

7. Series Expansions

7.0. Introduction

In this chapter we consider the problem of representing a given function as a power series. We prove that if a function is analytic at a point z_0 then it can be expanded as a power series called *Taylor's series* consisting of non-negative powers of $z - z_0$ and the expansion is valid in some neighbourhood of z_0 . We also prove that a function $f(z)$ which is analytic in an annular region $a < |z - z_0| < b$ can be expanded as a series called *Laurent's series* consisting of positive and negative powers of $z - z_0$. We also introduce the concept of *singular points* of a function and classify the singular points and discuss the behaviour of the function in the neighbourhood of a singularity.

7.1. Taylor's Series

Theorem 7.1. (Taylor's theorem)

Let $f(z)$ be analytic in a region D containing z_0 . Then $f(z)$ can be represented as a power series in $z - z_0$ given by

$$f(z) = f(z_0) + \frac{f'(z_0)}{1!}(z - z_0) + \frac{f''(z_0)}{2!}(z - z_0)^2 + \dots \\ \dots + \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n + \dots$$

The expansion is valid in the largest open disc with centre z_0 contained in D .

Proof. Let $r > 0$ be such that the disc $|z - z_0| < r$ is contained in D .

Let $0 < r_1 < r$. Let C_1 be the circle $|z - z_0| = r_1$.

By Cauchy's integral formula we have

$$f(z) = \frac{1}{2\pi i} \int_{C_1} \frac{f(\zeta)}{(\zeta - z)} d\zeta \quad \dots (1)$$

Also by theorem on higher derivatives we have

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_{C_1} \frac{f(\zeta) d\zeta}{(\zeta - z)^{n+1}} \quad \dots (2)$$

$$\begin{aligned}
 \text{Now, } \frac{1}{\zeta - z} &= \frac{1}{(\zeta - z_0) - (z - z_0)} \\
 &= \frac{1}{(\zeta - z_0) \left[1 - \frac{z - z_0}{\zeta - z_0} \right]} \\
 &= \frac{1}{\zeta - z_0} \left[1 + \left(\frac{z - z_0}{\zeta - z_0} \right) + \left(\frac{z - z_0}{\zeta - z_0} \right)^2 + \dots + \left(\frac{z - z_0}{\zeta - z_0} \right)^{n-1} \right. \\
 &\quad \left. + \frac{\left(\frac{z - z_0}{\zeta - z_0} \right)^n}{1 - \left(\frac{z - z_0}{\zeta - z_0} \right)} \right]
 \end{aligned}$$

$$\text{(using the identity } \frac{1}{1 - \alpha} = 1 + \alpha + \alpha^2 + \dots + \alpha^{n-1} + \frac{\alpha^n}{1 - \alpha} \text{)}$$

$$= \frac{1}{\zeta - z_0} + \frac{z - z_0}{(\zeta - z_0)^2} + \frac{(z - z_0)^2}{(\zeta - z_0)^3} + \dots + \frac{(z - z_0)^{n-1}}{(\zeta - z_0)^n} + \frac{(z - z_0)^n}{(\zeta - z_0)^n (\zeta - z)}$$

Now, multiplying throughout by $\frac{f(\zeta)}{2\pi i}$, integrating over C_1 and using (1) and (2) we get

$$\begin{aligned}
 f(z) &= f(z_0) + f'(z_0)(z - z_0) + \frac{f''(z_0)}{2!}(z - z_0)^2 + \dots \\
 &\quad \dots + \frac{f^{(n-1)}(z_0)}{(n-1)!}(z - z_0)^{n-1} + R_n \dots \dots (3)
 \end{aligned}$$

$$\text{where } R_n = \frac{(z - z_0)^n}{2\pi i} \int_{C_1} \frac{f(\zeta) d\zeta}{(\zeta - z)(\zeta - z_0)^n}$$

Here ζ lies on C_1 and z lies in the interior of C_1 so that $|\zeta - z_0| = r_1$ and $|z - z_0| < r_1$.

$$\therefore |\zeta - z| = |(\zeta - z_0) - (z - z_0)| \geq |\zeta - z_0| - |z - z_0| = r_1 - |z - z_0|.$$

$$\therefore \frac{1}{|\zeta - z|} \leq \frac{1}{r_1 - |z - z_0|}.$$

Let M denote the maximum value of $|f(z)|$ on C_1 .

$$\text{Then } |R_n| \leq \frac{|z - z_0|^n}{2\pi} \frac{M(2\pi r_1)}{(r_1 - |z - z_0|)r_1^n} \quad \text{(by theorem 6.2)}$$

$$= \frac{M|z - z_0|}{(r_1 - |z - z_0|)} \left(\frac{|z - z_0|}{r_1} \right)^{n-1}$$

Also $\left| \frac{z - z_0}{r_1} \right| < 1$. Hence $\lim_{n \rightarrow \infty} R_n = 0$.

$\therefore$ Taking limit as $n \rightarrow \infty$ in (3) we get

$$f(z) = f(z_0) + \frac{f'(z_0)}{1!}(z - z_0) + \frac{f''(z_0)}{2!}(z - z_0)^2 + \dots$$

$$+ \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n + \dots$$

Note 1. The above series is called the **Taylor series** of $f(z)$ about the point z_0 . Thus if $f(z)$ is analytic at a point z_0 then $f(z)$ can be represented as a Taylor's series about z_0 , which is a series in non negative powers of $z - z_0$. The expansion is valid in some neighbourhood of z_0 .

Note 2. The Taylor series expansion of $f(z)$ about the point zero is called the **Maclaurin's series**. Thus the Maclaurin's series of $f(z)$ is given by

$$f(z) = f(0) + \frac{z}{1!}f'(0) + \frac{z^2}{2!}f''(0) + \dots + \frac{z^n}{n!}f^{(n)}(0) + \dots$$

Example 1. The Taylor's series for $f(z) = \frac{1}{z}$ about $z = 1$ is given by

$$\frac{1}{z} = f(1) + \frac{f'(1)}{1!}(z - 1) + \frac{f''(1)}{2!}(z - 1)^2 + \frac{f'''(1)}{3!}(z - 1)^3 + \dots$$

$$\text{Now, } f(z) = \frac{1}{z} \Rightarrow f(1) = 1$$

$$f'(z) = -\frac{1}{z^2} \Rightarrow f'(1) = -1$$

$$f''(z) = \frac{2}{z^3} \Rightarrow f''(1) = 2$$

$$f'''(z) = -\frac{6}{z^4} \Rightarrow f'''(1) = -6$$

.....

$$\begin{aligned} \frac{1}{z} &= \frac{1}{z} \\ &= \frac{1}{1 + (z - 1)} \\ &= 1 - (z - 1) + (z - 1)^2 - (z - 1)^3 + \dots \end{aligned}$$

Hence the Taylor's series expansion for $\frac{1}{z}$ about 1 is

$$\frac{1}{z} = 1 - (z - 1) + (z - 1)^2 - (z - 1)^3 + \dots$$

This expansion is valid in the disc $|z - 1| < 1$.

Similarly the Taylor's series for $f(z) = \frac{1}{z}$ about $z = i$ is given by

$$\frac{1}{z} = \frac{1}{i} - \frac{z-i}{i^2} + \frac{(z-i)^2}{i^3} - \frac{(z-i)^3}{i^4} + \dots$$

and the expansion is valid in the disc $|z - i| < 1$. (verify)

Example 2. Let $f(z) = e^z$.

Then $f^{(n)}(z) = e^z$ for all n and hence $f^{(n)}(0) = 1$.

Hence the Maclaurin's series for e^z is given by

$$e^z = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \frac{z^3}{3!} + \dots + \frac{z^n}{n!} + \dots$$

and the expansion is valid in the entire complex plane.

Maclaurin's series expansion of some of the standard functions are given below.

1. $e^{-z} = 1 - \frac{z}{1!} + \frac{z^2}{2!} - \dots + (-1)^n \frac{z^n}{n!} + \dots (|z| < \infty)$
2. $\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \dots + (-1)^{n-1} \frac{z^{2n-1}}{(2n-1)!} + \dots (|z| < \infty)$
3. $\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \dots + (-1)^{n-1} \frac{z^{2n-2}}{(2n-2)!} + \dots (|z| < \infty)$
4. $\sinh z = \frac{z}{1!} + \frac{z^3}{3!} + \frac{z^5}{5!} + \dots + \frac{z^{2n-1}}{(2n-1)!} + \dots (|z| < \infty)$
5. $\cosh z = 1 + \frac{z^2}{2!} + \frac{z^4}{4!} + \dots + \frac{z^{2n}}{(2n)!} + \dots (|z| < \infty)$
6. $\frac{1}{1+z} = 1 - z + z^2 - z^3 + \dots + (-1)^n z^n + \dots (|z| < 1)$
7. $\frac{1}{1-z} = 1 + z + z^2 + z^3 + \dots + z^n + \dots (|z| < 1)$
8. $\log(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \dots + (-1)^{n-1} \frac{z^n}{n} + \dots (|z| < 1)$
9. $\log(1-z) = -z - \frac{z^2}{2} - \frac{z^3}{3} - \dots - \frac{z^n}{n} - \dots (|z| < 1).$

Solved problems

Problem 1. Expand $\cos z$ into a Taylor's series about the point $z = \pi/2$ and determine the region of convergence.

Solution. Let $f(z) = \cos z$

The Taylor's series for $f(z)$ about $z = \pi/2$ is

$$f(z) = f(\pi/2) + \frac{(z - \pi/2)}{1!} f'(\pi/2) + \frac{(z - \pi/2)^2}{2!} f''(\pi/2) + \frac{(z - \pi/2)^3}{3!} f'''(\pi/2) + \dots$$

Now $f(z) = \cos z$. Hence $f(\pi/2) = 0$.

$f'(z) = -\sin z$. Hence $f'(\pi/2) = -1$.

$f''(z) = -\cos z$. Hence $f''(\pi/2) = 0$

$f'''(z) = \sin z$. Hence $f'''(\pi/2) = 1$.

.....

$\therefore$ The Taylor's series for $\cos z$ about $z = \pi/2$ is

$$\cos z = -\frac{(z - \pi/2)}{1!} + \frac{(z - \pi/2)^3}{3!} - \frac{(z - \pi/2)^5}{5!} + \dots$$

The expansion is valid throughout the complex plane.

Problem 2. Expand $f(z) = \sin z$ in a Taylor's series about $z = \pi/4$ and determine the region of convergence of this series.

Solution. The Taylor's series for $f(z)$ about $z = \pi/4$ is

$$f(z) = f(\pi/4) + \frac{(z - \pi/4)}{1!} f'(\pi/4) + \frac{(z - \pi/4)^2}{2!} f''(\pi/4) + \dots$$

Here $f(z) = \sin z$. Hence $f(\pi/4) = \frac{1}{\sqrt{2}}$.

$f'(z) = \cos z$. Hence $f'(\pi/4) = \frac{1}{\sqrt{2}}$.

$f''(z) = -\sin z$. Hence $f''(\pi/4) = -\frac{1}{\sqrt{2}}$.

$f'''(z) = -\cos z$. Hence $f'''(\pi/4) = -\frac{1}{\sqrt{2}}$.

The Taylor's series for $\sin z$ about $z = \pi/4$ is

$$\begin{aligned} \sin z &= \frac{1}{\sqrt{2}} + \frac{(z - \pi/4)}{1!} \left(\frac{1}{\sqrt{2}} \right) - \frac{(z - \pi/4)^2}{2!} \left(\frac{1}{\sqrt{2}} \right) + \dots \\ &= \frac{1}{\sqrt{2}} \left[1 + \frac{(z - \pi/4)}{1!} - \frac{(z - \pi/4)^2}{2!} + \frac{(z - \pi/4)^3}{3!} + \dots \right] \end{aligned}$$

The expansion is valid in the entire complex plane.

Problem 3. Expand $f(z) = \frac{z-1}{z+1}$ as a Taylor's series

- (i) about the point $z = 0$.
 (ii) about the point $z = 1$. Determine the region of convergence in each case.

Solution.

(i) $f(z) = \frac{z-1}{z+1}$

$$= (z-1)(1+z)^{-1}$$

$$= (z-1)(1 - z + z^2 - z^3 + \dots) \text{ if } |z| < 1$$

$$= (z - z^2 + z^3 - \dots) - (1 - z + z^2 - z^3 + \dots)$$

$$= -1 + 2z - 2z^2 + 2z^3 + \dots$$

The region of convergence is $|z| < 1$.

(ii) $f(z) = \frac{z-1}{z+1}$

$$= \frac{z-1}{(2+z-1)}$$

$$= \frac{z-1}{2\left(1 + \frac{z-1}{2}\right)}$$

$$= \frac{z-1}{2} \left(1 + \frac{z-1}{2}\right)^{-1}$$

$$= \frac{z-1}{2} \left[1 - \frac{z-1}{2} + \left(\frac{z-1}{2}\right)^2 - \left(\frac{z-1}{2}\right)^3 + \dots \right] \text{ if } \left|\frac{z-1}{2}\right| < 1$$

$$= \frac{z-1}{2} - \frac{(z-1)^2}{2^2} + \frac{(z-1)^3}{2^3} - \dots$$

The region of convergence is given by $\left|\frac{z-1}{2}\right| < 1$ which is same as the circular disc $|z-1| < 2$.

Problem 4. Show that

(i) $\frac{1}{z^2} = 1 + \sum_{n=1}^{\infty} (n+1)(z+1)^n$ when $|z+1| < 1$.

(ii) $\frac{1}{z^2} = \frac{1}{4} + \frac{1}{4} \sum_{n=1}^{\infty} (-1)^n (n+1) \left(\frac{z-2}{2}\right)^n$ when $|z-2| < 2$.

Solution. (i) $\frac{1}{z^2} = \frac{1}{[1 - (z + 1)]^2}$
 $= [1 - (z + 1)]^{-2}$
 $= 1 + 2(z + 1) + 3(z + 1)^2 + 4(z + 1)^3 + \dots$ if $|z + 1| < 1$
 $= 1 + \sum_{n=1}^{\infty} (n + 1)(z + 1)^n$ when $|z + 1| < 1$.

(ii) $\frac{1}{z^2} = \frac{1}{(z - 2 + 2)^2}$
 $= \frac{1}{\left[2\left(1 + \frac{z - 2}{2}\right)\right]^2}$
 $= \frac{1}{4} \left(1 + \frac{z - 2}{2}\right)^{-2}$
 $= \frac{1}{4} \left[1 - 2\left(\frac{z - 2}{2}\right) + 3\left(\frac{z - 2}{2}\right)^2 - \dots\right]$ if $\left|\frac{z - 2}{2}\right| < 1$
 $= \frac{1}{4} - \frac{1}{4} \times 2\left(\frac{z - 2}{2}\right) + \frac{1}{4} \times 3\left(\frac{z - 2}{2}\right)^2 - \dots$
 $= \frac{1}{4} + \frac{1}{4} \sum_{n=1}^{\infty} (-1)^n (n + 1) \left(\frac{z - 2}{2}\right)^n$

Here the region of convergence is $\left|\frac{z - 2}{2}\right| < 1$ which is the same as the circular disc $|z - 2| < 2$.

Problem 5. Expand ze^{2z} in a Taylor's series about $z = -1$ and determine the region of convergence.

Solution. Let $f(z) = ze^{2z}$

$$\begin{aligned}
 &= ze^{2(z+1)} e^{-2} \\
 &= \frac{1}{e^2} \left[(z + 1)e^{2(z+1)} - e^{2(z+1)} \right] \\
 &= \frac{1}{e^2} \left[(z + 1) \left\{ 1 + \frac{2(z + 1)}{1!} + \frac{4(z + 1)^2}{2!} + \dots \right\} \right. \\
 &\quad \left. - \left\{ 1 + \frac{2(z + 1)}{1!} + \frac{4(z + 1)^2}{2!} + \dots \right\} \right]
 \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{e^2} \left[\left\{ (z+1) + \frac{2(z+1)^2}{1!} + \frac{2^2(z+1)^3}{2!} + \dots \right\} \right. \\
&\quad \left. - \left\{ 1 + \frac{2(z+1)}{1!} + \frac{2^2(z+1)^2}{2!} + \dots \right\} \right] \\
&= \frac{1}{e^2} \left[-1 + \left(1 - \frac{2}{1!}\right)(z+1) + \left(\frac{2}{1!} - \frac{2^2}{2!}\right)(z+1)^2 + \left(\frac{2^2}{2!} - \frac{2^3}{3!}\right)(z+1)^3 + \dots \right]
\end{aligned}$$

The expansion is valid throughout the complex plane.

Problem 6. Find the Taylor's series to represent $\frac{z^2 - 1}{(z+2)(z+3)}$ in $|z| < 2$

Solution. By partial fractions

$$\begin{aligned}
\frac{z^2 - 1}{(z+2)(z+3)} &= 1 + \frac{3}{z+2} - \frac{8}{z+3} \quad (\text{verify}) \\
&= 1 + \frac{3}{2\left(1 + \frac{z}{2}\right)} - \frac{8}{3\left(1 + \frac{z}{3}\right)} \\
&= 1 + \frac{3}{2} \left(1 + \frac{z}{2}\right)^{-1} - \frac{8}{3} \left(1 + \frac{z}{3}\right)^{-1} \\
&= 1 + \frac{3}{2} \left(1 - \frac{z}{2} + \frac{z^2}{2^2} - \frac{z^3}{2^3} + \dots\right) - \frac{8}{3} \left(1 - \frac{z}{3} + \frac{z^2}{3^2} - \frac{z^3}{3^3} + \dots\right) \\
&= \left(1 + \frac{3}{2} - \frac{8}{3}\right) + \left(-\frac{3}{2^2} + \frac{8}{3^2}\right)z + \left(\frac{3}{2 \cdot 2^2} - \frac{8}{3 \cdot 3^2}\right)z^2 + \dots \\
&= -\frac{1}{6} + \sum_{n=1}^{\infty} (-1)^{n+1} \left(\frac{8}{3^{n+1}} - \frac{3}{2^{n+1}}\right) z^n
\end{aligned}$$

and the expansion is valid in $|z| < 2$.

Exercises.

1. Verify all the ten expansions in 7.1
2. Expand $\frac{1}{z}$ about $z = -1$ and $z = 2$ as Taylor's series, stating the region of convergence.
3. Show that $\frac{1}{z^2} = 1 - 2(z-1) + 3(z-1)^2 - 4(z-1)^3 + \dots$ for all z in $|z-1| < 1$.

4. Expand $\frac{z}{z-3}$ as a Taylor's series about $z = 1$.
5. Find the Maclaurin's series for $\frac{1}{2-z}$. What is its radius of convergence.
6. Obtain the Taylor's series to represent $\frac{1}{(z+1)(z+3)}$ in $|z| < 1$.
7. Obtain the Taylor's series for $\frac{z}{z+2}$ about $z = 1$. State the region of validity.
8. Find the Taylor's series for ze^z about $z = 1$.
9. Show that $\sin z^2 = z^2 - \frac{z^6}{3!} + \frac{z^{10}}{5!} - \frac{z^{14}}{7!} + \dots$ for $|z| < \infty$.

Answers.

$$2. \frac{1}{z} = -1 - (z-1) - (z-1)^2 + \dots; |z+1| < 1$$

$$\frac{1}{z} = \frac{1}{2} - \frac{z-2}{2^2} + \frac{(z-2)^2}{2^3} - \frac{(z-2)^3}{2^4} + \dots; |z-2| < 2$$

$$4. -\frac{1}{2} - \frac{3}{2} \sum_{n=1}^{\infty} \left(\frac{z-1}{2}\right)^n \quad 5. \sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}}; 2$$

$$6. \frac{1}{2} \sum_{n=0}^{\infty} (-1)^n \left[1 - \frac{1}{3^{n+1}}\right] z^n$$

$$7. \frac{1}{3} + 2 \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(z-1)^n}{3^{n+1}}; |z-1| < 3$$

$$8. e \left[1 + \frac{2(z-1)}{1!} + \frac{3(z-1)^2}{2!} + \frac{4(z-1)^3}{3!} + \dots \right]$$

7.2. Laurent's Series

A series of the form $\sum_{n=1}^{\infty} \frac{b_n}{z^n}$... (1)

can be considered as an ordinary power series in the variable $\frac{1}{z}$. Hence if the radius of convergence of the power series $\sum_{n=1}^{\infty} b_n z^n$ is r and $r < \infty$ the series $\sum_{n=1}^{\infty} \frac{b_n}{z^n}$ converges

in the region $|z| > r$. The convergence is uniform in every region $|z| \geq \rho > r$ and the series represents an analytic function in $|z| > r$.

If the series (1) is combined with the usual power series we get a more general series of the form $\sum_{n=-\infty}^{\infty} a_n z^n$ (2)

This series is said to converge at a point if the part of the series consisting of the negative powers of z and the part of the series consisting of non-negative powers of z are separately convergent. We know that the series consisting of non-negative powers of z converges in a disc $|z| < r_2$ and the series consisting of negative powers of z converges in a region $|z| > r_1$.

$\therefore$ If $r_1 < r_2$ the series represented by (2) converges in the region $r_1 < |z| < r_2$ and in this *annulus region* it represents an analytic function.

We shall now prove that the converse situation is also true.

(i.e) any function which is analytic in a region containing the annulus $r_1 < |z - z_0| < r_2$ can be represented in a series of the form $\sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$.

Theorem 7.2. (Laurent's theorem)

Let C_1 and C_2 denote respectively the concentric circles $|z - z_0| = r_1$ and $|z - z_0| = r_2$ with $r_1 < r_2$. Let $f(z)$ be analytic in a region containing the circular annulus $r_1 < |z - z_0| < r_2$. Then $f(z)$ can be represented as a convergent series of positive and negative powers of $z - z_0$ given by

$$f(z) = \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n} + \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

where $b_n = \frac{1}{2\pi i} \int_{C_1} \frac{f(\zeta) d\zeta}{(\zeta - z_0)^{-n+1}}$ and $a_n = \frac{1}{2\pi i} \int_{C_2} \frac{f(\zeta) d\zeta}{(\zeta - z_0)^{n+1}}$

Proof. Let z be any point in the circular annulus $r_1 < |z - z_0| < r_2$.

Then by theorem 6.9 we have, $f(z) = \frac{1}{2\pi i} \int_{C_2} \frac{f(\zeta) d\zeta}{\zeta - z} - \frac{1}{2\pi i} \int_{C_1} \frac{f(\zeta) d\zeta}{\zeta - z}$.

$$\therefore f(z) = \frac{1}{2\pi i} \int_{C_2} \frac{f(\zeta) d\zeta}{\zeta - z} + \frac{1}{2\pi i} \int_{C_1} \frac{f(\zeta) d\zeta}{z - \zeta} \quad \dots (1)$$

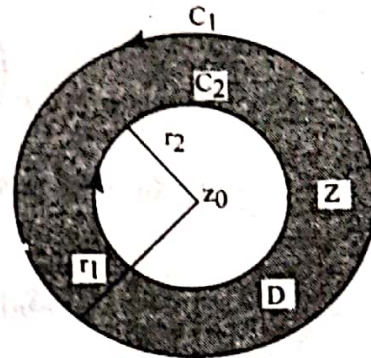
As in the proof of Taylor's theorem, we have

$$\begin{aligned} \frac{1}{2\pi i} \int_{C_2} \frac{f(\zeta) d\zeta}{\zeta - z} &= a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \dots \\ &\dots + a_{n-1}(z - z_0)^{n-1} + R_n(z) \end{aligned} \quad \dots (2)$$

where $a_n = \frac{1}{2\pi i} \int_{C_2} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} d\zeta$ and

$$R_n(z) = \frac{(z - z_0)^n}{2\pi i} \int_{C_2} \frac{f(\zeta) d\zeta}{(\zeta - z_0)^n (\zeta - z)}$$

$$\begin{aligned} \text{Now, } \frac{1}{z - \zeta} &= \frac{1}{z - z_0 + z_0 - \zeta} \\ &= \frac{1}{(z - z_0) - (\zeta - z_0)} \\ &= \frac{1}{(z - z_0) \left[1 - \frac{\zeta - z_0}{z - z_0} \right]} \end{aligned}$$



$$= \frac{1}{z - z_0} \left[1 + \left(\frac{\zeta - z_0}{z - z_0} \right) + \left(\frac{\zeta - z_0}{z - z_0} \right)^2 + \dots + \left(\frac{\zeta - z_0}{z - z_0} \right)^{n-1} + \frac{\left(\frac{\zeta - z_0}{z - z_0} \right)^n}{1 - \left(\frac{\zeta - z_0}{z - z_0} \right)} \right]$$

Multiplying by $\frac{f(\zeta)}{2\pi i}$ and integrating over C_1 we get

$$\int_{C_1} \frac{f(\zeta) d\zeta}{z - \zeta} = \frac{b_1}{z - z_0} + \frac{b_2}{(z - z_0)^2} + \dots + \frac{b_{n-1}}{(z - z_0)^{n-1}} + S_n(z) \quad \dots (3).$$

$$\text{where } b_n = \frac{1}{2\pi i} \int_{C_1} \frac{f(\zeta) d\zeta}{(\zeta - z_0)^{-n+1}}; S_n = \frac{1}{2\pi i (z - z_0)^n} \int_{C_1} \frac{f(\zeta) (\zeta - z_0)^n d\zeta}{z - \zeta}$$

From (1), (2) and (3) we get

$$\begin{aligned} f(z) &= a_0 + a_1(z - z_0) + \dots + a_{n-1}(z - z_0)^{n-1} \\ &\quad + \frac{b_1}{z - z_0} + \frac{b_2}{(z - z_0)^2} + \dots + \frac{b_{n-1}}{(z - z_0)^{n-1}} + R_n(z) + S_n(z) \quad \dots (4) \end{aligned}$$

The required result follows if we can prove that $R_n \rightarrow 0$ and $S_n \rightarrow 0$ as $n \rightarrow \infty$.

Now, if $\zeta \in C_1$ then $|\zeta - z_0| = r_1$ and

$$|z - \zeta| = |(z - z_0) - (\zeta - z_0)| \geq |z - z_0| - r_1.$$

If $\zeta \in C_2$ then $|\zeta - z_0| = r_2$ and

$$|\zeta - z| = |(\zeta - z_0) - (z - z_0)| \geq r_2 - |z - z_0|$$

Now let M denote the maximum value of $|f(z)|$ in $C_1 \cup C_2$.

Then $|R_n| \leq \frac{|z - z_0|^n}{2\pi} \frac{M(2\pi r_2)}{r_2^n(r_2 - |z - z_0|)}$ (by theorem 6.2)

$$\leq \frac{M|z - z_0|}{(r_2 - |z - z_0|)} \left(\frac{|z - z_0|}{r_2} \right)^{n-1}$$

Since $\frac{|z - z_0|}{r_2} < 1$, $R_n \rightarrow 0$ as $n \rightarrow \infty$.

Also $|S_n| \leq \frac{1}{|z - z_0|^n} \frac{Mr_1^n(2\pi r_1)}{2\pi(|z - z_0| - r_1)}$

$$\leq \frac{Mr_1}{(|z - z_0| - r_1)} \left(\frac{r_1}{|z - z_0|} \right)^n$$

Since $\frac{r_1}{|z - z_0|} < 1$, $S_n \rightarrow 0$ as $n \rightarrow \infty$.

Hence, by taking limit $n \rightarrow \infty$ in (4) we get

$$f(z) = \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n} + \sum_{n=0}^{\infty} a_n(z - z_0)^n.$$

Hence the theorem.

Remark. The formulae for the coefficients a_n and b_n in the Laurent's series expansion are given by $a_n = \frac{1}{2\pi i} \int_{C_2} \frac{f(\zeta)d\zeta}{(\zeta - z_0)^{n+1}}$... (1)

and $b_n = \frac{1}{2\pi i} \int_{C_1} \frac{f(\zeta)d\zeta}{(\zeta - z_0)^{-n+1}}$... (2)

Since the integrands in the integrals of (1) and (2) are analytic functions of ζ throughout the annular region, any simple closed curve C in the annulus can be used as the path of integration in place of C_1 and C_2 .

Hence Laurent's series can be written as

$$f(z) = \sum_{-\infty}^{\infty} A_n(z - z_0)^n, (r_1 < |z - z_0| < r_2) \text{ where } A_n = \frac{1}{2\pi i} \int_C \frac{f(\zeta)d\zeta}{(\zeta - z_0)^{n+1}}.$$

Solved Problems

Problem 1. Find the Laurent's series expansion of $f(z) = z^2 e^{1/z}$ about $z = 0$.

Solution. $f(z) = z^2 e^{1/z}$.

Clearly $f(z)$ is analytic at all points $z \neq 0$.

$$\begin{aligned}\text{Now, } f(z) &= z^2 \left[1 + \frac{1}{z} + \frac{1}{2!z^2} + \frac{1}{3!z^3} + \dots \right] \\ &= z^2 + z + \frac{1}{2} + \frac{1}{3!z} + \frac{1}{4!z^2} + \dots\end{aligned}$$

This is the required Laurent's series expansion for $f(z)$ at $z = 0$.

Problem 2. Expand $\frac{-1}{(z-1)(z-2)}$ as a power series in z in the regions

- (i) $|z| < 1$ (ii) $1 < |z| < 2$ (iii) $|z| > 2$.

Solution. Let $f(z) = \frac{-1}{(z-1)(z-2)}$.

By splitting into partial fractions, we have $f(z) = \frac{1}{z-1} - \frac{1}{z-2}$.

- (i) The only points where $f(z)$ is not analytic are 1 and 2. Hence $f(z)$ is analytic in $|z| < 1$ and hence can be represented as a Taylor's series in $|z| < 1$.

$$\begin{aligned}\therefore f(z) &= \frac{1}{z-1} - \frac{1}{z-2} \\ &= -\frac{1}{1-z} + \frac{1}{2-z} \\ &= -(1-z)^{-1} + \frac{1}{2} \left(1 - \frac{z}{2}\right)^{-1} \\ &= -(1+z+z^2+\dots+z^n+\dots) \\ &\quad + \frac{1}{2} \left(1 + \frac{z}{2} + \frac{z^2}{4} + \dots + \frac{z^n}{2^n} + \dots\right) \\ &= \sum_{n=0}^{\infty} \left[-z^n + \frac{1}{2} \left(\frac{z}{2}\right)^n \right] \\ &= \sum_{n=0}^{\infty} \left(\frac{1}{2^{n+1}} - 1 \right) z^n.\end{aligned}$$

- (ii) $f(z)$ is analytic in the annular region $1 < |z| < 2$ and hence can be expanded as a Laurent's series in this region.

$$\begin{aligned}
 f(z) &= \frac{1}{z-1} - \frac{1}{z-2} \\
 &= \frac{1}{z\left(1-\frac{1}{z}\right)} + \frac{1}{2\left(1-\frac{z}{2}\right)} \\
 &= \frac{1}{z}\left(1-\frac{1}{z}\right)^{-1} + \frac{1}{2}\left(1-\frac{z}{2}\right)^{-1} \\
 &= \frac{1}{z}\left[1 + \left(\frac{1}{z}\right) + \left(\frac{1}{z}\right)^2 + \dots\right] + \frac{1}{2}\left[1 + \left(\frac{z}{2}\right) + \left(\frac{z}{2}\right)^2 + \dots\right] \\
 &\quad \left(\text{since } \left|\frac{1}{z}\right| < 1 \text{ and } \left|\frac{z}{2}\right| < 1\right) \\
 &= \sum_{n=0}^{\infty} \frac{1}{z^{n+1}} + \sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}}.
 \end{aligned}$$

This gives the Laurent's series expansion in $1 < |z| < 2$.

- (iii) $f(z)$ is analytic in the domain $|z| > 2$ and in this domain we have $|2/z| < 1$. Hence

$$\begin{aligned}
 f(z) &= \frac{1}{z}\left[\frac{1}{1-(1/z)}\right] - \frac{1}{z}\left[\frac{1}{1-(2/z)}\right] \\
 &= \frac{1}{z}[1-(1/z)]^{-1} - \frac{1}{z}[1-(2/z)]^{-1} \\
 &= \frac{1}{z}\left[\left(1 + \left(\frac{1}{z}\right) + \left(\frac{1}{z}\right)^2 + \dots\right) - \left(1 + \left(\frac{2}{z}\right) + \left(\frac{2}{z}\right)^2 + \dots\right)\right] \\
 &= \sum_{n=0}^{\infty} \frac{1-2^n}{z^{n+1}}.
 \end{aligned}$$

Problem 3. Expand $\frac{1}{z(z-1)}$ as Laurent's series (i) about $z = 0$ in powers of z and (ii) about $z = 1$ in powers $z-1$. Also state the region of validity.

Solution. (i) The only points where $f(z)$ is not analytic are 0 and 1. Hence $f(z)$ can be expanded as a Laurent's series in the annulus $0 < |z| < 1$.

$$\begin{aligned}
 f(z) &= \frac{1}{z(z-1)} \\
 &= -\frac{1}{z}(1-z)^{-1} \\
 &= -\frac{1}{z}(1+z+z^2+\dots+z^n+\dots) \text{ (since } |z| < 1) \\
 &= -\left(\frac{1}{z} + 1 + z + z^2 + \dots + z^n + \dots\right).
 \end{aligned}$$

This is the Laurent's series expansion of $f(z)$ in $0 < |z| < 1$.

- (ii) $f(z)$ is analytic in $0 < |z-1| < 1$ and hence can be expanded as a Laurent's series in powers of $z-1$ in this region.

$$\begin{aligned}
 \frac{1}{z(z-1)} &= \frac{1}{z-1} \left[\frac{1}{1+(z-1)} \right] \\
 &= \frac{1}{z-1} [1+(z-1)]^{-1} \\
 &= \frac{1}{z-1} [1-(z-1)+(z-1)^2-(z-1)^3+\dots] \\
 &\quad \text{(since } |z-1| < 1) \\
 &= \frac{1}{z-1} - 1 + (z-1) - (z-1)^2 + \dots
 \end{aligned}$$

(Handwritten notes: $(1+x)^{-1} = 1 - x + x^2 - x^3 + \dots$, $(1+x)^{-1} < 1$)

This gives the Laurent's series expansion in $0 < |z-1| < 1$.

Problem 4. Find the Laurent's series for $\frac{z}{(z+1)(z+2)}$ about $z = -2$.

Solution. Let $f(z) = \frac{z}{(z+1)(z+2)}$

$$= \frac{-1}{z+1} + \frac{2}{z+2} \text{ (verify)}$$

$$= \frac{-1}{(z+2)-1} + \frac{2}{z+2}$$

$$= [1-(z+2)]^{-1} + \frac{2}{z+2}$$

$$= [1+(z+2)+(z+2)^2+\dots] + \frac{2}{z+2}$$

$$= \frac{2}{z+2} + 1 + (z+2) + (z+2)^2 + \dots$$

$$\begin{aligned}
 f(z) &= \frac{1}{z(z-1)} \\
 &= -\frac{1}{z}(1-z)^{-1} \\
 &= -\frac{1}{z}(1+z+z^2+\dots+z^n+\dots) \text{ (since } |z| < 1) \\
 &= -\left(\frac{1}{z} + 1 + z + z^2 + \dots + z^n + \dots\right).
 \end{aligned}$$

This is the Laurent's series expansion of $f(z)$ in $0 < |z| < 1$.

- (ii) $f(z)$ is analytic in $0 < |z-1| < 1$ and hence can be expanded as a Laurent's series in powers of $z-1$ in this region.

$$\begin{aligned}
 \frac{1}{z(z-1)} &= \frac{1}{z-1} \left[\frac{1}{1+(z-1)} \right] \\
 &= \frac{1}{z-1} [1+(z-1)]^{-1} \\
 &= \frac{1}{z-1} [1 - (z-1) + (z-1)^2 - (z-1)^3 + \dots] \\
 &\quad \text{(since } |z-1| < 1) \\
 &= \frac{1}{z-1} - 1 + (z-1) - (z-1)^2 + \dots
 \end{aligned}$$

(Handwritten notes: $(1+x)^{-1} = 1 - x + x^2 - x^3 + \dots$, $(1+x)^{-1} < 1$)

This gives the Laurent's series expansion in $0 < |z-1| < 1$.

Problem 4. Find the Laurent's series for $\frac{z}{(z+1)(z+2)}$ about $z = -2$.

Solution. Let $f(z) = \frac{z}{(z+1)(z+2)}$

$$= \frac{-1}{z+1} + \frac{2}{z+2} \text{ (verify)}$$

$$= \frac{-1}{(z+2)-1} + \frac{2}{z+2}$$

$$= [1 - (z+2)]^{-1} + \frac{2}{z+2}$$

$$= [1 + (z+2) + (z+2)^2 + \dots] + \frac{2}{z+2}$$

Problem 5. Expand $f(z) = \frac{z}{(z-1)(2-z)}$ in a Laurent's series valid for (i) $|z| < 1$ (ii) $1 < |z| < 2$ (iii) $|z| > 2$ (iv) $|z-1| > 1$ and (v) $0 < |z-2| < 1$.

Solution. Let $f(z) = \frac{z}{(z-1)(2-z)}$.

$$\therefore f(z) = \frac{1}{z-1} + \frac{2}{2-z} \text{ (by partial fractions).}$$

(i) $|z| < 1$.

$$f(z) = \frac{-1}{1-z} + \frac{2}{2(1-z/2)} = -(1-z)^{-1} + (1-z/2)^{-1}.$$

Since $|z| < 1$, $f(z)$ can be expanded in series as

$$\begin{aligned} f(z) &= -[1 + z + z^2 + z^3 + \dots] + \left[1 + \left(\frac{z}{2}\right) + \left(\frac{z}{2}\right)^2 + \left(\frac{z}{2}\right)^3 + \dots\right] \\ &= -\frac{z}{2} - \frac{3z^2}{4} - \frac{7z^3}{8} - \dots \end{aligned}$$

(ii) $1 < |z| < 2$.

$$f(z) = \frac{1}{z(1-1/z)} + \frac{2}{2(1-z/2)} = \frac{1}{z}(1-1/z)^{-1} + (1-z/2)^{-1}.$$

Now $1 < |z| < 2 \Rightarrow \left|\frac{1}{z}\right| < 1$ and $\left|\frac{z}{2}\right| < 1$. Hence we have

$$\begin{aligned} f(z) &= \frac{1}{z} \left[1 + \frac{1}{z} + \left(\frac{1}{z}\right)^2 + \left(\frac{1}{z}\right)^3 + \dots\right] \\ &\quad + \left[1 + \left(\frac{z}{2}\right) + \left(\frac{z}{2}\right)^2 + \left(\frac{z}{2}\right)^3 + \dots\right] \\ &= \dots + \left(\frac{1}{z}\right)^3 + \left(\frac{1}{z}\right)^2 + \frac{1}{z} + 1 + \left(\frac{z}{2}\right) + \left(\frac{z}{2}\right)^2 + \left(\frac{z}{2}\right)^3 + \dots \end{aligned}$$

(iii) $|z| > 2$. Hence $\left|\frac{2}{z}\right| < 1$ and $\left|\frac{1}{z}\right| < 1$.

$$\begin{aligned}
 f(z) &= \frac{1}{z(1-1/z)} - \frac{2}{z(1-2/z)} \\
 &= \frac{1}{z}(1-1/z)^{-1} - \frac{2}{z}(1-2/z)^{-1} \\
 &= \frac{1}{z} \left(1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \dots \right) - \frac{2}{z} \left(1 + \frac{2}{z} + \left(\frac{2}{z}\right)^2 + \left(\frac{2}{z}\right)^3 + \dots \right) \\
 &= -\frac{1}{z} - \frac{3}{z^2} - \frac{7}{z^3} - \frac{15}{z^4} - \dots
 \end{aligned}$$

(i) $|z-1| > 1$, Hence $\frac{1}{|z-1|} < 1$.

$$\begin{aligned}
 f(z) &= \frac{1}{z-1} - \frac{2}{z-2} \\
 &= \frac{1}{z-1} - \frac{2}{z-1-1} \\
 &= \frac{1}{z-1} - \frac{2}{(z-1)\left(1-\frac{1}{z-1}\right)} \\
 &= \frac{1}{z-1} - \frac{2}{z-1} \left(1-\frac{1}{z-1}\right)^{-1} \\
 &= \frac{1}{z-1} - \frac{2}{z-1} \left[1 + \frac{1}{z-1} + \left(\frac{1}{z-1}\right)^2 + \left(\frac{1}{z-1}\right)^3 + \dots \right] \\
 &= -\frac{1}{z-1} - \frac{2}{(z-1)^2} - \frac{2}{(z-1)^3} - \dots
 \end{aligned}$$

(ii) $0 < |z-2| < 1$.

$$\begin{aligned}
 f(z) &= \frac{1}{z-2+1} - \frac{2}{z-2} \\
 &= [1+(z-2)]^{-1} - \frac{2}{z-2} \\
 &= [1-(z-2)+(z-2)^2-(z-2)^3+\dots] - \frac{2}{z-2} \\
 &= \frac{-2}{z-2} + 1 - (z-2) + (z-2)^2 - (z-2)^3 + \dots
 \end{aligned}$$

Problem 6. Expand $\frac{1}{z^2 - 3z + 2}$ in Laurent's series valid in the region $1 < |z| < 2$.

Solution. Let $f(z) = \frac{1}{z^2 - 3z + 2}$. Then

$$f(z) = \frac{(z-2) - (z-1)}{(z-2)(z-1)} = \frac{1}{z-1} - \frac{1}{z-2}.$$

$f(z)$ is analytic in the region $1 < |z| < 2$.

Hence $f(z)$ can be expanded in Laurent's series in that region. Now

$$f(z) = \frac{1}{-2\left(1 - \frac{z}{2}\right)} - \frac{1}{z\left(1 - \frac{1}{z}\right)} = -\frac{1}{2}\left(1 - \frac{z}{2}\right)^{-1} - \frac{1}{z}\left(1 - \frac{1}{z}\right)^{-1}.$$

In the region $1 < |z| < 2$, we have $\left|\frac{z}{2}\right| < 1$ and $\left|\frac{1}{z}\right| < 1$. Hence $f(z)$ can be expanded in Laurent's series as

$$\begin{aligned} f(z) &= -\frac{1}{2}\left[1 + \frac{z}{2} + \left(\frac{z}{2}\right)^2 + \left(\frac{z}{2}\right)^3 + \dots\right] \\ &\quad - \frac{1}{z}\left[1 + \frac{1}{z} + \left(\frac{1}{z}\right)^2 + \left(\frac{1}{z}\right)^3 + \dots\right] \\ &= -\frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n - \frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^n \\ &= -\sum_{n=0}^{\infty}\frac{z^n}{2^{n+1}} - \sum_{n=0}^{\infty}\frac{1}{z^{n+1}}. \end{aligned}$$

Problem 7. If $f(z) = \frac{z+4}{(z+3)(z-1)^2}$ find Laurent's series expansions in

(i) $0 < |z-1| < 4$ and (ii) $|z-1| > 4$.

Solution. Let $f(z) = \frac{z+4}{(z+3)(z-1)^2}$.

By expressing $f(z)$ into partial fractions we get

$$f(z) = \frac{1}{16(z+3)} - \frac{1}{16(z-1)} + \frac{5}{4(z-1)^2}.$$

(i) $0 < |z-1| < 4$. Hence $0 < \left|\frac{z-1}{4}\right| < 1$.

$$\begin{aligned}
 f(z) &= \frac{1}{16(z-1+4)} - \frac{1}{16(z-1)} + \frac{5}{4(z-1)^2} \\
 &= \frac{1}{64 \left(1 + \frac{z-1}{4}\right)} - \frac{1}{16(z-1)} + \frac{5}{4(z-1)^2} \\
 &= \frac{1}{64} \left(1 + \frac{z-1}{4}\right)^{-1} - \frac{1}{16(z-1)} + \frac{5}{4(z-1)^2}.
 \end{aligned}$$

Since $\left|\frac{z-1}{4}\right| < 1$, we have

$$\begin{aligned}
 f(z) &= \frac{1}{64} \left[1 - \left(\frac{z-1}{4}\right) + \left(\frac{z-1}{4}\right)^2 - \left(\frac{z-1}{4}\right)^3 + \dots \right] \\
 &\quad - \frac{1}{16(z-1)} + \frac{5}{4(z-1)^2} \\
 &= \frac{5}{4(z-1)^2} - \frac{1}{16(z-1)} + \frac{1}{64} - \frac{1}{64} \left[\frac{z-1}{4} - \left(\frac{z-1}{4}\right)^2 + \dots \right].
 \end{aligned}$$

This is the required Laurent's series expansion for $f(z)$ in $0 < |z-1| < 4$.

(ii) $|z-1| > 4$. Hence $\left|\frac{4}{z-1}\right| < 1$.

$$\begin{aligned}
 \text{Now } f(z) &= \frac{1}{16(z-1) \left(1 + \frac{4}{z-1}\right)} - \frac{1}{16(z-1)} + \frac{5}{4(z-1)^2} \\
 &= \frac{1}{16(z-1)} \left[1 - \left(\frac{4}{z-1}\right) + \left(\frac{4}{z-1}\right)^2 - \left(\frac{4}{z-1}\right)^3 + \dots \right] \\
 &\quad - \frac{1}{16(z-1)} + \frac{5}{4(z-1)^2} \\
 &= \frac{1}{(z-1)^2} + \frac{1}{(z-1)^3} - \frac{4}{(z-1)^4} + \frac{4^2}{(z-1)^5} - \dots.
 \end{aligned}$$

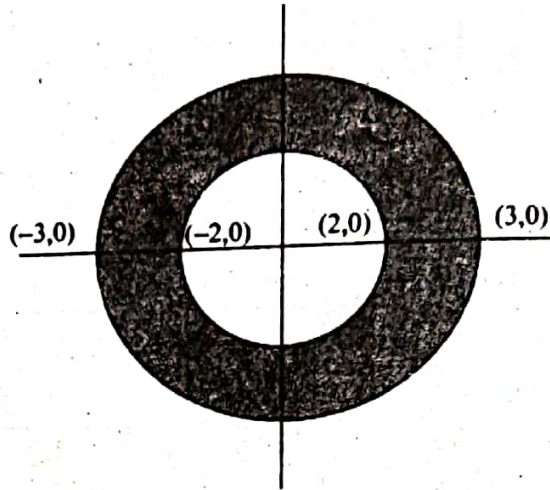
Problem 8. Find the Laurent's series expansion of the function $\frac{z^2-1}{(z+2)(z+3)}$ valid in the annular region $2 < |z| < 3$.

Solution. Let $f(z) = \frac{z^2-1}{(z+2)(z+3)}$.

By splitting $f(z)$ into partial fractions, we get

$$f(z) = 1 + \frac{3}{z+2} - \frac{8}{z+3}.$$

$f(z)$ is analytic in the annular region $2 < |z| < 3$.



Hence $f(z)$ can be expanded as a Laurent's series in that region.

$$\begin{aligned} f(z) &= 1 + \frac{3}{z \left(1 + \frac{2}{z}\right)} - \frac{8}{3 \left(1 + \frac{z}{3}\right)} \\ &= 1 + \frac{3}{z} \left(1 + \frac{2}{z}\right)^{-1} - \frac{8}{3} \left(1 + \frac{z}{3}\right)^{-1} \\ &= 1 + \frac{3}{z} \left[1 - \frac{2}{z} + \left(\frac{2}{z}\right)^2 - \left(\frac{2}{z}\right)^3 + \dots \right] \\ &\quad - \frac{8}{3} \left[1 - \frac{z}{3} + \left(\frac{z}{3}\right)^2 - \left(\frac{z}{3}\right)^3 + \dots \right] \\ &= 1 + \frac{3}{z} \sum_{n=0}^{\infty} (-1)^n \left(\frac{2}{z}\right)^n - \frac{8}{3} \sum_{n=0}^{\infty} (-1)^n \left(\frac{z}{3}\right)^n \\ &= 1 + 3 \sum_{n=0}^{\infty} \frac{(-1)^n 2^n}{z^{n+1}} - 8 \sum_{n=0}^{\infty} \frac{(-1)^n z^n}{3^{n+1}}. \end{aligned}$$

Problem 9. For the function $f(z) = \frac{2z^3 + 1}{z(z+1)}$ find (i) a Taylor's series valid in a neighbourhood of $z = i$ and (ii) a Laurent's series valid within an annulus of which centre is the origin.

Solution. (i) $f(z) = \frac{2z^3 + 1}{z(z+1)}$

$$= 2z - 2 + \frac{1}{z} + \frac{1}{z+1} \text{ (by partial fractions)}$$

$$= 2(z-1) + \frac{1}{z} + \frac{1}{z+1} \quad \dots (1)$$

$$= g(z) + h(z) + j(z)$$

where $g(z) = 2(z-1)$, $h(z) = \frac{1}{z}$ and $j(z) = \frac{1}{z+1}$.

Taylor's expansion for $g(z)$ about $z = i$ is obviously $2(i-1) + 2(z-i)$.

Taylor's expansion for $h(z)$ about $z = i$ is given by

$$h(z) = h(i) + \sum_{n=1}^{\infty} \frac{h^{(n)}(i)}{n!} (z-i)^n.$$

Here $h(i) = \frac{1}{i}$; $h^{(n)}(z) = \frac{(-1)^n n!}{z^{n+1}}$ so that $h^{(n)}(i) = \frac{(-1)^n n!}{i^{n+1}}$.

$$\therefore h(z) = \frac{1}{i} + \sum_{n=1}^{\infty} \frac{(-1)^n n!}{i^{n+1} n!} (z-i)^n = \sum_{n=0}^{\infty} \frac{(-1)^n}{i^{n+1}} (z-i)^n.$$

Similarly we can prove that $j(z) = \sum_{n=0}^{\infty} \frac{(-1)^n (z-i)^n}{(1+i)^{n+1}}$.

Hence the Taylor's expansion for $f(z)$ is

$$f(z) = 2(i-1) + 2(z-i) + \sum_{n=0}^{\infty} \left[\frac{(-1)^n}{i^{n+1}} + \frac{(-1)^n}{(1+i)^{n+1}} \right] (z-i)^n.$$

(ii) $f(z) = 2z - 2 + \frac{1}{z} + (1+z)^{-1}$ (from (1))

$$= 2z - 2 + \frac{1}{z} + (1 - z + z^2 - z^3 + \dots) \text{ if } |z| < 1.$$

$\therefore$ In the annulus $0 < |z| < 1$ the Laurent's expansion is given by

$$f(z) = \frac{1}{z} - 1 + z + z^2 - z^3 + z^4 - \dots$$

Problem 10. Expand $f(z) = \frac{e^{2z}}{(z-1)^3}$ about $z = 1$ as a Laurent's series. Also indicate the region of convergence of the series.

Solution.

$$\begin{aligned}
 f(z) &= \frac{e^{2(z-1)+2}}{(z-1)^3} \\
 &= \frac{e^2 e^{2(z-1)}}{(z-1)^3} \\
 &= \frac{e^2}{(z-1)^3} \left[1 + \frac{2(z-1)}{1!} + \frac{2^2(z-1)^2}{2!} + \frac{2^3(z-1)^3}{3!} + \dots \right] \\
 &= e^2 \left[\frac{1}{(z-1)^3} + \frac{2}{(z-1)^2} + \frac{2}{(z-1)} + \frac{4}{3} + \frac{2^4}{4!}(z-1) + \dots \right].
 \end{aligned}$$

This series converges for all values of z except $z = 1$.

Exercises

1. Prove that $\frac{1+2z}{z^2+z^3} = \frac{1}{z^2} + \frac{1}{z} - 1 + z - z^2 + z^3 - \dots$ where $0 < |z| < 1$.
2. Find a Laurent's series expansions in powers of z of the function $f(z) = \frac{1}{z(1+z^2)}$.
3. Find two different Laurent's series for $\frac{1}{z^2(1-z)}$ about $z = 0$ and state the regions of validity.
4. Expand $\frac{1}{(z-1)(z-2)}$ as a power series in z valid in
 - (i) $|z| < 1$
 - (ii) $1 < |z| < 2$
 - (iii) $|z| > 2$
5. Expand $\frac{1}{z(z^2-3z+2)}$ as a power series valid in $1 < |z| < 2$.
6. Expand $f(z) = \frac{1}{(z+1)(z+3)}$ in Laurent's series valid for
 - (i) $1 < |z| < 3$
 - (ii) $|z| > 3$
 - (iii) $0 < |z+1| < 2$
 - (iv) $|z| < 1$
7. Expand in Laurent's series $\frac{1}{z(z-1)^2}$ at the point $z = 1$.

8. Find a Laurent's series expansion in powers of z for the function $f(z) = \frac{1}{z(1+z^2)}$.
9. Expand $\frac{1}{z^2(z-3)^2}$ as a Laurent's series at $z = 3$ and state the region of validity.
10. Expand $\frac{1}{z(z^2 - 3z + 2)}$ in powers of z in the regions
(i) $0 < |z| < 1$ (ii) $1 < |z| < 2$ (iii) $|z| > 2$
11. Expand $f(z) = \frac{z+3}{z(z^2 - z - 2)}$ in powers of z
(i) within the unit circle about the origin;
(ii) within the annular region between the concentric circle about the origin having radii 1 and 2 respectively;
(iii) the exterior to the circle with centre as origin and radius 2.
12. Represent $f(z) = \frac{z}{(z-1)(z-3)}$ by a series of powers of $z-1$ in $0 < |z-1| < 2$.
13. Give two Laurent's series expansion in powers of z for the function $f(z) = \frac{1}{z^2(1-z)}$ and specify the regions in which the expansions are valid.
14. Represent the function $\frac{z+1}{z-1}$ by
(i) its Taylor's series in powers of z and give the region of validity;
(ii) its Laurent's series in powers of z for the region $|z| > 1$.
15. Obtain the Laurent's series of the function $f(z) = \frac{1}{(z-1)(z-3)}$ valid in the region of (i) $1 < |z| < 3$ (ii) $|z| > 3$.
16. Obtain the Laurent's series expansions for $\frac{1}{z(1-z)^2}$ about $z = 0$ and specify the regions in which the expansions are valid.
17. Find the expansions in powers of z for $\frac{1}{(z^2+1)(z^2+2)}$ when
(i) $|z| < 1$ (ii) $1 < |z| < \sqrt{2}$ (iii) $|z| > \sqrt{2}$

Answers.

$$2. f(z) = \frac{1}{z} - z + z^3 - z^5 + \dots; 0 < |z| < 1$$

$$f(z) = \frac{1}{z^3} - \frac{1}{z^5} + \frac{1}{z^7} \dots; |z| > 1.$$

3. (i) $\frac{1}{z^2} + \frac{1}{z} + 1 + z + z^2 + \dots; 0 < |z| < 1$

(ii) $-\left(\frac{1}{z^3} + \frac{1}{z^4} + \frac{1}{z^5} + \dots\right); |z| > 1$

4. (i) $\sum_{n=0}^{\infty} \left(1 - \frac{1}{2^{n+1}}\right) z^n$ (ii) $-\sum_{n=1}^{\infty} \frac{1}{z^n} - \sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}}$ (iii) $\sum_{n=1}^{\infty} \frac{2^n - 1}{z^{n+1}}$

5. $-\sum_{n=0}^{\infty} \left[\frac{1}{z^{n+2}} + \frac{z^{n-1}}{2^{n+1}}\right]$

6. (i) $-\frac{1}{2z^4} + \frac{1}{2z^3} - \frac{1}{2z^2} + \frac{1}{2z} - \frac{1}{6} + \frac{z}{18} - \frac{z^2}{54} + \dots$

(ii) $\frac{1}{z^2} - \frac{4}{z^3} + \frac{13}{z^4} - \frac{40}{z^5} + \dots$

(iii) $\frac{1}{2(z+1)} - \frac{1}{4} + \frac{1}{8}(z+1) - \frac{1}{16}(z+1)^2 + \dots$

(iv) $\frac{1}{3} - \frac{4z}{9} + \frac{13}{27}z^2 - \frac{40}{81}z^3 + \dots$

7. $\sum_{n=0}^{\infty} (-1)^n (z-1)^{n-2}$

8. $\frac{1}{z} + \sum_{n=1}^{\infty} (-1)^n z^{2n-1}$ if $0 < |z| < 1$

9. $\frac{1}{9(z-3)^2} - \frac{2}{27(z-3)} + \frac{1}{27} - \frac{4(z-3)}{243} + \dots; 0 < |z-3| < 3$

10. (i) $\frac{1}{2z} + \frac{3}{4} + \frac{7z}{8} + \frac{15z^2}{16} + \frac{31z^3}{32} + \dots$

(ii) $\left(-\frac{1}{2z} - \frac{1}{z^2} - \frac{1}{z^3} - \dots\right) - \frac{1}{4} \left(1 + \frac{z}{2} + \frac{z^2}{2^2} + \dots\right)$

(iii) $(2-1)\frac{1}{z^3} + (2^2-1)\frac{1}{z^4} + (2^3-1)\frac{1}{z^5} + \dots$

11. (i) $-\frac{3}{2z} + \sum_{n=0}^{\infty} \left[\frac{2}{3}(-1)^n - \frac{5}{12}\left(\frac{1}{2}\right)^n\right] z^n$

(ii) $-\frac{3}{2z} + \frac{2}{3z} \sum_{n=0}^{\infty} \frac{(-1)^n}{z^n} - \frac{5}{12} \sum_{n=0}^{\infty} \left(\frac{z}{2}\right)^n$

$$(iii) -\frac{3}{2z} + \frac{2}{3z} \sum_{n=0}^{\infty} \frac{(-1)^n}{z^n} + \frac{5}{6z} \sum_{n=0}^{\infty} \left(\frac{2}{z}\right)^n$$

$$12. -\frac{1}{2(z-1)} - \frac{3}{4} \sum_{n=0}^{\infty} \left(\frac{z-1}{2}\right)^n$$

$$13. (i) \sum_{n=0}^{\infty} z^{n-2}; 0 < |z| < 1$$

$$(ii) \sum_{n=0}^{\infty} z^{-n-3}; |z| > 1$$

$$14. (i) -1 - 2 \sum_{n=0}^{\infty} z^n; |z| < 1$$

$$(ii) 1 + 2 \sum_{n=0}^{\infty} z^{-n}; |z| > 1$$

$$15. (i) -\frac{1}{2(z-1)} - \frac{1}{4} \sum_{n=0}^{\infty} \left(\frac{z-1}{2}\right)^n$$

$$(ii) \frac{1}{2(z-3)} - \frac{1}{4} \sum_{n=0}^{\infty} (-1)^n \left(\frac{z-3}{2}\right)^n$$

$$16. (i) \sum_{n=0}^{\infty} (n+1)z^{n-1}; |z| < 1$$

$$(ii) \sum_{n=0}^{\infty} \left(\frac{n+1}{z^{n+3}}\right); |z| < 1$$

$$17. (i) \sum_{n=0}^{\infty} (-1)^n \left(1 - \frac{1}{2^{n+1}}\right) z^{2n}$$

$$(ii) \sum_{n=1}^{\infty} \frac{(-1)^n}{z^{2n+2}} + \sum_{n=0}^{\infty} \frac{(-1)^{n+1} z^{2n}}{2^{n+1}}$$

$$(iii) \sum_{n=0}^{\infty} \frac{(-1)^n (1-2^n)}{z^{2n+2}}$$

7.3. Zeros of an Analytic Function

Definition. Let $f(z)$ be a function which is analytic in a region D . Let $a \in D$. Then a is said to be a **zero of order r** (where r is a positive integer) for $f(z)$ if $f(z) = (z-a)^r \varphi(z)$ where $\varphi(z)$ is analytic at a and $\varphi(a) \neq 0$.

Example 1. Consider $f(z) = \sin z$

$$\begin{aligned}\text{we know that } \sin z &= z - \frac{z^3}{3!} + \frac{z^5}{5!} - \dots \\ &= z \left(1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \dots \right) \\ &= z \varphi(z).\end{aligned}$$

$$\text{where } \varphi(z) = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \dots$$

Obviously $\varphi(z)$ is analytic and $\varphi(0) = 1 \neq 0$.

$z = 0$ is a zero of order 1 for $\sin z$.

Example 2. Let $f(z) = (z - 2i)^2(z + 3)^3e^z$

$2i$ is a zero of order 2 and -3 is a zero of order 3 for $f(z)$.

Example 3. Let $f(z) = z^2 \sin z$

$$\begin{aligned}\text{Then } f(z) &= z^2 \left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \dots \right) \\ &= z^3 \left(1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \dots \right) \\ &= z^3 \varphi(z)\end{aligned}$$

$$\text{where } \varphi(z) = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \dots$$

Obviously $\varphi(z)$ is analytic and $\varphi(0) \neq 0$.

$\therefore z = 0$ is a zero of order 3 for $f(z) = z^2 \sin z$.

Example 4. Let $f(z) = \frac{z^3 - 1}{z^3 + 1}$.

$$\begin{aligned}f(z) = 0 &\Rightarrow z^3 - 1 = 0 \\ &\Rightarrow (z - 1)(z^2 + z + 1) = 0\end{aligned}$$

Hence the zeros of $f(z)$ are $1, \frac{-1 + i\sqrt{3}}{2}, \frac{-1 - i\sqrt{3}}{2}$ and each one is a zero of order 1.

Theorem 7.3. Suppose $f(z)$ is analytic in a region D and is not identically zero in D . Then the set of all zeros of $f(z)$ is isolated.

Proof. Let $a \in D$ be a zero for $f(z)$. We shall prove that there exists a neighbourhood $|z - a| < \delta$ such that this neighbourhood does not contain any other zero for $f(z)$.

Suppose a is a zero of order r for $f(z)$.

$$\text{Then } f(z) = (z - a)^r \varphi(z) \quad \dots (1)$$

where $\varphi(z)$ is analytic at a and $\varphi(a) \neq 0$.

Now, since φ is analytic at a , φ is continuous at a .

$\therefore$ We can find a $\delta > 0$ such that

$$|z - a| < \delta \Rightarrow |\varphi(z) - \varphi(a)| < \frac{|\varphi(a)|}{2}.$$

We claim that the neighbourhood $|z - a| < \delta$ does not contain any other zero of $f(z)$.

Suppose $b \neq a$ is another zero for $f(z)$ in this neighbourhood. Then $|b - a| < \delta$ and $f(b) = 0$.

$$\therefore (b - a)^r \varphi(b) = 0 \quad (\text{from (1)})$$

Now, since $b \neq a$, $(b - a)^r \neq 0$.

$$\therefore \varphi(b) = 0$$

$$\text{Further } |b - a| < \delta \Rightarrow |\varphi(b) - \varphi(a)| < \frac{|\varphi(a)|}{2}$$

$$\Rightarrow |\varphi(a)| < \frac{|\varphi(a)|}{2} \text{ which is a contradiction.}$$

Thus the neighbourhood $|z - a| < \delta$ contains no other zero of $f(z)$ and hence the set of all zeros of $f(z)$ is isolated.

Corollary 1. Let $f(z)$ be analytic in a region D . Suppose $f(z) \equiv 0$ on a subset of D which has a limit point in D . Then $f(z)$ is identically zero in D .

Corollary 2. Let $f(z)$ and $g(z)$ be two functions which are analytic in a region D . Suppose $f(z) = g(z)$ on a subset of D which has a limit point in D . Then $f(z) = g(z)$ in D .

(consider the function $f(z) - g(z)$ and the result follows from corollary 1)

Exercises.

- Find all the zeros of the following functions.

(a) $\cos z$

(b) $\frac{(z+1)^2(iz+2)^3}{z+7}$

- Prove that there is no analytic functions whose zeros are precisely the points $1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{n}, \dots$

Answers.

1. (a) $(2n + 1)\pi/2; n \in \mathbb{Z}$ (b) -1 and $-2/i$.

7.4. Singularities

Definition. A point a is called a **singular point** or a **singularity** of a function $f(z)$ if $f(z)$ is not analytic at a and f is analytic at some point of every disc $|z - a| < r$.

Example 1. Consider the function $f(z) = \frac{1}{z}$.

Then $f'(z) = -\frac{1}{z^2}$ for all $z \neq 0$.

Thus $f(z)$ is analytic except at $z = 0$.

$\therefore z = 0$ is a singular point of $f(z)$.

Example 2. Consider the function $f(z) = \frac{1}{z(z-i)}$.

0 and i are singular points for $f(z)$.

Definition. A point a is called an **isolated singularity** for $f(z)$ if

(i) $f(z)$ is not analytic at $z = a$ and

(ii) there exists $r > 0$ such that $f(z)$ is analytic in $0 < |z - a| < r$.

(i.e) the neighbourhood $|z - a| < r$ contains no singularity of $f(z)$ except a .

Example 1. $f(z) = \frac{z+1}{z^2(z^2+1)}$ has three isolated singularities $z = 0, i, -i$.

Example 2. Consider the principal branch of logarithm given by $\log re^{i\theta} = \log r + i\theta$ where $-\pi < \theta \leq \pi$.

All points on the negative real axis are singular points of this function. These singularities are not isolated.

Example 3. Consider the function $f(z) = \frac{1}{\sin z}$. The singular points are $0, \pm\pi, \pm2\pi, \dots$ and these are isolated singular points.

We now proceed to classify the isolated singularities of a function.

Let a be an isolated singularity for a function $f(z)$. Let $r > 0$ be such that $f(z)$ is analytic in $0 < |z - a| < r$. In this domain the function $f(z)$ can be represented as a Laurent series given by

$$f(z) = \sum_{n=1}^{\infty} \frac{b_n}{(z-a)^n} + \sum_{n=0}^{\infty} a_n(z-a)^n \quad \text{where}$$

$$a_n = \frac{1}{2\pi i} \int_C \frac{f(\zeta) d\zeta}{(\zeta-a)^{n+1}} \quad \text{and} \quad b_n = \frac{1}{2\pi i} \int_C \frac{f(\zeta) d\zeta}{(\zeta-a)^{-n+1}}.$$

The series consisting of the negative powers of $z-a$ in the above Laurent series expansion of $f(z)$ is given by $\sum_{n=1}^{\infty} \frac{b_n}{(z-a)^n}$ and is called the *principal part or singular part* of $f(z)$ at $z=a$.

The singular part of $f(z)$ at $z=a$ determines the character of the singularity. There are three types of singularities. They are

- (i) **Removable singularities**
- (ii) **Poles**
- (iii) **Essential singularities.**

Definition. Let a be an isolated singularity for $f(z)$. Then a is called a **removable singularity** if the principal part of $f(z)$ at $z=a$ has no terms.

Note. If a is a removable singularity for $f(z)$ then the Laurent's series expansion of $f(z)$ about $z=a$ is given by

$$f(z) = \sum_{n=0}^{\infty} a_n(z-a)^n$$

$$= a_0 + a_1(z-a) + \dots + a_n(z-a)^n + \dots$$

$$\text{Hence } \lim_{z \rightarrow a} f(z) = a_0$$

Hence by defining $f(a) = a_0$ the function $f(z)$ becomes analytic at a .

Example 1. Let $f(z) = \frac{\sin z}{z}$. Clearly 0 is an isolated singular point for $f(z)$.

$$\text{Now, } \frac{\sin z}{z} = \frac{1}{z} \left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \dots \right)$$

$$= 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \dots$$

Here the principal part of $f(z)$ at $z=0$ has no terms.

Hence $z=0$ is a removable singularity.

Also $\lim_{z \rightarrow 0} \frac{\sin z}{z} = 1$. Hence the singularity can be removed by defining $f(0) = 1$ so that the extended function becomes analytic at $z=0$.

Example 2. Let $f(z) = \frac{z - \sin z}{z^3}$.

$z = 0$ is an isolated singularity.

$$\begin{aligned}\text{Further } \frac{z - \sin z}{z^3} &= \frac{1}{z^3} \left[z - \left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \dots \right) \right] \\ &= \frac{1}{3!} - \frac{z^2}{5!} + \frac{z^4}{7!} - \dots\end{aligned}$$

$\therefore z = 0$ is a removable singularity.

By defining $f(0) = \frac{1}{6}$ the function becomes analytic at $z = 0$.

Definition. Let a be an isolated singularity of $f(z)$. The point a is called a **pole** if the principal part of $f(z)$ at $z = a$ has a finite number of terms. If the principal part of $f(z)$ at $z = a$ is given by

$$\frac{b_1}{z-a} + \frac{b_2}{(z-a)^2} + \dots + \frac{b_r}{(z-a)^r}$$

where $b_r \neq 0$, we say that a is a **pole of order r** for $f(z)$.

Note. A pole of order 1 is called a **simple pole** and a pole of order 2 is called a **double pole**.

Example 1. Consider $f(z) = \frac{e^z}{z}$.

$$\frac{e^z}{z} = \frac{1}{z} + 1 + \frac{z}{2!} + \frac{z^2}{3!} + \dots$$

Here the principal part of $f(z)$ at $z = 0$ has a single term $\frac{1}{z}$. Hence $z = 0$ is a simple pole of $f(z)$.

Example 2. Let $f(z) = \tan z = \frac{\sin z}{\cos z}$. The singularities of $f(z)$ are $\frac{\pi}{2} + n\pi$, where $n \in \mathbb{Z}$. All the singularities are poles of order 1.

Example 3. $f(z) = \frac{\cos z}{z^2}$ has a double pole at $z = 0$.

$$\begin{aligned}\text{For, } \frac{\cos z}{z^2} &= \frac{1}{z^2} \left(1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \dots \right) \\ &= \frac{1}{z^2} - \frac{1}{2!} + \frac{z^2}{4!} - \dots\end{aligned}$$

Example 4. Let $f(z) = \frac{z^2 - 2z + 3}{z - 2}$

Then $f(z) = 2 + (z - 2) + \frac{3}{z - 2}$ (by partial fractions)

Here $f(z)$ has a simple pole at $z = 2$.

Definition. Let a be an isolated singularity of $f(z)$. The point a is called an **essential singularity** of $f(z)$ at $z = a$ if the principal part of $f(z)$ at $z = a$ has an infinite number of terms.

Example 1. Let $f(z) = e^{1/z}$. Obviously $z = 0$ is an isolated singularity for $f(z)$.

Further $e^{1/z} = 1 + \frac{1}{z} + \frac{1}{2!z^2} + \frac{1}{3!z^3} + \dots$. The principal part of $f(z)$ has infinite number of terms. Hence $e^{1/z}$ has an essential singularity at $z = 0$.

Example 2. Let $f(z) = z^2 \sin(1/z)$. $f(z)$ has essential singularity at $z = 0$.

In the following theorem we give equivalent characterisations for an isolated singular point a of $f(z)$ to be a removable singularity.

Theorem 7.4. Let $f(z)$ be a function defined in a region D of the complex plane except possibly at a point $a \in D$ and let a be an isolated singularity for $f(z)$. Then a is a removable singularity for $f(z)$ if and only if there exists a complex number a_0 such that by defining $f(a) = a_0$ the extended function becomes analytic at a .

Proof. Suppose a is a removable singularity for $f(z)$.

$$\begin{aligned} \text{Then } f(z) &= \sum_{n=0}^{\infty} a_n(z-a)^n, \quad 0 < |z-a| < r \\ &= a_0 + a_1(z-a) + a_2(z-a)^2 + \dots \end{aligned}$$

$\therefore$ By defining $f(a) = a_0$, f becomes analytic at a .

Conversely, suppose there exists a complex number a_0 such that by defining $f(a) = a_0$, f becomes analytic in $|z-a| < r$.

Hence f can be represented as a Taylor's series, in power of $z-a$ in this neighbourhood, given by $f(z) = \sum_{n=0}^{\infty} a_n(z-a)^n$. This shows that the principal part of $f(z)$ at $z = a$ has no terms. Hence a is a removable singularity for $f(z)$. $|f(z)| \leq M$

Theorem 7.5. (Riemann's theorem) Let f be a function which is bounded and analytic throughout a domain $0 < |z - z_0| < \delta$. Then either f is analytic at z_0 or else z_0 is a removable singular point of f .

Proof. Consider the Laurent's series for the function in the given domain about z_0 . The coefficient b_n of $\frac{1}{(z - z_0)^n}$ is given by $b_n = \frac{1}{2\pi i} \int_C \frac{f(z)dz}{(z - z_0)^{-n+1}}$ where C is the circle

$$|z - z_0| = r \text{ where } r < \delta.$$

Now, since f is bounded there exists a positive real number M such that $|f(z)| \leq M$ in $0 < |z - z_0| < \delta$.

$$\begin{aligned} \therefore |b_n| &\leq \frac{1}{2\pi} \frac{M(2\pi r)}{r^{-n+1}} \text{ (by theorem 6.2)} \\ &= Mr^n \end{aligned}$$

Since it is true for every r such that $0 < r < \delta$, taking limit $r \rightarrow 0$ we get $b_n = 0$. Hence the Laurent's series for $f(z)$ has no principal part. Hence the theorem follows.

Theorem 7.6. Let $f(z)$ be a function having a as an isolated singular point. The following are equivalent.

- (i) a is a pole of order r for $f(z)$.
- (ii) $f(z)$ can be written in the form $f(z) = \frac{1}{(z - a)^r} \theta(z)$ where $\theta(z)$ has a removable singularity at $z = a$ and $\lim_{z \rightarrow a} \theta(z) \neq 0$.
- (iii) a is a zero of order r for $\frac{1}{f(z)}$.

Proof. (i) $\Rightarrow$ (ii). Let a be a pole order r for $f(z)$. Then the Laurent's series expansion of $f(z)$ about a is given by $f(z) = \sum_{n=1}^r \frac{b_n}{(z - a)^n} + \sum_{n=0}^{\infty} a_n (z - a)^n$ where $b_r \neq 0$.

$$\begin{aligned} \therefore f(z) &= \frac{1}{(z - a)^r} [b_r + b_{r-1}(z - a) + \dots + b_0(z - a)^{r-1} + a_0(z - a)^r + \dots] \\ &= \frac{1}{(z - a)^r} \theta(z) \text{ where } \theta(z) = b_r + b_{r-1}(z - a) + \dots \end{aligned}$$

Clearly $\lim_{z \rightarrow a} \theta(z) = b_r \neq 0$ and $\theta(z)$ has a removable singularity at $z = a$.

(ii) $\Rightarrow$ (iii) Let $f(z) = \frac{1}{(z - a)^r} \theta(z)$ and by suitably defining $\theta(a)$ we may assume that $\theta(z)$ is analytic at a and $\theta(a) \neq 0$.

$$\therefore \frac{1}{f(z)} = (z - a)^r \frac{1}{\theta(z)} \text{ and } \frac{1}{\theta(z)} \text{ is analytic at } a \text{ and } \frac{1}{\theta(a)} \neq 0.$$

Hence a is a zero of order r for $\frac{1}{f(z)}$.

(iii) $\Rightarrow$ (i) Let a be a zero of order r for $\frac{1}{f(z)}$.

Then $\frac{1}{f(z)} = (z-a)^r g(z)$ where $g(z)$ is analytic at a and $g(a) \neq 0$.

$\therefore f(z) = \frac{g_1(z)}{(z-a)^r}$ where $g_1(z)$ is analytic at a and $g_1(a) \neq 0$.

Let $g_1(z) = a_0 + a_1(z-a) + \dots + a_n(z-a)^n + \dots$ so that $a_0 \neq 0$.

$$\therefore f(z) = \frac{a_0}{(z-a)^r} + \frac{a_1}{(z-a)^{r-1}} + \dots + a_r + a_{r+1}(z-a) + \dots$$

in $0 < |z-a| < r$

$\therefore$ The principal part of $f(z)$ at $z=a$ is

$$\frac{a_0}{(z-a)^r} + \frac{a_1}{(z-a)^{r-1}} + \dots + \frac{a_{r-1}}{z-a} \text{ and } a_0 \neq 0.$$

$\therefore a$ is a pole of order r for $f(z)$.

Theorem 7.7. An isolated singularity a of $f(z)$ is a pole if and only if $\lim_{z \rightarrow a} f(z) = \infty$.

Proof. If a is a pole of order r for $f(z)$ then $f(z) = \frac{g(z)}{(z-a)^r}$ with $g(a) \neq 0$.

$$\therefore \lim_{z \rightarrow a} f(z) = \infty.$$

Conversely let a be an isolated singularity for $f(z)$ and let $\lim_{z \rightarrow a} f(z) = \infty$.

$$\text{Let } \theta(z) = \frac{1}{f(z)}.$$

$$\text{Then } \lim_{z \rightarrow a} \theta(z) = 0.$$

Hence a is a removable singularity for $\theta(z)$ and by defining $\theta(z) = 0$, θ becomes analytic at a . Let a be a zero of order r for the function $\theta(z)$. Then a is a pole of order r for $f(z)$.

Definition. A function $f(z)$ is said to be a **meromorphic function** if it is analytic except at a finite number of points and these finite set of points are poles.

Example 1. Let $f(z) = \frac{1}{z(z-1)^2}$.

$f(z)$ is analytic except at $z=0$ and $z=1$. Also 0 and 1 are poles of order 1 and 2 respectively. Hence $f(z)$ is a meromorphic function.

Example 2. $\frac{e^z}{z} = \frac{1}{z} + 1 + \frac{z}{2!} + \frac{z^2}{3!} + \dots$ is a meromorphic function.

Example 3. $e^{1/z}$ is not a meromorphic function since $z = 0$ is an essential singularity for $e^{1/z}$.

The following theorem due to Weierstrass describes the behaviour of a function in the neighbourhood of an essential singularity.

Theorem 7.8. Let z_0 be an essential singularity for a function $f(z)$. Let c be any complex number. Then given $\varepsilon, \delta > 0$ there exists a point z such that $|z - z_0| < \delta$ and $|f(z) - c| < \varepsilon$.

(i.e) The function $f(z)$ comes arbitrarily close to any complex number c in every neighbourhood of an essential singularity.

Proof. Suppose the theorem is false. Then there exist $\delta, \varepsilon > 0$ such that for every point z satisfying $0 < |z - z_0| < \delta$ we have $|f(z) - c| \geq \varepsilon$.

Now consider the function $g(z) = \frac{1}{f(z) - c}$.

$$\therefore |g(z)| = \frac{1}{|f(z) - c|} \leq \frac{1}{\varepsilon}.$$

Hence $g(z)$ is bounded and further $g(z)$ is analytic in $0 < |z - z_0| < \delta$.

Hence by Riemann's theorem $z = z_0$ is a removable singularity for $g(z)$.

Now, if $g(z_0) \neq 0$ then $\frac{1}{g(z)} = f(z) - c$ is analytic at z_0 .

$\therefore$ By suitably defining $g(z_0)$, the function $g(z)$ becomes analytic at z_0 .

If $g(z_0) = 0$ then let z_0 be a zero of order r for $g(z)$.

Then z_0 is a pole of order r for $\frac{1}{g(z)} = f(z) - c$.

Thus $f(z)$ is either analytic at z_0 or else z_0 is a pole of $f(z)$ which is a contradiction to the hypothesis that z_0 is an essential singularity for $f(z)$.

Hence the theorem.

Solved problems

Problem 1. Determine and classify the singular points of $f(z) = \frac{z}{e^z - 1}$.

Solution. The singularities of $f(z)$ are given by the values of z for which $e^z - 1 = 0$. Hence $z = 2n\pi i, n \in \mathbb{Z}$, are the singularities of $f(z)$.

$$\begin{aligned} \text{Now, } e^z - 1 &= \left(1 + z + \frac{z^2}{2!} + \dots + \frac{z^n}{n!} + \dots \right) - 1 \\ &= z + \frac{z^2}{2!} + \dots + \frac{z^n}{n!} + \dots \end{aligned}$$

$$\therefore \lim_{z \rightarrow 0} \frac{z}{e^z - 1} = 1$$

Hence 0 is a removable singularity for $f(z)$.

Also $\lim_{z \rightarrow 2n\pi i} \left(\frac{z}{e^z - 1} \right) = \infty$ if $n \neq 0$ and hence $2n\pi i, n \neq 0$, are simple poles of $f(z)$.

Problem 2. Determine and classify the singularities of $f(z) = \sin(1/z)$.

Solution. Clearly 0 is the only singularity of $f(z)$.

$$\text{Also } f(z) = \frac{1}{z} - \frac{1}{3!z^3} + \frac{1}{5!z^5} - \dots$$

Thus the principal part of $f(z)$ at $z = 0$ has infinitely many terms and hence 0 is an essential singularity for $f(z)$.

Problem 3. Determine and classify the singular points of $\frac{1}{(2 \sin z - 1)^2}$.

Solution. The singularities of $f(z)$ are given by the values of z for which $2 \sin z - 1 = 0$.

$\therefore$ The singularities of $f(z)$ are given by $z = \frac{\pi}{6} + 2n\pi, n \in \mathbf{Z}$, and they are double poles.

Exercises.

1. Find the singularities of the following functions and classify the singularities.

(i) $ze^{1/z}$

(ii) $\frac{z^2}{1+z}$

(iii) $\frac{\sin z}{z}$

(iv) $\frac{e^{1/z}}{z-1}$

(v) $\frac{z}{e^{1/z} - 1}$

(vi) $\sin\left(\frac{1}{1-z}\right)$

(vii) $\frac{z^2 - 2z + 3}{z - 2}$

(viii) $(z - i) \sin\left(\frac{1}{z + 2i}\right)$

2. Show that the singular points of each of the following functions are poles. Determine the order of each pole.

(i) $\frac{z+1}{z^2 - 2z}$

(ii) $\tanh z$

(iii) $\frac{1 - e^{2z}}{z^4}$

(iv) $\frac{e^{2z}}{(z-1)^2}$

(v) $\frac{1}{z^2 + 1}$

(vi) $\frac{1}{z^2(z-3)^2}$

$$(vii) \frac{1}{z^4 + 2z^2 + 1}$$

$$(ix) \frac{z^2 - 2z + 3}{z - 2}$$

$$(viii) \frac{z(1+z)}{1 - \cos z}$$

$$(x) (z - i) \sin \left(\frac{1}{z + 2i} \right)$$

3. Find the order of the pole $z = 0$ for the following functions.

$$(i) \frac{e^z}{z}$$

$$(ii) \frac{e^z}{z^2}$$

$$(iii) \frac{1 - \sin z}{z^5}$$

4. Let f and g have a pole of order m and n respectively at a . What can be said about the order of pole of (i) $f + g$ (ii) fg (iii) f/g at a .

5. Show that if f has an essential singularity at a so does f^2 .

Answers

2. (i) 0 and 2 are simple poles (ii) 0 is a simple pole (iii) 0 is a pole of order 3. (iv) 1 is a double pole (v) i and $-i$ are simple poles (vi) 0, 3 are double poles (vii) $i, -i$ are double poles (viii) 0 is a simple pole. (ix) 2 is a pole (x) $-2i$ is an essential singularity.

3. (i) 1 (ii) 2 (iii) 5.

8. Calculus of Residues

8.0. Introduction

In this chapter we introduce the concept of the residue of a function $f(z)$ at an isolated singular point and prove Cauchy's residue theorem. Using this theorem we evaluate certain types of real definite integrals.

8.1. Residues

Definition. Let a be an isolated singularity for $f(z)$. Then the **residue** of $f(z)$ at a is defined to be the coefficient of $\frac{1}{z-a}$ in the Laurent's series expansion of $f(z)$ about a and is denoted by $\text{Res}\{f(z); a\}$.

Thus $\text{Res}\{f(z); a\} = \frac{1}{2\pi i} \int_C f(z) dz = b_1$, where C is a circle $|z-a| = r$ such that f is analytic in $0 < |z-a| < r$.

Example. Consider $f(z) = \frac{e^z}{z^2}$

$$\begin{aligned}\frac{e^z}{z^2} &= \frac{1}{z^2} \left(1 + \frac{z}{1!} + \frac{z^2}{2!} + \dots \right) \\ &= \frac{1}{z^2} + \frac{1}{z} + \frac{1}{2!} + \frac{z}{3!} + \frac{z^2}{4!} + \dots\end{aligned}$$

$\therefore f(z)$ has a double pole at $z = 0$.

$\therefore \text{Res}\{f(z); 0\} = \text{coefficient of } \frac{1}{z} = 1$.

The following lemmas provide methods for calculation of residues:

Lemma 1. If $z = a$ is a simple pole for $f(z)$ then

$$\text{Res}\{f(z); a\} = \lim_{z \rightarrow a} (z-a)f(z).$$

Proof. Since $z = a$ is a simple pole for $f(z)$ the Laurent's series expansion for $f(z)$ about $z = a$ is given by $f(z) = \frac{b_1}{z-a} + a_0 + a_1(z-a) + \dots$

$$\text{Now, } (z-a)f(z) = b_1 + a_0(z-a) + a_1(z-a)^2 + \dots$$

$$\therefore \lim_{z \rightarrow a} (z - a) f(z) = b_1 \\ = \text{Res} \{f(z); a\}.$$

Lemma 2. If a is a simple pole for $f(z)$ and $f(z) = \frac{g(z)}{z - a}$ where $g(z)$ is analytic at a and $g(a) \neq 0$ then $\text{Res} \{f(z); a\} = g(a)$.

Proof. By Lemma 1, $\text{Res} \{f(z); a\} = \lim_{z \rightarrow a} (z - a) f(z) = \lim_{z \rightarrow a} g(z) = g(a)$.

Lemma 3. If a is a simple pole for $f(z)$ and if $f(z)$ is of the form $\frac{h(z)}{k(z)}$ where $h(z)$ and $k(z)$ are analytic at a and $h(a) \neq 0$ and $k(a) = 0$ then

$$\text{Res} \{f(z); a\} = \frac{h(a)}{k'(a)}.$$

Proof. $\text{Res} \{f(z); a\} = \lim_{z \rightarrow a} (z - a) f(z)$

$$= \lim_{z \rightarrow a} (z - a) \frac{h(z)}{k(z)}$$

$$= \lim_{z \rightarrow a} h(z) \lim_{z \rightarrow a} \frac{(z - a)}{k(z)}$$

$$= \lim_{z \rightarrow a} h(z) \lim_{z \rightarrow a} \left[\frac{z - a}{k(z) - k(a)} \right] \quad (\text{since } k(a) = 0)$$

$$= h(a) \left[\frac{1}{k'(a)} \right]$$

$$= \frac{h(a)}{k'(a)}.$$

Lemma 4. Let a be a pole of order $m > 1$ for $f(z)$ and let $f(z) = \frac{g(z)}{(z - a)^m}$ where $g(z)$ is analytic at a and $g(a) \neq 0$. Then

$$\text{Res} \{f(z); a\} = \frac{g^{(m-1)}(a)}{(m-1)!}.$$

Proof. $g^{(m-1)}(a) = \frac{(m-1)!}{2\pi i} \int_C \frac{g(z) dz}{(z - a)^m}$ (by theorem on higher derivatives) where C is a circle $|z - a| = r$ such that $f(z)$ is analytic in $0 < |z - a| < r$.

$$\therefore \frac{g^{(m-1)}(a)}{(m-1)!} = \frac{1}{2\pi i} \int_C f(z) dz = \text{Res} \{f(z); a\}.$$

Solved Problems

Problem 1. Calculate the residue of $\frac{z+1}{z^2-2z}$ at its poles.

Solution. Let $f(z) = \frac{z+1}{z^2-2z} = \frac{z+1}{z(z-2)}$.

$z=0$ and $z=2$ are simple poles for $f(z)$.

$$\begin{aligned} \text{Res} \{f(z); 0\} &= \lim_{z \rightarrow 0} (z-0) \left[\frac{z+1}{z(z-2)} \right] \\ &= \lim_{z \rightarrow 0} \frac{z+1}{z-2} = -\frac{1}{2} \end{aligned}$$

$$\begin{aligned} \text{Res} \{f(z); 2\} &= \lim_{z \rightarrow 2} (z-2) \left[\frac{z+1}{z(z-2)} \right] \\ &= \lim_{z \rightarrow 2} \frac{z+1}{z} = \frac{3}{2}. \end{aligned}$$

Aliter. $f(z)$ can be written as $f(z) = \frac{h(z)}{k(z)}$ where $h(z) = z+1$ and $k(z) = z^2-2z$ so that $k'(z) = 2z-2$.

$$\begin{aligned} \therefore \text{Res} \{f(z); 0\} &= \frac{h(0)}{k'(0)} \text{ (by Lemma 3)} \\ &= -\frac{1}{2}. \end{aligned}$$

$$\text{Res} \{f(z); 2\} = \frac{h(2)}{k'(2)} = \frac{3}{2}.$$

Problem 2. Find the residue at $z=0$ of $\frac{1+e^z}{z \cos z + \sin z}$.

Solution. Let $f(z) = \frac{1+e^z}{z \cos z + \sin z}$.

Clearly 0 is a pole of order 1 for $f(z)$.

$$\therefore \text{Res} \{f(z); 0\} = \lim_{z \rightarrow 0} \frac{h(z)}{k'(z)} \text{ where } h(z) = 1+e^z \text{ and } k(z) = z \cos z + \sin z.$$

$$\text{Now } k'(z) = -z \sin z + \cos z + \cos z = -z \sin z + 2 \cos z.$$

$$\therefore \operatorname{Res}\{f(z); 0\} = \frac{2}{2} = 1.$$

Problem 3. Find the residue of $\frac{1}{(z^2 + a^2)^2}$ at $z = ai$.

Solution. Let $f(z) = \frac{1}{(z^2 + a^2)^2} = \frac{1}{(z + ai)^2(z - ai)^2}$.

$z = ai$ and $z = -ai$ are poles of order 2 for $f(z)$.

$$\text{Let } g(z) = \frac{1}{(z + ai)^2}.$$

$$\therefore g'(z) = \frac{-2}{(z + ai)^3}.$$

$$\begin{aligned} \therefore \operatorname{Res}\{f(z); ai\} &= g'(ai) = \frac{-2}{(ai + ai)^3} = \frac{-2}{8a^3i^3} = \frac{2}{8a^3i} \\ &= \frac{-i}{4a^3} \end{aligned}$$

Problem 4. Find the poles of $f(z) = \frac{z^2 + 4}{z^3 + 2z^2 + 2z}$ and determine the residues at the poles.

Solution. $f(z) = \frac{z^2 + 4}{z^3 + 2z^2 + 2z} = \frac{z^2 + 4}{z(z + 1 - i)(z + 1 + i)}$.

$\therefore 0, i - 1$ and $-1 - i$ are simple poles for $f(z)$.

Here $f(z) = \frac{h(z)}{k(z)}$ where $h(z) = z^2 + 4$ and $k(z) = z^3 + 2z^2 + 2z$.

Hence $k'(z) = 3z^2 + 4z + 2$.

$$\operatorname{Res}\{f(z); 0\} = \frac{h(0)}{k'(0)} = \frac{4}{2} = 2.$$

$$\operatorname{Res}\{f(z); i - 1\} = \frac{h(i - 1)}{k'(i - 1)}$$

$$= \frac{(i - 1)^2 + 4}{3(i - 1)^2 + 4(i - 1) + 2}$$

$$= \frac{3i - 1}{2} \text{ (after simplification).}$$

Similarly $\text{Res}\{f(z); -1 - i\} = \frac{-(1 + 3i)}{2}$.

Problem 5. Find the residue of $\cot z$ at $z = 0$.

Solution. $z = 0$ is a simple pole for $\cot z$. Let $f(z) = \frac{\cos z}{\sin z} = \frac{h(z)}{k(z)}$.

$$\therefore \text{Res}\{f(z); 0\} = \frac{h(0)}{k'(0)} = \frac{\cos 0}{\cos 0} = 1.$$

Problem 6. Find the residue of $\frac{e^z}{z^2(z^2 + 9)}$ at its poles.

Solution. Let $f(z) = \frac{e^z}{z^2(z^2 + 9)}$.

Here $z = 0$ is a double pole and $z = 3i$ and $z = -3i$ are simple poles for $f(z)$.

To find the $\text{Res}\{f(z); 0\}$, let $g(z) = \frac{e^z}{z^2 + 9}$.

Clearly $g(z)$ is analytic at $z = 0$ and $g(0) \neq 0$.

$$\text{Also } g'(z) = e^z \left[\frac{(z^2 + 9) - 2z}{(z^2 + 9)^2} \right]$$

$$\begin{aligned} \therefore \text{Res}\{f(z); 0\} &= \frac{g'(0)}{1!} \text{ (by Lemma 4)} \\ &= \frac{1}{9}. \end{aligned}$$

Now, to find $\text{Res}\{f(z); 3i\}$; let $f(z) = \frac{h(z)}{k(z)}$ so that $h(z) = e^z$ and $k(z) = z^2(z^2 + 9)$.

$$\text{Then } k'(z) = 4z^3 + 18z$$

$$\begin{aligned} \therefore \text{Res}\{f(z); 3i\} &= \frac{h(3i)}{k'(3i)} \\ &= \frac{e^{3i}}{4(3i)^3 + 18(3i)} \\ &= \frac{e^{3i}}{-108i + 54i} \\ &= -\frac{e^{3i}}{54i} \\ &= \frac{i(\cos 3 + i \sin 3)}{54} \end{aligned}$$

Similarly $\text{Res}\{f(z); -3i\} = -\frac{(\sin 3 + i \cos 3)}{54}$.

Problem 7. Use Laurent's series to find the residue of $\frac{e^{2z}}{(z-1)^2}$ at $z = 1$.

Solution. Let $f(z) = \frac{e^{2z}}{(z-1)^2}$.

First we expand $f(z)$ as Laurent's series at $z = 1$.

$$\begin{aligned} f(z) &= \frac{e^{2(z-1)+2}}{(z-1)^2} \\ &= \frac{e^2 e^{2(z-1)}}{(z-1)^2} \\ &= \frac{e^2}{(z-1)^2} \left[1 + \frac{2(z-1)}{1!} + \frac{2^2(z-1)^2}{2!} + \frac{2^3(z-1)^3}{3!} + \dots \right] \\ &= e^2 \left[\frac{1}{(z-1)^2} + \frac{2}{z-1} + 2 + \frac{4}{3}(z-1) + \dots \right] \end{aligned}$$

This is Laurent's series expansion for $f(z)$ at $z = 1$.

$$\begin{aligned} \text{Res}\{f(z); 1\} &= \text{coefficient of } \frac{1}{z-1} \text{ in Laurent's expansion} \\ &= 2e^2. \end{aligned}$$

Note. Without expanding in Laurent's series the residue at $z = 1$ can be found as follows. Since $f(z)$ has a pole of order 2 at $z = 1$ we choose $g(z) = e^{2z}$.

$$\therefore \text{Res}\{f(z); 1\} = \frac{g'(1)}{1!} = \left[\frac{2e^{2z}}{1!} \right]_{z=1} = 2e^2.$$

Problem 8. Find the residue of $\frac{ze^z}{(z-1)^3}$ at its pole.

Solution. Let $f(z) = \frac{ze^z}{(z-1)^3}$.

$z = 1$ is a pole of order 3 for $f(z)$.

Let $g(z) = ze^z$ so that $g'(z) = e^z(z+1)$ and $g''(z) = e^z(z+2)$.

$$\text{Then } \text{Res}\{f(z); 1\} = \frac{g''(1)}{2!} = \frac{3e}{2}.$$

Problem 9. Find the residue of $\frac{1}{z - \sin z}$ at its pole.

Solution. Let $f(z) = \frac{1}{z - \sin z}$.

$$\begin{aligned}\text{Now } z - \sin z &= z - \left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \dots \right) \\ &= \frac{z^3}{3!} - \frac{z^5}{5!} + \dots \\ &= z^3 \left(\frac{1}{3!} - \frac{z^2}{5!} + \dots \right).\end{aligned}$$

$z = 0$ is a pole of order 3 for $f(z)$ and $f(z) = \frac{1}{z^3 \left(\frac{1}{3!} - \frac{z^2}{5!} + \dots \right)}$.

$$\text{Now let } g(z) = \frac{1}{\left(\frac{1}{3!} - \frac{z^2}{5!} + \dots \right)}.$$

Then $\text{Res} \{f(z); 0\} = \frac{g''(0)}{2!}$. Clearly $g(0) = 6$.

Now $\frac{1}{g(z)} = \frac{1}{3!} - \frac{z^2}{5!} + \frac{z^4}{7!} - \dots$. Differentiating with respect to z , we have

$$-\frac{g'(z)}{[g(z)]^2} = -\frac{2z}{5!} + \frac{4z^3}{7!} - \dots$$

Hence $g'(0) = 0$.

Again differentiating with respect to z we have

$$\frac{[g(z)]^2[-g''(z)] + g'(z)2g(z)g'(z)}{[g(z)]^4} = \frac{-2}{5!} + \frac{12z^2}{7!} - \dots$$

Putting $z = 0$ and using $g(0) = 6$ and $g'(0) = 0$ we get $\frac{-g''(0)}{36} = \frac{-2}{5!}$.

Hence $g''(0) = \frac{3}{5}$.

$$\therefore \text{Res} \{f(z); 0\} = \frac{g''(0)}{2!} = \frac{3}{10}.$$

Exercises

1. Find the order of each pole and find the residue at the poles for each of the following functions.

(i) $\frac{z}{z^2 + 1}$

(ii) $\frac{z+1}{z^2 - 2z}$

(iii) $\frac{2z+3}{z(z^2+1)}$

(iv) $\frac{z^2}{z^2 + a^2}$

(v) $\frac{1}{z^4 + 2z^2 + 1}$

(vi) $\frac{1}{z^2 e^z}$

(vii) $\frac{1}{z^3(z+4)}$

(viii) $\frac{z+1}{z^2(z-2)}$

(ix) $\frac{\cos z}{z^3}$

(x) $\frac{\sin z}{z^4}$

(xi) $\left(\frac{z+1}{z-1}\right)^2$

(xii) $\frac{e^{2z}}{(z-1)^2}$

(xiii) $\frac{e^z}{z^2 + \pi^2}$

(xiv) $\frac{1 - e^{2z}}{z^4}$

(xv) $\frac{e^{iz}}{z^2 + a^2}$ (a real)

(xvi) $\frac{2z}{(z+4)(z-1)^2}$

2. Find the residue of $\frac{z^2 - 2z}{(z+1)^2(z^2+4)}$ at all its poles.

3. Find the residue of $\frac{1 + e^z}{\sin z + z \cos z}$ at the pole $z = 0$.

4. Calculate the residue of $\sec^2 z$ at $z = \frac{\pi}{2}$.

(Hint: $\sec^2 z = \frac{1}{\cos^2 z} = \frac{2}{1 + \cos 2z}$).

5. Prove that the residue of $\frac{z^{2n}}{(1+z)^n}$ where $n \in \mathbb{N}$ at $z = -1$ is $\frac{(-1)^{n+1}(2n)!}{(n-1)!(n+1)!}$.

6. Find the residue of $\frac{1}{(1+z^2)^n}$ at $z = i$.

7. Prove that (i) $\text{Res} \{ \tan z, \pi/2 \} = -1$; (ii) $\text{Res} \left\{ \frac{1 - \cos z}{z^3}, 0 \right\} = \frac{1}{2}$.

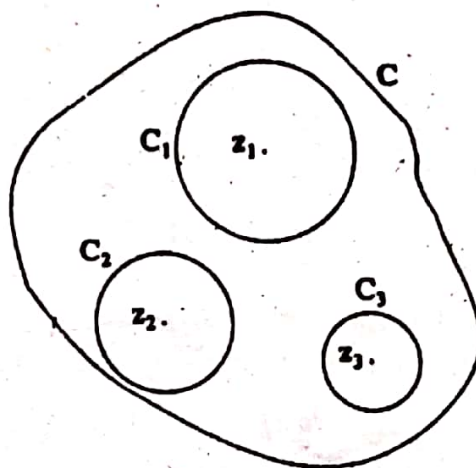
8. Show that all the singular points of $\frac{1}{z(e^z - 1)}$ are poles. Find the order of poles and find the radius at the poles.

Answers.

1. (i) simple pole; Res $\frac{1}{2}$; $-i$ simple pole; Res $\frac{1}{2}$ (ii) $z = 0$, 2 simple pole; Res $-1/2$, $3/2$ (iii) $0, i, -i$ simple poles; Res 3 , $(2i+3)/2$, $(2i-3)/2$ (iv) $ai, -ai$ simple poles; Res $ai/2, -ai/2$ (v) $i, -i$ order 2; Res $-i/4, i/4$ (vi) 0 pole of order 2; Res -1 (vii) 0 pole of order 3; Res $1/64$; -4 simple pole; Res $-1/64$ (viii) 0 pole of order 2; Res $-3/4$, 2 simple pole; Res $3/4$ (ix) 0 pole of order 3; Res $-1/2$ (x) 0 pole of order 4; Res -6 (xi) 1 simple pole; Res 4 (xii) 1 pole of order 2; Res $2e^2$ (xiii) $-\pi i$ simple pole; Res $1/2\pi i$ and πi simple pole; Res $-1/2\pi i$ (xiv) 0 pole of order 3; Res $-4/3$ (xv) $ai, -ai$ simple poles; Res $e^{-a}/2ai, e^{-a}/2ai$ (xvi) -4 simple pole; Res $8/25$, 1 pole of order 2; Res $8/25$ 2. $z = -1$; $-14/25$; $z = 2i$; $\frac{7+i}{25}$; $z = -2i$; $\frac{7-i}{25}$ 3. 1
4. 0 6. $\frac{-i(2n+2)!}{2^{2n-1}[(n-1)!]^2}$ 8. $z = 0$ pole of order 2; Res $-1/2$.

8.2. Cauchy's Residue Theorem**Theorem 8.1. (Cauchy's residue theorem)**

Let $f(z)$ be a function which is analytic inside and on a simple closed curve C except for a finite number of singular points $z_1, z_2, \dots, z_n$ inside C .



$$\text{Then } \int_C f(z)dz = 2\pi i \sum_{j=1}^n \text{Res} \{f(z); z_j\}.$$

Proof. Let $C_1, C_2, \dots, C_n$ be circles with centres $z_1, z_2, \dots, z_n$ respectively such that all circles are interior to C and are disjoint with each other. (refer figure).

By Cauchy's theorem for multiply connected regions we have

$$\int_C f(z)dz = \int_{C_1} f(z)dz + \int_{C_2} f(z)dz + \dots + \int_{C_n} f(z)dz.$$

$$= 2\pi i \operatorname{Res}\{f(z); z_1\} + 2\pi i \operatorname{Res}\{f(z); z_2\} + \cdots + 2\pi i \operatorname{Res}\{f(z); z_n\}$$

(by definition of residue)

$$= 2\pi i \sum_{j=1}^n \operatorname{Res}\{f(z); z_j\}. \text{ Hence the theorem.}$$

Example. Evaluate $\int_C \frac{z^2 dz}{(z-2)(z+3)}$ where C is the circle $|z| = 4$.

$$\text{Let } f(z) = \frac{z^2}{(z-2)(z+3)}.$$

$z = 2$ and $z = -3$ are simple poles for $f(z)$ and both of them lie inside $|z| = 4$.

$$\text{Now, } \operatorname{Res}\{f(z); 2\} = \lim_{z \rightarrow 2} (z-2) \left[\frac{z^2}{(z-2)(z+3)} \right] = \frac{4}{5}.$$

$$\operatorname{Res}\{f(z); -3\} = \lim_{z \rightarrow -3} (z+3) \left[\frac{z^2}{(z-2)(z+3)} \right] = -\frac{9}{5}.$$

$$\therefore \text{ By Residue theorem } \int_C f(z) dz = 2\pi i \left[\frac{4}{5} + \left(-\frac{9}{5} \right) \right]$$

$$= -2\pi i.$$

$$\therefore \int_C \frac{z^2 dz}{(z-2)(z+3)} = -2\pi i.$$

Theorem 8.2. (Argument theorem) Let f be a function which is analytic inside and on a simple closed curve C except for a finite number of poles inside C . Also let $f(z)$ have no zeros on C . Then $\frac{1}{2\pi i} \int_C \frac{f'(z)}{f(z)} dz = N - P$ where N is the number of zeros of $f(z)$ inside C and P is the number of poles of $f(z)$ inside C . (A pole or zero of order m is counted m times).

Solution. We observe that the singularities of the function $\frac{f'(z)}{f(z)}$ inside C are the poles and zeros of $f(z)$ lying inside C .

Let z_0 be a zero of order n for $f(z)$. Let C_1 be a circle with centre z_0 such that it is the only zero of $f(z)$ inside C_1 .

Then $f(z) = (z - z_0)^n g(z)$ where $g(z)$ is analytic and nonzero inside C_1 . Hence $f'(z) = n(z - z_0)^{n-1} g(z) + (z - z_0)^n g'(z)$.

$$\therefore \frac{f'(z)}{f(z)} = \frac{n}{z - z_0} + \frac{g'(z)}{g(z)} \quad \dots (1)$$

Since $g(z)$ is analytic and non zero inside C , $\frac{g'(z)}{g(z)}$ is also analytic and hence can be expanded as a Taylor's series about z_0 .

$$\therefore \operatorname{Res} \left\{ \frac{f'(z)}{f(z)} ; z_0 \right\} = \text{coefficient of } \frac{1}{z - z_0} \text{ in (1)}$$

$$f(z) = \frac{1}{(z - z_0)^n} \cdot g(z) \Rightarrow f(z) = (z - z_0)^{-n} g(z)$$

Similarly if z_1 is a pole of order p for $f(z)$, then $\operatorname{Res} \left\{ \frac{f'(z)}{f(z)} ; z_1 \right\} = -p$.

Hence by Cauchy's residue theorem, $\frac{1}{2\pi i} \int_C \frac{f'(z)}{f(z)} dz = N - P$ where N is the number of zeros and P is the number of poles of $f(z)$ within C .

Corollary. If $f(z)$ is analytic inside and on C and not zero on C , then $\frac{1}{2\pi i} \int_C \frac{f'(z)}{f(z)} dz = N$ where N is the number of zeros lying inside C .

Proof. Since the number of poles is zero we have $P = 0$.
Hence the result follows.

Theorem 8.3. (Rouché's theorem) If $f(z)$ and $g(z)$ are analytic inside and on a simple closed curve C and if $|g(z)| < |f(z)|$ on C then $f(z) + g(z)$ and $f(z)$ have the same number of zeros inside C .

Proof. $f(z) + g(z) = f(z) \left[1 + \frac{g(z)}{f(z)} \right] = f(z)\varphi(z)$ where $\varphi(z) = 1 + \frac{g(z)}{f(z)}$.

$$\text{Hence } [f(z) + g(z)]' = f'(z) + g'(z) = f'(z)\varphi(z) + f(z)\varphi'(z).$$

$$\begin{aligned} \therefore \frac{f'(z) + g'(z)}{f(z) + g(z)} &= \frac{f'(z)\varphi(z) + f(z)\varphi'(z)}{f(z)\varphi(z)} \\ &= \frac{f'(z)}{f(z)} + \frac{\varphi'(z)}{\varphi(z)} \end{aligned}$$

$$\therefore \frac{1}{2\pi i} \int_C \left[\frac{f'(z) + g'(z)}{f(z) + g(z)} \right] dz = \frac{1}{2\pi i} \int_C \frac{f'(z)}{f(z)} dz + \frac{1}{2\pi i} \int_C \frac{\varphi'(z)}{\varphi(z)} dz. \quad \dots (1)$$

Now, by hypothesis $|g(z)| < |f(z)|$ and hence $\left| \frac{g(z)}{f(z)} \right| < 1$ on C .

$$\therefore |\varphi(z) - 1| < 1 \text{ on } C.$$

Hence by maximum modulus theorem, $|\varphi(z) - 1| < 1$ for every point z inside C .

$\therefore \varphi(z) \neq 0$ for every point inside C .

$$\begin{aligned} \text{Hence } \int_C \frac{\varphi'(z)}{\varphi(z)} dz &= \text{Number of zeros of } \varphi(z) \text{ within } C. \\ &= 0. \end{aligned}$$

$$\text{Hence from (1), we have } \frac{1}{2\pi i} \int_C \left[\frac{f'(z) + g'(z)}{f(z) + g(z)} \right] dz = \frac{1}{2\pi i} \int_C \frac{f'(z)}{f(z)} dz$$

$\therefore N_1 = N_2$ where N_1 and N_2 denote respectively the number of zeros of $f(z) + g(z)$ and $f(z)$ inside C . Hence the theorem.

Remark. We can deduce the Fundamental theorem of Algebra from Rouché's theorem.

Theorem 8.4. (Fundamental theorem of algebra)

A polynomial of degree n with complex coefficients has n zeros in C .

Proof. Let $a_0 + a_1z + a_2z^2 + \cdots + a_nz^n$, where $a_n \neq 0$, be a polynomial of degree n . Let $f(z) = a_nz^n$ and $g(z) = a_0 + a_1z + \cdots + a_{n-1}z^{n-1}$.

$$\text{Clearly } \lim_{z \rightarrow \infty} \frac{g(z)}{f(z)} = 0.$$

Hence there exists a positive real number r such that $\left| \frac{g(z)}{f(z)} \right| < 1$ for all z with $|z| > r$.

Hence by Rouché's theorem $f(z)$ and $f(z) + g(z)$ have the same number of zeros inside the circle $|z| = r + 1$. But 0 is a zero of multiplicity n for $f(z)$. Hence the given polynomial $f(z) + g(z)$ also has n zeros.

Solved Problems

Problem 1. Evaluate $\int_C \frac{dz}{2z + 3}$ where C is $|z| = 2$.

Solution. $z = -\frac{3}{2}$ is the simple pole of $f(z)$ which lies inside the circle $|z| = 2$.

$$\text{Res} \left\{ f(z); -\frac{3}{2} \right\} = \lim_{z \rightarrow -3/2} \frac{h(z)}{k'(z)} \text{ where } h(z) = 1 \text{ and } k(z) = 2z + 3.$$

$$\therefore \text{Res} \left\{ f(z); -\frac{3}{2} \right\} = \frac{1}{2}.$$

∴ By residue theorem $\int_C f(z)dz = 2\pi i \left(\frac{1}{2}\right) = \pi i$.

Aliter.
$$\begin{aligned}\int_C \frac{dz}{2z+3} &= \int_C \frac{dz}{2\left(z+\frac{3}{2}\right)} \\ &= \frac{1}{2} \int_C \frac{dz}{\left(z+\frac{3}{2}\right)} \\ &= \frac{1}{2}(2\pi i) \left(\because \int_C \frac{dz}{z-a} = 2\pi i\right) \\ &= \pi i.\end{aligned}$$

Problem 2. Evaluate $\int_C \frac{dz}{z^2 e^z}$ where $C = \{z : |z| = 1\}$.

Solution. Given integral can be written as $\int_C f(z)dz$ where $f(z) = \frac{e^{-z}}{z^2}$.

$f(z)$ has pole of order 2 at $z = 0$ which lies inside the circle $|z| = 1$.

Let $g(z) = e^{-z}$. Hence $g'(z) = -e^{-z}$.

∴ By Lemma 4, $\text{Res}\{f(z); 0\} = \frac{g'(0)}{1!} = -1$.

By residue theorem $\int_C f(z)dz = 2\pi i(-1) = -2\pi i$.

Problem 3. Evaluate $\int_C \frac{2+3\sin \pi z}{z(z-1)^2} dz$ where C is the square having vertices

$3+3i, 3-3i, -3+3i, -3-3i$.

Solution. Let $f(z) = \frac{2+3\sin \pi z}{z(z-1)^2}$. Here $z = 0$ is a simple pole and $z = 1$ is a double pole for $f(z)$ and both of them lie within C .

$$\text{Res}\{f(z); 0\} = \lim_{z \rightarrow 0} z \left(\frac{2+3\sin \pi z}{z(z-1)^2} \right) = 2.$$

$$\text{Res}\{f(z); 1\} = \frac{g'(1)}{1!} \text{ where } g(z) = \frac{2+3\sin \pi z}{z}.$$

$$g'(z) = \frac{z3\pi \cos \pi z - (2 + 3 \sin \pi z)}{z^2}.$$

$$\therefore g'(1) = -3\pi - 2.$$

$$\therefore \operatorname{Res}\{f(z); 1\} = -3\pi - 2.$$

$$\therefore \text{By residue theorem } \int_C f(z) dz = 2\pi i(2 - 3\pi - 2) = -6\pi^2 i.$$

Problem 4. Evaluate $\int_C \tan z dz$ where C is $|z| = 2$.

Solution. Let $f(z) = \tan z = \frac{\sin z}{\cos z} = \frac{h(z)}{k(z)}.$

$$\cos z \text{ has zeros at } z = \frac{(2n+1)\pi}{2}, n \in \mathbb{N}.$$

$\therefore f(z)$ has simple poles at $z = -\frac{\pi}{2}$ and $z = \frac{\pi}{2}$ inside the circle $|z| = 2$.

$$\operatorname{Res}\{f(z); \pi/2\} = \frac{h(\pi/2)}{k'(\pi/2)} = \frac{\sin(\pi/2)}{-\sin(\pi/2)} = -1.$$

$$\operatorname{Res}\{f(z); -\pi/2\} = \frac{h(-\pi/2)}{k'(-\pi/2)} = \frac{\sin(-\pi/2)}{-\sin(-\pi/2)} = -1.$$

$$\text{By residue theorem } \int_C \tan z dz = 2\pi i[(-1) + (-1)] = -4\pi i.$$

Problem 5. Prove that $\int_C \frac{e^{2z}}{(z+1)^3} dz = \frac{4\pi i}{e^2}$ where C is $|z| = \frac{3}{2}$.

Solution. Let $f(z) = \frac{e^{2z}}{(z+1)^3}.$

$f(z)$ has a pole of order 3 at $z = -1$.

$$\operatorname{Res}\{f(z); -1\} = \frac{g''(-1)}{2!} \text{ where } g(z) = e^{2z}.$$

$$\text{Now } g'(z) = 2e^{2z} \text{ and } g''(z) = 4e^{2z}.$$

$$\therefore \text{Res}\{f(z); -1\} = \frac{4e^{-2}}{2!} = \frac{2}{e^2}.$$

$$\therefore \text{By residue theorem } \int_C f(z) dz = 2\pi i \left(\frac{2}{e^2} \right) = \frac{4\pi i}{e^2}.$$

Problem 6. Evaluate, using (i) Cauchy's integral formula (ii) residue theorem $\int_C \frac{z+1}{z^2+2z+4} dz$ where C is the circle $|z+1+i|=2$.

Solution. Clearly C is a circle with centre $a = -(1+i)$ and radius 2.

$$\begin{aligned} \text{Now } \frac{z+1}{z^2+2z+4} &= \frac{z+1}{(z+1)^2 + (\sqrt{3})^2} \\ &= \frac{z+1}{(z+1+i\sqrt{3})(z+1-i\sqrt{3})} \\ &= \frac{z+1}{[z-(-1-i\sqrt{3})][z-(-1+i\sqrt{3})]}. \end{aligned}$$

$z_0 = -1+i\sqrt{3}$ and $z_1 = -1-i\sqrt{3}$ are the singular points of the given integrand $\frac{z+1}{z^2+2z+4}$.

$$\text{Now } |z_0 - a| = |i(\sqrt{3}+1)| = \sqrt{3}+1 > 2$$

$$\text{and } |z_1 - a| = |-i(\sqrt{3}-1)| = \sqrt{3}-1 < 2.$$

$\therefore z_1 = -1-i\sqrt{3}$ lies inside C .

(i) By using Cauchy integral formula.

$$\text{Consider } f(z) = \frac{z+1}{z-(-1-i\sqrt{3})}.$$

We note that $f(z)$ is analytic at all points inside C .

$$\therefore \text{By Cauchy's integral formula } \frac{1}{2\pi i} \int_C \frac{f(z)}{z-z_1} dz = f(z_1);$$

$$\text{(i.e.) } \frac{1}{2\pi i} \int_C \frac{(z+1)dz}{[z-(1-i\sqrt{3})][z-(-1+i\sqrt{3})]} = f(-1-i\sqrt{3})$$

$$\text{(i.e.) } \frac{1}{2\pi i} \int_C \frac{(z+1) dz}{z^2+2z+4} = \frac{(-1-i\sqrt{3})+1}{(-1-i\sqrt{3})-(-1+i\sqrt{3})}$$

$$= \frac{-i\sqrt{3}}{-2i\sqrt{3}}$$

$$= \frac{1}{2}.$$

$$\therefore \int_C \frac{(z+1)dz}{z^2 + 2z + 4} = \frac{1}{2}(2\pi i) = \pi i.$$

(ii) By using residue theorem.

$$f(z) = \frac{z+1}{z^2 + 2z + 4}.$$

Since $z = -1 - i\sqrt{3}$ lies inside C

$$\text{Res}\{f(z); -1 - i\sqrt{3}\} = \frac{h(-1 - i\sqrt{3})}{k'(-1 - i\sqrt{3})} \text{ where } h(z) = z + 1 \text{ and}$$

$$k(z) = z^2 + 2z + 4 \text{ so that } k'(z) = 2z + 2.$$

$$\therefore \text{Res}\{f(z); -1 - i\sqrt{3}\} = \frac{-1 - i\sqrt{3} + 1}{2(-1 - i\sqrt{3}) + 2} = \frac{-i\sqrt{3}}{-i2\sqrt{3}} = \frac{1}{2}.$$

$$\text{By residue theorem } \int_C f(z)dz = \frac{2\pi i}{2} = \pi i.$$

Problem 7. Use residue calculus to evaluate $\int_C \frac{3 \cos z}{2i - 3z} dz$ where C is the unit circle.

Solution. Let $f(z) = \frac{3 \cos z}{2i - 3z}.$

Here $z = \frac{2i}{3}$ is a simple pole and lies within C .

$$\text{Res}\left\{f(z); \frac{2i}{3}\right\} = \lim_{z \rightarrow 2i/3} \frac{h(z)}{k'(z)} \text{ where } h(z) = 3 \cos z \text{ and } k(z) = 2i - 3z \text{ so that } k'(z) = -3.$$

$$\therefore \text{Res}\{f(z); 2i/3\} = \frac{3 \cos(2i/3)}{-3} = -\cos(2i/3) = -\cosh(2/3).$$

$$\text{By residue theorem } \int_C f(z)dz = 2\pi i[-\cosh(2/3)].$$

$$\text{(i.e.) } \int_C \frac{3 \cos z}{2i - 3z} dz = -2\pi i \cosh(2/3).$$

Problem 8. Use residue theorem to evaluate $\int \frac{3z^2 + z - 1}{(z^2 - 1)(z - 3)} dz$ around the circle $|z| = 2$.

Solution. Let $f(z) = \frac{3z^2 + z - 1}{(z^2 - 1)(z - 3)}$.

$f(z)$ has simple poles 1, -1, 3 and only 1, -1 lie inside $|z| = 2$.

$\text{Res}\{f(z); 1\} = \frac{h(1)}{k'(1)}$ where $h(z) = 3z^2 + z - 1$ and $k(z) = z^3 - 3z^2 - z + 3$ so that $k'(z) = 3z^2 - 6z - 1$.

$$\therefore \text{Res}\{f(z); 1\} = \frac{3 + 1 - 1}{3 - 6 - 1} = \frac{-3}{4}.$$

$$\text{Similarly Res}\{f(z); -1\} = \frac{3 - 1 - 1}{3 + 6 - 1} = \frac{1}{8}.$$

$\therefore$ By residue theorem,

$$\int \frac{3z^2 + z - 1}{(z^2 - 1)(z - 3)} dz = 2\pi i \left(-\frac{3}{4} + \frac{1}{8} \right) = 2\pi i \left(\frac{-5}{8} \right) = \frac{-5\pi i}{4}.$$

Problem 9. Evaluate $\int_C \frac{e^z dz}{(z + 2)(z - 1)}$ where C is the circle $|z - 1| = 1$.

Solution. $f(z) = \frac{e^z}{(z + 2)(z - 1)}$.

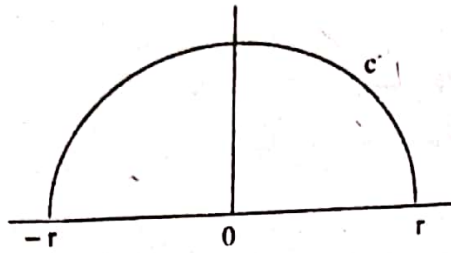
$f(z)$ has simple poles at 1, -2; the pole 1 is inside the circle $|z - 1| = 1$ and $z = -2$ lies outside the circle.

$$\text{Res}\{f(z); 1\} = \lim_{z \rightarrow 1} (z - 1) \left(\frac{e^z}{(z + 2)(z - 1)} \right) = \frac{e}{3}.$$

By residue theorem $\int_C f(z) dz = 2\pi i \left(\frac{e}{3} \right).$

$$\therefore \int \frac{e^z}{(z + 2)(z - 1)} dz = \frac{i2\pi e}{3}.$$

Problem 10. Show that the function $2 + z^2 - e^{iz}$ has precisely one zero in the open upper half plane.



Solution. Take $f(z) = 2 + z^2$ and $g(z) = -e^{iz}$. Let C be the simple closed curve consisting of the semi circle $|z| = r$ in the upper half plane together with the interval $[-r, r]$ on the real axis.

If $z \in [-r, r]$ then $|g(z)| = 1$ and $|f(z)| \geq 2$.

Hence $|f(z)| > |g(z)|$.

Now, if $z = re^{i\theta}$, $0 < \theta < \pi$ then $|f(z)| = |2 + z^2| \geq |z^2| - 2 = r^2 - 2$

Also $|g(z)| = |-e^{ire^{i\theta}}| = e^{-r \sin \theta}$.

Hence for sufficiently large value of r we have $|f(z)| > |g(z)|$.

Hence by Rouché's theorem $f(z) + g(z) = 2 + z^2 - e^{iz}$ and $f(z) = 2 + z^2$ have the same number of zeros in the upper half plane. Also $2 + z^2$ has exactly one zero in the upper half of the plane namely $i\sqrt{2}$.

Hence $2 + z^2 - e^{iz}$ has exactly one root in the upper half plane.

Problem 11. Let $f(z) = \frac{(z^2 + 1)}{(z^2 + 2z + 2)^2}$. Evaluate $\frac{1}{2\pi i} \int_C \frac{f'(z)}{f(z)} dz$ where C is the circle $|z| = 4$.

Solution. i and $-i$ are zeros of order 1 and $-1 + i$ and $-1 - i$ are poles of order 2 for $f(z)$. Also these zeros and poles lie inside C .

Hence number of zeros of $f(z) = N = 2$ and number of poles of $f(z) = P = 4$. (Poles are counted according to their multiplicity)

$$\therefore \text{By Argument theorem } \frac{1}{2\pi i} \int_C \frac{f'(z)}{f(z)} dz = N - P = 2 - 4 = -2$$

Exercises

1. Evaluate the following integrals.

(i) $\int_C \frac{3z - 4}{z(z - 1)} dz$ where C is the circle $|z| = 2$.

(ii) $\int_C \frac{(3z - 4)}{z(z - 1)(z - 2)} dz$ where C is the circle $|z| = \frac{3}{2}$

(iii) $\int_C \frac{2z^2 + 4}{z^2 - 1} dz$ where (a) C is the circle $|z| = 2$ (b) C is the circle

$$|z - 1| = 1.$$

(iv) $\int_C \frac{3dz}{z + 1}$ where C is the circle $|z| = 2$.

(v) $\int_C \frac{3 + z}{z} dz$ where C is the circle $|z| = 1$.

(vi) $\int_C \frac{z + 1}{z^2 - 2z} dz$ where C is the circle $|z| = 3$.

(vii) $\int_C \frac{3z^3 + 2}{(z - 1)(z^2 + 4)} dz$ where C is (a) $|z - 2| = 2$ (b) $|z| = 4$.

(viii) $\int_C \frac{dz}{z^3(z - 1)}$ where C is the circle $|z| = 3$.

(ix) $\int_C \frac{e^{-z}}{z^2} dz$ where C is the circle $|z| = 1$.

(x) $\int_C \frac{dz}{z^3(z + 4)}$ where C is (a) $|z| = 2$ (b) $|z + 2| = 3$.

(xi) $\int_C \frac{e^z dz}{z(z - 1)^2}$ where C is the circle $|z| = 2$.

(xii) $\int_C \frac{dz}{z^2(z + 1)}$ where C is the circle $|z| = \frac{1}{2}$.

(xiii) $\int_C \frac{e^{zt}}{z^2(z^2 + 2z + 2)} dz$ where C is the circle $|z| = 3$.

(xiv) $\int_C \frac{e^{zt}}{z(z^2 + 1)} dz$ ($t > 0$) where C is the square with vertices at $1 + i, -1 + i, -1 - i, 1 - i$.

(xv) $\int_C \frac{\cosh z dz}{z^3}$ where C is the square with vertices $\pm 2 \pm 2i$.

2. Prove that $\int_C \coth z dz = 0$ where C is the circle $|z| = 1$.

3. Prove that $\int_C z e^{1/z} dz = \pi i$ where C is the circle $|z| = 5$.

4. Prove that $\int_C \frac{e^z dz}{\cosh z} = 8\pi i$ where C is the circle $|z| = 5$.

Answers.

1. (i) $6\pi i$ (ii) $-2\pi i$ (iii) (a) $2\pi i$ (b) $3\pi i$ (iv) $-2\pi i$ (v) $6\pi i$ (vi) $2\pi i$ (vii) (a) πi (b) $6\pi i$ (viii) $4\pi i$ (ix) $-2\pi i$ (x) (a) $\frac{\pi i}{32}$ (b) 0 (xi) $2\pi i$ (xii) $-2\pi i$
 (xiii) $2\pi i \left[\frac{t-1}{2} + \frac{e^{-t}}{4} (\sin t + \cos t) \right]$ (xiv) $2\pi i (1 - \cos t)$ (xv) πi

8.3. Evaluation of Definite Integrals

We use Cauchy's residue theorem for evaluating certain types of real definite integrals.

TYPE 1. $\int_0^{2\pi} f(\cos \theta, \sin \theta) d\theta$ where $f(\cos \theta, \sin \theta)$ is a rational function of $\cos \theta$ and $\sin \theta$.

To evaluate this type of integral we substitute $z = e^{i\theta}$. As θ varies from 0 to 2π , z describes the unit circle $|z| = 1$

$$\text{Also, } \cos \theta = \frac{e^{i\theta} + e^{-i\theta}}{2} = \frac{z + z^{-1}}{2} \text{ and}$$

$$\sin \theta = \frac{e^{i\theta} - e^{-i\theta}}{2i} = \frac{z - z^{-1}}{2i}$$

Substituting these values in the given integrand the integral is transformed into $\int_C \theta(z) dz$ where $\theta(z) = f\left[\frac{z + z^{-1}}{2}, \frac{z - z^{-1}}{2i}\right]$ and C is the positively oriented unit circle $|z| = 1$. The integral $\int_C \theta(z) dz$ can be evaluated using the residue theorem.

Solved problems

Problem 1. Evaluate $\int_0^{2\pi} \frac{d\theta}{5 + 4 \sin \theta}$

Solution. Let $I = \int_0^{2\pi} \frac{d\theta}{5 + 4 \sin \theta}$.

Put $z = e^{i\theta}$.

Then $dz = iz d\theta$ and $\sin \theta = \frac{z - z^{-1}}{2i}$.

The given integral is transformed to $I = \int_C \frac{dz}{iz \left[5 + 4 \left(\frac{z - z^{-1}}{2i} \right) \right]}$ where C is the unit

circle $|z| = 1$.

$$= \int_C \frac{1}{iz} \frac{dz}{(5 + 2z - 2/z)} = \int_C \frac{dz}{2z^2 + 5iz - 2}$$

Let $f(z) = \frac{1}{2z^2 + 5iz - 2} = \frac{1}{2(z + 2i)(z + i/2)}$

$\therefore -2i$ and $-i/2$ are simple poles of $f(z)$ and the pole $-i/2$ lies inside C .

Also $\text{Res}\{f(z); -i/2\} = \lim_{z \rightarrow -i/2} \frac{1}{2(z + 2i)} = \frac{1}{3i}$.

Hence by Cauchy's residue theorem $I = 2\pi i \left(\frac{1}{3i} \right) = \frac{2\pi}{3}$.

$$f(z) = \frac{1}{2(z + i/2)(z + 2i)} = \frac{1}{2(z + i/2)(z + 2i)}$$

Problem 2. Prove that $\int_0^{2\pi} \frac{d\theta}{1+a \sin \theta} = \frac{2\pi}{\sqrt{1-a^2}}, (-1 < a < 1)$

Solution. Put $z = e^{i\theta}$. Then $\sin \theta = \frac{z - z^{-1}}{2i}$ and $dz = iz d\theta$.

$$\int_0^{2\pi} \frac{d\theta}{1+a \sin \theta} = \int_C \frac{dz}{iz \left[1 + a \left(\frac{z - z^{-1}}{2i} \right) \right]} \quad \text{where } C \text{ is the unit circle.}$$

$$= \int_C \frac{2dz}{z[2i + a(z - z^{-1})]}$$

$$= \int_C \frac{2dz}{az^2 + 2iz - a}$$

Let $f(z) = \frac{2}{az^2 + 2iz - a}$

The poles of $f(z)$ are given by $z = \frac{-2i \pm \sqrt{-4 + 4a^2}}{2a}$
 $= \frac{-i \pm i\sqrt{1-a^2}}{a}$ (since $-1 < a < 1$)

Let $z_1 = \frac{-i + i\sqrt{1-a^2}}{a}$ and $z_2 = \frac{-i - i\sqrt{1-a^2}}{a}$

We note that $|z_2| = \frac{1 + \sqrt{1-a^2}}{|a|} > 1$ (since $-1 < a < 1$)

Also, since $|z_1 z_2| = 1$ it follows the $|z_1| < 1$. Hence there are no singular points on C and $z = z_1$ is the only simple pole inside C

$$\begin{aligned} \text{Res} \{f(z); z_1\} &= \lim_{z \rightarrow z_1} (z - z_1) \left[\frac{2/a}{(z - z_1)(z - z_2)} \right] \\ &= \frac{2/a}{z_1 - z_2} \\ &= \frac{1}{i\sqrt{1-a^2}} \end{aligned}$$

By residue theorem
$$\int_0^{2\pi} \frac{d\theta}{1+a\sin\theta} = 2\pi i \left[\frac{1}{i\sqrt{1-a^2}} \right]$$

$$= \frac{2\pi}{\sqrt{1-a^2}}$$

Problem 3. Prove that $I = \int_0^\pi \frac{a d\theta}{a^2 + \sin^2 \theta} = \frac{\pi}{\sqrt{a^2+1}} (a > 0)$

Solution.
$$I = \int_0^\pi \frac{a d\theta}{a^2 + \left(\frac{1 - \cos 2\theta}{2}\right)}$$

$$= \int_0^\pi \frac{2a d\theta}{2a^2 + 1 - \cos 2\theta}$$

$$= \int_0^{2\pi} \frac{a d\varphi}{2a^2 + 1 - \cos \varphi}$$

$$= \frac{1}{i} \int_C \frac{adz}{z \left[2a^2 + 1 - \frac{(z+z^{-1})}{2} \right]}$$

$$= \frac{2a}{i} \int_C \frac{dz}{[2(2a^2+1)z - z^2 - 1]}$$

$$= 2ai \int_C \frac{dz}{z^2 - 2(2a^2+1)z + 1}$$

$$= 2ai \int_C f(z) dz$$

$2\theta = \phi \quad 2d\theta = d\phi$
 $\theta = 0, \quad \phi = 0$
 $\theta = \pi, \quad \phi = 2\pi$

(putting $2\theta = \varphi$)

(putting $z = e^{i\varphi}$)

$$z^2 - (4a^2+2)z + 1 = 0$$

$$z = \frac{4a^2+2 \pm \sqrt{16a^4+4+16a^2+4}}{2}$$

$$z = \frac{(4a^2+2) \pm 4a\sqrt{a^2+1}}{2} \dots (1)$$

where $f(z) = \frac{1}{z^2 - 2(2a^2+1)z + 1}$ and C is the unit circle $|z| = 1$.

$$z = (2a^2+1) \pm 2a\sqrt{a^2+1}$$

Poles of $f(z)$ are the roots of $z^2 - 2(2a^2+1)z + 1 = 0$.

$$\therefore z = (2a^2+1) \pm 2a\sqrt{a^2+1}$$

$$\text{Let } z_1 = (2a^2+1) + 2a\sqrt{a^2+1}; z_2 = (2a^2+1) - 2a\sqrt{a^2+1}$$

Clearly $|z_1| > 1$ and $|z_1 z_2| = 1$ so that $|z_2| < 1$.

Hence the only pole inside C is $z = z_2$.

$$\text{Res } \{f(z); z_2\} = \lim_{z \rightarrow z_2} (z - z_2) \frac{1}{(z - z_1)(z - z_2)}$$

$$= \frac{1}{z_2 - z_1}$$

$$= \frac{1}{(-4a)\sqrt{a^2 + 1}}$$

$$\text{From (1), } I = 2\pi i \left[\frac{2ai}{-4a\sqrt{a^2 + 1}} \right]$$

$$= \frac{\pi}{\sqrt{a^2 + 1}}$$

Problem 4. Using Cantour integration evaluate $\int_0^{2\pi} \frac{d\theta}{13 + 5 \sin \theta}$.

Solution. Let $I = \int_0^{2\pi} \frac{d\theta}{13 + 5 \sin \theta}$.

Put $z = e^{i\theta}$. Then $dz = ie^{i\theta} d\theta = iz d\theta$.

$$\text{Also } \sin \theta = \frac{z - z^{-1}}{2i}.$$

$\therefore$ The given integral is transformed to

$$I = \int_C \frac{dz}{iz \left[13 + 5 \left(\frac{z - z^{-1}}{2i} \right) \right]} \quad (\text{where } C \text{ is the circle } |z| = 1)$$

$$= \int_C \frac{dz}{iz \left[13 + 5 \left(\frac{z^2 - 1}{i2z} \right) \right]}$$

$$= \int_C \frac{2dz}{5z^2 + 26iz - 5}$$

$$\text{Let } f(z) = \frac{2}{5z^2 + i26z - 5} = \frac{2}{(z + 5i)(5z + i)}.$$

$-\frac{i}{5}$ and $-5i$ are simple poles of $f(z)$ and the pole $-\frac{i}{5}$ lies inside the unit circle.

$$\text{Res} \left\{ f(z); -\frac{i}{5} \right\} = \lim_{z \rightarrow -i/5} \frac{h(z)}{k'(z)} \quad (\text{where } h(z) = 2 \text{ and } k(z) = 5z^2 + i26z - 5)$$

$$= \lim_{z \rightarrow -i/5} \left(\frac{2}{10z + i26} \right)$$

$$= \frac{2}{-2i + 26i}$$

$$= \frac{1}{12i}.$$

Hence by Cauchy's residue theorem $I = 2\pi i \left(\frac{1}{12i} \right) = \frac{\pi}{6}$.

Problem 5. Use Contour integration technique to find the value of $\int_0^{2\pi} \frac{d\theta}{2 + \cos \theta}$.

Solution. Let $I = \int_0^{2\pi} \frac{d\theta}{2 + \cos \theta}$.

Put $z = e^{i\theta}$. Then $dz = ie^{i\theta} d\theta = iz d\theta$.

$$\text{Also } \cos \theta = \frac{z + z^{-1}}{2}.$$

$\therefore$ The given integral is transformed to $I = \int_C \frac{dz}{iz \left[2 + \frac{z + z^{-1}}{2} \right]}$ where C is the unit

circle $|z| = 1$.

$$\therefore I = \int_C \frac{dz}{iz \left[2 + \left(\frac{z^2 + 1}{2z} \right) \right]}$$

$$= \int_C \frac{2dz}{i(4z + z^2 + 1)}$$

$$= \int_C \frac{-2idz}{z^2 + 4z + 1}.$$

$$\text{Let } f(z) = \frac{-2i}{z^2 + 4z + 1}$$

$$= \frac{-2i}{(z+2)^2 - 3}$$

$$= \frac{-2i}{(z+2-\sqrt{3})(z+2+\sqrt{3})}$$

$\therefore -2 + \sqrt{3}$ and $-2 - \sqrt{3}$ are simple poles of $f(z)$; the pole $-2 + \sqrt{3}$ lies inside C .

$$\text{Res} \left\{ f(z); -2 + \sqrt{3} \right\} = \lim_{z \rightarrow -2 + \sqrt{3}} \left(\frac{-2i}{2z + 4} \right)$$

$$= \frac{-2i}{-4 + 2\sqrt{3} + 4}$$

$$= \frac{-i}{\sqrt{3}}$$

Hence by Cauchy's residue theorem $I = 2\pi i \left(\frac{-i}{\sqrt{3}} \right) = \frac{2\pi}{\sqrt{3}}$.

Exercises

1. Show that $\int_0^{2\pi} \frac{d\theta}{5 + 3 \cos \theta} = \frac{\pi}{2}$.

2. Show that $\int_0^{2\pi} \frac{\cos 2\theta d\theta}{5 + 4 \cos \theta} = \frac{\pi}{6}$.

3. Show that $\int_0^{2\pi} \frac{d\theta}{2 + \cos \theta} = \frac{2\pi}{\sqrt{3}}$.

4. Show that $\int_0^{2\pi} \frac{\cos^2 3\theta d\theta}{5 - 4 \cos 2\theta} = \frac{3\pi}{8}$.

5. Show that $\int_0^{2\pi} \frac{d\theta}{\cos \theta + 2 \sin \theta + 4} = \frac{2\pi}{\sqrt{11}}.$

6. Show that $\int_0^{\pi} \frac{d\theta}{a + \cos \theta} = \frac{\pi}{\sqrt{a^2 - 1}} (a > 1).$

7. Show that $\int_0^{2\pi} \frac{d\theta}{a + b \cos \theta} = \frac{2\pi}{\sqrt{a^2 - b^2}} (a > b > 0).$

8. Show that $\int_0^{2\pi} \frac{d\theta}{1 + a \sin \theta} = \frac{2\pi}{\sqrt{1 - a^2}} (a^2 < 1).$

9. Show that $\int_0^{\pi} \frac{d\theta}{(a + \cos \theta)^2} = \frac{2a\pi}{(a^2 - 1)^{3/2}} (a > 1).$

10. Show that $\int_0^{\pi} \frac{1 + 2 \cos \theta}{5 + 4 \cos \theta} d\theta = 0.$

11. Show that $\int_0^{2\pi} \frac{d\theta}{a^2 - 2a \cos \theta + 1} = \frac{2\pi}{1 - a^2} (0 < a < 1).$

TYPE 2. $\int_{-\infty}^{\infty} f(x) dx$ where $f(x) = \frac{g(x)}{h(x)}$ and $g(x), h(x)$ are polynomials in x and the degree of $h(x)$ exceeds that of $g(x)$ by at least two. $h(x) \geq g(x) + 2.$

To evaluate this type of integral we take $f(z) = \frac{g(z)}{h(z)}.$

The poles of $f(z)$ are determined by the zeros of the equation $h(z) = 0.$

Case (i) No pole of $f(z)$ lies on the real axis.

We choose the curve C consisting of the interval $[-r, r]$ on the real axis and the semi circle $|z| = r$ lying in the upper half of the plane.

Here r is chosen sufficiently large so that all the poles lying in the upper half of the plane are in the interior of C . Then we have

$$\int_C f(z) dz = \int_{-r}^r f(x) dx + \int_{C_1} f(z) dz$$

where C_1 is the semi circle.

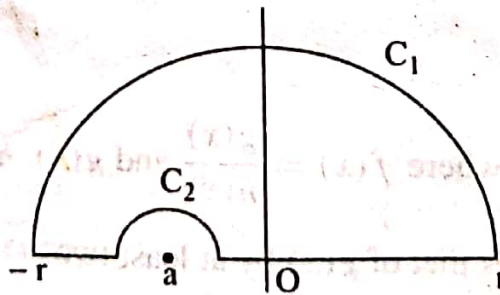
Since $\deg h(x) - \deg g(x) \geq 2$ it follows that $\int_{C_1} f(z) dz \rightarrow 0$ as $r \rightarrow \infty$ and hence

$$\int_C f(z) dz = \int_{-\infty}^{\infty} f(x) dx.$$

$\therefore \int_{-\infty}^{\infty} f(x) dx$ can be evaluated by evaluating $\int_C f(z) dz$ which in turn can be evaluated by using Cauchy's residue theorem.

Case (ii) $f(z)$ has poles lying on the real axis.

Suppose a is a pole lying on the real axis. In this case we indent the real axis by a semi-circle C_2 of radius ε with centre a lying in the upper half plane where ε is chosen to be sufficiently small (refer figure).



Such an indenting must be done for every pole of $f(z)$ lying on the real axis.

It can be proved that $\int_{C_2} f(z) dz = -\pi i \operatorname{Res} \{f(z); a\}$. By taking limit as $r \rightarrow \infty$ and

$\varepsilon \rightarrow 0$ we obtain the value of $\int_{-\infty}^{\infty} f(x) dx$.

Solved Problems

Problem 1. Use Contour integration method to evaluate $\int_{-\infty}^{\infty} \frac{dx}{1+x^4}$.

Solution. Let $f(z) = \frac{1}{1+z^4}$. $\Rightarrow z^4 = -1 \Rightarrow e^{i\pi} = z^4 \Rightarrow z = [\cos(\pi + 2n\pi) + i\sin(\pi + 2n\pi)]^{1/4}$

The poles of $f(z)$ are given by the roots of the equation $z^4 + 1 = 0$, which are the four fourth roots of -1 .

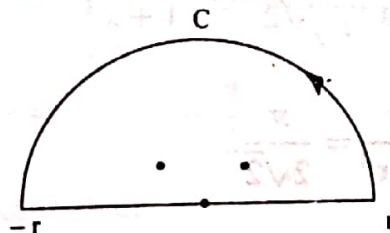
By De Moivre's theorem they are given by $e^{i\pi/4}$; $e^{i3\pi/4}$; $e^{i5\pi/4}$; $e^{i7\pi/4}$ and all are simple poles.

We choose the contour C consisting of the interval $[-r, r]$ on the real axis and the upper semi-circle $|z| = r$ which we denote by C_1 .

$$\therefore \int_C f(z)dz = \int_{-r}^r f(x)dx + \int_{C_1} f(z)dz \quad \dots (1)$$

The poles of $f(z)$ lying inside the contour C are obviously $e^{i\pi/4}$ and $e^{i3\pi/4}$ only.

We find the residues of $f(z)$ at these points.



$$\text{Res} \left\{ f(z); e^{i\pi/4} \right\} = \frac{h(e^{i\pi/4})}{k'(e^{i\pi/4})} \text{ where } h(z) = 1 \text{ and } k(z) = z^4 + 1 \text{ so that } k'(z) = 4z^3.$$

$$\therefore \text{Res} \left\{ f(z); e^{i\pi/4} \right\} = \frac{1}{4e^{i3\pi/4}} = \frac{e^{-i3\pi/4}}{4}.$$

$$\text{Similarly Res} \left\{ f(z); e^{i3\pi/4} \right\} = \frac{e^{-i9\pi/4}}{4}.$$

By residue theorem

$$\int_C f(z)dz = 2\pi i (\text{sum of the residues at the poles})$$

$$= 2\pi i \left[\frac{e^{-i3\pi/4}}{4} + \frac{e^{-i9\pi/4}}{4} \right]$$

$$\begin{aligned}
&= \frac{\pi i}{2} [(\cos(3\pi/4) - i \sin(3\pi/4)) + (\cos(9\pi/4) - i \sin(9\pi/4))] \\
&= \frac{\pi i}{2} \left[\left(-\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}} \right) + \left(\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}} \right) \right] \\
&= \frac{\pi i}{2} \left(-\frac{2i}{\sqrt{2}} \right) = \frac{\pi}{\sqrt{2}}.
\end{aligned}$$

From (1) $\int_{-r}^r \frac{dx}{1+x^4} + \int_{C_1} f(z)dz = \frac{\pi}{2}.$

As $r \rightarrow \infty$, $\int_{C_1} f(z)dz \rightarrow 0.$

$$\therefore \int_{-\infty}^{\infty} \frac{dx}{1+x^4} = \frac{\pi}{\sqrt{2}}.$$

$$\therefore 2 \int_0^{\infty} \frac{dx}{1+x^4} = \frac{\pi}{\sqrt{2}} \quad (\because \frac{1}{1+x^4} \text{ is an even function}).$$

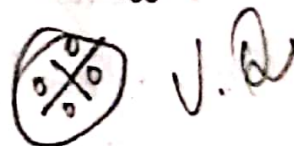
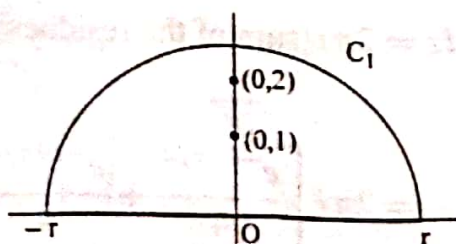
$$\therefore \int_0^{\infty} \frac{dx}{1+x^4} = \frac{\pi}{2\sqrt{2}}.$$

Problem 2. Using the method of contour integration evaluate $\int_{-\infty}^{\infty} \frac{x^2}{(x^2+1)(x^2+4)} dx.$

Solution. Let $f(z) = \frac{z^2}{(z^2+1)(z^2+4)}.$

The poles of $f(z)$ are $i, -i, 2i, -2i.$

Choose the contour C as shown in the figure.



The poles i and $2i$ lie within C . By residue theorem.

$$\int_C f(z) dz = 2\pi i (\text{sum of the residues of } f(z)) \quad \dots (1)$$

We find the residues of $f(z)$.

$$\text{Res } \{f(z); i\} = \frac{h(i)}{k'(i)} \text{ where } h(z) = z^2 \text{ and } k(z) = (z^2 + 1)(z^2 + 4) = z^4 + 5z^2 + 4$$

$$\text{so that } k'(z) = 4z^3 + 10z.$$

$$\therefore \text{Res } \{f(z); i\} = \frac{-1}{-4i + 10i} = \frac{-1}{6i} = \frac{i}{6}$$

$$\text{Res } \{f(z); 2i\} = -\frac{i}{3} \text{ (verify).}$$

$$\therefore \text{From (1)} \int_C f(z) dz = 2\pi i \left(\frac{i}{6} - \frac{i}{3} \right)$$

$$2\pi i \left[\frac{a}{2(a^2 - b^2)} - \frac{b}{2(a^2 - b^2)} \right] = \frac{2\pi i}{2(a^2 - b^2)} [a - b] = \frac{\pi i}{a^2 - b^2} = \frac{\pi}{3}$$

$\therefore$ Also (1) can be written, using (2), as

$$\int_{C_1} f(z) dz + \int_{-r}^r \frac{x^2}{(x^2 + 1)(x^2 + 4)} dx = \frac{\pi}{3} \quad \dots (3)$$

Further the integral $\int_{C_1} f(z) dz \rightarrow 0$ as $r \rightarrow \infty$.

$$\therefore \int_{-\infty}^{\infty} \frac{x^2}{(x^2 + 1)(x^2 + 4)} dx = \frac{\pi}{3}$$

Problem 3. Evaluate $\int_{-\infty}^{\infty} \frac{x^2}{(x^2 + a^2)(x^2 + b^2)} dx = \frac{\pi}{a + b}$.

Solution. Proceed as in previous problem.

Problem 4. Evaluate $\int_{-\infty}^{\infty} \frac{x^2 - x + 2}{x^4 + 10x^2 + 9} dx$.

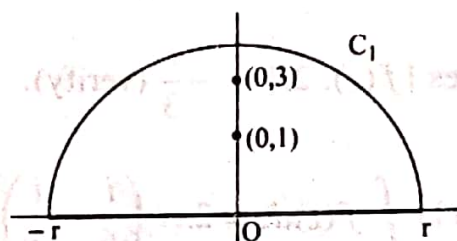
Solution. Let $f(z) = \frac{z^2 - z + 2}{z^4 + 10z^2 + 9}$.

Poles of $f(z)$ are the zeros of $z^4 + 10z^2 + 9 = 0$.

$$z^4 + 10z^2 + 9 = (z^2 + 9)(z^2 + 1).$$

$\therefore z = \pm 3i; \pm i$. Hence $z = 3i, -3i, i, -i$ are the simple poles of $f(z)$.

Choose the contour C as shown in the figure.



$$\int_C f(z) dz = \int_{-r}^r f(x) dx + \int_{C_1} f(z) dz. \quad \dots (1)$$

The poles of $f(z)$ lying within C are i and $3i$ and both of them are simple poles.

$$\text{Res } \{f(z); i\} = \frac{h(i)}{k'(i)} \text{ where } h(z) = z^2 - z + 2 \text{ and}$$

$$k(z) = z^4 + 10z^2 + 9 \text{ so that } k'(z) = 4z^3 + 20z.$$

$$\therefore \text{Res } \{f(z); i\} = \frac{-1 - i + 2}{-4i + 20i} = \frac{1 - i}{16i}.$$

$$\text{Similarly Res } \{f(z); 3i\} = \frac{7 + 3i}{48i} \text{ (verify).}$$

$$\therefore \int_C dz = 2\pi i \text{ (sum of the residues at the poles)}$$

$$= 2\pi i \left(\frac{1 - i}{16i} + \frac{7 + 3i}{48i} \right)$$

$$= 2\pi i \left(\frac{10}{48i} \right) = \frac{5\pi}{12}.$$

From (1) $\int_{-r}^r \frac{x^2 - x + 2}{x^4 + 10x^2 + 9} dx + \int_{C_1} \frac{z^2 - z + 2}{z^4 + 10z^2 + 9} dz = \frac{5\pi}{12}.$

Now as $r \rightarrow \infty$ the integral over $C_1 \rightarrow 0$.

$$\therefore \int_{-\infty}^{\infty} \frac{x^2 - x + 2}{x^4 + 10x^2 + 9} dx = \frac{5\pi}{12}.$$

Problem 5. Evaluate $I = \int_0^{\infty} \frac{dx}{(x^2 + a^2)^2}.$

Solution. Since $\frac{1}{(x^2 + a^2)^2}$ is an even function we have

$$\int_{-\infty}^{\infty} \frac{dx}{(x^2 + a^2)^2} = 2 \int_0^{\infty} \frac{dx}{(x^2 + a^2)^2}.$$

Let $f(z) = \frac{1}{(z^2 + a^2)^2}.$

Poles of $f(z)$ are the roots of $(z^2 + a^2)^2 = 0$.

Now, $(z^2 + a^2)^2 = (z + ai)^2(z - ai)^2.$

$\therefore ai$ and $-ai$ are double poles of $f(z)$.

Choose the contour C consisting of the interval $[-r, r]$ on the real axis and the semi circle C_1 with centre 0 and radius r that lies in the upper half plane.

$$\therefore \int_{-r}^r f(x) dx + \int_{C_1} f(z) dz = \int_C f(z) dz. \quad \dots (1)$$

The poles of $f(z)$ lying within C is $z = ai$.

$\text{Res}\{f(z); ai\} = \frac{1}{1!} g'(ai)$ where $g(z) = \frac{1}{(z + ai)^2}.$

Now $g'(z) = -2(z + ai)^{-3}.$

$\therefore g'(ai) = \frac{1}{4a^3i}.$

$$\begin{aligned} &= \frac{-2}{(z+ai)^3} = \frac{-2}{(ai)^3} = \frac{-2}{8a^3i} = \frac{+2}{16a^3i} = \frac{1}{8a^3i} \end{aligned}$$

$$\therefore \operatorname{Res}\{f(z); a_i\} = \frac{1}{4a^3i}.$$

$$\therefore \int_C f(z) dz = 2\pi i \left(\frac{1}{4a^3i} \right) = \frac{\pi}{2a^3}.$$

$$\therefore \int_{-r}^r \frac{dx}{(x^2 + a^2)^2} + \int_{C_1} f(z) dz = \frac{\pi}{2a^3}.$$

When $r \rightarrow \infty$ the integral over $C_1 \rightarrow 0$.

$$\therefore \int_{-\infty}^{\infty} \frac{dx}{(x^2 + a^2)^2} = \frac{\pi}{2a^3}.$$

$$\therefore \int_0^{\infty} \frac{dx}{(x^2 + a^2)^2} = \frac{\pi}{4a^3}.$$

Problem 6. Prove that $\int_0^{\infty} \frac{dx}{x^6 + 1} = \frac{\pi}{3}.$

Solution. Since $\frac{1}{x^6 + 1}$ is an even function we have

$$\int_{-\infty}^{\infty} \frac{dx}{x^6 + 1} = 2 \int_0^{\infty} \frac{dx}{x^6 + 1}.$$

Now, let $f(z) = \frac{1}{z^6 + 1}.$

The poles of $f(z)$ are given by the roots of the equation $z^6 + 1 = 0$, which are the sixth roots of -1 .

By De Moivre's theorem they are given by $e^{i\pi/6}, e^{i3\pi/6}, e^{i5\pi/6}, e^{i7\pi/6}$ and $e^{i11\pi/6}$ and they are simple poles.

Now choose the contour C consisting of the interval $[-r, r]$ on the real axis and the upper semi circle $|z| = r$, which we denote by C_1 .

The poles of $f(z)$ lying inside C are $e^{i\pi/6}, e^{i3\pi/6}$ and $e^{i5\pi/6}.$

$$\begin{aligned} \operatorname{Res} \left\{ f(z); e^{i\pi/6} \right\} &= \frac{h(e^{i\pi/6})}{k'(e^{i\pi/6})} \text{ where } h(z) = 1 \text{ and } k(z) = z^6 + 1 \\ &\text{so that } k'(z) = 6z^5 \\ &= \frac{1}{6e^{i5\pi/6}} = \frac{1}{6}e^{-i5\pi/6}. \end{aligned}$$

Similarly $\text{Res} \left\{ f(z); e^{i3\pi/6} \right\} = \frac{1}{6} e^{-i5\pi/2}$ and

$$\text{Res} \left\{ f(z); e^{i5\pi/6} \right\} = \frac{1}{6} e^{-i25\pi/6}.$$

$\therefore$ By residue theorem

$$\int_C f(z) dz = 2\pi i (\text{sum of the residues at the 3 poles})$$

$$= 2\pi i \left[\frac{1}{6} e^{5i\pi/6} + \frac{1}{6} e^{-5i\pi/2} + \frac{1}{6} e^{-25i\pi/6} \right]$$

$$= \frac{2\pi i}{6} \left[\left(\cos \frac{5\pi}{6} - i \sin \frac{5\pi}{6} \right) + \left(\cos \frac{5\pi}{2} - i \sin \frac{5\pi}{2} \right) + \left(\cos \frac{25\pi}{6} - i \sin \frac{25\pi}{6} \right) \right]$$

$$= \frac{\pi i}{3} \left[\left(-\frac{\sqrt{3}}{2} - \frac{i}{2} \right) + (0 - i) + \left(\frac{\sqrt{3}}{2} - \frac{i}{2} \right) \right]$$

$$= \frac{\pi i}{3} (-i - i) = \frac{2\pi}{3}.$$

$$= \frac{\pi i}{3} (-2i) = \frac{2\pi i^2}{3} = -\frac{2\pi}{3}$$

$$\therefore \text{From (1), } \int_{-r}^r \frac{dx}{x^6+1} + \int_{C_1} f(z) dz = \frac{2\pi}{3}.$$

As $r \rightarrow \infty$ the integral over $C_1 \rightarrow 0$.

$$\therefore \int_{-\infty}^{\infty} \frac{dx}{x^6+1} = \frac{2\pi}{3}.$$

$$\therefore \int_0^{\infty} \frac{dx}{x^6+1} = \frac{\pi}{3}.$$

Problem 7. Prove that $\int_0^{\infty} \frac{x^4 dx}{x^6-1} = \frac{\pi\sqrt{3}}{6}.$

Solution. Let $f(z) = \frac{z^4}{z^6-1}.$

$$z^6 - 1 = 0$$

$$z^6 = 1$$

$$z^6 = \cos 2n\pi + i \sin 2n\pi$$

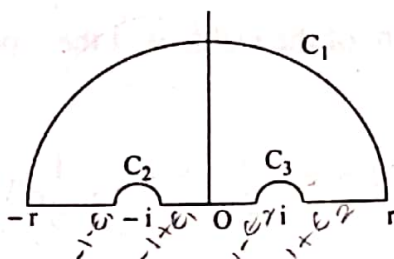
$$z = (\cos 2n\pi + i \sin 2n\pi)^{1/6}$$

$$z = \cos 2n\pi/6 + i \sin 2n\pi/6$$

The poles of $f(z)$ are given by the sixth roots of unity, namely $e^{2n\pi i/6}$; $n = 0, 1, \dots, 5$.

$\therefore f(z)$ has 2 simple poles on the real axis, viz., 1 and -1 and the two poles $e^{\pi i/3}$ and $e^{2\pi i/3}$ lie on the upper half of the plane.

Now choose the contour C as shown in the figure.



$$\begin{aligned} \therefore \int_C f(z) dz &= \int_{C_1} f(z) dz + \int_{-r}^{-1-\epsilon_1} f(x) dx + \int_{C_2} f(z) dz \\ &\quad + \int_{-1+\epsilon_1}^{1-\epsilon_2} f(x) dx + \int_{C_3} f(z) dz + \int_{1+\epsilon_2}^r f(x) dx \quad \dots (1) \end{aligned}$$

$$\text{Now, } \int_{C_2} f(z) dz = -\pi i \operatorname{Res}\{f(z); -1\}$$

$$= -\pi i \left(\frac{h(-1)}{k'(-1)} \right) \quad \text{where } h(z) = z^4 \text{ and } k(z) = z^6 - 1$$

$$= -\pi i(-1/6)$$

$$= \pi i/6. \quad \dots (2)$$

$$\text{Similarly } \int_{C_3} f(z) dz = -\pi i \operatorname{Res}\{f(z); 1\}$$

$$= -\pi i \left(\frac{h(1)}{k'(1)} \right)$$

$$= -\pi i(1/6)$$

$$= -\pi i/6. \quad \dots (3)$$

$$\text{Also } \int_C f(z) dz = 2\pi i \left[\text{Res} \left\{ f(z); e^{\pi i/3} \right\} + \text{Res} \left\{ f(z); e^{2\pi i/3} \right\} \right]$$

$$= 2\pi i \left[\frac{h(e^{\pi i/3})}{6e^{5\pi i/3}} + \frac{e^{i8\pi/3}}{6e^{i10\pi/3}} \right]$$

$$= 2\pi i \left[\frac{e^{i4\pi/3}}{6e^{5\pi i/3}} + \frac{e^{i8\pi/3}}{6e^{i10\pi/3}} \right]$$

$$= \frac{\pi i}{3} (e^{-i\pi/3} + e^{-i2\pi/3})$$

$$= \frac{\pi i}{3} (e^{-i\pi/3} - e^{i\pi/3}) \quad \cos 2\pi/3 - i \sin 2\pi/3$$

$$= \frac{\pi i}{3} (-2i \sin \pi/3) = \frac{1/2 - i\sqrt{3}/2}{6}$$

$$= \frac{\pi\sqrt{3}}{3} = \frac{1}{6} \{ -1/2 + i\sqrt{3}/2 \} \dots (4)$$

Substituting (2), (3), (4) in (1) and taking limits as $\epsilon_1, \epsilon_2 \rightarrow 0$ and $r \rightarrow \infty$ we get

$$\int_{-\infty}^{\infty} \frac{x^4}{x^6 - 1} dx + \frac{\pi i}{6} - \frac{\pi i}{6} = \frac{\pi\sqrt{3}}{3}$$

$$\therefore 2 \int_0^{\infty} \frac{x^4}{x^6 - 1} dx = \frac{\pi\sqrt{3}}{3}$$

$$\therefore \int_0^{\infty} \frac{x^4}{x^6 - 1} dx = \frac{\pi\sqrt{3}}{6}$$

Exercises

1. Prove the following by using Cauchy's residue theorem.

$$(i) \int_0^{\infty} \frac{dx}{x^2 + 1} = \frac{\pi}{2}$$

$$(ii) \int_0^{\infty} \frac{dx}{(x^2 + 1)^2} = \frac{\pi}{4}$$

$$(iii) \int_0^{\infty} \frac{x^2 dx}{(x^2 + 1)^2} = \frac{\pi}{4}$$

$$(iv) \int_0^{\infty} \frac{dx}{x^4 + x^2 + 1} = \frac{\pi\sqrt{3}}{6}$$

$$(v) \int_0^{\infty} \frac{(2x^2 - 1)dx}{x^4 + 5x^2 + 4} = \frac{\pi}{4}$$

$$(vi) \int_0^{\infty} \frac{dx}{(x^2 + 1)^3} = \frac{\pi}{16}$$

$$(vii) \int_0^{\infty} \frac{x^2 dx}{(x^2 + a^2)^3} = \frac{\pi}{16a^3}$$

$$(viii) \int_0^{\infty} \frac{x^2 dx}{(x^2 + 1)(x^2 + 4)} = \frac{\pi}{6}$$

$$(ix) \int_0^{\infty} \frac{x^2 dx}{(x^2 + 4)(x^2 + 9)} = \frac{\pi}{200}$$

$$(x) \int_0^{\infty} \frac{dx}{x^4 + 1} = \frac{\pi}{2\sqrt{2}}$$

2. Show that $\int_{-\infty}^{\infty} \frac{dx}{(x-1)(x^2+2)} = \frac{-\sqrt{2}\pi}{6}$.

TYPE 3. $\int_{-\infty}^{\infty} \frac{g(x)}{h(x)} \cos ax dx$ or $\int_{-\infty}^{\infty} \frac{g(x)}{h(x)} \sin ax dx$ where $g(x)$ and $h(x)$ are real polynomials such that degree of $h(x)$ exceeds that of $g(x)$ by at least one and $a > 0$.

Case (i) $h(x)$ has no zeros on the real axis.

In this case take $f(z) = \frac{g(z)}{h(z)} e^{iaz}$.

$\therefore f(z)$ has no poles on the real axis.

Choose the contour as in Type 2 and proceeding as in Type 2 we get the value of $\int_{-\infty}^{\infty} \frac{g(x)}{h(x)} e^{iax} dx$.

Taking the real and imaginary parts of $\frac{g(x)}{h(x)} e^{iax} dx$ we obtain the values of $\int_{-\infty}^{\infty} \frac{g(x)}{h(x)} \cos ax dx$ and $\int_{-\infty}^{\infty} \frac{g(x)}{h(x)} \sin ax dx$

Case (ii) $h(x)$ has zeros of order one on the real axis.

Take $f(z) = \frac{g(z)}{h(z)} e^{iaz}$. We notice that $f(z)$ has real poles. Suppose a is a real zero of $h(x)$ on the real axis. In this case we indent the real axis as Case (ii) of Type 2 and evaluate the integral.

To prove that the integral over the upper semicircle tends to zero as $r \rightarrow \infty$, we use the following lemma.

Jordan's Lemma. Let $f(z)$ be a function of the complex variable z satisfying the following conditions.

- (i) $f(z)$ is analytic in upper half plane except at a finite number of poles.
- (ii) $f(z) \rightarrow 0$ uniformly as $|z| \rightarrow \infty$ with $0 \leq \arg z \leq \pi$.
- (iii) a is a positive integer.

Then $\lim_{r \rightarrow \infty} \int_C f(z) e^{iaz} dz = 0$ where C is the semi circle with centre at the origin and radius r .

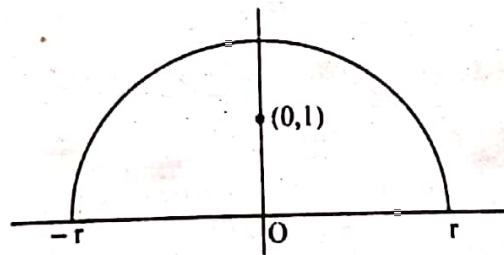
Solved Problems

Problem 1. Prove that $\int_0^{\infty} \frac{\cos x}{1+x^2} dx = \frac{\pi}{2e}$.

Solution. Let $f(z) = \frac{e^{iz}}{1+z^2}$.

The poles of $f(z)$ are given by i and $-i$.

Choose the contour C as shown in the figure.



The pole of $f(z)$ that lies within C is i . Hence by residue theorem

$$\int_C f(z) dz = 2\pi i \operatorname{Res}\{f(z); i\}$$

$$= 2\pi i \frac{h(i)}{k'(i)} \text{ where } h(z) = e^{iz} \text{ and } k(z) = 1+z^2$$

$$= \frac{2\pi i e^{-1}}{2i} = \frac{\pi}{e}$$

$$\therefore \int_{-r}^r \frac{e^{iax}}{x^2 + 1} dx + \int_{C_1} \frac{e^{iaz}}{z^2 + 1} dz = \frac{\pi}{e}.$$

When $r \rightarrow \infty$ the integral over C_1 tends to zero.

$$\therefore \int_{-\infty}^{\infty} \frac{e^{iax}}{x^2 + 1} dx = \frac{\pi}{e}.$$

Equating real parts we get $\int_{-\infty}^{\infty} \frac{\cos x}{1 + x^2} dx = \frac{\pi}{e}.$

$$\therefore 2 \int_0^{\infty} \frac{\cos x}{1 + x^2} dx = \frac{\pi}{e} \text{ (since } \frac{\cos x}{1 + x^2} \text{ is an even function)}$$

$$\therefore \int_0^{\infty} \frac{\cos x}{1 + x^2} dx = \frac{\pi}{2e}.$$

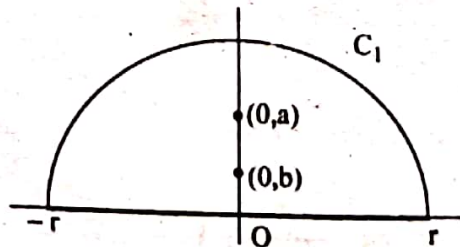
Problem 2. Using the method of contour integration evaluate

$$\int_{-\infty}^{\infty} \frac{\cos x}{(x^2 + a^2)(x^2 + b^2)} dx \quad (a > b > 0).$$

Solution. Let $f(z) = \frac{e^{iz}}{(z^2 + a^2)(z^2 + b^2)}.$

The poles of $f(z)$ are $ia, -ia, ib, -ib.$

Choose the contour C as shown in the figure.



The poles of $f(z)$ which lie within C are ia and $ib.$

Hence by residue theorem

$$\int_C f(z) dz = 2\pi i (\text{sum of the residues of } f(z)). \quad \dots (1)$$

We find the residues of $f(z).$

$$\text{Res}\{f(z); a_i\} = \frac{h(ai)}{k'(ai)} \text{ where } h(z) = e^{iz} \text{ and}$$

$$k(z) = (z^2 + a^2)(z^2 + b^2) = z^4 + (a^2 + b^2)z^2 + a^2b^2 \text{ so that}$$

$$k'(z) = 4z^3 + 2(a^2 + b^2)z.$$

$$\therefore \text{Res}\{f(z); ai\} = \frac{e^{-a}}{4(ia)^3 + 2(a^2 + b^2)(ia)}$$

$$= \frac{e^{-a}}{i2a[(a^2 + b^2) - 2a^2]}$$

$$= \frac{-ie^{-a}}{2a(b^2 - a^2)}$$

$$= \frac{ie^{-a}}{2a(a^2 - b^2)}.$$

$$\begin{aligned} & 4(ia)^3 + 2(a^2 + b^2)ia \\ &= -4ia^3 + 2ia(a^2 + b^2) \\ &= -2ia[2a^2 - (a^2 + b^2)] \\ &= -2ia[a^2 - b^2] \\ &= 2ia[b^2 - a^2] \end{aligned}$$

$$\begin{aligned} \text{Similarly, Res}\{f(z); bi\} &= \frac{ie^{-b}}{2b(b^2 - a^2)} \\ &= \frac{-ie^{-b}}{2b(a^2 - b^2)}. \end{aligned}$$

$$\text{From (1)} \int_C f(z)dz = 2\pi i \left[\frac{i}{2(a^2 - b^2)} \left(\frac{e^{-a}}{a} - \frac{e^{-b}}{b} \right) \right]$$

$$= \frac{-\pi}{a^2 - b^2} \left(\frac{e^{-a}}{a} - \frac{e^{-b}}{b} \right)$$

$$= \frac{\pi}{a^2 - b^2} \left(\frac{e^{-b}}{b} - \frac{e^{-a}}{a} \right). \quad \dots (2)$$

Also (1) can be written using (2) as

$$\int_{C_1} f(z) dz + \int_{-r}^r \frac{e^{ix}}{(x^2 + a^2)(x^2 + b^2)} dx = \frac{\pi}{a^2 - b^2} \left(\frac{e^{-b}}{b} - \frac{e^{-a}}{a} \right). \quad \dots (3)$$

Further the integral over C_1 tends to 0 as $r \rightarrow \infty$.

∴ (3) becomes

$$\int_{-\infty}^{\infty} \frac{e^{ix}}{(x^2 + a^2)(x^2 + b^2)} dx = \frac{\pi}{a^2 - b^2} \left(\frac{e^{-b}}{b} - \frac{e^{-a}}{a} \right).$$

Equating real parts on both sides we get

$$\int_{-\infty}^{\infty} \frac{\cos x}{(x^2 + a^2)(x^2 + b^2)} dx = \frac{\pi}{a^2 - b^2} \left(\frac{e^{-b}}{b} - \frac{e^{-a}}{a} \right).$$

Problem 3. Prove that $\int_0^{\infty} \frac{\cos ax \, dx}{(x^2 + 1)^2} = \frac{\pi}{4}(a + 1)e^{-a}$ where $a > 0$.

Solution. Let $f(z) = \frac{e^{iaz}}{(z^2 + 1)^2}$.

The poles of $f(z)$ are given by i and $-i$ which are double poles. Now choose the contour as in Problem 1. The pole of $f(z)$ that lies within C is i .

Now, $\text{Res}\{f(z); i\} = \frac{1}{1!} g'(i)$ where $g(z) = (z - i)^2 f(z) \doteq \frac{e^{iaz}}{(z + i)^2}$.

$$\therefore g'(z) = \frac{(z + i)^2 i a e^{iaz} - e^{iaz} 2(z + i)}{(z + i)^4}.$$

$$\therefore \text{Res}\{f(z); i\} = \frac{-4ia e^{-a} - e^{-a}(4i)}{16} = \frac{-ie^{-a}(a + 1)}{4}.$$

Hence by Cauchy's residue theorem

$$\int_C f(z) \, dz = 2\pi i \left(\frac{-ie^{-a}(a + 1)}{4} \right) = \frac{\pi(a + 1)e^{-a}}{2}.$$

$$\therefore \int_{-r}^r f(x) \, dx + \int_{C_1} f(z) \, dz = \frac{\pi(a + 1)e^{-a}}{2}.$$

As $r \rightarrow \infty$, the integral over C_1 tends to zero.

$$\therefore \int_{-\infty}^{\infty} f(x) \, dx = \frac{\pi(a + 1)e^{-a}}{2}.$$

Equating real parts $\int_{-\infty}^{\infty} \frac{\cos ax}{(x^2 + 1)^2} dx = \frac{\pi(a+1)e^{-a}}{2}.$

$$\therefore \int_0^{\infty} \frac{\cos ax}{(x^2 + 1)^2} dx = \frac{\pi(a+1)e^{-a}}{4}.$$

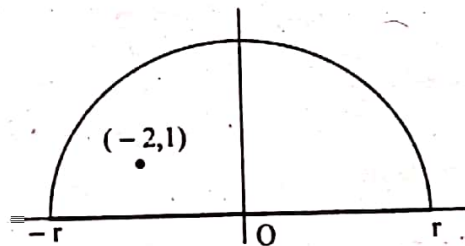
Problem 4. Prove that $\int_{-\infty}^{\infty} \frac{\sin x \, dx}{x^2 + 4x + 5} = -\frac{\pi \sin 2}{e}.$

Solution. Let $f(z) = \frac{e^{iz}}{z^2 + 4z + 5}.$

The poles of $f(z)$ are the roots of the equation $z^2 + 4z + 5 = 0$. They are given by

$$z = \frac{-4 \pm \sqrt{16 - 20}}{2} = -2 \pm i.$$

Now, choose the contour C as shown in the figure.



$-2 + i$ is the only pole of $f(z)$ that lies within C and it is a simple pole.

Hence by Cauchy's residue theorem,

$$\begin{aligned} \int_C f(z) \, dz &= 2\pi i \operatorname{Res}\{f(z); -2 + i\} \\ &= (2\pi i) \frac{h(-2 + i)}{k'(-2 + i)} \quad \text{where } h(z) = e^{iz} \text{ and } k(z) = z^2 + 4z + 5 \end{aligned}$$

$$\therefore \int_{-r}^r f(x) \, dx + \int_{C_1} f(z) \, dz = \frac{\pi e^{-2i}}{e}.$$

Since the integral over C_1 tends to zero as $r \rightarrow \infty$, we have

$$\int_{-\infty}^{\infty} f(x) \, dx = \frac{\pi e^{-2i}}{e} = \frac{\pi}{e} (\cos 2 - i \sin 2).$$

Equating imaginary parts we get

$$\int_{-\infty}^{\infty} \frac{\sin x dx}{x^2 + 4x + 5} = -\frac{\pi \sin 2}{e} \quad \dots (4)$$

Problem 5. Evaluate $\int_0^{\infty} \frac{x \sin x}{x^2 + a^2} dx$.

Solution. Let $f(z) = \frac{ze^{iz}}{z^2 + a^2}$

The poles of $f(z)$ are given by ia and $-ia$ which are simple poles. Choose the contour C as in Problem 1.

Only the pole $z = ia$ lies inside C .

$\text{Res}\{f(z); ia\} = \frac{h(ia)}{k'(ia)}$ where $h(z) = ze^{iz}$ and $k(z) = z^2 + a^2$ so that $k'(z) = 2z$.

$$\therefore \text{Res}\{f(z); ia\} = \frac{iae^{i(ia)}}{2(ia)} = \frac{e^{-a}}{2}.$$

Hence by Cauchy's residue theorem

$$\int_C f(z) dz = 2\pi i \left(\frac{e^{-a}}{2} \right) = \pi i e^{-a}.$$

$$\therefore \int_{-r}^r f(x) dx + \int_{C_1} f(z) dz = \pi i e^{-a}.$$

When $r \rightarrow \infty$, $\int_{C_1} f(z) dz = 0$.

$$\therefore \int_{-\infty}^{\infty} f(x) dx = \pi i e^{-a}.$$

$$\therefore \int_{-\infty}^{\infty} \frac{xe^{ix}}{x^2 + a^2} dx = \pi i e^{-a}.$$

$$(i.e.) \int_{-\infty}^{\infty} \frac{x(\cos x + i \sin x) dx}{x^2 + a^2} = \pi i e^{-a}.$$

$$(i.e.) \int_{-\infty}^{\infty} \frac{x(\cos x + i \sin x) dx}{x^2 + a^2} = \pi i e^{-a}.$$

Equating imaginary parts on both sides we get

$$\int_{-\infty}^{\infty} \frac{x \sin x}{x^2 + a^2} dx = \pi e^{-a}.$$

Since the above integrand is an even function we have

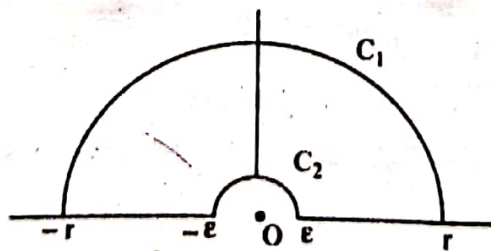
$$2 \int_0^{\infty} \frac{x \sin x}{x^2 + a^2} dx = \pi e^{-a}.$$

$$\therefore \int_0^{\infty} \frac{x \sin x}{x^2 + a^2} dx = \frac{\pi e^{-a}}{2}.$$

Problem 6. Prove that $\int_0^{\infty} \frac{\sin x}{x} dx = \frac{\pi}{2}.$

Solution. Let $f(z) = \frac{e^{iz}}{z}.$

The only singular point of $f(z)$ is 0 which is a simple pole and it lies on the real axis. Now choose the contour C as shown in the figure.



$$\begin{aligned} \text{Then } \int_C f(z) dz &= \int_{-r}^{-\epsilon} f(x) dx + \int_{C_2} f(z) dz \\ &\quad + \int_{\epsilon}^r f(x) dx + \int_{C_1} f(z) dz. \end{aligned} \quad \dots (1)$$

Since $f(z)$ is analytic within C , $\int_C f(z) dz = 0$... (2)

$$\begin{aligned}\text{Also } \int_{C_2} f(z) dz &= -\pi i \operatorname{Res}\{f(z); 0\} \\ &= -\pi i e^0 \\ &= -\pi i. \quad \dots (3)\end{aligned}$$

Further the integral over C_1 tends to 0 as $r \rightarrow \infty$.

Hence using (2) and (3) in (1) and taking limit as $r \rightarrow \infty$ we get

$$0 = \int_{-\infty}^0 f(x) dx - \pi i + \int_0^{\infty} f(x) dx.$$

$$\therefore \int_{-\infty}^{\infty} f(x) dx = \pi i.$$

Equating the imaginary parts we get $\int_{-\infty}^{\infty} \frac{\sin x}{x} dx = \pi$.

$$\therefore \int_0^{\infty} \frac{\sin x}{x} dx = \frac{\pi}{2} \text{ (since } \frac{\sin x}{x} \text{ is an even function).}$$

Exercises

1. Establish the following with the help of residues.

$$(i) \int_{-\infty}^{\infty} \frac{\sin x dx}{x^2 + 4x + 5} = -\frac{\pi \sin 2}{e}$$

$$(ii) \int_{-\infty}^{\infty} \frac{\sin x dx}{x^2 - 2x + 5} = \frac{\pi \sin 1}{2e^2}$$

$$(iii) \int_0^{\infty} \frac{\cos ax dx}{(x^2 + b^2)^2} = \frac{\pi}{4b^3} (1 + ab) e^{-ab} \quad (a > 0, b > 0).$$

$$(iv) \int_0^{\infty} \frac{x \sin x dx}{x^2 + a^2} = \frac{\pi e^{-a}}{2} \quad (a > 0).$$