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Class : III B.Sc Maths

Sub : Abstract Algebra

Unit - II

1 If $a \in R^*$, prove that $\{a^n / n \in z\}$ is a subgroup of R^*.

Sol:

Let H is a subgroup of G.

clearly H is non-empty.

Now, Let $x, y \in H$, then $x = a^s$ and $y = a^t$ where $s, t \in z$

$\therefore xy^{-1} = a^s (a^t)^{-1} = a^{s-t} \in H$

Hence H is a subgroup of R^*.

Unit - IV

2 P.T the set R of all matrices of the form $\begin{bmatrix} a & b \\ -b & a \end{bmatrix}$ where $a, b \in R$ is a ring under matrix addition and multiplication.

Proof:

Let $A = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$ and $B = \begin{pmatrix} c & d \\ -d & c \end{pmatrix} \in R$

Then $A+B = \begin{pmatrix} a & b \\ -b & a \end{pmatrix} + \begin{pmatrix} c & d \\ -d & c \end{pmatrix}$

$$= \begin{pmatrix} a+c & b+d \\ -(b+d) & a+c \end{pmatrix} \in R$$

$$AB = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}\begin{pmatrix} c & d \\ -d & c \end{pmatrix}$$

$$= \begin{pmatrix} ac-bd & ad+bc \\ -(ad+ac) & ac-bd \end{pmatrix} \in R$$

Clearly matrix addition is commutative and associative.

$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \in R$ is the zero element.

$\begin{pmatrix} -a & -b \\ b & -a \end{pmatrix}$ is the inverse of the matrix $\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$

Further matrix multiplication is associative and the distributive laws are vadid for 2×2 matrices

Hence R is a ring.

Hence the proof//

3 3. P.T the set F of all real numbers of the form $a+b\sqrt{2}$ where $a, b \in \mathbb{Q}$ is a field under the usual addition and multiplication of real numbers.

Sol: Obviously, $(F, +)$ is an abelian group with 0 as the zero element.

Now, let $a+b\sqrt{2}$ and $c+d\sqrt{2} \in F$.

Then $(a+b\sqrt{2})(c+d\sqrt{2}) = (ac+2bd)+(ad+bc)\sqrt{2} \in F$

Since the two binary operations are the usual addition and multiplication of real numbers, multiplication is associative and commutative and the two distributive laws are true.

$1 = 1+0\sqrt{2} \in F$ and is the multiplicative identity.

Now, Let $a+b\sqrt{2} \in F - \{0\}$. Then a and b are not simultaneously zero.

Also $\dfrac{1}{a+b\sqrt{2}} = \dfrac{a-b\sqrt{2}}{a-b\sqrt{2}} \times \dfrac{1}{a+b\sqrt{2}}$

$$\Rightarrow \frac{1}{a+b\sqrt{2}} = \frac{a-b\sqrt{2}}{a^2-2b^2}$$

we claim that $a^2-2b^2 \neq 0$

~~Claim~~: case (i) $a \neq 0$ and $b=0$, then

$$a^2-2b^2 = a^2 \neq 0$$

case (ii) $a=0$ and $b \neq 0$, then $a^2-2b^2 =$

$$-2b^2 \neq 0$$

case (iii) $a \neq 0$ and $b \neq 0$.

Suppose $a^2-2b^2=0$

$\Rightarrow a^2 = 2b^2$ so that $a^2/b^2 = 2$.

$\Rightarrow a/b = \pm\sqrt{2}$

Now, $a/b \in \mathbb{Q}$ and $\sqrt{2} \notin \mathbb{Q}$. This is $\Rightarrow\Leftarrow$

Hence $a^2-2b^2 \neq 0$.

$$\therefore \frac{1}{a+b\sqrt{2}} = \left(\frac{a}{a^2-2b^2}\right) - \left(\frac{b}{a^2-2b^2}\right)\sqrt{2} \in F$$

and is the inverse of $a+b\sqrt{2}$

Hence F is a field

Hence the proof //.

4. Prove that the only isomorphism $f: \mathbb{Q} \to \mathbb{Q}$ is the identity map.

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Sol: Since f is an isomorphism $f(0) = 0$ and $f(1) = 1$

Now, Let 'n' be a positive integer.

$$f(n) = f(1+1+\cdots+1) \text{ (written } n \text{ times)}$$
$$= f(1)+f(1)+\cdots+f(1) \text{ ('')}$$
$$= 1+1+\cdots+ \text{ (written } n \text{ times)}$$

$\Rightarrow f(n) = n$

Now, if 'n' is a negative integer, let $n = -m$ where $m \in \mathbb{Z}$. Then

$$f(n) = f(-m) = -f(m)$$
$$= -m$$
$$= n$$

Thus for any integer n, $f(n) = n$.

Now, let $a \in \mathbb{Q}$. Then $a = p/q$ where $p, q \in \mathbb{Z}$

Hence $f(a) = f(p/q) = f(pq^{-1}) = f(p) \cdot f(q^{-1})$
$$= f(p)\left[f(q)\right]^{-1}$$
$$= pq^{-1} = p/q = a$$

Hence f is the identity map.//

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