TRIGONOMETRY

CHAPTER I

EXPANSION IN SERIES

When n is a positive integer, expand $\cos^n \theta$ in a series of cosines of multiples of θ.

Let $x = \cos\theta + i\sin\theta$

Then $\frac{1}{x} = x^{-1} = (\cos\theta + i\sin\theta)^{-1} = \cos\theta - i\sin\theta$

$\therefore \quad x + \frac{1}{x} = 2\cos\theta.$ ...(*i*)

Again, since $x = \cos\theta + i\sin\theta$, $x^n = (\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$.

and $\frac{1}{x^n} = x^{-n} = (\cos\theta + i\sin\theta)^{-n} = \cos n\theta - i\sin n\theta$

$\therefore \quad x^n + \frac{1}{x^n} = 2\cos n\theta$ ...(*ii*)

From (*i*), $(2\cos\theta)^n = \left(x + \frac{1}{x}\right)^n$

$$= x^n + {}^nC_1 x^{n-1}\frac{1}{x} + ... + {}^nC_{n-1}x.\frac{1}{x^{n-1}} + \frac{1}{x^n},$$

using the binomial theorem.

But ${}^nC_{n-1} = {}^nC_1; {}^nC_{n-2} = {}^nC_2.........$

$\therefore \quad (2\cos\theta)^n = x^n + {}^nC_1 x^{n-2} + {}^nC_2 x^{n-4} + ...$

$$+ {}^nC_2\frac{1}{x^{n-4}} + {}^nC_1\frac{1}{x^{n-2}} + \frac{1}{x^n}$$

i.e., $2^n\cos^n\theta = \left(x^n + \frac{1}{x^n}\right) + {}^nC_1\left(x^{n-2} + \frac{1}{x^{n-2}}\right)$

$$+ {}^nC_2\left(x^{n-4} + \frac{1}{x^{n-4}}\right) + \text{...(iii)}$$

In (*ii*), if we replace n by $n-2$; $n-4$ etc, we get,

$$x^{n-2} + \frac{1}{x^{n-2}} = 2\cos(n-2)\theta\ ;\ x^{n-4} + \frac{1}{x^{n-4}} = 2\cos(n-4)\theta\ ;\ \text{etc.}$$

Using these in (*iii*), we have

$2^n\cos^n\theta = 2\cos n\theta + {}^nC_1.2\cos(n-2)\theta + {}^nC_2.\ 2\cos(n-4)\theta + ...$

$$\therefore \cos^n \theta = \frac{1}{2^{n-1}} [\cos n\theta + {}^nC_1 \cos (n-2)\theta + {}^nC_2 \cos (n-4)\theta + ...]$$

Note 1. It can be shown that when n is odd, the last term in the bracket is $\dfrac{n!}{\left(\frac{n-1}{2}\right)!\left(\frac{n+1}{2}\right)!} \cos \theta$, while when n is even, that term is $\dfrac{1}{2} \dfrac{n!}{\left\{\left(\frac{n}{2}\right)!\right\}^2}$.

Note 2. The expansion for a powr of a sine will be in a series of sines or cosines according as the power is odd or even; this will be seen from the following examples.

Ex. 1. *Using De Moivre's theorem, expand $\cos^8 \theta$ in a series of cosines of multiples of θ.*

Sol. Let $x = \cos \theta + i \sin \theta$. Then as before, $x + \dfrac{1}{x} = 2 \cos \theta$ ...(*i*)

and $$x^n + \frac{1}{x^n} = 2 \cos n\theta \quad ...(ii)$$

From (*i*), $$(2 \cos \theta)^8 = \left(x + \frac{1}{x}\right)^8$$

$$= x^8 + {}^8C_1 x^7 . \frac{1}{x} + {}^8C_2 x^6 . \frac{1}{x^2} + {}^8C_3 x^5 . \frac{1}{x^3} + {}^8C_4 x^4 \frac{1}{x^4} + {}^8C_5 x^3 . \frac{1}{x^5} + {}^8C_6 x^2 \frac{1}{x^6} + {}^8C_7 x \frac{1}{x^7} + \frac{1}{x^8}$$

$$= x^8 + 8x^6 + 28x^4 + 56x^2 + 70 + 56 . \frac{1}{x^2} + 28 . \frac{1}{x^4} + 8 . \frac{1}{x^6} + \frac{1}{x^8}$$

i.e., $$(2 \cos \theta)^8 = \left(x^8 + \frac{1}{x^8}\right) + 8\left(x^6 + \frac{1}{x^6}\right) + 28\left(x^4 + \frac{1}{x^4}\right) + 56\left(x^2 + \frac{1}{x^2}\right) + 70 \quad ...(iii)$$

In (*ii*), replacing n by 8, 6, 4 and 2, we get,

$$x^8 + \frac{1}{x^8} = 2 \cos 8\theta \,;\, x^6 + \frac{1}{x^6} = 2 \cos 6\theta \,;\, x^4 + \frac{1}{x^4} = 2 \cos 4\theta,$$

and $$x^2 + \frac{1}{x^2} = 2 \cos 2\theta.$$

Using these in (*iii*), we have

$$2^8 \cos^8 \theta = 2 \cos 8\theta + 8 \times 2 \cos 6\theta + 28 \times 2 \cos 4\theta + 56 \times 2 \cos 2\theta + 70$$

$$= 2[\cos 8\theta + 8 \cos 6\theta + 28 \cos 4\theta + 56 \cos 2\theta + 35]$$

$$\therefore \quad \cos^8 \theta = \frac{1}{2^7} [\cos 8\theta + 8 \cos 6\theta + 28 \cos 4\theta + 56 \cos 2\theta + 35]$$

Ex. 2. *Expand* $\sin^8\theta$ *in a series of cosines of multiples of* θ.

Sol. Let $x = \cos\theta + i\sin\theta$.

Then $\frac{1}{x} = x^{-1} = (\cos\theta + i\sin\theta)^{-1} = \cos\theta - i\sin\theta$

$\therefore \quad x - \frac{1}{x} = 2i\sin\theta$ ...(*i*)

Again as before $x^n + \frac{1}{x^n} = 2\cos n\theta$. ...(*ii*)

From (*i*), $(2i\sin\theta)^8 = \left(x - \frac{1}{x}\right)^8$

$$= x^8 - {}^8C_1x^7.\frac{1}{x} + {}^8C_2x^6.\frac{1}{x^2} - {}^8C_3x^5.\frac{1}{x^3} + {}^8C_4x^4.\frac{1}{x^4}$$

$$- {}^8C_5x^3.\frac{1}{x^5} + {}^8C_6x^2.\frac{1}{x^6} - {}^8C_7x.\frac{1}{x^7} + \frac{1}{x^8}$$

$$= x^8 - 8x^6 + 28x^4 - 56x^2 + 70 - 56.\frac{1}{x^2} + 28.\frac{1}{x^4} - 8.\frac{1}{x^6} + \frac{1}{x^8}$$

i.e., $2^8\sin^8\theta = \left(x^8 + \frac{1}{x^8}\right) - 8\left(x^6 + \frac{1}{x^6}\right) + 28\left(x^4 + \frac{1}{x^4}\right)$

$$-56\left(x^2 + \frac{1}{x^2}\right) + 70 \quad \text{...(iii)}$$

In (*ii*) replacing n by 8, 6, 4 and 2 we get,

$$x^8 + \frac{1}{x^8} = 2\cos 8\theta \,;\, x^6 + \frac{1}{x^6} = 2\cos 6\theta \,;\, x^4 + \frac{1}{x^4} = 2\cos 4\theta,$$

and $x^2 + \frac{1}{x^2} = 2\cos 2\theta$. Using these values in (*iii*),

$$2^8\sin^8\theta = 2\cos 8\theta - 8\times 2\cos 6\theta + 28$$

$$\times 2\cos 4\theta - 56\times 2\cos 2\theta + 70$$

$$= 2[\cos 8\theta - 8\cos 6\theta + 28\cos 4\theta - 56\cos 2\theta + 35]$$

$$\therefore \quad \sin^8\theta = \frac{1}{2^7}[\cos 8\theta - 8\cos 6\theta + 28\cos 4\theta - 56\cos 2\theta + 35].$$

Ex. 3. *Expand* $\cos^5\theta\,\sin^7\theta$ *in a series of sines of multiples of* θ.

Sol. Let $x = \cos\theta + i\sin\theta$. Then, working as in the previous problems,

$x + \frac{1}{x} = 2\cos\theta$ (*i*) $x - \frac{1}{x} = 2i\sin\theta$... (*ii*) and $x^n - \frac{1}{x^n} = 2i\sin n\theta$... (*iii*)

From (*i*) and (*ii*), we get $(2\cos\theta)^5(2i\sin\theta)^7 = \left(x + \frac{1}{x}\right)^5\left(x - \frac{1}{x}\right)^7$

$$= \left\{\left(x+\frac{1}{x}\right)\left(x-\frac{1}{x}\right)\right\}^5\left(x-\frac{1}{x}\right)^2 = \left(x^2-\frac{1}{x^2}\right)^5\left(x-\frac{1}{x}\right)^2$$

$$= \left(x^{10}-5x^6+10x^2-10.\frac{1}{x^2}+5.\frac{1}{x^6}-\frac{1}{x^{10}}\right)\left(x^2-2+\frac{1}{x^2}\right)$$

$$= x^{12}-2x^{10}-4x^8+10x^6+5x^4-20x^2+20.\frac{1}{x^2}-5.\frac{1}{x^4}-10.\frac{1}{x^6}$$

$$+4.\frac{1}{x^8}+2.\frac{1}{x^{10}}-\frac{1}{x^{12}}.$$

i.e., $2^{12}\, i^7 \cos^5\theta \sin^7\theta = \left(x^{12}-\frac{1}{x^{12}}\right)-2\left(x^{10}-\frac{1}{x^{10}}\right)-4\left(x^8-\frac{1}{x^8}\right)$

$$+10\left(x^6-\frac{1}{x^6}\right)+5\left(x^4-\frac{1}{x^4}\right)-20\left(x^2-\frac{1}{x^2}\right) \quad ...(iii)$$

But $i^7 = -i$, further in (*iii*) replacing n by 12, 10..........and 2, we get,

$x^{12}-\frac{1}{x^{12}} = 2i \sin 12\theta$; $x^{10}-\frac{1}{x^{10}} = 2i\sin 10\theta$.....and $x^2-\frac{1}{x^2} = 2i\sin 2\theta$.

Using these in (*iii*), $-2^{12}\, i\cos^5\theta\sin^7\theta = 2i\sin 12\theta - 2\times 2i\sin 10\theta$

$- 4\times 2i\sin 8\theta + 10\times 2i\sin 6\theta + 5\times 2i\sin 4\theta - 20\times 2i\sin 2\theta$

$\therefore \cos^5 \sin^7\theta = -\frac{1}{2^{11}}[\sin 12\theta - 2\sin 10\theta - 4\sin 8\theta + 10\sin 6\theta$

$+ 5\sin 4\theta - 20\sin 2\theta]$.

Note. Whether the expansion be that of a sine alone or the product of a cosine and sine, the series will be in sines, if the power of the given sine is odd. In all other cases, the series will be in cosines.

[Hint: Integrate and put ...]

Expansion of cos $n\theta$ and sin $n\theta$ in powers of sin θ and cos θ, when n is a positive integer.

Since n is a positive integer, by the binomial theorem,

$$(\cos\theta + i\sin\theta)^n = \cos^n\theta + {}^nC_1(\cos^{n-1}\theta)\, i\sin\theta + {}^nC_2(\cos^{n-2}\theta)i^2\sin^2\theta$$
$$+ {}^nC_3(\cos^{n-3}\theta)i^3\sin^3\theta + {}^nC_4(\cos^{n-4}\theta)i^4\sin^4\theta + \ldots \quad \ldots(i)$$

Using De Moivre's theorem,

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta \quad \ldots(ii)$$

Equating the right hand side of (*ii*) and (*i*) and remembering that $i^2 = -1$; $i^3 = -i$; $i^4 = +1$ etc., we get,

$$\cos n\theta + i\sin n\theta = \cos^n\theta + {}^nC_1 \times i\cos^{n-1}\theta\sin\theta$$
$${}^nC_2\cos^{n-2}\theta\sin^2\theta - {}^nC_3 \times i\cos^{n-3}\theta\sin^3\theta + {}^nC_4\cos^{n-4}\theta\sin^4\theta + \ldots(iii)$$

Equating the real parts.

$$\cos n\theta = \cos^n\theta - {}^nC_2\cos^{n-2}\theta\sin^2\theta + {}^nC_4\cos^{n-4}\theta\sin^4\theta - \ldots$$

Equating the imaginary parts in (*iii*),

$$\sin n\theta = {}^nC_1\cos^{n-1}\theta\sin\theta - {}^nC_3\cos^{n-3}\theta\sin^3\theta$$
$$+ {}^nC_5\cos^{n-5}\theta\sin^5\theta - \ldots$$

These are the required expansions.

Note. Replacing $\sin^2\theta$ by $1 - \cos^2\theta$, $\sin^4\theta$ by $(1 - \cos^2\theta)^2$ etc., $\cos n\theta$ and $\dfrac{\sin n\theta}{\sin\theta}$ can be expressed in terms of powers of cos θ alone.

$$\tan n\theta = \frac{\sin n\theta}{\cos n\theta} = \frac{{}^nC_1\cos^{n-1}\theta\sin\theta - {}^nC_3\cos^{n-3}\theta\sin^3\theta + \ldots}{\cos^n\theta - {}^nC_2\cos^{n-2}\theta\sin^2\theta + {}^nC_4\cos^{n-4}\theta\sin^4\theta - \ldots}$$

Dividing both the numerator and the denominator by $\cos^n\theta$, we get,

$$\tan n\theta = \frac{{}^nC_1 \tan\theta - {}^nC_3 \tan^3\theta + {}^nC_5 \tan^5\theta - \ldots}{1 - {}^nC_2 \tan^2\theta + {}^nC_4 \tan^4\theta - \ldots}$$

Ex. 4. *Express* $\cos 6\theta$ *and* $\frac{\sin 6\theta}{\sin\theta}$ *in series of powers of* $\cos\theta$.

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Sol. Using the binomial theorem,

$$(\cos\theta + i\sin\theta)^6 = \cos^6\theta + 6\cos^5\theta\,(i\sin\theta) + 15\cos^4\theta\,(i\sin\theta)^2 + 20\cos^3\theta\,(i\sin\theta)^3 + 15\cos^2\theta\,(i\sin\theta)^4 + 6\cos\theta\,(i\sin\theta)^5 + (i\sin\theta)^6$$

Applying De Moivre's theorem to the L.H.S.,

$$\cos 6\theta + i\sin 6\theta = \cos^6\theta + 6i\cos^5\theta\sin\theta - 15\cos^4\theta\sin^2\theta - 20\,i\cos^3\theta\sin^3\theta + 15\cos^2\theta\sin^4\theta + 6i\cos\theta\sin^5\theta - \sin^6\theta \quad \ldots(i)$$

Equating the real parts,

$$\begin{aligned}\cos 6\theta &= \cos^6\theta - 15\cos^4\theta\sin^2\theta + 15\cos^2\theta\sin^4\theta - \sin^6\theta \\ &= \cos^6\theta - 15\cos^4\theta\,(1-\cos^2\theta) + 15\cos^2\theta\,(1-\cos^2\theta)^2 - (1-\cos^2\theta)^3 \\ &= \cos^6\theta - 15\cos^4\theta + 15\cos^6\theta + 15\cos^2\theta\,(1 - 2\cos^2\theta + \cos^4\theta) \\ &\qquad - (1 - 3\cos^2\theta + 3\cos^4\theta - \cos^6\theta) \\ &= 32\cos^6\theta - 48\cos^4\theta + 18\cos^2\theta - 1.\end{aligned}$$

Equating the imaginary parts of (*i*),

$$\sin 6\theta = 6\cos^5\theta\sin\theta - 20\cos^3\theta\sin^3\theta + 6\cos\theta\sin^5\theta$$

$$\begin{aligned}\therefore \quad \frac{\sin 6\theta}{\sin\theta} &= 6\cos^5\theta - 20\cos^3\theta\sin^2\theta + 6\cos\theta\sin^4\theta \\ &= 6\cos^5\theta - 20\cos^3\theta\,(1-\cos^2\theta) + 6\cos\theta\,(1-\cos^2\theta)^2 \\ &= 6\cos^5\theta - 20\cos^3\theta + 20\cos^5\theta \\ &\qquad + 6\cos\theta\,(1 - 2\cos^2\theta + \cos^4\theta) \\ &= 32\cos^5\theta - 32\cos^3\theta + 6\cos\theta.\end{aligned}$$

Ex. 5. *If* $\cos^5\theta = A\cos\theta + B\cos 3\theta + C\cos 5\theta$*, prove that*

$\sin^5\theta = A\sin\theta - B\sin 3\theta + C\sin 5\theta$.

Sol. In the given relation replace θ by $\frac{\pi}{2} - \theta$.

$$\cos^5\left(\frac{\pi}{2} - \theta\right) = A\cos\left(\frac{\pi}{2} - \theta\right) + B\cos\left(\frac{3\pi}{2} - 3\theta\right) + C\cos\left(\frac{5\pi}{2} - 5\theta\right)$$

$$\sin^5\theta = A\sin\theta - B\sin 3\theta + C\sin 5\theta.$$

Ex. 6. *Express* $\tan 6\theta$ *in terms of* $\tan\theta$.

Sol. From example 4,

$$\tan 6\theta = \frac{\sin 6\theta}{\cos 6\theta} = \frac{6\cos^5\theta \sin\theta - 20\cos^3\theta \sin^3\theta + 6\cos\theta \sin^5\theta}{\cos^6\theta - 15\cos^4\theta \sin^2\theta + 15\cos^2\theta \sin^4\theta - \sin^6\theta}$$

Dividing Numerator and Denominator by $\cos^6\theta$,

$$\tan 6\theta = \frac{6\tan\theta - 20\tan^3\theta + 6\tan^5\theta}{1 - 15\tan^2\theta + 15\tan^4\theta - \tan^6\theta}$$

Ex. 7. *If $x = 2\cos\theta$, prove that $2(1 + \cos 8\theta) = (x^4 - 4x^2 + 2)^2$*

Sol. $2(1 + \cos 8\theta) = 4\cos^2 4\theta = (2\cos 4\theta)^2$

$$= [2(2\cos^2 2\theta - 1)]^2 = [4\cos^2 2\theta - 2)]^2$$

$$= [4(2\cos^2\theta - 1)^2 - 2]^2 = [16\cos^4\theta - 16\cos^2\theta + 2]^2$$

$$= [x^4 - 4x^2 + 2]^2 \qquad \left[\text{since } \cos\theta = \frac{x}{2}\right]$$

$$a_0 = 0 ; a_1 = 1 ; a_2 = 0 ; a_3 = -1 ; \ldots$$ The law of formation is as follows : the coefficient of even powers of θ in (*i*) are zero and the coefficients of odd powers are alternately + 1 and – 1. Substituting these values, the R.H.S sides of

(*i*) and (*ii*) are $\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots$ and $1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots$

These series can be proved to be convergent. Hence from (*i*) and (*ii*),

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots \text{ and } \cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots$$

The expansion for tan θ. The expansion for tan θ up to infinity is difficult, but a few terms of the expansion can be had as follows :

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots\cdots}{1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots\cdots}$$

$$= \left(\theta - \frac{\theta^3}{6} + \frac{\theta^5}{120} - \ldots\right)\left\{1 - \left(\frac{\theta^2}{2} - \frac{\theta^4}{24} + \ldots\right)\right\}^{-1}$$

Using the binomial theorem to expand the last factor,

$$\tan\theta = \left(\theta - \frac{\theta^3}{6} + \frac{\theta^5}{120} - \ldots\right)$$
$$\times\left\{1 + \left(\frac{\theta^2}{2} - \frac{\theta^4}{24} + \ldots\right) + \left(\frac{\theta^2}{2} - \frac{\theta^4}{24} + \ldots\right)^2 + \ldots\right\}$$

Expanding and omitting powers of θ higher than the fifth,

$$\tan\theta = \left(\theta - \frac{\theta^3}{6} + \frac{\theta^5}{120} - \ldots\right)\left(1 + \frac{\theta^2}{2} - \frac{\theta^4}{24} + \ldots + \frac{\theta^4}{4} + \ldots\right)$$

$$= \theta - \frac{\theta^3}{6} + \frac{\theta^5}{120} - \ldots + \frac{\theta^3}{2} - \frac{\theta^5}{12} + \ldots - \frac{\theta^5}{24} + \ldots + \frac{\theta^5}{4} - \ldots$$

$$= \theta + \frac{\theta^3}{6}(-1 + 3) + \frac{\theta^5}{120}(1 - 10 - 5 + 30) + \cdots$$

$$= \theta + \frac{2}{6}\theta^3 + \frac{16}{120}\theta^5 + \ldots \textit{i.e.}, \tan\theta = \theta + \frac{1}{3}\theta^3 + \frac{2}{15}\theta^5 + \cdots$$

Aliter. Since $\tan(-x) = -\tan x$, $\tan x$ is *odd* function of x ; *i.e.*, the expansion will have odd powers of x only. Assuming that $\tan x$ can be expanded in powers of x, the expansion must be of the form.

$$\tan x = ax + bx^3 + cx^5 + \ldots \qquad \ldots(i)$$

But $\tan x . \cos x \equiv \sin x$. In this identity using the known expansion for $\sin x$ and $\cos x$, and using (*i*), we get

$$(ax + bx^3 + cx^5 + \ldots)\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots\right) \equiv \frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots$$

Equating the coefficients of x, x^3 and x^5 on either side, $a = 1$;

$b - \frac{1}{2}a = -\frac{1}{6}$ and $c - \frac{1}{2}b + \frac{1}{24}a = \frac{1}{120}$; these give us $a = 1, b = \frac{1}{3}$, and $c = \frac{2}{15}$. Using these values in (i), $\tan x = x + \frac{x^3}{3} + \frac{2}{15}x^5 + \ldots$

Ex. 8. *If* $\frac{\sin\theta}{\theta} = \frac{5045}{5046}$, *prove that the angle* θ *is* $1° 58'$ *nearly.*

Sol. If θ is expressed in radian measure, $\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \ldots$

Hence $\frac{\sin\theta}{\theta} = 1 - \frac{\theta^2}{3!} + \frac{\theta^4}{5!} - \ldots$ ...(i)

By hypothesis, $\frac{\sin\theta}{\theta} = \frac{5045}{5046} = 1 - \frac{1}{5046}$; this number is very nearly equal to 1. Hence from the R.H.S. of (i), we see that θ is a very small angle. Therefore omitting powers of θ higher than the second, the equation takes the form, $1 - \frac{\theta^2}{3!} = 1 - \frac{1}{5046} \therefore \frac{\theta^2}{3!} = \frac{1}{5046}$

Hence $\theta^2 = \frac{6}{5046} = \frac{1}{841}$

$\therefore \theta = \frac{1}{29}$ radians $= \frac{1}{29} \times 57.29$ degrees, nearly $= 1° 58'$ nearly.

Ex. 9. *Solve approximately :* $\cos\left(\frac{\pi}{3} + \theta\right) = 0.49$.

Sol. We know that $\cos\frac{\pi}{3} = 0.5$ but by hypothesis, $\cos\left(\frac{\pi}{3} + \theta\right) = 0.49$.

Thus the cosines of $\frac{\pi}{3}$ and $\frac{\pi}{3} + \theta$ are very nearly equal. Hence the above angles themselves are very nearly equal. Therefore θ is a small angle. Hence for an approximate solution, we can omit powers of θ higher than the first.

$$\cos\left(\frac{\pi}{3} + \theta\right) = \cos\frac{\pi}{3}\cos\theta - \sin\frac{\pi}{3}\sin\theta$$

$$= \frac{1}{2}\left(1 - \frac{\theta^2}{2!} + \ldots\right) - \frac{\sqrt{3}}{2}\left(\theta - \frac{\theta^3}{3!} + \ldots\right)$$

$= \frac{1}{2} - \frac{\sqrt{3}}{2}\theta$, omitting powers of θ higher than the first.

Therefore, the given equation takes the form of $\frac{1}{2} - \frac{\sqrt{3}}{2}\theta = 0.49$

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i.e., $\frac{1}{2}-\frac{\sqrt{3}}{2}\theta = \frac{49}{100}$ $\therefore \frac{1}{2}-\frac{49}{100}=\frac{\sqrt{3}}{2}\theta$ i.e., $\frac{\sqrt{3}}{2}\theta=\frac{1}{100}$

Hence $\theta = \frac{1}{50\sqrt{3}}$ radian $= \frac{\sqrt{3}}{150}$ radian

$= \frac{1\cdot732}{150}$ radian $= \cdot0115$ radian $=$ 40 minutes (nearly).

Ex. 10. *Expand cos 7θ in powers of cos θ and sin 7θ in powers of sin θ. Also find tan 7θ in powers of tan θ.*

Sol. $(\cos 7\theta + i \sin 7\theta) = (\cos\theta + i\sin\theta)^7$

$= (\cos\theta)^7 + {}^7C_1(\cos\theta)^6(i\sin\theta) + {}^7C_2(\cos\theta)^5(i\sin\theta)^2$
$+ {}^7C_3(\cos\theta)^4(i\sin\theta)^3 + {}^7C_4(\cos\theta)^3(i\sin\theta)^4$
$+ {}^7C_5(\cos\theta)^2(i\sin\theta)^5 + {}^7C_6(\cos\theta)(i\sin\theta)^6 + (i\sin\theta)^7$

$= \cos^7\theta + 7\cos^6\theta(i\sin\theta) - 21\cos^5\theta\sin^2\theta$
$- 35\cos^4\theta(i\sin^3\theta) + 35\cos^3\theta\sin^4\theta$
$+ 21\cos^2\theta(i\sin^5\theta) - 7\cos\theta\sin^6\theta - i\sin^7\theta$........ ...(1)

Equating real parts,

$\cos 7\theta = \cos^7\theta - 21\cos^5\theta\sin^2\theta + 35\cos^3\theta\sin^4\theta - 7\cos\theta\sin^6\theta$ ...(2)

$= \cos^7\theta - 21\cos^5\theta(1-\cos^2\theta) + 35\cos^3\theta(1-\cos^2\theta)^2 - 7\cos\theta(1-\cos^2\theta)^3$

$= \cos^7\theta - 21\cos^5\theta + 21\cos^7\theta + 35\cos^3\theta(1-2\cos^2\theta+\cos^4\theta)$
$-7\cos\theta(1-3\cos^2\theta+3\cos^4\theta-\cos^6\theta)$

$\therefore \cos 7\theta = 64\cos^7\theta - 112\cos^5\theta + 56\cos^3\theta - 7\cos\theta$

Equating imaginary parts from (i)

$\sin 7\theta = 7\cos^6\theta\sin\theta - 35\cos^4\theta\sin^3\theta + 21\cos^2\theta\sin^5\theta - \sin^7\theta$...(3)

$= 7(1-\sin^2\theta)^3\sin\theta - 35(1-\sin^2\theta)^2\sin^3\theta + 21(1-\sin^2\theta)\sin^5\theta - \sin^7\theta$

$= 7(1-3\sin^2\theta+3\sin^4\theta-\sin^6\theta)\sin\theta$
$-35(1-2\sin^2\theta+\sin^4\theta)\sin^3\theta + 21(\sin^5\theta-\sin^7\theta) - \sin^7\theta$

$\sin 7\theta = 7\sin\theta - 56\sin^3\theta + 112\sin^5\theta - 64\sin^7\theta$

Using (2) and (3),

$$\tan 7\theta = \frac{\sin 7\theta}{\cos 7\theta} = \frac{7\cos^6\theta\sin\theta - 35\cos^4\theta\sin^3\theta + 21\cos^2\theta\sin^5\theta - \sin^7\theta}{\cos^7\theta - 21\cos^5\theta\sin^2\theta + 35\cos^3\theta\sin^4\theta - 7\cos\theta\sin^6\theta}$$

Divide Numerators and denominator by $\cos^7\theta$.

$$\therefore \tan 7\theta = \frac{7\tan\theta - 35\tan^3\theta + 21\tan^5\theta - \tan^7\theta}{1 - 21\tan^2\theta + 35\tan^4\theta - 7\tan^6\theta}.$$

Note. Aliter. In the formula for cos 7θ, replace θ by $\frac{\pi}{2}-\theta$; we get sin 7θ.

$$\cos 7\theta = 64\cos^7\theta - 112\cos^5\theta + 56\cos^3\theta - 7\cos\theta$$

$$\cos 7\left(\frac{\pi}{2}-\theta\right) = 64\cos^7\left(\frac{\pi}{2}-\theta\right) - 112\cos^5\left(\frac{\pi}{2}-\theta\right) + 56\cos^3\left(\frac{\pi}{2}-\theta\right) - 7\cos\left(\frac{\pi}{2}-\theta\right); \quad \text{...(iv)}$$

Since $\cos\left(\frac{\pi}{2}-\theta\right) = \sin\theta$ and $\cos 7\left(\frac{\pi}{2}-\theta\right) = \cos\left(\frac{7\pi}{2}-7\theta\right) = -\sin 7\theta$

Equation (4) becomes

$$-\sin 7\theta = 64\sin^7\theta - 112\sin^5\theta + 56\sin^3\theta - 7\sin\theta$$

i.e., $$\sin 7\theta = 7\sin\theta - 56\sin^3\theta + 112\sin^5\theta - 64\sin^7\theta.$$

Ex. 11. *Prove (i)* $\cos^5\theta = \frac{1}{16}(\cos 5\theta + 5\cos 3\theta + 10\cos\theta)$

(ii) $\sin^5\theta = \frac{1}{16}(\sin 5\theta - 5\sin 3\theta + 10\sin\theta)$

(iii) $\sin^4\theta\cos^2\theta = \frac{1}{32}[\cos 6\theta - 2\cos 4\theta - \cos 2\theta + 2]$

Sol. Let $x = \cos\theta + i\sin\theta \therefore \frac{1}{x} = \cos\theta - i\sin\theta$

$$\therefore \quad x + \frac{1}{x} = 2\cos\theta,\ x - \frac{1}{x} = 2i\sin\theta$$

$$x^n + \frac{1}{x^n} = 2\cos n\theta;\ x^n - \frac{1}{x^n} = 2i\sin n\theta.$$

(i) $$(2\cos\theta)^5 = \left(x + \frac{1}{x}\right)^5$$

$$32\cos^5\theta = x^5 + {}^5C_1 x^4\left(\frac{1}{x}\right) + {}^5C_2 x^3\left(\frac{1}{x}\right)^2 + {}^5C_3 x^2\left(\frac{1}{x}\right)^3 + {}^5C_4 x\left(\frac{1}{x}\right)^4 + \left(\frac{1}{x}\right)^5$$

$$= x^5 + 5x^3 + 10x + \frac{10}{x} + \frac{5}{x^3} + \frac{1}{x^5}$$

$$= \left(x^5 + \frac{1}{x^5}\right) + 5\left(x^3 + \frac{1}{x^3}\right) + 10\left(x + \frac{1}{x}\right)$$

$$= 2\cos 5\theta + 5(2\cos 3\theta) + 10(2\cos\theta)$$

$$\therefore \quad \cos^5\theta = \frac{1}{16}(\cos 5\theta + 5\cos 3\theta + 10\cos\theta)$$

(ii) $$(2i\sin\theta)^5 = \left(x - \frac{1}{x}\right)^5$$

$$32i\sin^5\theta = x^5 - {}^5C_1 x^4\left(\frac{1}{x}\right) + {}^5C_2 x^3\left(\frac{1}{x}\right)^2 - {}^5C_3 x^2\left(\frac{1}{x}\right)^3 + {}^5C_4 x\left(\frac{1}{x}\right)^4 - \left(\frac{1}{x}\right)^5$$

$$= x^5 - 5x^3 + 10x - \frac{10}{x} + 5.\frac{1}{x^3} - \frac{1}{x^5}$$

$$= \left(x^5 - \frac{1}{x^5}\right) - 5\left(x^3 - \frac{1}{x^3}\right) + 10\left(x - \frac{1}{x}\right)$$

$$= (2i \sin 5\theta) - 5\,(2i \sin 3\theta) + 10\,(2i \sin \theta)$$

$$\therefore \quad \sin^5\theta = \frac{1}{16}[\sin 5\theta - 5\sin 3\theta + 10\sin\theta]$$

(*iii*) $(2i \sin\theta)^4 (2\cos\theta)^2 = \left(x - \frac{1}{x}\right)^4 \left(x + \frac{1}{x}\right)^2$

$$= \left(x^2 - \frac{1}{x^2}\right)^2 \left(x - \frac{1}{x}\right)^2$$

$$= \left(x^4 - 2 + \frac{1}{x^4}\right)\left(x^2 - 2 + \frac{1}{x^2}\right)$$

$$= x^6 - 2x^4 + x^2 - 2x^2 + 4 - \frac{2}{x^2} + \frac{1}{x^2} - \frac{2}{x^4} + \frac{1}{x^6}$$

$$= \left(x^6 + \frac{1}{x^6}\right) - 2\left(x^4 + \frac{1}{x^4}\right) - \left(x^2 + \frac{1}{x^2}\right) + 4$$

$$= (2\cos 6\theta) - 2\,(2\cos 4\theta) - (2\cos 2\theta) + 4$$

$$\therefore \quad \sin^4\theta\cos^2\theta = \frac{1}{32}(\cos 6\theta - 2\cos 4\theta - \cos 2\theta + 2)$$

Note. Changing θ to $\frac{\pi}{2} - \theta$, we get,

$$\cos^4\theta\sin^2\theta = \frac{1}{32}(-\cos 6\theta - 2\cos 4\theta + \cos 2\theta + 2)$$

Ex. 12. *Evaluate*

(*i*) $\underset{x\to 0}{Lt}\ \frac{1-\cos x}{x^2}$ (*ii*) $\underset{x\to 0}{Lt}\ \frac{\sin x - x}{x^3}$

Sol. (*i*) $\underset{x\to 0}{\text{Lt}}\ \frac{1-\cos x}{x^2} = \underset{x\to 0}{\text{Lt}}\ \frac{1 - \left[1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots\right]}{x^2}$

$$= \underset{x\to 0}{\text{Lt}}\ \frac{\frac{x^2}{2} - \frac{x^4}{24} + \dots}{x^2}$$

$$= \underset{x\to 0}{\text{Lt}}\left(\frac{1}{2} - \frac{x^2}{24} + \dots\right) = \frac{1}{2}$$

(ii) $$\underset{x\to 0}{\text{Lt}} \frac{\sin x - x}{x^3} = \underset{x\to 0}{\text{Lt}} \frac{\left(x - \frac{x^3}{3!} + \frac{x^5}{5!}\right) - x}{x^3}$$

$$= \underset{x\to 0}{\text{Lt}} \left(-\frac{1}{6} + \frac{x^2}{5!} + \dots\right)$$

$$= -\frac{1}{6}$$

Ex. 13. *Evaluate* $\underset{x\to 0}{Lt} \frac{5 \sin x - \sin 5x}{5(\cos x - \cos 5x)}$.

Sol. $$\underset{x\to 0}{\text{Lt}} \frac{5 \sin x - \sin 5x}{5(\cos x - \cos 5x)}$$

$$= \underset{x\to 0}{\text{Lt}} \frac{5\left[x - \frac{x^3}{3!} + \dots\right] - \left[(5x) - \frac{(5x)^3}{3!} + \dots\right]}{5\left[\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots\right) - \left\{1 - \frac{(5x)^2}{2!} + \dots\right\}\right]}$$

$$= \underset{x\to 0}{\text{Lt}} \frac{\frac{120}{3!}x^3 + \dots\dots}{60x^2 + \dots\dots}$$

$$= \underset{x\to 0}{\text{Lt}} \left[\frac{20x + \dots\dots}{60 + (\ )x^2 + \dots}\right] = 0$$

Ex. 14. *Evaluate* $\underset{x\to 0}{Lt} \frac{x \log(1+x)}{1 - \cos x}$

Sol. $$\underset{x\to 0}{\text{Lt}} \frac{x \log(1+x)}{1 - \cos x} = \underset{x\to 0}{\text{Lt}} \frac{x\left[x - \frac{x^2}{2} + \frac{x^3}{3} - \dots\right]}{1 - \left[1 - \frac{x^2}{2!} + \dots\right]}$$

$$= \underset{x\to 0}{\text{Lt}} \frac{x^2\left[1 - \frac{x}{2} + \dots\right]}{\frac{x^2}{2} - \frac{x^4}{24} + \dots}$$

$$= \underset{x\to 0}{\text{Lt}} \frac{1 - \frac{x}{2} + \dots}{\frac{1}{2} - \frac{x^2}{24} + \dots}$$

$$= 2$$

Ex. 15. *Expand* $\sin^3 x \cos x$ *in powers of* x*, giving the general term.*

Sol. $\sin 3x = 3 \sin x - 4 \sin^3 x \quad \therefore \quad 4 \sin^3 x = 3 \sin x - \sin 3x$

Hence $\sin^3 x = \frac{3}{4}\sin x - \frac{1}{4}\sin 3x$

Multiplying both the sides by $\cos x$,

$$\sin^3 x\cos x = \frac{3}{4}\sin x\cos x - \frac{1}{4}\sin 3x\cos x$$

$$= \frac{3}{8}(2\sin x\cos x) - \frac{1}{8}(2\sin 3x\cos x)$$

$$= \frac{3}{8}\sin 2x - \frac{1}{8}(\sin 4x + \sin 2x)$$

$$= \frac{1}{4}\sin 2x - \frac{1}{8}\sin 4x$$

$$= \frac{1}{4}\left\{2x - \frac{(2x)^3}{3!} + \frac{(2x)^5}{5!} - \ldots\right\} - \frac{1}{8}\left\{4x - \frac{(4x)^3}{3!} + \frac{(4x)^5}{5!} - \ldots\right\}$$

The general term

$$= (-1)^{n-1}\left\{\frac{1}{4}\frac{(2x)^{2n-1}}{(2n-1)!} - \frac{1}{8}\frac{(4x)^{2n-1}}{(2n-1)!}\right\}$$

$$= (-1)^n \frac{(2x)^{2n-1}}{4\{(2n-1)!\}}\left\{\frac{1}{2}\cdot 2^{2n-1} - 1\right\}$$

$$= (-1)^n \frac{2^{2n-3}\,x^{2n-1}}{(2n-1)!}\left\{2^{2n-2} - 1\right\}$$

Ex. 16. *Determine a, b, c so that* $\underset{\theta \to 0}{Lt} \frac{\theta(a + b\cos\theta) - c\sin\theta}{\theta^5} = 1$:

Sol. $\underset{\theta \to 0}{\text{Lt}} \frac{\theta(a + b\cos\theta) - c\sin\theta}{\theta^5}$

$$= \underset{\theta \to 0}{\text{Lt}} \frac{\theta\left[a + b\left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!}\ldots\right)\right] - c\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!}\ldots\right)}{\theta^5}$$

$$= \underset{\theta \to 0}{\text{Lt}} \frac{(a + b - c)\theta + \left(-\frac{b}{2} + \frac{c}{6}\right)\theta^3 + \left(\frac{b}{24} - \frac{c}{120}\right)\theta^5 + \ldots}{\theta^5}$$

Since the limit tends to 1, coefficient of θ, θ^3 must vanish and coefficient of θ^5 must be 1 (in the numerator)

$\therefore\ a + b - c = 0;\ -\frac{b}{2} + \frac{c}{6} = 0$ and $\frac{b}{24} - \frac{c}{120} = 1$

$\therefore\ a + b - c = 0$ ...(1) $c = 3b$ ... (2) $5b - c = 120$ .. (3)

using (2) in (3) $2b = 120\ \therefore\ b = 60$; hence $c = 180$

using in (1), $a = 120$ $\therefore$ $a = 120, b = 60, c = 180$.

Ex. 17. *Find a and b if*

$$\underset{x \to 0}{Lt} \frac{a \sin 3x + bx \cos x + \tan 2x}{x^5} \text{ may be finite.}$$

Sol. Exp $= \underset{x \to 0}{\text{Lt}} \frac{1}{x^5}\left[a\left\{(3x) - \frac{(3x)^3}{3!} + \frac{(3x)^5}{5!} \ldots\right\} + bx\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots\right)\right]$

$$+ (2x) + \frac{1}{3}(2x)^3 + \frac{2}{15}(2x)^5 + \ldots\Bigg]$$

$$= \underset{x \to 0}{\text{Lt}} \frac{1}{x^5}\left[(3a + b + 2)\,x + \left(-\frac{9}{2}a - \frac{b}{2} + \frac{8}{3}\right)x^3 + \left(\quad\right)x^5 + \ldots\right]$$

Since this limit is finite, coefficient of x, x^3 must be zero.

$$\therefore \quad 3a + b + 2 = 0 \text{ and } -\frac{9a}{2} - \frac{b}{2} + \frac{8}{3} = 0$$

Solving these equations $a = \frac{11}{9}$, $b = -17/3$

EXPONENTIAL SERIES AND HYPERBOLIC FUNCTIONS

Exponential series. If z is a complex number, the series

$$1+\frac{z}{1!}+\frac{z^2}{2!}+\frac{z^3}{3!}+\ldots\frac{z^n}{n!}+\text{ to }\infty \qquad \ldots(i)$$

is known to be convergent. The sum of this series is denoted by exp. (z), or $E(z)$.

If x is real, we know that the series $1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots+\text{ to }\infty$ ...(ii)

is convergent, and that the sum of their series is e^x, where e stands for the sum of $1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\ldots+\text{ to }\infty$ ...(iii)

Since the series (i) is of a form similar to (ii), it is customary to write e^z for the sum of (i). In other words, we write e^z for exp. (z). When z_1 and z_2 are complex, it is proved that $e^{z_1}.e^{z_2}=e^{z_1+z_2}$.

Periodicity. A real number is after all a special or particular case of a complex number. Hence, if x is real,

$$\text{exp. }(x) = 1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\ldots=e^x \qquad \ldots(i)$$

$$\text{exp. }(iy) = 1+\frac{iy}{1!}+\frac{(iy)^2}{2!}+\frac{(iy)^3}{3!}+\frac{(iy)^4}{4!}+\ldots$$

$$= 1+\frac{iy}{1!}-\frac{y^2}{2!}-\frac{iy^3}{3!}+\frac{y^4}{4!}+\ldots$$

$$= 1-\frac{y^2}{2!}+\frac{y^4}{4!}-\ldots.+i\left(\frac{y}{1!}-\frac{y^3}{3!}+\frac{y^5}{5!}\ldots.\right)$$

$$= \cos y + i\sin y \qquad \ldots(ii)$$

Hence, exp. $(x+iy)$ = exp. (x) exp. (iy)

Hence using (i) and (ii), exp. $(x+iy)=e^x(\cos y+i\sin y)$ ...(iii)

Now using (iii),

$$\text{exp. }\{x+i(y+2n\pi)\} = e^x\{\cos(y+2n\pi)+i\sin(y+2n\pi)\}$$

$$= e^x\{\cos y+i\sin y\} = \text{exp. }(x+iy),\ \text{using }(iii)$$

i.e., $\exp.(x + iy + i2n\pi) = \exp.(x + iy)$

Thus the function exp. (z) is periodic, with the period $2\pi i$; in other words exp. (z), exp. ($z + 2\pi i$), exp. ($z + 4\pi i$), exp. ($z + 6\pi i$), ... have the same value.

Circular functions in terms of exponentials

In (ii) of the above article, we have seen that

$$e^{iy} = \cos y + i \sin y \quad \text{...}(i)$$

$$\therefore \quad e^{-iy} = e^{i(-y)} = \cos(-y) + i \sin(-y)$$

i.e.,
$$e^{-iy} = \cos y - i \sin y \quad \text{...}(ii)$$

$\therefore$ Adding (i) and (ii) $e^{iy} + e^{-iy} = 2\cos y$. Hence $\cos y = \dfrac{e^{iy} + e^{-iy}}{2}$ ...(iii)

(i) – (ii) gives, $e^{iy} - e^{-iy} = 2i \sin y$. Hence $\sin y = \dfrac{e^{iy} - e^{-iy}}{2i}$

Results (i) to (iv) are extremely useful and are to be committed to memory.

Hyperbolic Functions.

Whether x be real or complex, $\dfrac{e^{x} - e^{-x}}{2}$ is defined as hyperbolic sine of x and is written as sinh x of sh x. Similarly : $\dfrac{e^{x} + e^{-x}}{2}$ is called the hyperbolic cosine of x and is written as cosh x or ch x. Expanding e^{x}, we get,

$$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} \ldots \text{ and } \cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots.$$

The hyperbolic tangent, hyperbolic cotangent, hyperbolic secant and hyperbolic cosecant which are written as tanh x (or th x) coth x, sech x and cosech x are given by the relations,

$$\tanh x = \frac{\sinh x}{\cosh x};\ \coth x = \frac{\cosh x}{\sinh x};\ \text{sech } x = \frac{1}{\cosh x} \text{ and}$$

$$\text{cosech } x = \frac{1}{\sinh x}$$

The following facts are noteworthy :

1. $\cosh 0 = 1$; $\sinh 0 = 0$ and $\tanh 0 = 0$. 2. When x is real, $\cosh x > 1$

Proof. $\cosh x = 1 + \dfrac{x^2}{2!} + \dfrac{x^4}{4!} + \ldots$; here x^2, x^4 are positive ; hence the result.

3.
$$\cosh^2 x = \frac{(e^{x} + e^{-x})^2}{4} = \frac{e^{2x} + e^{-2x} + 2}{4}$$

$$\sinh^2 x = \frac{(e^{x} - e^{-x})^2}{4} = \frac{e^{2x} + e^{-2x} - 2}{4} \quad \therefore \sinh^2 x < \cosh^2 x$$

Hence sinh x is numerically < cosh x $\therefore$ tanh x is numerically < 1.

But $\tanh \infty = +1$ and $\tanh(-\infty) = -1$. Hence $y = \pm 1$ are the asymptotes of the curve $y = \tanh x$.

4. $\cosh x + \sinh x = e^x$; $\cosh x - \sinh x = e^{-x}$

Exercise. *Prove that $(\cosh x + \sinh x)^n = \cosh nx + \sinh nx$.*

Relations connecting hyperbolic functions and circular functions.

$$\sin iz = iz - \frac{(iz)^3}{3!} + \frac{(iz)^5}{5!} - \frac{(iz)^7}{7!} + \dots$$

$$= i\left(z + \frac{z^3}{3!} + \frac{z^5}{5!} + \frac{z^7}{7!} + \dots\right)$$

$$\sin iz = i \sinh z.$$

$$\therefore \quad \cos iz = 1 - \frac{(iz)^2}{2!} + \frac{(iz)^4}{4!} - \frac{(iz)^6}{6!} + \dots$$

$$= 1 + \frac{z^2}{2!} + \frac{z^4}{4!} + \frac{z^6}{6!} + \dots = \cosh z$$

$$\therefore \quad \cos iz = \cosh z$$

Hence by division, $\dfrac{\sin iz}{\cos iz} = \dfrac{i \sinh z}{\cosh z}$ i.e. $\tan iz = i \tanh z$.

Also,
$$\sinh iz = \frac{e^{iz} - e^{-iz}}{2}$$

$$= \frac{(\cos z + i\sin z) - (\cos z - i\sin z)}{2} = \frac{2i\sin z}{2} = i\sin z$$

$$\cosh iz = \frac{e^{iz} + e^{-iz}}{2}$$

$$= \frac{(\cos z + i\sin z) + (\cos z - i\sin z)}{2} = \frac{2\cos z}{2} = \cos z$$

The above formulae enable us to derive formulae connecting hyperbolic functions from those of circular functions. A few results are proved below.

$$\cosh^2 z - \sinh^2 z = (\cos iz)^2 - \left(\frac{\sin iz}{i}\right)^2$$

$$= \cos^2 iz + \sin^2 iz \quad \text{i.e., } \cosh^2 z - \sinh^2 z = 1$$

This can also be proved as follows :

$$\cosh^2 z - \sinh^2 z = \frac{(e^z + e^{-z})^2}{4} - \frac{(e^z - e^{-z})^2}{4} = \frac{4}{4} = 1.$$

Again,
$$\sinh(x + y) = \frac{1}{i}\sin\{i(x + y)\}$$

$$= \frac{1}{i}(\sin ix \cos iy + \cos ix \sin iy)$$

$$= \frac{1}{i}\{i \sinh x \cosh y + (\cosh x)\, i \sinh y\}$$

i.e., $\sinh(x+y) = \sinh x \cosh y + \cosh x \sinh y$

This result can be proved directly as follows :

$\sinh x \cosh y + \cosh x \sinh y$

$$= \frac{e^x - e^{-x}}{2} \cdot \frac{e^y + e^{-y}}{2} + \frac{e^x + e^{-x}}{2} \cdot \frac{e^y - e^{-y}}{2}$$

$$= \frac{e^{x+y} - e^{-x+y} + e^{x-y} - e^{-x-y} + e^{x+y} + e^{-x+y} - e^{x-y} - e^{-x-y}}{4}$$

$$= \frac{2\{e^{x+y} - e^{-(x+y)}\}}{4} = \frac{e^{x+y} - e^{-(x+y)}}{2} = \sinh(x+y)$$

The following working rule, known as Osborn's rule, helps us to derive a formula in hyperbolic functions, if we know the corresponding law in circular functions : In any formula connecting the circular functions of general angles, replace each circular function by the corresponding hyperbolic function and change the sign of every product (or implied product) of two sines or tans.

Thus from $\tan 3A = \dfrac{3\tan A - \tan^3 A}{1 - 3\tan^2 A}$

we get that $\tanh 3A = \dfrac{3\tanh A + \tanh^3 A}{1 + 3\tanh^2 A}$

We have seen that $\cosh^2\theta - \sinh^2\theta = 1$. Hence the point $(a\cosh\theta, b\sinh\theta)$ lies on $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$. Because of this characteristic, $\cosh\theta$ and $\sinh\theta$ are called hyperbolic functions.

Periods of Hyperbolic Functions

$$\sinh z = \frac{1}{i}\sin iz = \frac{1}{i}\sin(iz + 2n\pi) = \frac{1}{i}\sin\{i(z - 2n\pi i)\}$$

i.e., $\sinh z = \sinh(z - 2n\pi i)$

Thus, whatever be the value of z, $\sinh(z - 2n\pi i)$ and $\sinh z$ have the same value. This idea is expressed by saying that $\sinh z$ is periodic with the period $2\pi i$.

Again, $\cosh z = \cos iz = \cos(iz + 2n\pi) = \cos\{i(z - 2n\pi i)\}$

i.e., $\cosh z = \cosh(z - 2n\pi i)$.

Therefore, $\cosh z$ is also having a period $2\pi i$.

Similarly, the student can prove that $\tanh z$ is periodic with the period πi.

Ex. 1. *Prove that*

$$64\, ch^7 x = ch\, 7x + 7\, ch\, 5x + 21\, ch\, 3x + 35\, ch\, x.$$

Sol. L.H.S. $64\left(\dfrac{e^x + e^{-x}}{2}\right)^7$

$$= \frac{64}{128}\{(e^x)^7 + 7(e^x)^6(e^{-x}) + 21(e^x)^5(e^{-x})^2 + 35(e^x)^4(e^{-x})^3 + 35(e^x)^3(e^{-x})^4 + 21(e^x)^2 + (e^{-x})^5 + 7e^x(e^{-x})^6 + (e^{-x})^7\}$$

$$= \frac{1}{2}\{e^{7x} + 7e^{5x} + 21e^{3x} + 35e^x + 35e^{-x} + 21e^{-3x} + 7e^{-5x} + e^{-7x}\}$$

$$= \frac{e^{7x} + e^{-7x}}{2} + 7\frac{e^{5x} + e^{-5x}}{2} + 21\frac{e^{3x} + e^{-3x}}{2} + 35\frac{e^x + e^{-x}}{2}$$

$$= \text{ch } 7x + 7 \text{ ch } 5x + 21 \text{ ch } 3x + 35 \text{ ch } x.$$

Ex. 2. *If $x + iy = \sin(A + iB)$, prove that*

$$\frac{x^2}{\cosh^2 B} + \frac{y^2}{\sinh^2 B} = 1, \text{ and } \frac{x^2}{\sin^2 A} - \frac{y^2}{\cos^2 A} = 1.$$

Sol.
$$x + iy = \sin(A + iB) = \sin A \cos iB + \cos A \sin iB$$
$$= \sin A \cosh B + (\cos A)\, i \sinh B$$

i.e.,
$$x + iy = \sin A \cosh B + i \cos A \sinh B$$

Equating the real and imaginary parts, we have

$$\left.\begin{aligned} x &= \sin A \cosh B\,; \\ y &= \cos A \sinh B \end{aligned}\right\} \quad \ldots(i)$$

$$\therefore \quad \frac{x}{\cosh B} = \sin A, \text{ and } \frac{y}{\sinh B} = \cos A.$$

But $\sin^2 A + \cos^2 A = 1.$

Hence substituting, $\dfrac{x^2}{\cosh^2 B} + \dfrac{y^2}{\sinh^2 B} = 1.$

But $\cosh^2 B - \sinh^2 B = 1.$

Hence substituting, $\dfrac{x^2}{\sin^2 A} - \dfrac{y^2}{\cos^2 A} = 1.$

Ex. 3. *If $x + iy = \tan(A + iB)$, prove that*

$$x^2 + y^2 + 2x \cot 2A = 1, \text{ and } x^2 + y^2 - 2y \coth 2B + 1 = 0.$$

Sol.
$$\tan 2A = \tan\{(A + iB) + (A - iB)\}$$

i.e.,
$$\tan 2A = \frac{\tan(A + iB) + \tan(A - iB)}{1 - \tan(A + iB)\tan(A - iB)} \quad \ldots(i)$$

$$\left.\begin{aligned} &\text{But by hypothesis, } \tan(A + iB) = x + iy\,; \\ &\text{hence } \tan(A - iB) = x - iy. \end{aligned}\right\} \quad \ldots(ii)$$

Substituting these in (i), we get,

$$\tan 2A = \frac{(x+iy)+(x-iy)}{1-(x+iy)(x-iy)} = \frac{2x}{1-(x^2-i^2y^2)}$$

i.e., $$\tan 2A = \frac{2x}{1-(x^2+y^2)}$$

$\therefore$ $$1-(x^2+y^2) = 2x \cot 2A \text{ or } 1 = x^2+y^2+2x\cot 2A$$

Again, $$\tan 2(Bi) = \tan\{(A+iB)-(A-iB)\}$$

i.e., $$\tan(2Bi) = \frac{\tan(A+iB)-\tan(A-iB)}{1+\tan(A+iB)\tan(A-iB)}$$

Now using (*ii*), $\tan(2Bi) = \dfrac{(x+iy)-(x-iy)}{1+(x+iy)(x-iy)}$

i.e., $$i\tanh 2B = \frac{2iy}{1+(x^2-i^2y^2)}$$

$\therefore$ $$\tanh 2B = \frac{2y}{1+x^2+y^2} \quad \therefore \quad 1+x^2+y^2 = (2y)\frac{1}{\tanh 2B}$$

i.e., $x^2+y^2+1 = 2y\coth 2B$. Hence $x^2+y^2+1-2y\coth 2B = 0$.

Ex. 4. *Separate into real and imaginary parts of coth* $(\alpha+i\beta)$.

Sol. Let $\coth(\alpha+i\beta) = x+iy$

i.e., $$x+iy = \frac{\cosh(\alpha+i\beta)}{\sinh(\alpha+i\beta)} = \frac{i\cos\{i(\alpha+i\beta)\}}{\sin\{i(\alpha+i\beta)\}}$$

i.e., $$x+iy = \frac{i\cos(i\alpha-\beta)}{\sin(i\alpha-\beta)}$$

The product of $\sin(i\alpha-\beta)$ and $\sin(i\alpha+\beta)$ will be real ; further if the denominator is real, it becomes simple. Hence we multiply the numerator and denominator by $2\sin(i\alpha+\beta)$.

Then $$x+iy = \frac{2i\cos(i\alpha-\beta)\sin(i\alpha+\beta)}{2\sin(i\alpha-\beta)\sin(i\alpha+\beta)}$$

$$= \frac{i\{\sin(2i\alpha)+\sin 2\beta\}}{\cos 2\beta-\cos(2i\alpha)} = \frac{i\{i\sinh 2\alpha+\sin 2\beta\}}{\cos 2\beta-\cosh 2\alpha}$$

i.e., $$x+iy = \frac{-\sinh 2\alpha+i\sin 2\beta}{\cos 2\beta-\cosh 2\alpha}$$

Hence $$x = \frac{-\sinh 2\alpha}{\cos 2\beta-\cosh 2\alpha} \text{ and } y = \frac{\sin 2\beta}{\cos 2\beta-\cosh 2\alpha}$$

Ex. 5. *If* $\cos(A+iB) = \cos\theta + i\sin\theta$, *prove that*

$$\cos 2A + \cosh 2B = 2.$$

Sol. Since, $$\cos(A+iB) = \cos\theta+i\sin\theta \quad \text{...}(i)$$

$$\cos(A-iB) = \cos\theta-i\sin\theta \quad \text{...}(ii)$$

Multiplying (*i*) and (*ii*),

$$2\cos(A+iB)\cos(A-iB) = 2(\cos\theta+i\sin\theta)(\cos\theta-i\sin\theta)$$
$$\cos 2A+\cos 2iB = 2(\cos^2\theta+\sin^2\theta)$$
$$\cos 2A+\cosh 2B = 2.$$

Ex. 6. *Prove sinh (A + B) = sinh A cosh B + cosh A sinh B.*

Sol.
$$\sin(iA+iB) = \sin iA\cos iB+\cos iA\sin iB$$
$$i\sinh(A+B) = i\sinh A\cosh B+\cosh A\,(i\sinh B).$$
$$\therefore\quad \sinh(A+B) = \sinh A\cosh B+\cosh A\sinh B.$$

Ex. 7. *Find real and imaginary parts of cot (x + iy).*

If $u+iv = \cot(x+iy)$ *prove* $\frac{u}{v}=-\frac{\sin 2x}{\sinh 2y}$

Sol.
$$\cot(x+iy) = \frac{2\cos(x+iy)}{2\sin(x+iy)}\cdot\frac{\sin(x-iy)}{\sin(x-iy)}=\frac{\sin 2x-\sin(2iy)}{\cos 2iy-\cos 2x}$$
$$u+iv = \frac{\sin 2x-i\sinh 2y}{\cosh 2y-\cos 2x}$$
$$u = \text{real part}=\frac{\sin 2x}{\cosh 2y-\cos 2x}$$
$$v=\text{Imaginary part} = \frac{-\sinh 2y}{\cosh 2y-\cos 2x}\quad\therefore\ \frac{u}{v}=-\frac{\sin 2x}{\sinh 2y}$$

Ex. 8. *If* $\tan\frac{x}{2}=\tanh\frac{y}{2}$ *prove that* $\cos x\cosh y = 1$.

Sol.
$$\cos x = \frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}=\frac{1-\tanh^2\frac{y}{2}}{1+\tanh^2\frac{y}{2}}=\frac{1}{\cosh y}$$
$$\therefore\quad \cos x\cosh y = 1.$$

Inverse Hyperbolic Functions

If $\sinh x = y$ then x is defined as $\sinh^{-1} y$.

Similarly, if $\cosh\theta = x$ then $\theta=\cosh^{-1} x$.

Prove that $\sinh^{-1} x = \log_e(x+\sqrt{x^2+1})$

Proof. Let $\sinh^{-1} x = y$

$$\text{Then, } x = \sinh y=\frac{e^y-e^{-y}}{2}\quad i.e., x=\frac{z-\frac{1}{z}}{2}\ \text{where } z=e^y.$$
$$2x = z-\frac{1}{z}$$
$$z^2-2xz-1=0.\ \text{Solving for } z,\ z=\frac{2x\pm\sqrt{4x^2+4}}{2}=x\pm\sqrt{x^2+1}$$
$$e^y = z=x\pm\sqrt{x^2+1}$$

e^y is always positive and $x-\sqrt{x^2+1}$ is negative.

$$\therefore \qquad e^y \neq x-\sqrt{x^2+1}$$

$$\text{Hence} \qquad e^y = x+\sqrt{x^2+1}$$

$$\sinh^{-1} x = y = \log_e (x+\sqrt{x^2+1})$$

Another form for $\cosh^{-1} y$:

Prove that $\cosh^{-1} y = \pm \log (y+\sqrt{y^2-1})$

Proof. Let $x = \cosh^{-1} y \quad \therefore \quad y = \cosh x$

$$\text{i.e.,} \qquad y = \frac{e^x+e^{-x}}{2}$$

Hence $2y = e^x + e^{-x} \quad \therefore \quad 2ye^x = e^{2x}+1 \quad \therefore \quad 0 = e^{2x}+1-2ye^x$

$$\text{i.e.,} \qquad (e^x)^2 - 2ye^x + 1 = 0.$$

$$\text{Hence} \qquad e^x = \frac{2y \pm \sqrt{(4y^2-4)}}{2} = y \pm \sqrt{(y^2-1)}$$

$$\text{i.e.,} \qquad e^x = y+\sqrt{(y^2-1)} \text{ or } \frac{1}{y+\sqrt{(y^2-1)}}$$

$$\therefore \qquad x = \pm \log \{y+\sqrt{(y^2-1)}\}$$

Both the values are admissible; but the positive one is called the principal value. If we take the principal value,

$$\cosh^{-1} y = \log \{y+\sqrt{(y^2-1)}\}$$

Another form for $\tanh^{-1} y$:

Prove $\tanh^{-1} y = \frac{1}{2}\log\left(\frac{1+y}{1-y}\right)$

Proof. Let $x = \tan h^{-1} y$

Then $y = \tanh x \quad$ i.e., $\frac{y}{1} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$

Hence $\frac{1+y}{1-y} = \frac{2e^x}{2e^{-x}} \quad$ i.e., $\frac{1+y}{1-y} = e^{2x}$

Taking logarithms, $\log\left(\frac{1+y}{1-y}\right) = 2x$

Hence $x = \frac{1}{2}\log\left(\frac{1+y}{1-y}\right) \quad$ i.e., $\tanh^{-1} y = \frac{1}{2}\log\left(\frac{1+y}{1-y}\right)$.

Ex. 9. *Separate into real and imaginary parts of* $\tan^{-1}(x+iy)$.

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Sol. Let $\tan^{-1}(x+iy) = A + iB$. Then $\tan(A+iB) = x+iy$

Working as in example 3 we get

$$\tan 2A = \frac{2x}{1-(x^2+y^2)} \text{ and } \tanh 2B = \frac{2y}{1+x^2+y^2}$$

Hence,
$$2A = \tan^{-1}\frac{2x}{1-(x^2+y^2)} \text{ and } 2B = \tanh^{-1}\frac{2y}{1+x^2+y^2}$$

$$A = \frac{1}{2}\tan^{-1}\frac{2x}{1-(x^2+y^2)} \text{ and } B = \frac{1}{2}\tanh^{-1}\frac{2y}{1+x^2+y^2}.$$

Ex. 10. *Prove* $\sinh 3A = 3\sinh A + 4\sinh^3 A$.

Sol. We known, $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$. Replace θ by iA.

$$\sin i3A = 3\sin iA - 4\sin^3(iA)$$

$$i\sinh 3A = i.3\sinh A - i^3.4\sinh^3 A$$

$$\therefore \quad \sinh 3A = 3\sinh A + 4\sinh^3 A$$

(equating imaginary parts on both sides)

Ex. 11. *If* $\cosh x = \sec\alpha$, *express* $\sinh x$ *and* $\tanh x$ *in terms of* α.

Sol. $\cosh^2 x - \sinh^2 x = 1$

$$\therefore \quad \sinh^2 x = \cosh^2 x - 1 = \sec^2\alpha - 1 = \tan^2\alpha \quad \therefore \sinh x = \tan\alpha;$$

$$\tanh x = \frac{\tan\alpha}{\sec\alpha} = \sin\alpha$$

Ex. 12. *If* $u = \log\tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right)$, *prove* $\tanh\frac{u}{2} = \tan\frac{\theta}{2}$ **(MS Ap. 94)**

Sol.
$$u = \log\left(\frac{\tan\frac{\pi}{4}+\tan\theta/2}{1-\tan\frac{\pi}{4}\tan\frac{\theta}{2}}\right) = \log\frac{1+\tan\theta/2}{1-\tan\theta/2}$$

$$\therefore \quad \frac{1+\tan\theta/2}{1-\tan\theta/2} = \frac{e^u}{1}$$

Hence by Componendo et dividendo.

$$\tan\theta/2 = \frac{e^u-1}{e^u+1}; \tan\frac{\theta}{2} = \frac{e^{u/2}-e^{-u/2}}{e^{u/2}+e^{-u/2}} = \tanh\frac{u}{2}.$$