CHAPTER III

EXPANSIONS

§ 1. Expansions of cos $n\theta$ and sin $n\theta$.

We have

$$(\cos n\theta + i \sin n\theta) = (\cos\theta + i\sin\theta)^n.$$

If n is a positive integer, the expression on the right-hand side can be expanded by Binomial Theorem. Hence

$$\cos n\theta + i\sin n\theta = \cos^n\theta + n\cos^{n-1}\theta\,(i\sin\theta) + \frac{n(n-1)}{2!}\cos^{n-2}\theta\,(i\sin\theta)^2 + \frac{n(n-1)(n-2)}{3!}\cos^{n-3}\theta\,(i\sin\theta)^3 + \dots$$

$$i^2 = -1,\ i^3 = -i,\ i^4 = 1,\ i^5 = i,\ \dots$$

Hence we have

$$\cos n\theta + i\sin n\theta = \cos^n\theta - \frac{n(n-1)}{2!}\cos^{n-2}\theta\sin^2\theta + \frac{n(n-1)(n-2)(n-3)}{4!}\cos^{n-4}\theta\sin^4\theta + \dots$$
$$+ i\left\{n\cos^{n-1}\theta\sin\theta - \frac{n(n-1)(n-2)}{3!}\cos^{n-3}\theta\sin^3\theta + \dots\right\}.$$

Equating real and imaginary parts, we have

$$\cos n\theta = \cos^n\theta - \frac{n(n-1)}{2!}\cos^{n-2}\theta\sin^2\theta + \dots$$

$$\sin n\theta = n\cos^{n-1}\theta\sin\theta - \frac{n(n-1)(n-2)}{3!}\cos^{n-3}\theta\sin^3\theta + \dots$$

Note.—(1) The terms are alternately positive and negative.

(2) Each series continues till one of the factors in the numerator is zero and then ceases.

(3) The sum of the powers of cos θ and sin θ in every term of the expansions equals n.

(4) Both the series are in descending powers of cos θ and in ascending powers of sin θ.

Cor. 1. $\frac{\sin n\theta}{\sin\theta} = n\cos^{n-1}\theta - \frac{n(n-1)(n-2)}{3!}\cos^{n-3}\theta\sin^2\theta + \ldots$

$$= n\cos^{n-1}\theta - \frac{n(n-1)(n-2)}{3!}\cos^{n-3}\theta(1-\cos^2\theta)$$

$$+\frac{n(n-1)(n-2)(n-3)(n-4)}{5!}\cos^{n-5}\theta(1-\cos^2\theta)^2$$

Similarly in the expansion of $\cos n\theta$, by putting

$$\sin^2\theta = 1-\cos^2\theta,$$

$\cos n\theta$ can be expressed in a series containing powers of $\cos\theta$.

Cor. 2. Coefficient of $\cos^{n-1}\theta$ in the expansion

$$\frac{\sin n\theta}{\sin\theta} = {}^nc_1 + {}^nc_3 + {}^nc_5 \ldots\ldots = 2^{n-1}.$$

Cor. 3. Coefficient of $\cos^n\theta$ in the expansion

$$\cos n\theta = {}^nc_0 + {}^nc_2 + {}^nc_4 + \ldots\ldots = 2^{n-1}.$$

§ 2. Expansion of tan nθ in powers of tan θ.

$$\tan n\theta = \frac{\sin n\theta}{\cos n\theta}$$

$$= \frac{{}^nc_1\cos^{n-1}\theta\,.\,\sin\theta - {}^nc_3\cos^{n-3}\theta\sin^3\theta + \ldots\ldots}{\cos^n\theta - {}^nc_2\cos^{n-2}\theta\sin^2\theta + {}^nc_4\cos^{n-4}\sin^4\theta\ldots\ldots}$$

$$= \frac{{}^nc_1\tan\theta - {}^nc_3\tan^3\theta + \ldots\ldots}{1 - {}^nc_2\tan^2\theta + {}^nc_4\tan^4\theta - \ldots\ldots}$$

on dividing both numerator and denominator by $\cos^n\theta$.

§ 3. Expansion of tan (A + B + C + ...).

$$\cos A + i\sin A = \cos A\,(1 + i\tan A)$$
$$\cos B + i\sin B = \cos B\,(1 + i\tan B)$$
$$\cos C + i\sin C = \cos C\,(1 + i\tan C)$$

....

....

$\therefore$ $(\cos A + i\sin A)(\cos B + i\sin B)(\cos C + i\sin C)\ldots$

$=\cos A\cos B\cos C\ldots.(1+i\tan A)(1+i\tan B)(1+i\tan C)\ldots$

$=\cos A\cos B\cos C\ldots.[1 + i\,\Sigma\tan A + i^2\,\Sigma\tan A\tan B$

$\qquad + i^3\,\Sigma\tan A\tan B\tan C\ldots]$

$= \cos A\cos B\cos C\ldots.\,[1 + iS_1 - S_2 - iS_3 + \ldots.]$

where S_r is the sum of the products taken r at a time of tan A, tan B, tan C ...

Equating real and imaginary parts on both sides, we have

$$\cos(A+B+C+\dots) = \cos A \cos B \cos C \dots (1 - S_2 + S_4 \dots)$$

$$\sin(A+B+C+\dots) = \cos A \cos B \cos C \dots (S_1 - S_3 + S_5 \dots).$$

$$\therefore \tan(A+B+C+\dots) = \frac{S_1 - S_3 + S_5 \dots}{1 - S_2 + S_4 \dots}.$$

Cor. Putting $A = B = C = \dots \theta$, taking n angles

$$\tan n\theta = \frac{S_1 - S_3 + S_5 \dots}{1 - S_2 + S_4 \dots},$$

where S_r is the sum of the products taken r at a time of $\tan A$, ..., n terms.

Hence $S_1 = n \tan\theta$, $S_2 = nc_2 \tan^2\theta$, $S_3 = nc_3 \tan^3\theta \dots$

$$\therefore \tan n\theta = \frac{nc_1 \tan\theta - nc_3 \tan^2\theta + \dots}{1 - nc_2 \tan^2\theta + nc_4 \tan^4\theta \dots}.$$

Examples.

Ex. 1. Express $\cos 8\theta$ in terms of $\sin\theta$.

We have

$$(\cos 8\theta + i \sin 8\theta) = (\cos\theta + i\sin\theta)^8$$

$$= \cos^8\theta + 8c_1 \cos^7\theta\,(i\sin\theta) + 8c_2 \cos^6\theta\,(i\sin\theta)^2 + \dots$$

$$= \cos^8\theta + 8c_1\cos^7\theta + 8c_4 \cos^4\theta \sin^4\theta$$

$$= (\cos^8\theta - 8c_2\cos^6\theta\sin^2\theta + 8c_4\cos^4\theta\sin^4\theta - 8c_6\cos^2\theta\sin^6\theta + 8c_8\sin^8\theta) + i\,(8c_1\cos^7\theta\sin\theta \dots).$$

Equating the real parts, we have

$$\cos 8\theta = \cos^8\theta - 8c_2\cos^6\theta\sin^2\theta + 8c_4\cos^4\theta\sin^4\theta - 8c_6\cos^2\theta\sin^6\theta + 8c_8\sin^8\theta$$

$$= (1-\sin^2\theta)^4 - 28\,(1-\sin^2\theta)^3\sin^2\theta + 70\,(1-\sin^2\theta)^2\sin^4\theta - 28\,(1-\sin^2\theta)\sin^6\theta + \sin^8\theta$$

$$= 128\sin^8\theta - 256\sin^6\theta + 160\sin^4\theta - 32\sin^2\theta + 1.$$

Ex. 2. Express $\dfrac{\sin 6\theta}{\sin\theta}$ in terms of $\cos\theta$.

$$\cos 6\theta + i\sin 6\theta = (\cos\theta + i\sin\theta)^6$$

$$= \cos^6\theta + 6c_1\cos^5\theta\, i\sin\theta + 6c_2\cos^4\theta\,(i\sin\theta)^2$$

$$+ 6c_3\cos^3\theta\,(i\sin\theta)^3 + 6c_4\cos^2\theta\,(i\sin\theta)^4$$

$$+ 6c_5\cos\theta\,(i\sin\theta)^5 + (i\sin\theta)^6$$

$$= \cos^6\theta - 6c_2\cos^4\theta\sin^2\theta + 6c_4\cos^2\theta\sin^4\theta - \sin^6\theta$$

$$+ i\,(6c_1\cos^5\theta\sin\theta - 6c_3\cos^3\theta\sin^3\theta + 6c_5\cos\theta\sin^5\theta).$$

Equating the imaginary parts on both sides, we

$$\sin 6\theta = 6c_1 \cos^5\theta \sin\theta - 6c_3 \cos^3\theta \sin^3\theta + 6c_5 \cos\theta \ldots$$

$$= 6\cos^5\theta \sin\theta - 20\cos^3\theta \sin^3\theta + 6\cos\theta \sin^5\theta.$$

$$\therefore \frac{\sin 6\theta}{\sin\theta} = 6\cos^5\theta - 20\cos^3\theta \sin^2\theta + 6\cos\theta \ldots$$

$$= 6\cos^5\theta - 20\cos^3\theta\,(1-\cos^2\theta) + 6\cos\theta\,(1-\cos \ldots$$

$$= 32\cos^5\theta - 32\cos^3\theta + 6\cos\theta.$$

Ex. 3. If α, β, γ be the roots of the equ... $x^3 + px^2 + qx + p = 0$, prove that

$$\tan^{-1}\alpha + \tan^{-1}\beta + \tan^{-1}\gamma = n\pi \text{ radians}$$

except when $q = 1$.

Since α, β, γ are the roots of the equation, we have

$$\alpha + \beta + \gamma = -p$$

$$\alpha\beta + \beta\gamma + \gamma\alpha = q$$

$$\alpha\beta\gamma = -p$$

Let $\tan^{-1}\alpha$, $\tan^{-1}\beta$, $\tan^{-1}\gamma$ be respectively equal ... x_1, x_2, x_3.

Then $\alpha = \tan x_1$, $\beta = \tan x_2$, $\gamma = \tan x_3$.

Equations (1), (2), (3) then become

$$S_1 = \tan x_1 + \tan x_2 + \tan x_3 = -p.$$

$$S_2 = \tan x_1 \tan x_2 + \tan x_2 \tan x_3 + \tan x_3 \tan x_1 = \ldots$$

$$S_3 = \tan x_1 \tan x_2 \tan x_3 = -p.$$

$$\tan(x_1 + x_2 + x_3) = \frac{S_1 - S_3}{1 - S_2} = \frac{-p+p}{1-q}.$$

Hence if $q \neq 1$, $\tan(x_1 + x_2 + x_3) = 0$.

$$\therefore x_1 + x_2 + x_3 = n\pi$$

i.e., $\tan^{-1}\alpha + \tan^{-1}\beta + \tan^{-1}\gamma = n\pi.$

Ex. 4. Prove that the equation

$$\frac{ah}{\cos\theta} - \frac{bk}{\sin\theta} = a^2 - b^2$$

has four roots and that the sum of the four values of θ which satisfy it is equal to an odd ... radians

Rewriting the given equation after substitutions,

$$\sin\theta = \frac{2\tan\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}}, \quad \cos\theta = \frac{1-\tan^2\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}}$$

$$= \frac{2t}{1+t^2} \qquad\qquad = \frac{1-t^2}{1+t^2},$$

where $t = \tan\frac{\theta}{2}$.

We have $\frac{ah(1+t^2)}{1-t^2} - \frac{bk(1+t^2)}{2t} = a^2 - b^2.$

Simplifying, this equation reduces to

$$bk\,t^4 + 2[ab + a^2 - b^2]\,t^3 + 2(ah - a^2 + b^2)\,t - bk = 0.$$

Let $t_1 = \tan\frac{\theta_1}{2}$, $t_2 = \tan\frac{\theta_2}{2}$, $t_3 = \tan\frac{\theta_3}{2}$; $t_4 = \tan\frac{\theta_4}{2}$ be the four roots of the equation in t.

$\therefore\ \Sigma t_1 t_2 = 0,\ t_1 t_2 t_3 t_4 = -1.$

$$\tan\left(\frac{\theta_1}{2}+\frac{\theta_2}{2}+\frac{\theta_3}{2}+\frac{\theta_4}{2}\right) = \frac{\Sigma t_1 - \Sigma t_1 t_2 t_3}{1 - \Sigma t_1 t_2 + t_1 t_2 t_3 t_4}.$$

The denominator $= 1 - 0 - 1 = 0.$

$$\therefore\ \tan\left(\frac{\theta_1}{2}+\frac{\theta_2}{2}+\frac{\theta_3}{2}+\frac{\theta_4}{2}\right) = \infty$$

i.e., $\frac{\theta_1}{2}+\frac{\theta_2}{2}+\frac{\theta_3}{2}+\frac{\theta_4}{2} = (2n+1)\frac{\pi}{2}$

i.e., $\theta_1+\theta_2+\theta_3+\theta_4 = (2n+1)\pi.$

Cor. Students familiar with Analytical Geometry will recollect that the above result proves that the sum of the eccentric angles of the feet of the four normals drawn from a given point to an ellipse is an odd multiple of π.

If $\alpha, \beta, \gamma, \delta$ are the independent roots of the equation $a\ \cos\ 2\theta + b\ \sin\ 2\theta + c\ \cos\theta + d\ \sin\theta = 0$, show that $\tan(\alpha + \beta + \gamma + \delta) = \frac{b}{a}$.

Prove that the equation $a^2\cos^2\theta + b^2\sin^2\theta + 2g\,a\cos\theta + 2fb\sin\theta + c = 0$ has four roots and that the sum of the values of θ which satisfy it is an even multiple of π radians.

Show that $\frac{\tan x}{\tan 3x}$ never lies between $\frac{1}{3}$ and 3 whatever the value of x may be.

Prove that

$$64(\cos^8\theta + \sin^8\theta) = \cos 8\theta + 28\cos 4\theta + 35.$$

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By solving the equation $\cos 3\theta + \sin 3\theta = 0$, show that the roots of equation $x^2 + 4x + 1 = 0$ are $-\tan\left(\frac{\pi}{12}\right)$ and $-\tan\left(\frac{5\pi}{12}\right)$.

Examples on formation of equations.

Ex. 1. Expand $\sin 7\theta$ as a polynomial in $\sin\theta$.

Hence obtain the cubic equation whose roots are

$$\sin^2\frac{\pi}{7},\ \sin^2\frac{2\pi}{7},\ \sin^2\frac{3\pi}{7}.$$

We can easily show that

$$\sin 7\theta = 7\sin\theta - 56\sin^3\theta + 112\sin^5\theta - 64\sin^7\theta.$$

If $\theta = 0, \frac{2\pi}{7}, \frac{4\pi}{7}, \frac{6\pi}{7}, \frac{8\pi}{7}, \frac{10\pi}{7}, \frac{12\pi}{7}$, $\sin 7\theta = 0$.

Hence these seven values of θ are the roots of the equation $7\sin\theta - 56\sin^3\theta + 112\sin^5\theta - 64\sin^7\theta = 0$.

Putting $\sin\theta = x$, $7x - 56x^3 + 112x^5 - 64x^7 = 0$, has roots 0, $\sin\frac{2\pi}{7}, \sin\frac{4\pi}{7}, \sin\frac{6\pi}{7}, \sin\frac{8\pi}{7}, \sin\frac{10\pi}{7}, \sin\frac{12\pi}{7}$.

$\therefore\ 64x^6 - 112x^4 + 56x^2 - 7 = 0\ (1)$ has roots

$\sin\frac{4\pi}{7}, \sin\frac{6\pi}{7}, \sin\frac{8\pi}{7}, \sin\frac{10\pi}{7}, \sin\frac{12\pi}{7}.$

We have $\sin\frac{12\pi}{7} = \sin\left(2\pi - \frac{2\pi}{7}\right) = -\sin\frac{2\pi}{7}$

$$\sin\frac{10\pi}{7} = \sin\left(2\pi - \frac{4\pi}{7}\right) = -\sin\frac{4\pi}{7}$$

$$\sin\frac{8\pi}{7} = \sin\left(2\pi - \frac{6\pi}{7}\right) = -\sin\frac{6\pi}{7}.$$

Hence equation (1) has roots $\pm\sin\frac{2\pi}{7}, \pm\sin\frac{4\pi}{7}, \pm$

Put $x^2 = y$, in equation (1). Then

$64y^3 - 112y^2 + 56y - 7 = 0$ has roots $\sin^2\frac{2\pi}{7},$

$\sin^2\frac{6\pi}{7}.$

Ex. 2. Find the equation whose roots are 2 cos

$2\cos\frac{4\pi}{7},\ 2\cos\frac{6\pi}{7}.$

We have shown that

$$\frac{\sin 7\theta}{\sin\theta} = 7 - 56\sin^2\theta + 112\sin^4\theta - 64\sin^6\theta$$

$$= 7 - 28(1 - \cos 2\theta) + 28(1 - \cos 2\theta)^2$$

$$- 8(1 - \cos$$

$$= 8\cos^3 2\theta + 4\cos^2 2\theta - 4\cos 2\theta - 1.$$

When $\theta = \pm\frac{\pi}{7}, \pm\frac{2\pi}{7}, \pm\frac{3\pi}{7}, \sin 7\theta = 0.$

$\therefore\ 8\cos^3 2\theta + 4\cos^2 2\theta - 4\cos 2\theta - 1 = 0$ has ro

$\pm\frac{\pi}{7}, \pm\frac{2\pi}{7}, \pm\frac{3\pi}{7}.$

Hence $8x^3 + 4x^2 - 4x - 1 = 0$ has roots

$$\cos\left(\pm\frac{2\pi}{7}\right), \cos\left(\pm\frac{4\pi}{7}\right), \cos\left(\pm\frac{6\pi}{7}\right)$$

i.e., $\cos\frac{2\pi}{7}, \cos\frac{4\pi}{7}, \cos\frac{6\pi}{7}$.

Put $2x = y$.

Then $y^3 + y^2 - 2y - 1 = 0$ has roots

$$2\cos\frac{2\pi}{7}, \; 2\cos\frac{4\pi}{7}, \; 2\cos\frac{6\pi}{7}.$$

Aliter. Put $\frac{2\pi}{7}$ or $\frac{4\pi}{7}$ or $\frac{6\pi}{7} = \theta$.

$$7\theta = \text{even } \pi$$

$$4\theta = \text{even } \pi - 3\theta.$$

$$\cos 4\theta = \cos 3\theta.$$

$$2\cos^2 2\theta - 1 = 4\cos^3\theta - 3\cos\theta.$$

$2(2\cos^2\theta - 1)^2 - 1 = 4\cos^3\theta - 3\cos\theta$; put $\cos\theta = x$,

$$2(2x^2 - 1)^2 - 1 = 4x^3 - 3x.$$

$$8x^4 - 4x^3 - 8x^2 + 3x + 1 = 0.$$

Remove $x - 1$ as factor ;

$$(x - 1)\;[8x^3 + 4x^2 - 4x - 1] = 0.$$

$\therefore\; 8x^3 + 4x^2 - 4x - 1 = 0$ has roots

$$\cos\frac{2\pi}{7}, \cos\frac{4\pi}{7}, \cos\frac{6\pi}{7}.$$

Put $2x = y$.

$y^3 + y^2 - 2y - 1 = 0$ has roots

$$2\cos\frac{2\pi}{7}, 2\cos\frac{4\pi}{7}, 2\cos\frac{6\pi}{7}.$$

Ex. 3. Show that $\cos\frac{\pi}{9} \cdot \cos\frac{2\pi}{9} \cdot \cos\frac{4\pi}{9} = \frac{1}{8}$.

Expanding $\cos 9\theta$, in powers of $\cos\theta$, we have

$$\cos 9\theta = 256\cos^9\theta - 576\cos^7\theta + 432\cos^5\theta - 120\cos^3\theta + 9\cos\theta$$

$\cos\left(\pm\frac{6\pi}{7}\right)$

$\cos\frac{6\pi}{7}$

$= 0$ has roots

$2\cos\frac{4\pi}{7},\ 2\cos\frac{6\pi}{7}$

$\frac{6\pi}{7} = 0.$

$7\theta =$ even π

$4\theta =$ even $\pi - 3\theta.$

$\cos 4\theta = \cos 3\theta.$

$= 4\cos^3\theta - 3\cos\theta.$

$= 4\cos^3\theta - 3\cos\theta$; put $\cos\theta = x$

$= 4x^3 - 3x.$

$+ 3x + 1 = 0.$

$x - 1$ as factor :

$(x-1)\,[8x^3 + 4x^2 - 4x - 1] = 0.$

$- 4x - 1 = 0$ has roots

$\cos\frac{4\pi}{7},\ \cos\frac{6\pi}{7}$

$- 1 = 0$ has roots

$2\cos\frac{2\pi}{7},\ 2\cos\frac{4\pi}{7},\ 2\cos\frac{6\pi}{7}$

has $\cos\frac{\pi}{9},\ \cos\frac{2\pi}{9}$

$\cos 9\theta$ in powers

$- 576\cos^7\theta$

But $\cos\frac{8\pi}{9} = -\cos\frac{\pi}{9}$.

Hence $\cos\frac{\pi}{9} \cdot \cos\frac{2\pi}{9} \cdot \cos\frac{4\pi}{9} = \frac{1}{8}$.

Ex. Find the equation whose roots are $\tan\frac{\pi}{5}$, $\tan\frac{2\pi}{5}$, $\tan\frac{3\pi}{5}$ and $\tan\frac{4\pi}{5}$.

$$\tan 5\theta = \frac{5\tan\theta - 5c_3\tan^3\theta + 5c_5\tan^5\theta}{1 - 5c_2\tan^2\theta + 5c_4\tan^4\theta}.$$

When $\theta = 0, \frac{\pi}{5}, \frac{2\pi}{5}, \frac{3\pi}{5}, \frac{4\pi}{5}$, $\tan 5\theta = 0$.

Hence $5\tan\theta - 5c_3\tan^3\theta + 5c_5\tan^5\theta = 0$ .. (1)

has roots $\tan\theta$, where θ is $0, \frac{\pi}{5}, \frac{2\pi}{5}, \frac{3\pi}{5}, \frac{4\pi}{5}$.

Put $\tan\theta = x$, then the equation (1) reduces to

$5x - 10x^3 + x^5 = 0$ .. (2)

Since 0 is a root of the equation, we have

$x^4 - 10x^2 + 5 = 0$ .. (3)

has roots $\tan\frac{\pi}{5}$, $\tan\frac{2\pi}{5}$, $\tan\frac{3\pi}{5}$, $\tan\frac{4\pi}{5}$.

Ex. Prove that

$$\tan\frac{\pi}{11} \cdot \tan\frac{2\pi}{11} \cdot \tan\frac{3\pi}{11} \cdot \tan\frac{4\pi}{11} \cdot \tan\frac{5\pi}{11} = \sqrt{11}.$$

$$\tan 11\theta = \frac{11\tan\theta - 11c_3\tan^3\theta + \ldots - \tan^{11}\theta}{1 - 11c_2\tan^2\theta \ldots - 11\tan^{10}\theta}.$$

If we put $\tan 11\theta = 0$, the equation

$11\tan\theta - 11c_3\tan^3\theta + \ldots - \tan^{11}\theta = 0$ .. (1)

has roots $\tan\theta$, where θ is $0, \frac{\pi}{11}, \frac{2\pi}{11}, \frac{3\pi}{11}, \frac{4\pi}{11}, \frac{5\pi}{11}, \ldots \frac{10\pi}{11}$.

Put $\tan\theta = x$, then the equation (1) reduces to

$11x - 55x^3 + 462x^5 - 330x^7 + 55x^9 - x^{11} = 0$ .. (2)

Hence equation (2) has roots

$0, \tan\frac{\pi}{11}, \tan\frac{2\pi}{11}, \tan\frac{3\pi}{11}, \ldots \tan\frac{9\pi}{11}, \tan\frac{10\pi}{11}$.

Since $\tan\frac{10\pi}{11} = -\tan\frac{\pi}{11}$, $\tan\frac{9\pi}{11} = -\tan\frac{2\pi}{11}$

$\tan\frac{8\pi}{11} = -\tan\frac{3\pi}{11}$, $\tan\frac{7\pi}{11} = -\tan\frac{4\pi}{11}$,

$\tan\frac{6\pi}{11} = -\tan\frac{5\pi}{11}$,

$$x^{10} - 55x^8 + 330x^6 - 462x^4 + 55x^2 - 11 = 0$$

has roots $\pm\tan\frac{\pi}{11}, \pm\tan\frac{2\pi}{11}, \pm\tan\frac{3\pi}{11}, \pm\tan\frac{4\pi}{11}, \pm\tan\frac{5\pi}{11}$

Put $x^2 = y$, then equation (3) reduces to

$$y^5 - 55y^4 + 330y^3 - 462y^2 + 55y - 11 = 0$$

This equation has roots

$$\tan^2\frac{\pi}{11}, \tan^2\frac{2\pi}{11}, \tan^2\frac{3\pi}{11}, \tan^2\frac{4\pi}{11}, \tan^2\frac{5\pi}{11}.$$

$$\therefore \tan^2\frac{\pi}{11}\cdot\tan^2\frac{2\pi}{11}\cdot\tan^2\frac{3\pi}{11}\cdot\tan^2\frac{4\pi}{11}\cdot\tan^2\frac{5\pi}{11} = 11$$

i.e., $\tan\frac{\pi}{11}\cdot\tan\frac{2\pi}{11}\cdot\tan\frac{3\pi}{11}\cdot\tan\frac{4\pi}{11}\cdot\tan\frac{5\pi}{11} = \sqrt{11}$.

The negative sign is discarded, since all the terms of the expression on the left side are positive, each angle involved being acute.

Ex. 6. Expand $\tan 4\theta$ in terms of $\tan\theta$ and show that $\tan\frac{\pi}{16}, \tan\frac{5\pi}{16}, \tan\frac{9\pi}{16}, \tan\frac{13\pi}{16}$ are roots of the equation $x^4 + 4x^3 - 6x^2 - 4x + 1 = 0$.

$$\tan 4\theta = \frac{s_1 - s_3}{1 - s_2 + s_4} = \frac{4c_1\tan\theta - 4c_3\tan^3\theta}{1 - 4c_2\tan^2\theta + 4c_4\tan^4\theta}$$

$$= \frac{4x - 4x^3}{1 - 6x^2 + x^4}, \text{ where } x = \tan\theta.$$

Let θ denote any of the angles $\frac{\pi}{16}, \frac{5\pi}{16}, \frac{9\pi}{16}, \frac{13\pi}{16}$.

Then $4\theta = \frac{\pi}{4}, \frac{5\pi}{4}, \frac{9\pi}{4}, \frac{13\pi}{4}$

$$= \left(n\pi + \frac{\pi}{4}\right), \text{ where } n = 0, 1, 2, 3.$$

$$\therefore \tan 4\theta = \tan\left(n\pi + \frac{\pi}{4}\right) = 1.$$

We make use of these re... expand $\cos^n \theta$ and $\sin^n \theta$ in series of cosines and sines of multiples of θ.

Expansions of $\cos n\theta$ when n is a positive integer.

$$2 \cos \theta = x + \frac{1}{x}.$$

$$\therefore (2 \cos \theta)^n = \left(x + \frac{1}{x}\right)^n$$

$$= x^n + nc_1 . x^{n-1} . \frac{1}{x} + nc_2 x^{n-2} . \frac{1}{x^2} + \ldots$$

$$+ \ldots . + nc_{n-2}\, x^2 . \frac{1}{x^{n-2}} + nc_{n-1} . x . \frac{1}{x^{n-1}} + \frac{1}{x^n}$$

$$= \left(x^n + \frac{1}{x^n}\right) + nc_1 \left(x^{n-2} + \frac{1}{x^{n-2}}\right) + nc_2 \left(x^{n-4} + \frac{1}{x^{n-4}}\right) + \ldots .$$

Since $x^n + \frac{1}{x^n} = 2 \cos n\theta$, we have

$$2^n \cos^n \theta = 2 \cos n\theta + nc_1\, 2 \cos (n-2)\, \theta + nc_2\, 2 \cos (n-4)\, \theta + \ldots .$$

$$\therefore 2^{n-1} \cos^n \theta = \cos n\theta + nc_1 \cos (n-2)\, \theta + nc_2 \cos (n-4)\, \theta + \ldots .$$

Note.—(1) If n is odd, there will be $(n+1)$ terms in the expansion of $\left(x + \frac{1}{x}\right)^n$ and hence these can be grouped in pairs. Hence the last term contains $\cos \theta$. We can easily see that the coefficient of $\cos \theta$ in the expansion of $2^{n-1} \cos^n \theta$ is $nc_{n-1/2}$.

(2) When n is even, the number of terms in the expansion of $\left(x + \frac{1}{x}\right)^n$ is $n+1$ and the middle term is independent of x and is left over when all the other terms are grouped in pairs. Hence the last term in the expansion of $2^{n-1} \cos^n \theta$ is independent of θ and is equal to $\frac{1}{2}\, {}^n c_{n/2}$.

Example.

Expand $\cos^6 \theta$ and $\cos^5 \theta$ in series of cosines of multiples of θ.

Let $x = \cos \theta$ ...

Then $(2\cos\theta)^6 = \left(x + \frac{1}{x}\right)^6$

$= x^6 + {}_6c_1 x^5 \cdot \frac{1}{x} + {}_6c_2 x^4 \cdot \frac{1}{x^2} + {}_6c_3 \ldots$

$+ {}_6c_4 x^2 \cdot \frac{1}{x^4} + {}_6c_5 \cdot x \cdot \frac{1}{x^5} + \ldots$

$= \left(x^6 + \frac{1}{x^6}\right) + {}_6c_1\left(x^4 + \frac{1}{x^4}\right)$

$+ {}_6c_2\left(x^2 + \frac{1}{x^2}\right) + \ldots$

$= 2\cos 6\theta + {}_6c_1 (2\cos 4\theta)$

$+ {}_6c_2 (2\cos 2\theta) + \ldots$

$\therefore \quad 2^5 \cos^6\theta = \cos 6\theta + 6\cos 4\theta + 15\cos 2\theta + 10.$

$\therefore \quad \cos^6\theta = \frac{1}{32}(\cos 6\theta + 6\cos 4\theta + 15\cos 2\theta + 10).$

Again $(2\cos\theta)^5 = \left(x + \frac{1}{x}\right)^5$

$= x^5 + 5c_1\, x^4 \frac{1}{x} + 5c_2\, x^3 \frac{1}{x^2} + 5c_3\, x^2 \frac{1}{x^3} + 5c_4\, x \frac{1}{x^4} + \frac{1}{x^5}$

$= \left(x^5 + \frac{1}{x^5}\right) + 5c_1\left(x^3 + \frac{1}{x^3}\right) + 5c_2\left(x + \frac{1}{x}\right)$

$= 2\cos 5\theta + 5c_1 (2\cos 3\theta) + 5c_2 (2\cos\theta).$

$\therefore \quad 2^4 \cos^5\theta = \cos 5\theta + 5\cos 3\theta + 10\cos\theta.$

§ 4·1. Expansion of $\sin^n\theta$ when n is a positive integer.

$2i\sin\theta = x - \frac{1}{x}.$

$\therefore \quad (2i\sin\theta)^n = \left(x - \frac{1}{x}\right)^n$

$= x^n - nc_1\, x^{n-1}\frac{1}{x} + nc_2\, x^{n-2}\frac{1}{x^2} \ldots$

Case 1. *n is even.*

The number of terms in the expansions is odd. The signs of the terms are alternatively positive and negative and the last term is positive.

$$\therefore \quad (2i \sin \theta)^n = \left(x^n + \frac{1}{x^n}\right) - nc_1 \left(x^{n-2} + \frac{1}{x^{n-2}}\right)$$

$$\ldots\ldots\ldots\ldots\ldots + nc_2 \left(x^{n-4} + \frac{1}{x^{n-4}}\right)$$

$$\text{i.e.,} \quad 2^n (-1)^{n/2} \sin^n \theta = (2 \cos n\theta) - nc_1\, 2 \cos (n-2)\theta$$
$$+ nc_2\, 2 \cos (n-4)\, \theta \ldots.$$

$$\text{Hence } (-1)^{n/2}\, 2^{n-1} \sin^n \theta = \cos n\theta - nc_1 \cos (n-2)\, \theta$$
$$+ nc_2 \cos (n-4)\theta \ldots.$$

Case 2. *n is odd.*

$$(2i \sin \theta)^n = x^n - nc_1\, x^{n-2} + nc_2\, x^{n-4} \ldots. - \frac{1}{x^n}$$

$$= \left(x^n - \frac{1}{x^n}\right) - nc_1 \left(x^{n-2} - \frac{1}{x^{n-2}}\right)$$
$$+ nc_2 \left(x^{n-4} - \frac{1}{x^{n-4}}\right) \ldots.$$

$$= 2i \sin n\theta - nc_1\, 2i \sin (n-2)\, \theta$$
$$+ nc_2\, 2i \sin (n-4)\, \theta + \ldots.$$

$$\text{i.e.,} \quad 2^{n-1} (i)^{n-1} \sin^n \theta = \sin n\theta - nc_1 \sin (n-2)\, \theta$$
$$+ nc_2 \sin (n-4)\, \theta + \ldots.$$

$$\text{i.e.,} \quad 2^{n-1} (-1)^{(n-1)/2} \sin^n \theta = \sin n\theta - nc_1 \sin (n-2)\, \theta$$
$$+ nc_2 \sin (n-4)\, \theta + \ldots.$$

Examples.

Ex. 1. Expand $\sin^7 \theta$ in a series of sines of multiples of θ.

We have

$$\left(x - \frac{1}{x}\right)^7 = x^7 - 7x^5 + 21x^3 - 35x$$
$$+ \frac{35}{x} - \frac{21}{x^3} + \frac{7}{x^5} - \frac{1}{x^7}$$

$$= \left(x^7 - \frac{1}{x^7}\right) - 7\left(x^5 - \frac{1}{x^5}\right)$$
$$+ 21\left(x^3 - \frac{1}{x^3}\right) - 35\left(x - \frac{1}{x}\right).$$

Putting $x = \cos\theta + i\sin\theta$, so that $x^n - \frac{1}{x^n} = 2i\sin n\theta$ for all integral values of n, we have

$$(2i\sin\theta)^7 = 2i\sin 7\theta - 7(2i\sin 5\theta) + 21(2i\sin 3\theta) - 35(2i\sin\theta)$$

i.e., $$2^6(-1)^3\sin^7\theta = \sin 7\theta - 7\sin 5\theta + 21\sin 3\theta - 35\sin\theta$$

$$\therefore \quad \sin^7\theta = -\frac{1}{64}(\sin 7\theta - 7\sin 5\theta + 21\sin 3\theta - 35\sin\theta).$$

Ex. 2. Expand $\sin^6\theta$ in a series of cosines of multiples of θ.

We have

$$\left(x-\frac{1}{x}\right)^6 = x^6 - 6x^4 + 15x^2 - 20 + \frac{15}{x^2} - \frac{6}{x^4} + \frac{1}{x^6}$$

$$= \left(x^6+\frac{1}{x^6}\right) - 6\left(x^4+\frac{1}{x^4}\right) + 15\left(x^2+\frac{1}{x^2}\right) - 20.$$

Putting $x = \cos\theta + i\sin\theta$, $x - \frac{1}{x} = 2i\sin\theta$ and $x^n + \frac{1}{x^n} = 2\cos n\theta$ for all integral values of n.

$$\therefore \quad (2i\sin\theta)^6 = 2\cos 6\theta - 6(2\cos 4\theta) + 15(2\cos 2\theta) - 20$$

i.e., $$2^6(-1)^3\sin^6\theta = 2\cos 6\theta - 6(2\cos 4\theta) + 15(2\cos 2\theta) - 20.$$

$$\therefore \quad \sin^6\theta = -\frac{1}{32}(\cos 6\theta - 6\cos 4\theta + 15\cos 2\theta - 10).$$

Ex. 3. Expand $\sin^3\theta\cos^5\theta$ in a series of sines of multiples of θ.

$$(2i\sin\theta)^3(2\cos\theta)^5 = \left(x-\frac{1}{x}\right)^3\left(x+\frac{1}{x}\right)^5$$

$$= \left(x-\frac{1}{x}\right)^3\left(x+\frac{1}{x}\right)^3\left(x+\frac{1}{x}\right)^2$$

$$= \left(x^2-\frac{1}{x^2}\right)^3\left(x+\frac{1}{x}\right)^2$$

$$= \left(x^6 - 3x^2 + \frac{3}{x^2} - \frac{1}{x^6}\right)\left(x^2 + 2 + \frac{1}{x^2}\right)$$

$$= \left(x^8 - \frac{1}{x^8}\right) + 2\left(x^6 - \frac{1}{x^6}\right) - 2\left(x^4 - \frac{1}{x^4}\right) - 6\left(x^2 - \frac{1}{x^2}\right)$$

$$= (2i \sin 8\theta) + 2(2i \sin 6\theta) - 2(2i \sin 4\theta) - 6(2i \sin 2\theta)$$

i.e., $2^3(-i)\sin^3\theta \; 2^5 \cos^5\theta$
$= 2i(\sin 8\theta + 2\sin 6\theta - 2\sin 4\theta - 6\sin 2\theta)$.

$$\therefore \sin^3\theta \cos^5\theta = -\frac{1}{2^7}(\sin 8\theta + 2\sin 6\theta - 2\sin 4\theta - 6\sin 2\theta).$$

Ex. 4. Expand $\sin^4\theta \cos^2\theta$ in a series of cosines of multiples of θ.

$$(2i\sin\theta)^4 (2\cos\theta)^2 = \left(x - \frac{1}{x}\right)^4 \left(x + \frac{1}{x}\right)^2$$

$$= \left(x^2 - \frac{1}{x^2}\right)^2 \left(x - \frac{1}{x}\right)^2$$

$$= \left(x^4 - 2 + \frac{1}{x^4}\right)\left(x^2 - 2 + \frac{1}{x^2}\right)$$

$$= \left(x^6 + \frac{1}{x^6}\right) - 2\left(x^4 + \frac{1}{x^4}\right) - \left(x^2 + \frac{1}{x^2}\right) + 4$$

$$= 2\cos 6\theta - 2(2\cos 4\theta) - 2\cos 2\theta + 4$$

i.e., $2^4 \sin^4\theta \,.\, 2^2\cos^2\theta = 2\cos 6\theta - 2(2\cos 4\theta) - 2\cos 2\theta + 4$

$$\therefore \sin^4\theta\cos^2\theta = \frac{1}{2^6}(\cos 6\theta - 2\cos 4\theta - \cos 2\theta + 2).$$

Exercises XI

Prove the following results :—

1. $2^6 \cos^7\theta$ ...

HYPERBOLIC FUNCTIONS

§ 1. If θ is expressed in radians, $\cos\theta$ and $\sin\theta$ can be expanded in powers of θ, the results being

$$\cos\theta = 1 - \frac{\theta^2}{2\,!} + \frac{\theta^4}{4\,!} - \frac{\theta^6}{6\,!} + \ldots.\infty \qquad \ldots (1)$$

$$\sin\theta = \theta - \frac{\theta^3}{3\,!} + \frac{\theta^5}{5\,!} - \frac{\theta^7}{7\,!} + \ldots.\infty \qquad \ldots (2)$$

(For proofs of these, refer to § 5 on page 74. These expansions are valid for all values of θ, real or imaginary.)

The student is familiar with the exponential series, *viz.*, for all real values of x.

$$e^x = 1 + \frac{x}{1\,!} + \frac{x^2}{2\,!} + \ldots. + \frac{x^n}{n\,!} + \ldots.\infty \qquad \ldots (3)$$

$$\text{where } e = 1 + \frac{1}{1\,!} + \frac{1}{2\,!} + \ldots. + \frac{1}{n\,!} + \ldots.\infty.$$

Put $x = i\theta$ in (3). Then

$$e^{i\theta} = 1 + \frac{i\theta}{1\,!} + \frac{(i\theta)^2}{2\,!} + \frac{(i\theta)^3}{3\,!} + \ldots.\infty$$

$$= 1 + \frac{i\theta}{1\,!} - \frac{\theta^2}{2\,!} - \frac{i\theta^3}{3\,!} + \ldots.\infty$$

$$= \left(1 - \frac{\theta^2}{2\,!} + \frac{\theta^4}{4\,!} \ldots.\infty\right) + i\left(\frac{\theta}{1\,!} - \frac{\theta^3}{3\,!} + \frac{\theta^5}{5\,!} \ldots.\infty\right)$$

$= \cos\theta + i\sin\theta$ from (1) and (2).

(This formula is known as Euler's formula.)

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Put $x = -i\theta$ in (3). Then

$$e^{-i\theta} = 1 + \frac{(-i\theta)}{1\,!} + \frac{(-i\theta)^2}{2\,!} + \frac{(-i\theta)^3}{3\,!} + \ldots\ldots\infty$$

$$= 1 - \frac{i\theta}{1\,!} - \frac{\theta^2}{2\,!} + \frac{i\theta^3}{3\,!} + \frac{\theta^4}{4\,!} \ldots\ldots \infty$$

$$= \left(1 - \frac{\theta^2}{2\,!} + \frac{\theta^4}{4\,!} \ldots\ldots \infty\right)$$

$$- i\left(\frac{\theta}{1\,!} - \frac{\theta^3}{3\,!} \ldots\ldots \infty\right)$$

$$= \cos\theta - i\sin\theta.$$

Hence we get the relations

$$e^{i\theta} = \cos\theta + i\sin\theta$$

$$e^{-i\theta} = \cos\theta - i\sin\theta.$$

Adding $2\cos\theta = e^{i\theta} + e^{-i\theta}$

$$\text{i.e.,}\quad \cos\theta = \frac{e^{-i\theta} + e^{-i\theta}}{2} \qquad \ldots (4)$$

Subtracting we get the relation

$$2i\sin\theta = e^{-\theta} - e^{-i\theta}$$

$$\text{i.e.,}\quad \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i} \qquad \ldots (5)$$

§ 2. Hyperbolic functions.

The expression $\frac{1}{2}(e^x + e^{-x})$ and $\frac{1}{2}(e^x - e^{-x})$ are defined as hyperbolic cosine and sine respectively of the angle x and symbolically

$$\cosh x = \frac{e^x + e^{-x}}{2}$$

$$\sinh x = \frac{e^x - e^{-x}}{2}.$$

The hyperbolic tangent, secant, cosecant and cotangent are obtained from the hyperbolic sine and cosine just as the ... obtained

Thus $\tanh x = \dfrac{\sinh x}{\cosh x}$

$\operatorname{sech} x = \dfrac{1}{\cosh x}$

$\operatorname{cosech} x = \dfrac{1}{\sinh x}$

$\coth x = \dfrac{1}{\tanh x}$.

§ 2·1. Relations between hyperbolic functions.

(1) $\cosh^2 x - \sinh^2 x = \frac{1}{4}\{(e^x + e^{-x})^2 - (e^x - e^{-x})^2\}$

$= \frac{1}{4}\{e^{2x} + 2 + e^{-2x} - (e^{2x} - 2 + e^{-2x})\}$

$= 1.$

(2) $2 \sinh x \cosh x = 2 \cdot \dfrac{(e^x - e^{-x})}{2} \cdot \dfrac{(e^x + e^{-x})}{2}$

$= \dfrac{(e^{2x} - e^{-2x})}{2} = \sinh 2x.$

(3) $\cosh^2 x + \sinh^2 x = \{\frac{1}{4}(e^x + e^{-x})^2 + (e^x - e^{-x})^2\}$

$= \{\frac{1}{4}(e^{2x} + 2 + e^{-2x}) + (e^{2x} - 2 + e^{-2x})\}$

$= \dfrac{e^{2x} + e^{-2x}}{2} = \cosh 2x.$

(4) From the relation (3), we get the relations

$\cosh 2x = 2\cosh^2 x - 1$

$\cosh 2x = 1 + 2\sinh^2 x$

$\cosh^2 x = \frac{1}{2}(\cosh 2x + 1)$

$\sinh^2 x = \frac{1}{2}(\cosh 2x - 1).$

(5) The series for $\sinh x$ and $\cosh x$ are derived below :

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots.$$

$$e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} - \ldots.$$

Subtracting $e^x - e^{-x} = 2\left(x + \dfrac{x^3}{3!} \ldots \infty\right).$

$$\therefore \sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots \infty.$$

Adding, $e^x + e^{-x} = 2\left(1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots\ldots \infty\right).$

$\therefore \quad \cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots\ldots \infty.$

(6) We have seen that

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}, \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}.$$

Put $\theta = ix$ in these relations. We have

$$\cos(ix) = \frac{e^{-x} + e^{x}}{2} = \cosh x$$

$$\sin(ix) = \frac{e^{-x} - e^{x}}{2i} = (i)^2 \frac{\sinh x}{i}$$

$$= i \sin x.$$

$\therefore \quad \tan(ix) = i \tanh x.$

The following relations also hold good :—

$$\sinh(i\theta) = \frac{e^{i\theta} - e^{-i\theta}}{2} = i\sin\theta.$$

$$\cosh(i\theta) = \frac{e^{i\theta} + e^{-i\theta}}{2} = \cos\theta.$$

$$\tanh(i\theta) = i\tan\theta.$$

§ **2·2.** Using these relations, we can derive relations between hyperbolic functions corresponding to relations between circular functions. For example,

(*i*) $\sin^2\theta + \cos^2\theta = 1.$ Put $\theta = ix.$

$\therefore \quad \sin^2(ix) + \cos^2(ix) = 1$

i.e., $(i\sinh x)^2 + (\cosh x)^2 = 1$

i.e., $\cosh^2 x - \sinh^2 x = 1.$

(*ii*) $\cos 2\theta = \cos^2\theta - \sin^2\theta$

$$\cos(2ix) = \cos^2(ix) - \sin^2(ix)$$

$$= (\cosh x)^2 - (i\sinh x)^2$$

$$= \cosh^2 x + \sinh^2 x.$$

(iii) $\sin 2\theta = 2 \sin \theta \cos \theta.$

$\sin (2ix) = 2 \sin (ix) \cos (ix)$

i.e., $i \sinh 2x = 2i \sinh x \cosh x$

i.e., $\sinh 2x = 2 \sinh x \cosh x.$

(iv) $1 + \tan^2 \theta = \sec^2 \theta$

$1 + \tan^2 (ix) = \sec^2 (ix)$

$$1 + (i \tanh x)^2 = \frac{1}{(\cosh x)^2}$$

i.e., $1 - \tanh^2 x = \text{sech}^2 x.$

(v) $\sin (\theta + \phi) = \sin \theta \cos \phi + \cos \theta \sin \phi.$

Put $\theta = ix$, $\phi = iy$. Then

$\sin (ix + iy) = \sin (ix) \cos (iy) + \cos (ix) \sin (iy)$

i.e., $i \sinh (x + y) = i \sinh x \cosh y + (\cosh x)(i \sin y).$

$\therefore \sinh (x + y) = \sinh x \cosh y + \cosh x \sinh y.$

Similarly $\sinh (x - y) = \sinh x \cosh y - \cosh x \sinh y.$

(vi) $\cos (\theta + \phi) = \cos \theta \cos \phi - \sin \theta \sin \phi.$

Put $\theta = ix$, $\phi = iy$. Then

$\cos (ix + iy) = \cos ix \cos iy - \sin ix \sin iy$

$$\begin{aligned}\cosh (x + y) &= \cosh x \cosh y - (i \sinh x)(i \sinh y) \\ &= \cosh x \cosh y + \sinh x \sinh y.\end{aligned}$$

Similarly $\cosh (x - y) = \cosh x \cosh y - \sinh x \sinh y.$

(vii) $\tan 2\theta = \dfrac{2 \tan \theta}{1 - \tan^2 \theta}.$

Put $\theta = ix$.

$$\therefore \tan (2ix) = \frac{2 \tan (ix)}{1 - \tan^2 (ix)}$$

$$i \tanh 2x = \frac{2i \tanh x}{1 - (i \tanh x)^2}.$$

$$\therefore \tanh 2x = \frac{2 \tanh x}{1 + \tanh^2 x}.$$

§ 2·3. Inverse hyperbolic functions.

We can express $\sinh^{-1} x$, $\cos^{-1} x$, $\tanh^{-1} x$ in terms of logarithmic functions.

(*i*) Let $y = \sinh^{-1} x$. Then $x = \sinh y$.

$\therefore \quad \frac{1}{2}(e^y - e^{-y}) = x$

i.e., $e^{2y} - 1 = 2x\, e^y$

i.e., $e^{2y} - 2x\, e^y - 1 = 0$.

$$\therefore \quad e^y = \frac{2x \pm \sqrt{4x^2 + 4}}{2} = x \pm \sqrt{x^2 + 1}.$$

Since e^y is always positive $e^y = x + \sqrt{x^2 + 1}$.

Taking logarithms to the base e on both sides, we have

$$y = \log_e (x + \sqrt{x^2 + 1}).$$

$\therefore \quad \sinh^{-1} x = \log_e (x + \sqrt{x^2 + 1})$.

(*ii*) $y = \cosh^{-1} x$. Then $x = \cosh y$.

$$\therefore \quad \frac{1}{2}(e^y + e^{-y}) = x$$

i.e., $e^{2y} - 2x\, e^y + 1 = 0$.

$$\therefore \quad e^y = x \pm \sqrt{x^2 - 1}$$

$$= x + \sqrt{x^2 - 1} \text{ or } \frac{1}{x + \sqrt{x^2 - 1}}.$$

$$\therefore \quad y = \log_e (x + \sqrt{x^2 - 1}) \text{ or } - \log_e (x + \sqrt{x^2 - 1}).$$

$$= \pm \log_e (x + \sqrt{x^2 - 1}).$$

The positive sign is usually taken.

$\cosh^{-1} x = \log_e (x + \sqrt{x^2 - 1})$.

(*iii*) Let $y = \tanh^{-1} x$. Then $x = \tanh y$.

$$\therefore \quad \frac{e^y - e^{-y}}{e^y + e^{-y}} = x$$

i.e., $e^y - e^{-y} = x\,(e^y + e^{-y})$

i.e., $e^y (1 - x) = e^{-y} (1 + x)$

i.e., $$e^{2y} = \frac{1 + x}{1 - x}.$$

$\therefore \; 2y = \log_e\left(\dfrac{1+x}{1-x}\right)$

(ii) $y = \dfrac{1}{2}\log_e\left(\dfrac{1+x}{1-x}\right)$.

$\tanh^{-1} x = \dfrac{1}{2}\log_e\left(\dfrac{1+x}{1-x}\right)$.

Example

Ex. 1. If $\cosh u = \sec\theta$, show that

$$u = \log\tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right).$$

$\cosh u = \sec\theta$.

$$\begin{aligned}\therefore \; u &= \cosh^{-1}(\sec\theta)\\ &= \log_e\left(\sec\theta + \sqrt{\sec^2\theta - 1}\right)\\ &= \log_e(\sec\theta + \tan\theta)\\ &= \log_e\left(\frac{1+\sin\theta}{\cos\theta}\right)\end{aligned}$$

$$= \log_e\left\{1 + \frac{2\tan\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}}\right\} \Big/ \left\{\frac{1-\tan^2\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}}\right\}$$

$$= \log_e \frac{1+\tan\frac{\theta}{2}}{1-\tan\frac{\theta}{2}} = \log_e \tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right).$$

Ex. 2. If $\tan A = \tan\alpha \tanh\beta$, $\tan B = \cot\alpha \tanh\beta$, prove that $\tan(A+B) = \sinh 2\beta \operatorname{cosec} 2\alpha$.

$$\begin{aligned}\tan(A+B) &= \frac{\tan A + \tan B}{1 - \tan A \tan B}\\ &= \frac{\tan\alpha\tanh\beta + \cot\alpha\tanh\beta}{1 - \tan\alpha\tanh\beta \,.\, \cot\alpha\tanh\beta}\\ &= \frac{\tanh\beta(\tan\alpha + \cot\alpha)}{1-\tanh^2\beta}\\ &= \frac{\sinh\beta\cosh\beta}{\cosh^2\beta - \sinh^2\beta}\left(\frac{\sin\alpha}{\cos\alpha}+\frac{\cos\alpha}{\sin\alpha}\right)\end{aligned}$$

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$$= \frac{\sinh \beta \cosh \beta}{\sin \alpha \cos \alpha}$$

$$= \frac{\frac{1}{2} \sinh 2\beta}{\frac{1}{2} \sin 2\alpha}$$

$$= \sinh 2\beta \operatorname{cosec} 2\alpha.$$

Ex. 3. Express $\cosh^6 \theta$ in terms of hyperbolic cosines of multiples of θ.

$$\cosh^6 \theta = \left(\frac{e^{\theta} + e^{-\theta}}{2}\right)^6$$

$$= \frac{1}{2^6}\Big[e^{6\theta} + 6c_1 e^{4\theta} + 6c_2 e^{2\theta} + 6c_3 + 6c_4 e^{-2\theta} + 6c_5 e^{-4\theta} + 6c_6 e^{-6\theta} \Big] \text{ by the Binomial Theorem}$$

$$= \frac{1}{2^6}\Big[(e^{6\theta} + e^{-6\theta}) + 6c_1 (e^{4\theta} + e^{-4\theta}) + 6c_2 (e^{2\theta} + e^{-2\theta}) + 6c_3 \Big]$$

$$= \frac{1}{2^5}\Big[\cosh 6\theta + 6 \,.\, \cosh 4\theta + 15 \cosh 2\theta + 10 \Big].$$

Ex. 4. If $\cos \alpha \,.\, \cosh \beta = \cos \phi$, $\sin \alpha \sinh \beta = \sin \phi$, prove that $\sin \phi = \pm \sin^2 \alpha = \pm \sinh^2 \beta$.

$$\cosh \beta = \frac{\cos \phi}{\cos \alpha}, \quad \sinh \beta = \frac{\sin \phi}{\sin \alpha}$$

$$\cosh^2 \beta - \sinh^2 \beta = 1.$$

$$\therefore \quad \frac{\cos^2 \phi}{\cos^2 \alpha} - \frac{\sin^2 \phi}{\sin^2 \alpha} = 1$$

i.e., $\cos^2 \phi \sin^2 \alpha - \sin^2 \phi \cos^2 \alpha = \cos^2 \alpha \sin^2 \alpha$

i.e., $(1 - \sin^2 \phi) \sin^2 \alpha - \sin^2 \phi (1 - \sin^2 \alpha) = \sin^2 \alpha (1 - \sin^2 \alpha).$

Simplifying, $\sin^2 \phi = \sin^4 \alpha.$

$\therefore \quad \sin \phi = \pm \sin^2 \alpha.$

$\sin \alpha \sinh \beta = \sin \phi$

i.e., $\sin \alpha \sinh \beta = \pm \sin^2 \alpha$

i.e., $\sinh \beta = \pm \sin \alpha.$

$\therefore \quad \sinh^2 \beta = \sin^2 \alpha = \pm \sin \phi.$

Hence $\sin \phi = \pm \sin^2 \alpha = \pm \sinh^2 \beta$.

Ex. 5. If $\cos (x + iy) = \cos \theta + i \sin \theta$, prove that $\cos 2x + \cosh 2y = 2$.

We have $\cos \theta + i \sin \theta = \cos (x + iy)$

$$= \cos x \cos (iy) - \sin x \sin (iy)$$

$$= \cos x \cosh y - i \sin x \sinh y.$$

Equating the real and imaginary parts, we have

$$\cos \theta = \cos x \cosh y$$

$$\sin \theta = - \sin x \sinh y.$$

Squaring and adding,

$$\cos^2 x \cosh^2 y + \sin^2 x \sinh^2 y = 1$$

i.e., $\cos^2 x \cosh^2 y + (1 - \cos^2 x) \sinh^2 y = 1$

i.e., $\cos^2 x (\cosh^2 y - \sinh^2 y) + \sinh^2 y = 1$

i.e., $\cos^2 x + \sinh^2 y = 1$

i.e., $\dfrac{1 + \cos 2x}{2} + \dfrac{\cosh 2y - 1}{2} = 1.$

$\therefore \cos 2x + \cosh 2y = 2.$

Ex. 6. If $\sin (A + iB) = x + iy$, prove that

(1) $\dfrac{x^2}{\sin^2 A} - \dfrac{y^2}{\cos^2 A} = 1.$

(2) $\dfrac{x^2}{\cosh^2 B} + \dfrac{y^2}{\sinh^2 B} = 1.$

$$x + iy = \sin (A + iB)$$

$$= \sin A \cos (iB) + \cos A \sin (iB)$$

$$= \sin A \cosh B + i \cos A \sinh B.$$

Equating real and imaginary parts, we have

$$x = \sin A \cosh B$$

$$y = \cos A \sinh B.$$

$$\therefore \frac{x^2}{\sin^2 A} - \frac{y^2}{\cos^2 A} = \cosh^2 B - \sinh^2 B = 1.$$

$$\frac{x^2}{\cosh^2 B} + \frac{y^2}{\sinh^2 B} = \sin^2 A + \cos^2 A = 1.$$

Ex. 7. If $\cosh(a+ib)\cos(c+id)=1$, prove that

(1) $\cos b \cos c \cosh a \cosh d$

$$+\sin b \sin c \sinh a \sinh d = 1,$$

(2) $\tanh a \tan b = \tanh d \tan c.$

$$1 = \cosh(a+ib)\cos(c+id)$$

$$= \{\cosh a \cosh(ib) + \sinh a \sinh(ib)\}$$

$$\{\cos c \cos(id) - \sin c \sin(id)\}$$

$$\cosh(iy) = \cos y$$

and $\sinh(iy) = i \sin y$. [*Vide* § 2·1 (6).]

$\therefore\quad \cosh(ib) = \cos b$ and $\sinh(ib) = i \sin b.$

Substituting these values in equation (1), we have

$$1 = (\cosh a \cos b + i \sinh a \sin b)(\cos c \cosh d - i \sin c \sinh d)$$

$$= \cosh a \cos b \cos c \cosh d + \sinh a \sin b \sin c \sinh d$$

$$+ i(\sinh a \sin b \cos c \cosh d - \cosh a \cos b \sin c \sinh d).$$

Equating the real parts, we get result (1).

Equating the imaginary parts, we have

$$\sinh a \sin b \cos c \cosh d - \cosh a \cosh b \sin c \sinh d = 0$$

i.e., $$\frac{\sinh a \sin b}{\cosh a \cos b} - \frac{\sin c \sinh d}{\cos c \cosh d} = 0$$

i.e., $$\tanh a \tan b - \tan c \tanh d = 0.$$

Ex. 8. If $\tan(x+iy) = u + iv$, prove that

$$\frac{u}{v} = \frac{\sin 2x}{\sinh 2y}.$$

$$\tan(x+iy) = \frac{\sin(x+iy)}{\cos(x+iy)}$$

$$= \frac{2\cos(x-iy)\sin(x+iy)}{2\cos(x-iy)\cos(x+iy)}$$

$$= \frac{\sin(2x) + \sin(2iy)}{\cos 2x + \cosh 2iy}$$

$$= \frac{\sin 2x + i \sinh 2y}{\cos 2x + \cosh 2y}.$$

This expression is given as $u + iv$.

$$\therefore \quad u = \frac{\sin 2x}{\cos 2x + \cosh 2y}$$

$$v = \frac{\sinh 2y}{\cos 2x + \cosh 2y}.$$

$$\therefore \quad \frac{u}{v} = \frac{\sin 2x}{\sinh 2y}.$$

Ex. 9. Separate into real and imaginary parts

$$\tanh (1 + i).$$

It is known that $\tan (ix) = i \tanh x$.

Put $x = 1 + i$. $\therefore \quad i \tanh (1 + i) = \tan i (1 + i)$

$$= \tan (i - 1).$$

$$\therefore \quad i \tanh (1 + i) = \frac{\sin (i - 1)}{\cos (i - 1)}$$

$$= \frac{2 \cos (i + 1) \sin (i - 1)}{2 \cos (i + 1) \cos (i - 1)}$$

$$= \frac{\sin (2i) - \sin (2)}{\cos (2i) + \cos (2)}$$

$$= \frac{i \sinh 2 - \sin 2}{\cosh 2 + \cos (2)}.$$

$$\therefore \quad \tanh (1 + i) = \frac{\sinh (2) + i \sin (2)}{\cosh (2) + \cos (2)}.$$

Ex. 10. Separate into real and imaginary parts

$$\tan^{-1} (x + iy).$$

Let $\tan^{-1} (x + iy) = \alpha + i\beta$.

Then $\tan (\alpha + i\beta) = (x + iy)$.

We easily see that $\tan (\alpha - i\beta) = (x - iy)$.

$$\tan (2\alpha) = \tan (\alpha + i\beta + \alpha - i\beta)$$

$$= \frac{\tan (\alpha + i\beta) + \tan (\alpha - i\beta)}{1 - \tan (\alpha + i\beta) \tan (\alpha - i\beta)}$$

$$= \frac{x + iy + x - iy}{1 - (x + iy)(x - iy)}$$

$$= \frac{2x}{1 - (x^2 + y^2)}.$$

$$\therefore \quad \alpha = \frac{1}{2} \tan^{-1}\left(\frac{2x}{1 - x^2 - y^2}\right).$$

$$\tan(2\beta i) = \tan\left(\overline{\alpha + i\beta} - \overline{\alpha - i\beta}\right)$$

$$i.e., \; i \tanh 2\beta = \frac{\tan(\alpha + i\beta) - \tan(\alpha - i\beta)}{1 + \tan(\alpha + i\beta)\tan(\alpha - i\beta)}$$

$$= \frac{(x + iy) - (x - iy)}{1 + (x + iy)(x - iy)}$$

$$= \frac{2iy}{1 + x^2 + y^2}.$$

$$\therefore \quad \tanh 2\beta = \frac{2y}{1 + x^2 + y^2}.$$

$$\therefore \quad \beta = \frac{1}{2} \tanh^{-1}\left(\frac{2y}{1 + x^2 + y^2}\right).$$