x_2 \le 2$$
$$3x_1 + 4x_2 \ge 12$$
$$\text{and } x_1, x_2 \ge 0.$$

[MKU. M.Sc 1985, MU. BE. Apr 97]

Solution : By introducing the non-negative slack variable s_1 and surplus variable s_2, the standard form of the LPP becomes

$$\text{Maximize } Z = 3x_1 + 2x_2 + 0s_1 + 0s_2$$

subject to

$$2x_1 + x_2 + s_1 + 0s_2 = 2$$
$$3x_1 + 4x_2 + 0s_1 - s_2 = 12$$
$$x_1, x_2, s_1, s_2 \ge 0.$$

But this will not yield a basic feasible solution. To get the basic feasible solution, add the artificial variable R_1 to the left hand side of the constraint equation which does not possess the slack variable and assign − M to the artificial variable in the objective function. The LPP becomes

$$\text{Max } Z = 3x_1 + 2x_2 + 0s_1 + 0s_2 - MR_1$$

subject to

$$2x_1 + x_2 + s_1 + 0s_2 = 2$$
$$3x_1 + 4x_2 + 0s_1 - s_2 + R_1 = 12$$
$$x_1, x_2, s_1, s_2, R_1 \ge 0$$

The initial basic feasible solution is given by

$s_1 = 2$, $R_1 = 12$ (basic) ($x_1 = x_2 = s_2 = 0$, non-basic)

Initial iteration :

		C_j	(3	2	0	0	−M)	
C_B	Y_B	X_B	x_1	x_2	s_1	s_2	R_1	θ
0	s_1	2	2	(1)	1	0	0	$\frac{2}{1} = 2$
−M	R_1	12	3	4	0	−1	1	$\frac{12}{4} = 3$
$Z_j - C_j$		−12M	−3M −3	−4M−2	0	M	0	

Since there are some $(Z_j - C_j) < 0$, The current basic feasible solution is not optimal.

The non-basic variable x_2 enters into the basis and the basic variable s_1 leaves the basis.

First Iteration :

	C_j	(3	2	0	0	–M)
C_B Y_B	X_B	x_1	x_2	s_1	s_2	R_1
2 x_2	2	2	1	1	0	0
–M R_1	4	–5	0	–4	–1	1
$Z_j - C_j$	–4M + 4	5M + 1	0	4M + 2	M	0

Since all $(Z_j - C_j) \geq 0$, and an artificial variable R_1 appears in the basis at non-zero level, the given LPP does not possess any feasible solution. But the LPP possess a ***pseudo optimal solution.***

Example 2 : Solve the following problem by simplex method :

$$\text{Maximize } Z = x_1 + 2x_2 + 3x_3 - x_4$$

subject to

$$x_1 + 2x_2 + 3x_3 = 15$$
$$2x_1 + x_2 + 5x_3 \geq 20$$
$$x_1 + 2x_2 + x_3 + x_4 \geq 10$$
$$x_1, x_2, x_3, x_4 \geq 0.$$

[MU. BE. Apr 93]

Solution: By introducing the non-negative surplus variables s_1 and s_2, the standard form of the LPP becomes

$$\text{Maximize } Z = x_1 + 2x_2 + 3x_3 - x_4 + 0s_1 + 0s_2$$

subject to

$$x_1 + 2x_2 + 3x_3 + 0s_1 + 0s_2 = 15$$
$$2x_1 + x_2 + 5x_3 - s_1 + 0s_2 = 20$$
$$x_1 + 2x_2 + x_3 + x_4 + 0s_1 - s_2 = 10$$
$$x_1, x_2, x_3, x_4, s_1, s_2 \geq 0.$$

But this will not yield a basic feasible solution. To get the basic feasible solution, add the artificial variables R_1, R_2, R_3, to the left hand side of the constraint equations which does not possess the slack variables and assign –M to the artificial variables in the objective function. So the LPP becomes

$$\text{Max } Z = x_1 + 2x_2 + 3x_3 - x_4 + 0s_1 + 0s_2 - MR_1 - MR_2 - MR_3$$

subject to the constraints

$$x_1 + 2x_2 + 3x_3 + 0s_1 + 0s_2 + R_1 = 15$$
$$2x_1 + x_2 + 5x_3 - s_1 + 0s_2 + R_2 = 20$$
$$x_1 + 2x_2 + x_3 + x_4 + 0s_1 - s_2 + R_3 = 10$$
$$x_1, x_2, x_3, x_4, s_1, s_2, R_1, R_2, R_3 \geq 0.$$

The initial basic feasible solution is given by

$R_1 = 15$, $R_2 = 20$, $R_3 = 10$ (basic) ($x_1 = x_2 = x_3 = x_4 = s_1 = s_2 = 0$, non-basic)

Initial iteration :

	C_j	(1	2	3	–1	0	0	–M	–M	–M)	
C_B Y_B	X_B	x_1	x_2	x_3	x_4	s_1	s_2	R_1	R_2	R_3	θ
–M R_1	15	1	2	3	0	0	0	1	0	0	$\frac{15}{3} = 5$
–M R_2	20	2	1	(5)	0	–1	0	0	1	0	$\frac{20}{5} = 4$
–M R_3	10	1	2	1	1	0	–1	0	0	1	$\frac{10}{1} = 10$
$Z_j - C_j$	–45M	–4M –1	–5M –2	–9M –3	–M +1	M	M	0	0	0	

Since there are some $(Z_j - C_j) < 0$, the current basic feasible solution is not optimal.

The non-basic variable x_3 enters into the basis and the basic variable R_2 leaves the basis. Also since the artificial variable R_2 leaves the basis, we drop that artificial variable R_2 and omit all the entries corresponding to its column from the simplex table.

First iteration :

	C_j	(1	2	3	–1	0	0	–M	–M)	
C_B Y_B	X_B	x_1	x_2	x_3	x_4	s_1	s_2	R_1	R_3	θ
–M R_1	3	$\frac{-1}{5}$	$\left(\frac{7}{5}\right)$	0	0	$\frac{3}{5}$	0	1	0	$\frac{15}{7}$
3 x_3	4	$\frac{2}{5}$	$\frac{1}{5}$	1	0	$\frac{-1}{5}$	0	0	0	20
–M R_3	6	$\frac{3}{5}$	$\frac{9}{5}$	0	1	$\frac{1}{5}$	–1	0	1	$\frac{30}{9}$
$Z_j - C_j$	–9 M +12	$\frac{-2M+1}{5}$	$\frac{-16M-7}{5}$	0	–M +1	$\frac{-4M-3}{5}$	M	0	0	

Since there are some $(Z_j - C_j) < 0$, the current basic feasible solution is not optimal.

The non-basic variable x_2 enters into the basis and the basic variable R_1 leaves the basis.

Second iteration :

	C_j	1	2	3	–1	0	0	–M	
C_B Y_B	X_B	x_1	x_2	x_3	x_4	s_1	s_2	R_3	θ
2 x_2	$\frac{15}{7}$	$\frac{-1}{7}$	1	0	0	$\frac{3}{7}$	0	0	–
3 x_3	$\frac{25}{7}$	$\frac{3}{7}$	0	1	0	$\frac{-2}{7}$	0	0	–
–M R_3	$\frac{15}{7}$	$\frac{6}{7}$	0	0	(1)	$\frac{-4}{7}$	–1	1	$\frac{15}{7}$
$Z_j - C_j$	$\frac{-15M+105}{7}$	$\frac{-6M}{7}$	0	0	–M +1	$\frac{4M}{7}$	M	0	

Since there are some $(Z_j - C_j) < 0$, the current basic feasible solution is not optimal.

The non-basic variable x_4 enters into the basis and the basic variable R_3 leaves the basis.

Third iteration :

	C_j	1	2	3	–1	0	0	
C_B Y_B	X_B	x_1	x_2	x_3	x_4	s_1	s_2	θ
2 x_2	$\frac{15}{7}$	$\frac{-1}{7}$	1	0	0	$\frac{3}{7}$	0	–
3 x_3	$\frac{25}{7}$	$\frac{3}{7}$	0	1	0	$\frac{-2}{7}$	0	$\frac{25}{3}$
–1 x_4	$\frac{15}{7}$	$(\frac{6}{7})$	0	0	1	$\frac{-4}{7}$	–1	$\frac{15}{6}$
$Z_j - C_j$	$\frac{90}{7}$	$\frac{-6}{7}$	0	0	0	$\frac{4}{7}$	1	

Since there are some $(Z_j - C_j) = \frac{-6}{7} < 0$, the current basic feasible solution is not optimal.

The non-basic variable x_1 enters the basis and the basic variable x_4 leaves the basis.

Fourth iteration :

	C_j	1	2	3	–1	0	0
C_B Y_B	X_B	x_1	x_2	x_3	x_4	s_1	s_2
2 x_2	$\frac{5}{2}$	0	1	0	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{-1}{6}$
3 x_3	$\frac{5}{2}$	0	0	1	$\frac{-1}{2}$	0	$\frac{1}{2}$
1 x_1	$\frac{5}{2}$	1	0	0	$\frac{7}{6}$	$\frac{-2}{3}$	$\frac{-7}{6}$
$Z_j - C_j$	15	0	0	0	1	0	0

Since all $(Z_j - C_j) \geq 0$, the current basic feasible solution is optimal.

∴ The optimal solution is Max Z = 15, $x_1 = \frac{5}{2}$, $x_2 = \frac{5}{2}$, $x_3 = \frac{5}{2}$, $x_4 = 0$.

Example 3 : **Use Big – M method to solve**

$$\text{Minimize } Z = 4x_1 + 3x_2$$

subject to

$$2x_1 + x_2 \geq 10$$
$$-3x_1 + 2x_2 \leq 6$$
$$x_1 + x_2 \geq 6$$
$$\text{and } x_1, x_2 \geq 0.$$

[BRU M.Sc 1988]

Solution : Given $\text{Min } Z = 4x_1 + 3x_2$

subject to

$$2x_1 + x_2 \geq 10$$
$$-3x_1 + 2x_2 \leq 6$$
$$x_1 + x_2 \geq 6$$
$$x_1, x_2 \geq 0.$$

That is $\text{Max } Z^* = -4x_1 - 3x_2$

subject to

$$2x_1 + x_2 \geq 10$$
$$-3x_1 + 2x_2 \leq 6$$
$$x_1 + x_2 \geq 6$$
$$x_1, x_2 \geq 0.$$

By introducing the non-negative slack, surplus and artificial variables, the standard form of the LPP becomes

$$\text{Max } Z^* = -4x_1 - 3x_2 + 0s_1 + 0s_2 + 0s_3 - MR_1 - MR_2$$

subject to

$$2x_1 + x_2 - s_1 + 0s_2 + 0s_3 + R_1 = 10$$
$$-3x_1 + 2x_2 + 0s_1 + s_2 + 0s_3 = 6$$
$$x_1 + x_2 + 0s_1 + 0s_2 - s_3 + R_2 = 6$$
$$\text{and } x_1, x_2, x_3, s_1, s_2, s_3, R_1, R_2 \geq 0.$$

(Here : s_1, s_3 – surplus, s_2 – slack, R_1, R_2 – artificials)

The initial basic feasible solution is given by

$R_1 = 10,\ s_2 = 6, R_2 = 6$ (basic) ($x_1 = x_2 = s_1 = s_3 = 0$, non-basic)

Initial iteration :

	C_j	(–4	–3	0	0	0	–M	–M)	
C_B Y_B	X_B	x_1	x_2	s_1	s_2	s_3	R_1	R_2	θ
–M R_1	10	(2)	1	–1	0	0	1	0	$\frac{10}{2} = 5$
0 s_2	6	–3	2	0	1	0	0	0	–
–M R_2	6	1	1	0	0	–1	0	1	$\frac{6}{1} = 6$
$Z_j^* - C_j$	–16M	–3M+4	–2M+3	M	0	M	0	0	

Since there are some $(Z_j^* - C_j) < 0$, the current basic feasible solution is not optimal.

The non-basic variable x_1 enters into the basis and the basic variable R_1 leaves the basis.

First iteration :

	C_j	(–4	–3	0	0	0	–M)	
C_B Y_B	X_B	x_1	x_2	s_1	s_2	s_3	R_2	θ
–4 x_1	5	1	$^1/_2$	$-^1/_2$	0	0	0	10
0 s_2	21	0	$^7/_2$	$-^3/_2$	1	0	0	$\frac{42}{7}$
–M R_2	1	0	$(^1/_2)$	$^1/_2$	0	-1	1	2
$Z_j^* - C_j$	–M–20	0	$\frac{-M+2}{2}$	$\frac{-M+4}{2}$	0	M	0	

Since there are some $(Z_j^* - C_j) < 0$, the current basic feasible solution is not optimal.

The non-basic variable x_2 enters into the basis and the basic variable R_2 leaves the basis.

Second iteration :

	C_j	(–4	–3	0	0	0)
C_B Y_B	X_B	x_1	x_2	s_1	s_2	s_3
–4 x_1	4	1	0	–1	0	1
0 s_2	14	0	0	–5	1	7
–3 x_2	2	0	1	1	0	–2
$Z_j^* - C_j$	–22	0	0	1	0	2

Since all $(Z_j^* - C_j) \geq 0$, the current basic feasible solution is optimal.

$\therefore$ Max $Z^* = -22, x_1 = 4, x_2 = 2$

But Min $Z = -$ Max $(-Z) = -$Max $Z^* = -(-22) = 22$.

$\therefore$ The optimal solution is Min $Z = 22, x_1 = 4, x_2 = 2$

Example 4 : Use Penalty method to

$$\text{Maximize } Z = 2x_1 + x_2 + x_3$$

subject to

$$4x_1 + 6x_2 + 3x_3 \leq 8$$
$$3x_1 - 6x_2 - 4x_3 \leq 1$$
$$2x_1 + 3x_2 - 5x_3 \geq 4$$
$$\text{and } x_1, x_2, x_3 \geq 0.$$

[MU. BE. Apr 90]

Solution: By introducing the non-negative slack, surplus and artificial variables, the standard form of the LPP becomes

$$\text{Max } Z = 2x_1 + x_2 + x_3 + 0s_1 + 0s_2 + 0s_3 - MR_1$$

subject to

$$4x_1 + 6x_2 + 3x_3 + s_1 + 0s_2 + 0s_3 = 8$$
$$3x_1 - 6x_2 - 4x_3 + 0s_1 + s_2 + 0s_3 = 1$$
$$2x_1 + 3x_2 - 5x_3 + 0s_1 + 0s_2 - s_3 + R_1 = 4$$
$$\text{and } x_1, x_2, x_3, s_1, s_2, s_3, R_1 \geq 0.$$

(Here : s_1, s_2 – slack, s_3 – surplus, R_1 – artificial)

The initial basic feasible solution is given by

$s_1 = 8$, $s_2 = 1$, $R_1 = 4$ (basic) ($x_1 = x_2 = x_3 = s_3 = 0$, non-basic)

Initial iteration :

C_B	Y_B	C_j → X_B	2 x_1	1 x_2	1 x_3	0 s_1	0 s_2	0 s_3	–M R_1	θ
0	s_1	8	4	6	3	1	0	0	0	$\frac{8}{6} = 1.3$
0	s_2	1	3	–6	–4	0	1	0	0	–
–M	R_1	4	2	(3)	–5	0	0	–1	1	$\frac{4}{3} = 1.33$
$Z_j - C_j$		–4M	–2M–2	–3M–1	5M–1	0	0	M	0	

Since there are some $(Z_j - C_j) \nless 0$, the current basic feasible solution is not optimal.

The non-basic variable x_2 enters into the basis and the basic variable R_1 leaves the basis.

First iteration :

C_B	Y_B	C_j → X_B	2 x_1	1 x_2	1 x_3	0 s_1	0 s_2	0 s_3	θ
0	s_1	0	0	0	(13)	1	0	2	0
0	s_2	9	7	0	–14	0	1	–2	–
1	x_2	$\frac{4}{3}$	$\frac{2}{3}$	1	$\frac{-5}{3}$	0	0	$\frac{-1}{3}$	–
$Z_j - C_j$		$\frac{4}{3}$	$\frac{-4}{3}$	0	$\frac{-8}{3}$	0	0	$\frac{-1}{3}$	

Since there are some $(Z_j - C_j) < 0$, the current basic feasible solution is not optimal.

The non-basic variable x_3 enters into the basis and the basic variable s_1 leaves the basis.

Second iteration :

C_B	Y_B	C_j → X_B	x_1 (2)	x_2 (1)	x_3 (1)	s_1 (0)	s_2 (0)	s_3 (0)	θ
1	x_3	0	0	0	1	$\frac{1}{13}$	0	$\frac{2}{13}$	–
0	s_2	9	(7)	0	0	$\frac{14}{13}$	1	$\frac{2}{13}$	$\frac{9}{7}$
1	x_2	$\frac{4}{3}$	$\frac{2}{3}$	1	0	$\frac{5}{39}$	0	$\frac{-1}{13}$	2
$Z_j - C_j$		$\frac{4}{3}$	$\frac{-4}{3}$	0	0	$\frac{8}{39}$	0	$\frac{1}{13}$	

Since there are some $(Z_j - C_j) < 0$, the current basic feasible solution is not optimal.

The non-basic variable x_1 enters into the basis and the basic variable s_2 leaves the basis.

Third iteration :

C_B	Y_B	C_j → X_B	x_1 (2)	x_2 (1)	x_3 (1)	s_1 (0)	s_2 (0)	s_3 (0)	θ
1	x_3	0	0	0	1	$\frac{1}{13}$	0	$\frac{2}{13}$	
2	x_1	$\frac{9}{7}$	1	0	0	$\frac{2}{13}$	$\frac{1}{7}$	$\frac{2}{91}$	
1	x_2	$\frac{10}{21}$	0	1	0	$\frac{1}{39}$	$\frac{-2}{21}$	$\frac{-25}{273}$	
$Z_j - C_j$		$\frac{64}{21}$	0	0	0	$\frac{16}{39}$	$\frac{4}{21}$	$\frac{+29}{273}$	

Since all $(Z_j - C_j) \geq 0$, the current basic feasible solution is optimal.

$\therefore$ The optimal solution is Max $Z = \frac{64}{21}$, $x_1 = \frac{9}{7}$, $x_2 = \frac{10}{21}$, $x_3 = 0$.

3.2.2 The Two Phase Method

[BNU. BE. Nov 98]

The two phase method is another method to solve a given problem in which some artificial variables are involved. The solution is obtained in two phases as follows :

Phase I : In this phase, the simplex method is applied to a specially constructed ***auxiliary linear programming problem*** leading to a final simplex table containing a basic feasible solution to the original problem.

Step 1 : Assign a cost –1 to each artificial variable and a cost 0 to all other variables (in place of their original cost) in the objective function. Thus the new objective function is $Z^* = -R_1 - R_2 - R_3 - \ldots - R_n$

where R_i's are the artificial variables.

Step 2 : Construct the auxiliary LPP in which the new objective function Z* is to be maximized subject to the given set of constraints.

Step 3 : Solve the auxiliary LPP by simplex method until either of the following three possibilities arise.

(i) Max Z* < 0 and atleast one artificial variable appears in the optimum basis at a non–zero level. In this case the given LPP does not possess any feasible solution, stop the procedure.

(ii) Max Z* = 0 and atleast one artificial variable appears in the optimum basis at zero level. In this case proceed to phase – II.

(iii) Max Z* = 0 and no artificial variable appears in the optimum basis. In this case proceed to phase – II.

Phase II : Use the optimum basic feasible solution of Phase – I as a starting solution for the original LPP. Assign the actual costs to the variables in the objective function and a 0 cost to every artificial variable that appears in the basis at the zero level. Use simplex method to the modified simplex table obtained at the end of Phase – I, till an optimum basic feasible solution (if any) is obtained.

Note 1 : In Phase – I, the iterations are stopped as soon as the value of the new objective function becomes zero because this is its maximum value. There is no need to continue till the optimality is reached if this value becomes zero earlier than that.

Note 2 : The new objective function is always of maximization type regardless of whether the original problem is of maximization or minimization type.

Note 3 : Before starting phase – II, remove all artificial variables from the table which were non-basic at the end of phase – I.

Example 1 : **Use Two–phase simplex method to solve**

$$\text{Maximize } Z = 5x_1 + 8x_2$$

subject to the constraints

$$3x_1 + 2x_2 \geq 3$$
$$x_1 + 4x_2 \geq 4$$
$$x_1 + x_2 \leq 5$$
$$x_1, x_2 \geq 0.$$

Solution: By introducing the non-negative slack, surplus and artificial variables, the standard form of the LPP becomes

$$\text{Max } Z = 5x_1 + 8x_2 + 0s_1 + 0s_2 + 0s_3$$

subject to

$$3x_1 + 2x_2 - s_1 + 0s_2 + 0s_3 + R_1 = 3$$
$$x_1 + 4x_2 + 0s_1 - s_2 + 0s_3 + R_2 = 4$$
$$x_1 + x_2 + 0s_1 + 0s_2 + s_3 = 5$$
$$\text{and } x_1, x_2, s_1, s_2, s_3, R_1, R_2 \geq 0.$$

(*Here :* s_1, s_2 – surplus, s_3 – slack, R_1, R_2 – artificials)

The initial basic feasible solution is given by

$R_1 = 3,\ R_2 = 4,\ s_3 = 5$ (basic) ($x_1 = x_2 = s_1 = s_2 = 0$, non-basic)

Phase–I : Assigning a cost –1 to the artificial variables and costs 0 to all other variables, the objective function of the auxiliary LPP becomes

$$\text{Max } Z^* = -R_1 - R_2$$

subject to the given constraints.

The iterative simplex tables for the auxiliary LPP are :

Initial iteration :

	C_j	(0	0	0	0	0	–1	–1)	
C_B Y_B	X_B	x_1	x_2	s_1	s_2	s_3	R_1	R_2	θ
–1 R_1	3	3	2	–1	0	0	1	0	$\frac{3}{2}$
–1 R_2	4	1	(4)	0	–1	0	0	1	$\frac{4}{4}$
0 s_3	5	1	1	0	0	1	0	0	$\frac{5}{1}$
$Z_j^* - C_j$	–7	–4	–6	1	1	0	0	0	

Since there are some $(Z_j^* - C_j) < 0$, the current basic feasible solution is not optimal.

First iteration : Introduce x_2 and drop R_2.

	C_j	(0	0	0	0	0	–1	–1)	
C_B Y_B	X_B	x_1	x_2	s_1	s_2	s_3	R_1	R_2	θ
–1 R_1	1	$\left(\frac{5}{2}\right)$	0	–1	$\frac{1}{2}$	0	1	$\frac{-1}{2}$	$\frac{2}{5}$
0 x_2	1	$\frac{1}{4}$	1	0	$\frac{-1}{4}$	0	0	$\frac{1}{4}$	4
0 s_3	4	$\frac{3}{4}$	0	0	$\frac{1}{4}$	1	0	$\frac{-1}{4}$	$\frac{16}{3}$
$Z_j^* - C_J$	–1	$\frac{-5}{2}$	0	1	$\frac{-1}{2}$	0	0	$\frac{3}{2}$	

Since there are some $(Z_j^* - C_j) < 0$, the current basic feasible solution is not optimal.

Second iteration : Introduce x_1 and drop R_1.

C_B	Y_B	X_B	x_1 (0)	x_2 (0)	s_1 (0)	s_2 (0)	s_3 (0)	R_1 (−1)	R_2 (−1)
0	x_1	$\frac{2}{5}$	1	0	$\frac{-2}{5}$	$\frac{1}{5}$	0	$\frac{2}{5}$	$\frac{-1}{5}$
0	x_2	$\frac{9}{10}$	0	1	$\frac{1}{10}$	$\frac{-3}{10}$	0	$\frac{-1}{10}$	$\frac{3}{10}$
0	s_3	$\frac{37}{10}$	0	0	$\frac{3}{10}$	$\frac{1}{10}$	1	$\frac{-3}{10}$	$\frac{-1}{10}$
$Z_j^* - C_j$		0	0	0	0	0	0	1	1

Since all $(Z_j^* - C_j) \geq 0$, the current basic feasible solution is optimum. Furthermore, no artificial variable appears in the optimum basis so we proceed to phase –II.

Phase – II :

Here, we consider the actual costs associated with the original variables. The new objective function then becomes

$$\text{Max } Z = 5x_1 + 8x_2 + 0s_1 + 0s_2 + 0s_3$$

The initial basic feasible solution for this phase is the one obtained at the end of Phase – I.

The iterative simplex tables for this phase are :

Initial iteration :

C_B	Y_B	X_B	x_1 (5)	x_2 (8)	s_1 (0)	s_2 (0)	s_3 (0)	θ
5	x_1	$\frac{2}{5}$	1	0	$\frac{-2}{5}$	$(\frac{1}{5})$	0	2
8	x_2	$\frac{9}{10}$	0	1	$\frac{1}{10}$	$\frac{-3}{10}$	0	–
0	s_3	$\frac{37}{10}$	0	0	$\frac{3}{10}$	$\frac{1}{10}$	1	37
$(Z_j - C_j)$		$\frac{46}{5}$	0	0	$\frac{-6}{5}$	$\frac{-7}{5}$	0	

Since there are some $(Z_j - C_j) < 0$, the current basic feasible solution is not optimal.

First iteration : Introduce s_2 and drop x_1

	C_j	5	8	0	0	0	
C_B Y_B	X_B	x_1	x_2	s_1	s_2	s_3	θ
0 s_2	2	5	0	−2	1	0	–
8 x_2	$\frac{3}{2}$	$\frac{3}{2}$	1	$\frac{-1}{2}$	0	0	–
0 s_3	$\frac{7}{2}$	$\frac{-1}{2}$	0	$(\frac{1}{2})$	0	1	7
$(Z_j - C_j)$	12	7	0	−4	0	0	

Since there are some $(Z_j - C_j) < 0$, current basic feasible solution is not optimal.

Second iteration : Introduce s_1 and drop s_3

	C_j	5	8	0	0	0
C_B Y_B	X_B	x_1	x_2	s_1	s_2	s_3
0 s_2	16	3	0	0	1	4
8 x_2	5	1	1	0	0	1
0 s_1	7	−1	0	1	0	2
$(Z_J - C_J)$	40	3	0	0	0	8

Since all $(Z_J - C_J) \geq 0$, the current basic feasible solution is optimal.

$\therefore$ The optimal solution is Max $z = 40, x_1 = 0, x_2 = 5$.

Example 2 : Solve by two phase simplex method

$$\text{Maximize } X_0 = -4x_1 - 3x_2 - 9x_3$$

Subject to

$$2x_1 + 4x_2 + 6x_3 - s_1 + R_1 = 15$$
$$6x_1 + x_2 + 6x_3 - s_2 + R_2 = 12$$
$$x_1, x_2, x_3, s_1, s_2, R_1, R_2 \geq 0.$$

[MU. BE. Nov 92]

Solution : The initial basic feasible solution is given by

$R_1 = 15$, $R_2 = 12$, (basic) $(x_1 = x_2 = x_3 = s_1 = s_2 = 0$, non-basic)

Phase–I : Assigning a cost −1 to the artificial variables and costs 0 to all other variables, the objective function of the auxiliary LPP becomes

$$\text{Max } Z^* = -R_1 - R_2$$

The iterative simplex tables for the auxiliary LPP are :

Initial iteration :

C_B	Y_B	C_j → X_B	x_1 (0)	x_2 (0)	x_3 (0)	s_1 (0)	s_2 (0)	R_1 (−1)	R_2 (−1)	θ
−1	R_1	15	2	4	6	−1	0	1	0	$\frac{15}{6}$
−1	R_2	12	6	1	(6)	0	−1	0	1	$\frac{12}{6}$
$Z_j^* - C_j$		−27	−8	−5	−12	1	1	0	0	

Since there are some $(Z_j^* - C_j) < 0$, the current basic feasible solution is not optimal.

First iteration : Introduce x_3 and drop R_2.

C_B	Y_B	C_j → X_B	x_1 (0)	x_2 (0)	x_3 (0)	s_1 (0)	s_2 (0)	R_1 (−1)	R_2 (−1)	θ
−1	R_1	3	−4	(3)	0	−1	1	1	−1	$\frac{3}{3}$
0	x_3	2	1	$\frac{1}{6}$	1	0	$\frac{-1}{6}$	0	$\frac{1}{6}$	12
$Z_j^* - C_j$		−3	4	−3	0	1	−1	0	2	

Since there are some $(Z_j^* - C_j) < 0$, the current basic feasible solution is not optimal.

Second iteration : Introduce x_2 and drop R_1.

C_B	Y_B	C_j → X_B	x_1 (0)	x_2 (0)	x_3 (0)	s_1 (0)	s_2 (0)	R_1 (−1)	R_2 (−1)
0	x_2	1	$\frac{-4}{3}$	1	0	$\frac{-1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{-1}{3}$
0	x_3	$\frac{11}{6}$	$\frac{22}{18}$	0	1	$\frac{1}{18}$	$\frac{-4}{18}$	$\frac{-1}{18}$	$\frac{4}{18}$
$Z_j^* - C_j$		0	0	0	0	0	0	1	1

Since all $(Z_j^* - C_j) \geq 0$, the current basic feasible solution is optimal. Further, no artificial variable appears in the basis, so we proceed to phase – II.

Phase – II :

Here, we consider the actual costs associated with the original variables. The new objective function then becomes

$$\text{Max } X_0 = -4x_1 - 3x_2 - 9x_3 + 0s_1 + 0s_2$$

The initial basic feasible solution for this phase is the one obtained at the end of Phase – I.

The iterative simplex tables for this phase are :

Initial iteration :

	C_j	$(-4$	-3	-9	0	$0)$	
C_B Y_B	X_B	x_1	x_2	x_3	s_1	s_2	θ
-3 x_2	1	$\frac{-4}{3}$	1	0	$\frac{-1}{3}$	$\frac{1}{3}$	–
-9 x_3	$\frac{11}{6}$	$\left(\frac{22}{18}\right)$	0	1	$\frac{1}{18}$	$\frac{-4}{18}$	$\frac{3}{2}$
$(X_0 - C_j)$	$\frac{-39}{2}$	-3	0	0	$\frac{1}{2}$	1	

Since there are some $(X_0 - C_J) < 0$, the current basic feasible solution is not optimal.

First iteration : Introduce x_1 and drop x_3

	C_j	$(-4$	-3	-9	0	$0)$
C_B Y_B	X_B	x_1	x_2	x_3	s_1	s_2
-3 x_2	3	0	1	$\frac{12}{11}$	$\frac{-3}{11}$	$\frac{1}{11}$
-4 x_1	$\frac{3}{2}$	1	0	$\frac{18}{22}$	$\frac{1}{22}$	$\frac{-4}{22}$
$(X_0 - C_j)$	-15	0	0	$\frac{27}{11}$	$\frac{7}{11}$	$\frac{5}{11}$

Since all $(X_0 - C_j) \geq 0$, the current basic feasible solution is optimal.

$\therefore$ The optimal solution is Max $X_0 = -15, x_1 = \frac{3}{2}, x_2 = 3, x_3 = 0$

Example 3 : Use two–phase method to

$$\textbf{Maximize } Z = 2x_1 + x_2 + \frac{1}{4}x_3$$

subject to constraints

$$4x_1 + 6x_2 + 3x_3 \leq 8$$
$$3x_1 - 6x_2 - 4x_3 \leq 1$$
$$2x_1 + 3x_2 - 5x_3 \geq 4$$
$$x_1, x_2, x_3 \geq 0.$$

[MKU.B.Sc.1988]

Solution : By introducing slack, surplus and artificial variables, the standard form of the LPP becomes

$$\text{Max } Z = 2x_1 + x_2 + \frac{1}{4}x_3$$

subject to

$$4x_1 + 6x_2 + 3x_3 + s_1 + 0s_2 + 0s_3 = 8$$
$$3x_1 - 6x_2 - 4x_3 + 0s_1 + s_2 + 0s_3 = 1$$
$$2x_1 + 3x_2 - 5x_3 + 0s_1 + 0s_2 - s_3 + R_1 = 4$$
$$\text{and } x_1, x_2, x_3, s_1, s_2, s_3, R_1 \geq 0$$

(Here : s_1, s_2 – slack, s_3 – surplus, R_1 – artificial)

The initial basic feasible solution is given by

$s_1 = 8,\ s_2 = 1,\ R_1 = 4$ (basic) ($x_1 = x_2 = x_3 = s_3 = 0$, non-basic)

Phase – I : The objective function of the auxiliary LPP is

$$\text{Max } Z^* = -R_1$$

The iterative simplex tables for the auxiliary LPP are :

Initial iteration :

C_B	Y_B	C_j → X_B	x_1 (0)	x_2 (0)	x_3 (0)	s_1 (0)	s_2 (0)	s_3 (0)	R_1 (–1)	θ
0	s_1	8	4	(6)	3	1	0	0	0	$\frac{8}{6}$
0	s_2	1	3	–6	–4	0	1	0	0	–
–1	R_1	4	2	3	–5	0	0	–1	1	$\frac{4}{3}$
$Z_j^* - C_j$		–4	–2	–3	5	0	0	1	0	

Since there are some $(Z_j^* - C_j) < 0$, the current basic feasible solution is not optimal.

First iteration : Introduce x_2 and drop s_1.

C_B	Y_B	C_j → X_B	x_1 (0)	x_2 (0)	x_3 (0)	s_1 (0)	s_2 (0)	s_3 (0)	R_1 (–1)
0	x_2	$\frac{4}{3}$	$\frac{2}{3}$	1	$\frac{1}{2}$	$\frac{1}{6}$	0	0	0
0	s_2	9	7	0	–1	1	1	0	0
–1	R_1	0	0	0	$\frac{-13}{2}$	$\frac{-1}{2}$	0	–1	1
$Z_j^* - C_j$		0	0	0	$\frac{13}{2}$	$\frac{1}{2}$	0	1	0

Since all $(Z_j^* - C_j) \geq 0$, the current basic feasible solution is optimal for the auxiliary LPP.

But at the same time the artificial variable R_1 appears in the optimum basis at the zero level. This optimal solution may or may not be optimal to the given (original) LPP. So we proceed to phase – II.

Phase–II : Here, we consider the actual costs associated with the original variables and assign a cost 0 to the artificial variable R_1, which appeared at zero level in phase – I, in the objective function. The new objective function then becomes

$$\text{Max } Z = 2x_1 + x_2 + \frac{1}{4}x_3 + 0s_1 + 0s_2 + 0s_3$$

The initial basic feasible solution for this phase is the one obtained at the end of Phase – I.

The iterative simplex tables for this phase are :

Initial iteration :

C_B	Y_B	C_j → X_B	2 x_1	1 x_2	$\frac{1}{4}$ x_3	0 s_1	0 s_2	0 s_3	0 R_1	θ
1	x_2	$\frac{4}{3}$	$\frac{2}{3}$	1	$\frac{1}{2}$	$\frac{1}{6}$	0	0	0	$\frac{4}{2}$
0	s_2	9	(7)	0	–1	1	1	0	0	$\frac{9}{7}$
0	R_1	0	0	0	$\frac{-13}{2}$	$\frac{-1}{2}$	0	–1	1	–
$(Z_j - C_j)$		$\frac{4}{3}$	$\frac{-4}{3}$	0	$\frac{1}{4}$	$\frac{1}{6}$	0	0	0	

Since there are some $(Z_j - C_j) < 0$, the current basic feasible solution is not optimal.

First iteration : Introduce x_1 and drop s_2

C_B	Y_B	C_j → X_B	2 x_1	1 x_2	$\frac{1}{4}$ x_3	0 s_1	0 s_2	0 s_3	0 R_1
1	x_2	$\frac{10}{21}$	0	1	$\frac{25}{42}$	$\frac{1}{14}$	$\frac{-2}{21}$	0	0
2	x_1	$\frac{9}{7}$	1	0	$\frac{-1}{7}$	$\frac{1}{7}$	$\frac{1}{7}$	0	0
0	R_1	0	0	0	$\frac{-13}{2}$	$\frac{-1}{2}$	0	–1	1
$(Z_j - C_j)$		$\frac{64}{21}$	0	0	$\frac{5}{84}$	$\frac{5}{14}$	$\frac{4}{21}$	0	0

Since all $(Z_j - C_j) \geq 0$, the current basic feasible solution is optimal.

∴ The optimal solution is Max $Z = \frac{64}{21}$, $x_1 = \frac{9}{7}$, $x_2 = \frac{10}{21}$, $x_3 = 0$.

Example 4 : Use two phase simplex method to

$$\textbf{Maximize } Z = 5x_1 + 3x_2$$

subject to

$$2x_1 + x_2 \leq 1$$
$$x_1 + 4x_2 \geq 6$$
$$x_1, x_2 \geq 0.$$

Solution : By introducing slack, surplus and artificial variables, the standard form of the LPP becomes

$$\text{Max } Z = 5x_1 + 3x_2$$

subject to

$$2x_1 + x_2 + s_1 + 0s_2 = 1$$
$$x_1 + 4x_2 + 0s_1 - s_2 + R_1 = 6$$
$$\text{and } x_1, x_2, s_1, s_2, R_1 \geq 0.$$

(***Here :*** s_1 - slack, s_2 –surplus , R_1 – artificial)

The initial basic feasible solution is given by $s_1 = 1$, $R_1 = 6$ (basic) ($x_1 = x_2 = s_2 = 0$, non-basic)

Phase – I : The objective function of the auxiliary LPP is

$$\text{Max } Z^* = -R_1$$

The iterative simplex tables for the auxiliary LPP are :

Initial iteration :

	C_j	(0	0	0	0	–1)	
C_B Y_B	X_B	x_1	x_2	s_1	s_2	R_1	θ
0 s_1	1	2	(1)	1	0	0	1
–1 R_1	6	1	4	0	–1	1	$\frac{6}{4}$
$Z_j^* - C_j$	–6	–1	–4	0	1	0	

Since there are some $(Z_j^* - C_j) < 0$, the current basic feasible solution is not optimal.

First iteration : Introduce x_2 and drop s_1.

	C_j	(0	0	0	0	–1)
C_B Y_B	X_B	x_1	x_2	s_1	s_2	R_1
0 x_2	1	2	1	1	0	0
–1 R_1	2	–7	0	–4	–1	1
$(Z_j^* - C_j)$	–2	7	0	4	1	0

Since all $(Z_j^* - C_j) \geq 0$, the current basic feasible solution is optimal to the auxiliary LPP

But since, Max $Z^* < 0$ and one artificial variable R_1 appears in the optimum basis at non–zero level, the given (original) LPP has **no feasible solution.**

Example 5 : **Using simplex algorithm.**

Minimize $-2x_1 - x_2$ subject to the constraints

$$x_1 + x_2 \geq 2$$
$$x_1 + x_2 \leq 4$$
$$x_1, x_2 \geq 0.$$

[MU. BE. Oct 95]

Solution : Let $Z = -2x_1 - x_2$

$$\therefore \text{Min } Z = -2x_1 - x_2$$

subject to

$$x_1 + x_2 \geq 2$$
$$x_1 + x_2 \leq 4$$
$$x_1, x_2 \geq 0.$$

That is $\text{Max } Z^* = 2x_1 + x_2$

subject to

$$x_1 + x_2 \geq 2$$
$$x_1 + x_2 \leq 4$$
$$x_1, x_2 \geq 0.$$

By introducing slack, surplus and artificial variables, the standard form of the LPP becomes

$$\text{Max } Z^* = 2x_1 + x_2$$

subject to

$$x_1 + x_2 - s_1 + 0s_2 + R_1 = 2$$
$$x_1 + x_2 + 0s_1 + s_2 = 4$$
$$x_1, x_2, s_1, s_2, R_1 \geq 0$$

(Here : s_1 - surplus, s_2 –slack , R_1 – artificial)

The initial basic feasible solution is given by $R_1 = 2$, $s_2 = 4$ (basic) ($x_1 = x_2 = s_1 = 0$, non-basic)

Phase – I : The objective function of the auxiliary LPP is

$$\text{Max } Z^* = -R_1$$

The iterative simplex tables for the auxiliary LPP are :

Initial iteration :

	C_j	(0	0	0	0	–1)	
C_B Y_B	X_B	x_1	x_2	s_1	s_2	R_1	θ
–1 R_1	2	(1)	1	–1	0	1	$\frac{2}{1}$
0 s_2	4	1	1	0	1	0	$\frac{4}{1}$
$(Z_j^* - C_j)$	–2	–1	–1	1	0	0	

Since there are some $(Z_j^* - C_j) < 0$, the current basic feasible solution is not optimal.

First iteration : Introduce x_1 and drop R_1.

	C_j	(0	0	0	0	−1)
C_B Y_B	X_B	x_1	x_2	s_1	s_2	R_1
0 x_1	2	1	1	−1	0	1
0 s_2	2	0	0	1	1	−1
$(Z_j^* - C_j)$	0	0	0	0	0	1

Since all $(Z_j^* - C_j) \geq 0$, and no artificial variable appears in the optimum basis, the current basic feasible solution is optimal to the auxiliary LPP and we proceed to Phase – II.

Phase – II : Here, we consider the actual costs associated with the original variables. The new objective function then becomes

$$\text{Max } Z^* = 2x_1 + x_2 + 0s_1 + 0s_2$$

The initial basic feasible solution for this phase is the one obtained at the end of phase – I. The iterative simplex tables for this phase are :

Initial iteration :

	C_j	(2	1	0	0)	
C_B Y_B	X_B	x_1	x_2	s_1	s_2	θ
2 x_1	2	1	1	−1	0	–
0 s_2	2	0	0	(1)	1	$\frac{2}{1}$
$(Z_j^* - C_j)$	4	0	1	−2	0	

Since there are some $(Z_j^* - C_j) < 0$, the current basic feasible solution is not optimal.

First iteration : Introduce s_1 and drop s_2.

	C_j	(2	1	0	0)
C_B Y_B	X_B	x_1	x_2	s_1	s_2
2 x_1	4	1	1	0	1
0 s_1	2	0	0	1	1
$(Z_j^* - C_j)$	8	0	1	0	2

Since all $(Z_j^* - C_j) \geq 0$, the current basic feasible solution is optimal.

∴ The optimal solution is max $Z^* = 8$, $x_1 = 4$, $x_2 = 0$

But Min Z = – Max (–Z) = – Max Z* = –8

∴ Min Z = –8, $x_1 = 4$, $x_2 = 0$.

Disadvantage of Big–M method over Two – phase method :

Even though Big–M method can always be used to check the existence of a feasible solution, it may be computationally inconvenient especially when a digital computer is used because of the manipulation of the constant M. On the other hand, Two-phase method eliminates the constant M from calculations.

EXERCISE

1. **Explain briefly the term "Artificial" variables.**
   [MU. BE. 79, MU. MBA. Nov 96, Apr 97]
2. **Explain the use of artificial variables in LPP.**
3. **Describe briefly the Big–M method of solving a LPP with artificial variables.** *[MU. MCA. Nov 98]*
4. **Describe briefly the Two–phase method of solving a LPP with artificial variables.**
5. **Explain the disadvantage of Big–M method over Two –phase method.**
6. **Using simplex method, solve :**

$$\text{Maximize } Z = 5x_1 - 2x_2 + 3x_3$$

subject to

$$2x_1 + 2x_2 - x_3 \geq 2$$
$$3x_1 - 4x_2 \leq 3$$
$$x_2 + 3x_3 \leq 5$$
$$x_1, x_2, x_3 \geq 0.$$

[MU. BE. Nov 89, Apr 94, Apr 95]

7. **Solve by simplex method,**

$$\text{Max } Z = x_1 + 2x_2 + 3x_3 - x_4$$

subject to

$$x_1 + 2x_2 + 3x_3 = 15$$
$$2x_1 + x_2 + 5x_3 = 20$$
$$x_1 + 2x_2 + x_3 + x_4 = 10$$
$$x_1, x_2, x_3, x_4 \geq 0.$$

[MU. BE. Apr 91, Annamalai M.Sc.82]

8. **Solve the following LPP**

$$\text{Min } Z = 12x_1 + 20x_2$$

subject to

$$6x_1 + 8x_2 \geq 100$$
$$7x_1 + 12x_2 \geq 120$$
$$x_1, x_2 \geq 0.$$

[BRU. B.Sc 90]