7. Series Expansions

7.0. Introduction

In this chapter we consider the problem of representing a given function as a power series. We prove that if a function is analytic at a point z_0 then it can be expanded as a power series called *Taylor's series* consisting of non-negative powers of $z - z_0$ and the expansion is valid in some neighbourhood of z_0. We also prove that a function $f(z)$ which is analytic in an annular region $a < |z - z_0| < b$ can be expanded as a series called *Laurent's series* consisting of positive and negative powers of $z - z_0$. We also introduce the concept of *singular points* of a function and classify the singular points and discuss the behaviour of the function in the neighbourhood of a singularity.

7.1. Taylor's Series

Theorem 7.1. (Taylor's theorem)

Let $f(z)$ be analytic in a region D containing z_0. Then $f(z)$ can be represented as a power series in $z - z_0$ given by

$$f(z) = f(z_0) + \frac{f'(z_0)}{1!}(z - z_0) + \frac{f''(z_0)}{2!}(z - z_0)^2 + \cdots$$

$$\cdots + \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n + \cdots\cdots$$

The expansion is valid in the largest open disc with centre z_0 contained in D.

Proof. Let $r > 0$ be such that the disc $|z - z_0| < r$ is contained in D.

Let $0 < r_1 < r$. Let C_1 be the circle $|z - z_0| = r_1$.

By Cauchy's integral formula we have

$$f(z) = \frac{1}{2\pi i} \int_{C_1} \frac{f(\zeta)}{(\zeta - z)} d\zeta \qquad \ldots (1)$$

Also by theorem on higher derivatives we have

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_{C_1} \frac{f(\zeta) dr}{(\zeta - z)^{n+1}} \qquad \ldots (2)$$

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Now, $$\frac{1}{\zeta - z} = \frac{1}{(\zeta - z_0) - (z - z_0)}$$

$$= \frac{1}{(\zeta - z_0)\left[1 - \frac{z - z_0}{\zeta - z_0}\right]}$$

$$= \frac{1}{\zeta - z_0}\left[1 + \left(\frac{z - z_0}{\zeta - z_0}\right) + \left(\frac{z - z_0}{\zeta - z_0}\right)^2 + \cdots\cdots + \left(\frac{z - z_0}{\zeta - z_0}\right)^{n-1} + \frac{\left(\frac{z - z_0}{\zeta - z_0}\right)^n}{1 - \left(\frac{z - z_0}{\zeta - z_0}\right)}\right]$$

(using the identify $\frac{1}{1-\alpha} = 1 + \alpha + \alpha^2 + \ldots + \alpha^{n-1} + \frac{\alpha^n}{1-\alpha}$)

$$= \frac{1}{\zeta - z_0} + \frac{z - z_0}{(\zeta - z_0)^2} + \frac{(z - z_0)^2}{(\zeta - z_0)^3} + \cdots\cdots + \frac{(z - z_0)^{n-1}}{(\zeta - z_0)^n} + \frac{(z - z_0)^n}{(\zeta - z_0)^n(\zeta - z)}.$$

Now, multiplying throughout by $\frac{f(\zeta)}{2\pi i}$, integrating over C_1 and using (1) and (2) we get

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + \frac{f''(z_0)}{2!}(z - z_0)^2 + \cdots$$

$$\cdots + \frac{f^{(n-1)}(z_0)}{(n-1)!}(z - z_0)^{n-1} + R_n \cdots\cdots (3)$$

where $R_n = \frac{(z - z_0)^n}{2\pi i}\int_{C_1} \frac{f(\zeta)d\zeta}{(\zeta - z)(\zeta - z_0)^n}$.

Here ζ lies on C_1 and z lies in the interior of C_1 so that $|\zeta - z_0| = r_1$ and $|z - z_0| < r_1$.

$$\therefore \quad |\zeta - z| = |(\zeta - z_0) - (z - z_0)| \geq |\zeta - z_0| - |z - z_0| = r_1 - |z - z_0|.$$

$$\therefore \quad \frac{1}{|\zeta - z|} \leq \frac{1}{r_1 - |z - z_0|}.$$

Let M denote the maximum value of $|f(z)|$ on C_1.

Then $|R_n| \leq \frac{|z - z_0|^n}{2\pi}\frac{M(2\pi r_1)}{(r_1 - |z - z_0|)r_1^n}$ (by theorem 6.2)

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$$= \frac{M|z - z_0|}{(r_1 - |z - z_0|)} \left(\frac{|z - z_0|}{r_1} \right)^{n-1}$$

Also $\left| \frac{z - z_0}{r_1} \right| < 1$. Hence $\lim_{n \to \infty} R_n = 0$.

$\therefore$ Taking limit as $n \to \infty$ in (3) we get

$$f(z) = f(z_0) + \frac{f'(z_0)}{1!}(z - z_0) + \frac{f''(z_0)}{2!}(z - z_0)^2 + \cdots$$

$$+ \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n + \cdots$$

Note 1. The above series is called the **Taylor series** of $f(z)$ about the point z_0. Thus if $f(z)$ is analytic at a point z_0 then $f(z)$ can be represented as a Taylor's series about z_0, which is a series in non negative powers of $z - z_0$. The expansion is valid in some neighbourhood of z_0.

Note 2. The Taylor series expansion of $f(z)$ about the point zero is called the **Maclaurin's series.** Thus the Maclaurin's series of $f(z)$ is given by

$$f(z) = f(0) + \frac{z}{1!} f'(0) + \frac{z^2}{2!} f''(0) + \cdots + \frac{z^n}{n!} f^{(n)}(0) + \cdots$$

Example 1. The Taylor's series for $f(z) = \frac{1}{z}$ about $z = 1$ is given by

$$\frac{1}{z} = f(1) + \frac{f'(1)}{1!}(z - 1) + \frac{f''(1)}{2!}(z - 1)^2 + \frac{f'''(1)}{3!}(z - 1)^3 + \cdots$$

$$\text{Now, } f(z) = \frac{1}{z} \Rightarrow f(1) = 1$$

$$f'(z) = -\frac{1}{z^2} \Rightarrow f'(1) = -1$$

$$f''(z) = \frac{2}{z^3} \Rightarrow f''(1) = 2$$

$$f'''(z) = -\frac{6}{z^4} \Rightarrow f'''(1) = -6$$

....

Hence the Taylor's series expansion for $\frac{1}{z}$ about 1 is

$$\frac{1}{z} = 1 - (z - 1) + (z - 1)^2 - (z - 1)^3 + \cdots$$

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This expansion is valid in the disc $|z-1|<1$.

Similarly the Taylor's series for $f(z)=\frac{1}{z}$ about $z=i$ is given by

$$\frac{1}{z}=\frac{1}{i}-\frac{z-i}{i^2}+\frac{(z-i)^2}{i^3}-\frac{(z-i)^3}{i^4}+\cdots$$

and the expansion is valid in the disc $|z-i|<1$. (verify)

Example 2. Let $f(z)=e^z$.

Then $f^{(n)}(z)=e^z$ for all n and hence $f^{(n)}(0)=1$.

Hence the Maclaurin's series for e^z is given by

$$e^z=1+\frac{z}{1!}+\frac{z^2}{2!}+\frac{z^3}{3!}+\cdots+\frac{z^n}{n!}+\cdots$$

and the expansion is valid in the entire complex plane.

Maclaurin's series expansion of some of the standard functions are given below.

1. $e^{-z}=1-\frac{z}{1!}+\frac{z^2}{2!}-\cdots+(-1)^n\frac{z^n}{n!}+\cdots\ (|z|<\infty)$
2. $\sin z=z-\frac{z^3}{3!}+\frac{z^5}{5!}-\cdots+(-1)^{n-1}\frac{z^{2n-1}}{(2n-1)!}+\cdots\ (|z|<\infty)$
3. $\cos z=1-\frac{z^2}{2!}+\frac{z^4}{4!}-\cdots+(-1)^{n-1}\frac{z^{2n-2}}{(2n-2)!}+\cdots\ (|z|<\infty)$
4. $\sinh z=\frac{z}{1!}+\frac{z^3}{3!}+\frac{z^5}{5!}+\cdots+\frac{z^{2n-1}}{(2n-1)!}+\cdots\ (|z|<\infty)$
5. $\cosh z=1+\frac{z^2}{2!}+\frac{z^4}{4!}+\cdots+\frac{z^{2n}}{(2n)!}+\cdots\ (|z|<\infty)$
6. $\frac{1}{1+z}=1-z+z^2-z^3+\cdots+(-1)^nz^n+\cdots\ (|z|<1)$
7. $\frac{1}{1-z}=1+z+z^2+z^3+\cdots+z^n+\cdots\ (|z|<1)$
8. $\log(1+z)=z-\frac{z^2}{2}+\frac{z^3}{3}-\cdots(-1)^{n-1}\frac{z^n}{n}+\cdots\ (|z|<1)$
9. $\log(1-z)=-z-\frac{z^2}{2}-\frac{z^3}{3}-\cdots-\frac{z^n}{n}-\cdots\ (|z|<1).$

Solved problems

Problem 1. Expand $\cos z$ into a Taylor's series about the point $z=\pi/2$ and determine the region of convergence.

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Solution. Let $f(z) = \cos z$

The Taylor's series for $f(z)$ about $z = \pi/2$ is

$$f(z)=f(\pi/2)+\frac{(z-\pi/2)}{1!}f'(\pi/2)+\frac{(z-\pi/2)^2}{2!}f''(\pi/2)+\frac{(z-\pi/2)^3}{3!}f'''(\pi/2)+\cdots$$

Now $f(z) = \cos z$. Hence $f(\pi/2) = 0$.

$f'(z) = -\sin z$. Hence $f'(\pi/2) = -1$.

$f''(z) = -\cos z$. Hence $f''(\pi/2) = 0$

$f'''(z) = \sin z$. Hence $f'''(\pi/2) = 1$.

....

$\therefore$ The Taylor's series for $\cos z$ about $z = \pi/2$ is

$$\cos z = -\frac{(z-\pi/2)}{1!}+\frac{(z-\pi/2)^3}{3!}-\frac{(z-\pi/2)^5}{5!}+\cdots$$

The expansion is valid throughout the complex plane.

Problem 2. Expand $f(z) = \sin z$ in a Taylor's series about $z = \pi/4$ and determine the region of convergence of this series.

Solution. The Taylor's series for $f(z)$ about $z = \pi/4$ is

$$f(z) = f(\pi/4)+\frac{(z-\pi/4)}{1!}f'(\pi/4)+\frac{(z-\pi/4)^2}{2!}f''(\pi/4)+\cdots$$

Here $f(z) = \sin z$. Hence $f(\pi/4) = \frac{1}{\sqrt{2}}$.

$f'(z) = \cos z$. Hence $f'(\pi/4) = \frac{1}{\sqrt{2}}$.

$f''(z) = -\sin z$. Hence $f''(\pi/4) = -\frac{1}{\sqrt{2}}$.

$f'''(z) = -\cos z$. Hence $f'''(\pi/4) = -\frac{1}{\sqrt{2}}$.

The Taylor's series for $\sin z$ about $z = \pi/4$ is

$$\sin z = \frac{1}{\sqrt{2}}+\frac{(z-\pi/4)}{1!}\left(\frac{1}{\sqrt{2}}\right)-\frac{(z-\pi/4)^2}{2!}\left(\frac{1}{\sqrt{2}}\right)+\ldots\ldots$$

$$= \frac{1}{\sqrt{2}}\left[1+\frac{(z-\pi/4)}{1!}-\frac{(z-\pi/4)^2}{2!}-\frac{(z-\pi/4)^3}{3!}+\cdots\right]$$

The expansion is valid in the entire complex plane.

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Problem 3. Expand $f(z) = \dfrac{z-1}{z+1}$ as a Taylor's series

(i) about the point $z = 0$.

(ii) about the point $z = 1$. Determine the region of convergence in each case.

Solution.

(i) $$f(z) = \frac{z-1}{z+1}$$

$$= (z-1)(1+z)^{-1}$$

$$= (z-1)(1 - z + z^2 - z^3 + \cdots\cdots) \text{ if } |z| < 1$$

$$= (z - z^2 + z^3 - \cdots\cdots) - (1 - z + z^2 - z^3 \cdots\cdots)$$

$$= -1 + 2z - 2z^2 + 2z^3 + \cdots\cdots$$

The region of convergence is $|z| < 1$.

(ii) $$f(z) = \frac{z-1}{z+1}$$

$$= \frac{z-1}{(2+z-1)}$$

$$= \frac{z-1}{2\left(1+\frac{z-1}{2}\right)}$$

$$= \frac{z-1}{2}\left(1+\frac{z-1}{2}\right)^{-1}$$

$$= \frac{z-1}{2}\left[1 - \frac{z-1}{2} + \left(\frac{z-1}{2}\right)^2 - \left(\frac{z-1}{2}\right)^3 + \cdots\cdots\right] \text{ if } \left|\frac{z-1}{2}\right| < 1$$

$$= \frac{z-1}{2} - \frac{(z-1)^2}{2^2} + \frac{(z-1)^3}{2^3} - \cdots\cdots$$

The region of convergence is given by $\left|\dfrac{z-1}{2}\right| < 1$ which is same as the circular disc $|z-1| < 2$.

Problem 4. Show that

(i) $$\frac{1}{z^2} = 1 + \sum_{n=1}^{\infty}(n+1)(z+1)^n \text{ when } |z+1| < 1.$$

(ii) $$\frac{1}{z^2} = \frac{1}{4} + \frac{1}{4}\sum_{n=1}^{\infty}(-1)^n(n+1)\left(\frac{z-2}{2}\right)^n \text{ when } |z-2| < 2.$$

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Solution. (i) $\dfrac{1}{z^2} = \dfrac{1}{[1-(z+1)]^2}$

$$= [1-(z+1)]^{-2}$$

$$= 1 + 2(z+1) + 3(z+1)^2 + 4(z+1)^3 + \cdots\cdots \text{ if } |z+1| < 1$$

$$= 1 + \sum_{n=1}^{\infty}(n+1)(z+1)^n \text{ when } |z+1| < 1.$$

(ii) $\dfrac{1}{z^2} = \dfrac{1}{(z-2+2)^2}$

$$= \frac{1}{\left[2\left(1+\frac{z-2}{2}\right)\right]^2}$$

$$= \frac{1}{4}\left(1+\frac{z-2}{2}\right)^{-2}$$

$$= \frac{1}{4}\left[1 - 2\left(\frac{z-2}{2}\right) + 3\left(\frac{z-2}{2}\right)^2 - \cdots\right] \text{ if } \left|\frac{z-2}{2}\right| < 1$$

$$= \frac{1}{4} - \frac{1}{4}\times 2\left(\frac{z-2}{2}\right) + \frac{1}{4}\times 3\left(\frac{z-2}{2}\right)^2 - \cdots$$

$$= \frac{1}{4} + \frac{1}{4}\sum_{n=1}^{\infty}(-1)^n(n+1)\left(\frac{z-2}{2}\right)^n.$$

Here the region of convergence is $\left|\dfrac{z-2}{2}\right| < 1$ which is the same as the circular disc $|z-2| < 2$.

Problem 5. Expand ze^{2z} in a Taylor's series about $z = -1$ and determine the region of convergence.

Solution. Let $f(z) = ze^{2z}$

$$= ze^{2(z+1)}e^{-2}$$

$$= \frac{1}{e^2}\left[(z+1)e^{2(z+1)} - e^{2(z+1)}\right]$$

$$= \frac{1}{e^2}\left[(z+1)\left\{1 + \frac{2(z+1)}{1!} + \frac{4(z+1)^2}{2!} + \cdots\right\} - \left\{1 + \frac{2(z+1)}{1!} + \frac{4(z+1)^2}{2!} + \cdots\right\}\right]$$

$$= \frac{1}{e^2}\left[\left\{(z+1) + \frac{2(z+1)^2}{1!} + \frac{2^2(z+1)^3}{2!} + \cdots\right\} - \left\{1 + \frac{2(z+1)}{1!} + \frac{2^2(z+1)^2}{2!} + \cdots\right\}\right]$$

$$= \frac{1}{e^2}\left[-1 + \left(1 - \frac{2}{1!}\right)(z+1) + \left(\frac{2}{1!} - \frac{2^2}{2!}\right)(z+1)^2 + \left(\frac{2^2}{2!} - \frac{2^3}{3!}\right)(z+1)^3 + \ldots\right.$$

The expansion is valid throughout the complex plane.

Problem 6. Find the Taylor's series to represent $\dfrac{z^2-1}{(z+2)(z+3)}$ in $|z| < 2$

Solution. By partial fractions

$$\frac{z^2-1}{(z+2)(z+3)} = 1 + \frac{3}{z+2} - \frac{8}{z+3} \quad \text{(verify)}$$

$$= 1 + \frac{3}{2\left(1+\frac{z}{2}\right)} - \frac{8}{3\left(1+\frac{z}{3}\right)}$$

$$= 1 + \frac{3}{2}\left(1+\frac{z}{2}\right)^{-1} - \frac{8}{3}\left(1+\frac{z}{3}\right)^{-1}$$

$$= 1 + \frac{3}{2}\left(1 - \frac{z}{2} + \frac{z^2}{2^2} - \frac{z^3}{2^3} + \cdots\right) - \frac{8}{3}\left(1 - \frac{z}{3} + \frac{z^2}{3^2} - \frac{z^3}{3^3} + \cdots\right)$$

$$= \left(1 + \frac{3}{2} - \frac{8}{3}\right) + \left(-\frac{3}{2^2} + \frac{8}{3^2}\right)z + \left(\frac{3}{2\cdot 2^2} - \frac{8}{3\cdot 3^2}\right)z^2 + \cdots$$

$$= -\frac{1}{6} + \sum_{n=1}^{\infty}(-1)^{n+1}\left(\frac{8}{3^{n+1}} - \frac{3}{2^{n+1}}\right)z^n$$

and the expansion is valid in $|z| < 2$.

Exercises.

1. Verify all the ten expansions in 7.1
2. Expand $\dfrac{1}{z}$ about $z = -1$ and $z = 2$ as Taylor's series, stating the region of convergence.
3. Show that $\dfrac{1}{z^2} = 1 - 2(z-1) + 3(z-1)^2 - 4(z-1)^3 + \cdots$ for all z in $|z-1| < 1$.

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4. Expand $\dfrac{z}{z-3}$ as a Taylor's series about $z = 1$.

5. Find the Maclaurin's series for $\dfrac{1}{2-z}$. What is its radius of convergence.

6. Obtain the Taylor's series to represent $\dfrac{1}{(z+1)(z+3)}$ in $|z| < 1$.

7. Obtain the Taylor's series for $\dfrac{z}{z+2}$ about $z = 1$. State the region of validity.

8. Find the Taylor's series for ze^z about $z = 1$.

9. Show that $\sin z^2 = z^2 - \dfrac{z^6}{3!} + \dfrac{z^{10}}{5!} - \dfrac{z^{14}}{7!} + \cdots\cdots$ for $|z| < \infty$.

Answers.

2. $\dfrac{1}{z} = -1 - (z-1) - (z-1)^2 + \cdots;\ |z+1| < 1$

$\dfrac{1}{z} = \dfrac{1}{2} - \dfrac{z-2}{2^2} + \dfrac{(z-2)^2}{2^3} - \dfrac{(z-2)^3}{2^4} + \cdots;\ |z-2| < 2$

4. $-\dfrac{1}{2} - \dfrac{3}{2}\sum_{n=1}^{\infty}\left(\dfrac{z-1}{2}\right)^n$ 5. $\sum_{0}^{\infty}\dfrac{z^n}{2^{n+1}}$; 2

6. $\dfrac{1}{2}\sum_{n=0}^{\infty}(-1)^n\left[1 - \dfrac{1}{3^{n+1}}\right]z^n$

7. $\dfrac{1}{3} + 2\sum_{n=1}^{\infty}(-1)^{n+1}\dfrac{(z-1)^n}{3^{n+1}};\ |z-1| < 3$

8. $e\left[1 + \dfrac{2(z-1)}{1!} + \dfrac{3(z-1)^2}{2!} + \dfrac{4(z-1)^3}{3!} + \cdots\cdots\right]$

7.2. Laurent's Series

$$\text{A series of the form } \sum_{n=1}^{\infty}\frac{b_n}{z_n} \qquad \ldots (1)$$

can be considered as an ordinary power series in the variable $\dfrac{1}{z}$. Hence if the radius of convergence of the power series $\sum_{n=1}^{\infty} b_n z^n$ is r and $r < \infty$ the the series $\sum_{n=1}^{\infty}\dfrac{b_n}{z^n}$ converges

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in the region $|z| > r$. The convergence is uniform in every region $|z| \geq \rho > r$ and the series represents an analytic function in $|z| > r$.

If the series (1) is combined with the usual power series we get a more general series of the form $\sum_{-\infty}^{\infty} a_n z^n$. ... (2)

This series is said to converge at a point if the part of the series consisting of the negative powers of z and the part of the series consisting of non-negative powers of z are separately convergent. We know that the series consisting of non-negative powers of z converges in a disc $|z| < r_2$ and the series consisting of negative powers of z converges in a region $|z| > r_1$.

$\therefore$ If $r_1 < r_2$ the series represented by (2) converges in the region $r_1 < |z| < r_2$ and in this *annulus region* it represents an analytic function.

We shall now prove that the converse situation is also true.

(i.e) any function which is analytic in a region containing the annulus $r_1 < |z - z_0| < r_2$ can be represented in a series of the form $\sum_{-\infty}^{\infty} a_n (z - z_0)^n$.

Theorem 7.2. (Laurent's theorem)

Let C_1 and C_2 denote respectively the concentric circles $|z - z_0| = r_1$ and $|z - z_0| = r_2$ with $r_1 < r_2$. Let $f(z)$ be analytic in a region containing the circular annulus $r_1 < |z - z_0| < r_2$. Then $f(z)$ can be represented as a convergent series of positive and negative powers of $z - z_0$ given by

$$f(z) = \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n} + \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

$$\text{where } b_n = \frac{1}{2\pi i} \int_{C_1} \frac{f(\zeta) d\zeta}{(\zeta - z_0)^{-n+1}} \text{ and } a_n = \frac{1}{2\pi i} \int_{C_2} \frac{f(\zeta) d\zeta}{(\zeta - z_0)^{n+1}}$$

Proof. Let z be any point in the circular annulus $r_1 < |z - z_0| < r_2$.

Then by theorem 6.9 we have, $f(z) = \frac{1}{2\pi i} \int_{C_2} \frac{f(\zeta) d\zeta}{\zeta - z} - \frac{1}{2\pi i} \int_{C_1} \frac{f(\zeta) d\zeta}{\zeta - z}$.

$$\therefore \quad f(z) = \frac{1}{2\pi i} \int_{C_2} \frac{f(\zeta) d\zeta}{\zeta - z} + \frac{1}{2\pi i} \int_{C_1} \frac{f(\zeta) d\zeta}{z - \zeta} \quad \dots (1)$$

As in the proof of Taylor's theorem, we have

$$\frac{1}{2\pi i} \int_{C_2} \frac{f(\zeta)}{\zeta - z} d\zeta = a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \dots\dots$$

$$\dots\dots + a_{n-1}(z - z_0)^{n-1} + R_n(z) \quad \dots (2)$$

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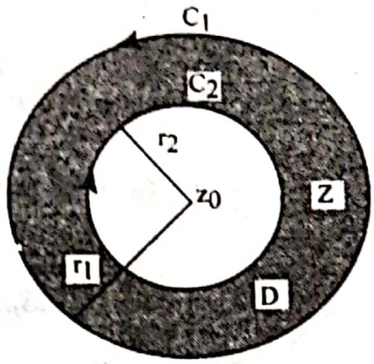
where $a_n = \frac{1}{2\pi i}\int\limits_{C_2} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} d\zeta$ and

$$R_n(z) = \frac{(z-z_0)^n}{2\pi i}\int\limits_{C_2} \frac{f(\zeta)d\zeta}{(\zeta - z_0)^n(\zeta - z)}$$

Now, $$\frac{1}{z-\zeta} = \frac{1}{z - z_0 + z_0 - \zeta}$$

$$= \frac{1}{(z-z_0) - (\zeta - z_0)}$$

$$= \frac{1}{(z-z_0)\left[1 - \frac{\zeta - z_0}{z - z_0}\right]}$$



$$= \frac{1}{z-z_0}\left[1 + \left[\frac{\zeta - z_0}{z-z_0}\right] + \left[\frac{\zeta - z_0}{z-z_0}\right]^2 + \cdots\cdots + \left(\frac{\zeta - z_0}{z-z_0}\right)^{n-1} + \frac{\left(\frac{\zeta - z_0}{z-z_0}\right)^n}{1 - \left(\frac{\zeta - z_0}{z-z_0}\right)}\right]$$

Multiplying by $\frac{f(\zeta)}{2\pi i}$ and integrating over C_1 we get

$$\int\limits_{C_1} \frac{f(\zeta)d\zeta}{z-\zeta} = \frac{b_1}{z-z_0} + \frac{b_2}{(z-z_0)^2} + \cdots\cdots + \frac{b_{n-1}}{(z-z_0)^{n-1}} + S_n(z) \qquad \ldots (3).$$

where $b_n = \frac{1}{2\pi i}\int\limits_{C_1} \frac{f(\zeta)d\zeta}{(\zeta - z_0)^{-n+1}}$; $S_n = \frac{1}{2\pi i(z-z_0)^n}\int\limits_{C_1} \frac{f(\zeta)(\zeta - z_0)^n d\zeta}{z - \zeta}$

From (1), (2) and (3) we get

$$f(z) = a_0 + a_1(z-z_0) + \cdots\cdots + a_{n-1}(z-z_0)^{n-1}$$

$$+ \frac{b_1}{z-z_0} + \frac{b_2}{(z-z_0)^2} + \cdots\cdots + \frac{b_{n-1}}{(z-z_0)^{n-1}} + R_n(z) + S_n(z) \qquad \ldots (4)$$

The required result follows if we can prove that $R_n \to 0$ and $S_n \to 0$ as $n \to \infty$.

Now, if $\zeta \in C_1$ then $|\zeta - z_0| = r_1$ and

$$|z - \zeta| = |(z-z_0) - (\zeta - z_0)| \geq |z - z_0| - r_1.$$

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If $\zeta \in C_2$ then $|\zeta - z_0| = r_2$ and

$$|\zeta - z| = |(\zeta - z_0) - (z - z_0)| \geq r_2 - |z - z_0|$$

Now let M denote the maximum value of $|f(z)|$ in $C_1 \cup C_2$.

$$\text{Then } |R_n| \leq \frac{|z - z_0|^n}{2\pi} \frac{M(2\pi r_2)}{r_2^n(r_2 - |z - z_0|)} \qquad \text{(by theorem 6.2)}$$

$$\leq \frac{M|z - z_0|}{(r_2 - |z - z_0|)} \left(\frac{|z - z_0|}{r_2}\right)^{n-1}.$$

Since $\dfrac{|z - z_0|}{r_2} < 1$, $R_n \to 0$ as $n \to \infty$.

$$\text{Also} \quad |S_n| \leq \frac{1}{|z - z_0|^n \, 2\pi} \frac{M r_1^n (2\pi r_1)}{(|z - z_0| - r_1)}$$

$$\leq \frac{M r_1}{(|z - z_0| - r_1)} \left(\frac{r_1}{|z - z_0|}\right)^n$$

Since $\dfrac{r_1}{|z - z_0|} < 1$, $S_n \to 0$ as $n \to \infty$.

Hence, by taking limit $n \to \infty$ in (4) we get

$$f(z) = \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n} + \sum_{n=0}^{\infty} a_n (z - z_0)^n.$$

Hence the theorem.

Remark. The formulae for the coefficients a_n and b_n in the Laurent's series expansion are given by $a_n = \dfrac{1}{2\pi i} \displaystyle\int_{C_2} \frac{f(\zeta)d\zeta}{(\zeta - z_0)^{n+1}}$ ... (1)

$$\text{and} \quad b_n = \frac{1}{2\pi i} \int_{C_1} \frac{f(\zeta)d\zeta}{(\zeta - z_0)^{-n+1}} \qquad \text{... (2)}$$

Since the integrands in the integrals of (1) and (2) are analytic functions of ζ throughout the annular region, any simple closed curve C in the annulus can be used as the path of integration in place of C_1 and C_2.

Hence Laurent's series can be written as

$$f(z) = \sum_{-\infty}^{\infty} A_n (z - z_0)^n, \; (r_1 < |z - z_0| < r_2) \text{ where } A_n = \frac{1}{2\pi i} \int_C \frac{f(\zeta)d\zeta}{(\zeta - z_0)^{n+1}}.$$

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Solved Problems

Problem 1. Find the Laurent's series expansion of $f(z) = z^2 e^{1/z}$ about $z = 0$.

Solution. $f(z) = z^2 e^{1/z}$.

Clearly $f(z)$ is analytic at all points $z \neq 0$.

$$\text{Now, } f(z) = z^2 \left[1 + \frac{1}{z} + \frac{1}{2!z^2} + \frac{1}{3!z^3} + \cdots\right]$$
$$= z^2 + z + \frac{1}{2} + \frac{1}{3!z} + \frac{1}{4!z^2} + \cdots.$$

This is the required Laurent's series expansion for $f(z)$ at $z = 0$.

Problem 2. Expand $\dfrac{-1}{(z-1)(z-2)}$ as a power series in z in the regions

(i) $|z| < 1$ **(ii)** $1 < |z| < 2$ **(iii)** $|z| > 2$.

Solution. Let $f(z) = \dfrac{-1}{(z-1)(z-2)}$.

By splitting into partial fractions, we have $f(z) = \dfrac{1}{z-1} - \dfrac{1}{z-2}$.

(i) The only points where $f(z)$ is not analytic are 1 and 2. Hence $f(z)$ is analytic in $|z| < 1$ and hence can be represented as a Taylor's series in $|z| < 1$.

$$\therefore\ f(z) = \frac{1}{z-1} - \frac{1}{z-2}$$
$$= -\frac{1}{1-z} + \frac{1}{2-z}$$
$$= -(1-z)^{-1} + \frac{1}{2}\left(1 - \frac{z}{2}\right)^{-1}$$
$$= -(1 + z + z^2 + \cdots + z^n + \cdots)$$
$$+ \frac{1}{2}\left(1 + \frac{z}{2} + \frac{z^2}{4} + \cdots + \frac{z^n}{2^n} + \cdots\right)$$
$$= \sum_{n=0}^{\infty}\left[-z^n + \frac{1}{2}\left(\frac{z}{2}\right)^n\right]$$
$$= \sum_{n=0}^{\infty}\left(\frac{1}{2^{n+1}} - 1\right) z^n.$$

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(ii) $f(z)$ is analytic in the annular region $1 < |z| < 2$ and hence can be expanded as a Laurent's series in this region.

$$f(z) = \frac{1}{z-1} - \frac{1}{z-2}$$

$$= \frac{1}{z\left(1-\frac{1}{z}\right)} + \frac{1}{2\left(1-\frac{z}{2}\right)}$$

$$= \frac{1}{z}\left(1-\frac{1}{z}\right)^{-1} + \frac{1}{2}\left(1-\frac{z}{2}\right)^{-1}$$

$$= \frac{1}{z}\left[1+\left(\frac{1}{z}\right)+\left(\frac{1}{z}\right)^2+\cdots\right]+\frac{1}{2}\left[1+\left(\frac{z}{2}\right)+\left(\frac{z}{2}\right)^2+\cdots\right]$$

$$\left(\text{since } \left|\frac{1}{z}\right| < 1 \text{ and } \left|\frac{z}{2}\right| < 1\right)$$

$$= \sum_{n=0}^{\infty} \frac{1}{z^{n+1}} + \sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}}.$$

This gives the Laurent's series expansion in $1 < |z| < 2$.

(iii) $f(z)$ is analytic in the domain $|z| > 2$ and in this domain we have $|2/z| < 1$. Hence

$$f(z) = \frac{1}{z}\left[\frac{1}{1-(1/z)}\right] - \frac{1}{z}\left[\frac{1}{1-(2/z)}\right]$$

$$= \frac{1}{z}[1-(1/z)]^{-1} - \frac{1}{z}[1-(2/z)]^{-1}$$

$$= \frac{1}{z}\left[\left(1+\left(\frac{1}{z}\right)+\left(\frac{1}{z}\right)^2+\cdots\right)-\left(1+\left(\frac{2}{z}\right)+\left(\frac{2}{z}\right)^2+\cdots\right)\right]$$

$$= \sum_{n=0}^{\infty} \frac{1-2^n}{z^{n+1}}.$$

Problem 3. Expand $\frac{1}{z(z-1)}$ as Laurent's series (i) about $z = 0$ in powers of z and (ii) about $z = 1$ in powers $z - 1$. Also state the region of validity.

Solution. (i) The only points where $f(z)$ is not analytic are 0 and 1. Hence $f(z)$ can be expanded as a Laurent's series in the annulus $0 < |z| < 1$.

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$$f(z) = \frac{1}{z(z-1)}$$

$$= -\frac{1}{z}(1-z)^{-1}$$

$$= -\frac{1}{z}(1 + z + z^2 + \cdots + z^n + \cdots) \text{ (since } |z| < 1)$$

$$= -\left(\frac{1}{z} + 1 + z + z^2 + \cdots + z^n + \cdots\right).$$

This is the Laurent's series expansion of $f(z)$ in $0 < |z| < 1$.

(ii) $f(z)$ is analytic in $0 < |z-1| < 1$ and hence can be expanded as a Laurent's series in powers of $z-1$ in this region.

$$\frac{1}{z(z-1)} = \frac{1}{z-1}\left[\frac{1}{1+(z-1)}\right]$$

$$= \frac{1}{z-1}[1+(z-1)]^{-1}$$

$$= \frac{1}{z-1}[1-(z-1)+(z-1)^2-(z-1)^3+\cdots]$$

(since $|z-1| < 1$)

$$= \frac{1}{z-1} - 1 + (z-1) - (z-1)^2 + \cdots.$$

This gives the Laurent's series expansion in $0 < |z-1| < 1$.

Problem 4. Find the Laurent's series for $\dfrac{z}{(z+1)(z+2)}$ about $z = -2$.

Solution. Let $f(z) = \dfrac{z}{(z+1)(z+2)}$

$$= \frac{-1}{z+1} + \frac{2}{z+2} \text{ (verify)}$$

$$= \frac{-1}{(z+2)-1} + \frac{2}{z+2}$$

$$= [1-(z+2)]^{-1} + \frac{2}{z+2}$$

$$= [1+(z+2)+(z+2)^2+\cdots] + \frac{2}{z+2}$$

$$= \frac{2}{z+2} + 1 + (z+2) + (z+2)^2 + \cdots.$$

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$$
\begin{aligned}
f(z) &= \frac{1}{z(z-1)} \\
&= -\frac{1}{z}(1-z)^{-1} \\
&= -\frac{1}{z}(1+z+z^2+\cdots+z^n+\cdots) \text{ (since } |z|<1) \\
&= -\left(\frac{1}{z}+1+z+z^2+\cdots+z^n+\cdots\right).
\end{aligned}
$$

This is the Laurent's series expansion of $f(z)$ in $0 < |z| < 1$.

(ii) $f(z)$ is analytic in $0 < |z-1| < 1$ and hence can be expanded as a Laurent's series in powers of $z-1$ in this region.

$$
\begin{aligned}
\frac{1}{z(z-1)} &= \frac{1}{z-1}\left[\frac{1}{1+(z-1)}\right] \\
&= \frac{1}{z-1}[1+(z-1)]^{-1} \\
&= \frac{1}{z-1}[1-(z-1)+(z-1)^2-(z-1)^3+\cdots] \\
&\qquad \text{(since } |z-1|<1) \\
&= \frac{1}{z-1} - 1 + (z-1) - (z-1)^2 + \cdots.
\end{aligned}
$$

This gives the Laurent's series expansion in $0 < |z-1| < 1$.

Problem 4. Find the Laurent's series for $\dfrac{z}{(z+1)(z+2)}$ about $z = -2$.

Solution. Let
$$
\begin{aligned}
f(z) &= \frac{z}{(z+1)(z+2)} \\
&= \frac{-1}{z+1} + \frac{2}{z+2} \text{ (verify)} \\
&= \frac{-1}{(z+2)-1} + \frac{2}{z+2} \\
&= [1-(z+2)]^{-1} + \frac{2}{z+2} \\
&= [1+(z+2)+(z+2)^2+\cdots] + \frac{2}{z+2}
\end{aligned}
$$

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Problem 5. Expand $f(z) = \dfrac{z}{(z-1)(2-z)}$ in a Laurent's series valid for (i) $|z| < 1$ (ii) $1 < |z| < 2$ (iii) $|z| > 2$ (iv) $|z-1| > 1$ and (v) $0 < |z-2| < 1$.

Solution. Let $f(z) = \dfrac{z}{(z-1)(2-z)}$.

$$\therefore \; f(z) = \frac{1}{z-1} + \frac{2}{2-z} \text{ (by partial fractions).}$$

(i) $|z| < 1$.

$$f(z) = \frac{-1}{1-z} + \frac{2}{2(1-z/2)} = -(1-z)^{-1} + (1-z/2)^{-1}.$$

Since $|z| < 1$, $f(z)$ can be expanded in series as

$$f(z) = -[1+z+z^2+z^3+\cdots] + \left[1 + \left(\frac{z}{2}\right) + \left(\frac{z}{2}\right)^2 + \left(\frac{z}{2}\right)^3 + \cdots\right]$$

$$= -\frac{z}{2} - \frac{3z^2}{4} - \frac{7z^3}{8} - \cdots.$$

(ii) $1 < |z| < 2$.

$$f(z) = \frac{1}{z(1-1/z)} + \frac{2}{2(1-z/2)} = \frac{1}{z}(1-1/z)^{-1} + (1-z/2)^{-1}.$$

Now $1 < |z| < 2 \Rightarrow \left|\dfrac{1}{z}\right| < 1$ and $\left|\dfrac{z}{2}\right| < 1$. Hence we have

$$f(z) = \frac{1}{z}\left[1 + \frac{1}{z} + \left(\frac{1}{z}\right)^2 + \left(\frac{1}{z}\right)^3 + \cdots\right]$$
$$+ \left[1 + \left(\frac{z}{2}\right) + \left(\frac{z}{2}\right)^2 + \left(\frac{z}{2}\right)^3 + \cdots\right]$$
$$= \cdots + \left(\frac{1}{z}\right)^3 + \left(\frac{1}{z}\right)^2 + \frac{1}{z} + 1 + \left(\frac{z}{2}\right) + \left(\frac{z}{2}\right)^2 + \left(\frac{z}{2}\right)^3 + \cdots.$$

(iii) $|z| > 2$. Hence $\left|\dfrac{2}{z}\right| < 1$ and $\left|\dfrac{1}{z}\right| < 1$.

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$$f(z) = \frac{1}{z(1-1/z)} - \frac{2}{z(1-2/z)}$$

$$= \frac{1}{z}(1-1/z)^{-1} - \frac{2}{z}(1-2/z)^{-1}$$

$$= \frac{1}{z}\left(1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \cdots\right) - \frac{2}{z}\left(1 + \frac{2}{z} + \left(\frac{2}{z}\right)^2 + \left(\frac{2}{z}\right)^3 \cdots\right)$$

$$= -\frac{1}{z} - \frac{3}{z^2} - \frac{7}{z^3} - \frac{15}{z^4} - \cdots.$$

v) $|z-1| > 1$. Hence $\dfrac{1}{|z-1|} < 1$.

$$f(z) = \frac{1}{z-1} - \frac{2}{z-2}$$

$$= \frac{1}{z-1} - \frac{2}{z-1-1}$$

$$= \frac{1}{z-1} - \frac{2}{(z-1)\left(1 - \frac{1}{z-1}\right)}$$

$$= \frac{1}{z-1} - \frac{2}{z-1}\left(1 - \frac{1}{z-1}\right)^{-1}$$

$$= \frac{1}{z-1} - \frac{2}{z-1}\left[1 + \frac{1}{z-1} + \left(\frac{1}{z-1}\right)^2 + \left(\frac{1}{z-1}\right)^3 + \cdots\right]$$

$$= -\frac{1}{z-1} - \frac{2}{(z-1)^2} - \frac{2}{(z-1)^3} - \cdots$$

(v) $0 < |z-2| < 1$.

$$f(z) = \frac{1}{z-2+1} - \frac{2}{z-2}$$

$$= [1 + (z-2)]^{-1} - \frac{2}{z-2}$$

$$= [1 - (z-2) + (z-2)^2 - (z-2)^3 + \cdots] - \frac{2}{z-2}$$

$$= \frac{-2}{z-2} + 1 - (z-2) + (z-2)^2 - (z-2)^3 + \cdots.$$

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Problem 6. Expand $\dfrac{1}{z^2-3z+2}$ in Laurent's series valid in the region $1 < |z| < 2$.

Solution. Let $f(z) = \dfrac{1}{z^2-3z+2}$. Then

$$f(z) = \frac{(z-2)-(z-1)}{(z-2)(z-1)} = \frac{1}{z-1} - \frac{1}{z-2}.$$

$f(z)$ is analytic in the region $1 < |z| < 2$.
Hence $f(z)$ can be expanded in Laurent's series in that region. Now

$$f(z) = \frac{1}{-2\left(1-\frac{z}{2}\right)} - \frac{1}{z\left(1-\frac{1}{z}\right)} = -\frac{1}{2}\left(1-\frac{z}{2}\right)^{-1} - \frac{1}{z}\left(1-\frac{1}{z}\right)^{-1}.$$

In the region $1 < |z| < 2$, we have $\left|\frac{z}{2}\right| < 1$ and $\left|\frac{1}{z}\right| < 1$. Hence $f(z)$ can be expanded in Laurent's series as

$$f(z) = -\frac{1}{2}\left[1+\frac{z}{2}+\left(\frac{z}{2}\right)^2+\left(\frac{z}{2}\right)^3+\cdots\right]$$
$$-\frac{1}{z}\left[1+\frac{1}{z}+\left(\frac{1}{z}\right)^2+\left(\frac{1}{z}\right)^3+\cdots\right]$$
$$= -\frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n - \frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^n$$
$$= -\sum_{n=0}^{\infty}\frac{z^n}{2^{n+1}} - \sum_{n=0}^{\infty}\frac{1}{z^{n+1}}.$$

Problem 7. If $f(z) = \dfrac{z+4}{(z+3)(z-1)^2}$ find Laurent's series expansions in

(i) $0 < |z-1| < 4$ and **(ii)** $|z-1| > 4$.

Solution. Let $f(z) = \dfrac{z+4}{(z+3)(z-1)^2}$.

By expressing $f(z)$ into partial fractions we get

$$f(z) = \frac{1}{16(z+3)} - \frac{1}{16(z-1)} + \frac{5}{4(z-1)^2}.$$

(i) $0 < |z-1| < 4$. Hence $0 < \left|\dfrac{z-1}{4}\right| < 1$.

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$$f(z) = \frac{1}{16(z-1+4)} - \frac{1}{16(z-1)} + \frac{5}{4(z-1)^2}$$

$$= \frac{1}{64\left(1+\frac{z-1}{4}\right)} - \frac{1}{16(z-1)} + \frac{5}{4(z-1)^2}$$

$$= \frac{1}{64}\left(1+\frac{z-1}{4}\right)^{-1} - \frac{1}{16(z-1)} + \frac{5}{4(z-1)^2}.$$

Since $\left|\frac{z-1}{4}\right| < 1$, we have

$$f(z) = \frac{1}{64}\left[1-\left(\frac{z-1}{4}\right)+\left(\frac{z-1}{4}\right)^2-\left(\frac{z-1}{4}\right)^3+\cdots\right] - \frac{1}{16(z-1)} + \frac{5}{4(z-1)^2}$$

$$= \frac{5}{4(z-1)^2} - \frac{1}{16(z-1)} + \frac{1}{64} - \frac{1}{64}\left[\frac{z-1}{4}-\left(\frac{z-1}{4}\right)^2+\cdots\right].$$

This is the required Laurent's series expansion for $f(z)$ in $0 < |z-1| < 4$.

(ii) $|z-1| > 4$. Hence $\left|\frac{4}{z-1}\right| < 1$.

Now $$f(z) = \frac{1}{16(z-1)\left(1+\frac{4}{z-1}\right)} - \frac{1}{16(z-1)} + \frac{5}{4(z-1)^2}$$

$$= \frac{1}{16(z-1)}\left[1-\left(\frac{4}{z-1}\right)+\left(\frac{4}{z-1}\right)^2-\left(\frac{4}{z-1}\right)^3+\cdots\right] - \frac{1}{16(z-1)} + \frac{5}{4(z-1)^2}$$

$$= \frac{1}{(z-1)^2} + \frac{1}{(z-1)^3} - \frac{4}{(z-1)^4} + \frac{4^2}{(z-1)^5} - \cdots.$$

Problem 8. Find the Laurent's series expansion of the function $\frac{z^2-1}{(z+2)(z+3)}$ valid in the annular region $2 < |z| < 3$.

Solution. Let $f(z) = \frac{z^2-1}{(z+2)(z+3)}$.

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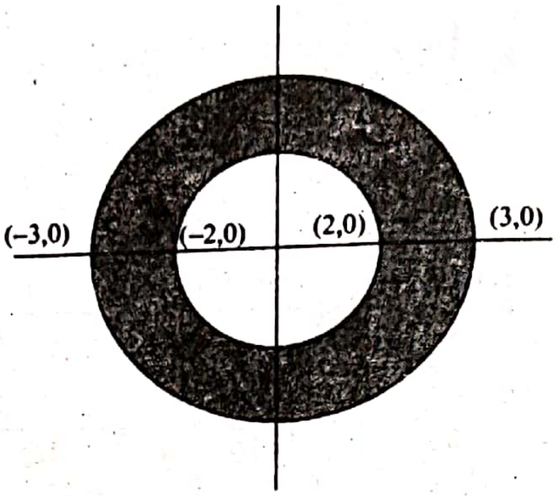
By splitting $f(z)$ into partial fractions, we get

$$f(z) = 1 + \frac{3}{z+2} - \frac{8}{z+3}.$$

$f(z)$ is analytic in the annular region $2 < |z| < 3$.



Hence $f(z)$ can be expanded as a Laurent's series in that region.

$$f(z) = 1 + \frac{3}{z\left(1+\frac{2}{z}\right)} - \frac{8}{3\left(1+\frac{z}{3}\right)}$$

$$= 1 + \frac{3}{z}\left(1+\frac{2}{z}\right)^{-1} - \frac{8}{3}\left(1+\frac{z}{3}\right)^{-1}$$

$$= 1 + \frac{3}{z}\left[1 - \frac{2}{z} + \left(\frac{2}{z}\right)^2 - \left(\frac{2}{z}\right)^3 + \cdots\right]$$

$$- \frac{8}{3}\left[1 - \frac{z}{3} + \left(\frac{z}{3}\right)^2 - \left(\frac{z}{3}\right)^3 + \cdots\right]$$

$$= 1 + \frac{3}{z}\sum_{n=0}^{\infty}(-1)^n\left(\frac{2}{z}\right)^n - \frac{8}{3}\sum_{n=0}^{\infty}(-1)^n\left(\frac{z}{3}\right)^n$$

$$= 1 + 3\sum_{n=0}^{\infty}\frac{(-1)^n 2^n}{z^{n+1}} - 8\sum_{n=0}^{\infty}\frac{(-1)^n z^n}{3^{n+1}}.$$

Problem 9. For the function $f(z) = \dfrac{2z^3+1}{z(z+1)}$ find **(i)** a Taylor's series valid in a neighbourhood of $z = i$ and **(ii)** a Laurent's series valid within an annulus of which centre is the origin.

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Solution. (i) $f(z) = \dfrac{2z^3+1}{z(z+1)}$

$$= 2z - 2 + \frac{1}{z} + \frac{1}{z+1} \text{ (by partial fractions)}$$

$$= 2(z-1) + \frac{1}{z} + \frac{1}{z+1} \qquad \ldots (1)$$

$$= g(z) + h(z) + j(z)$$

where $g(z) = 2(z-1)$, $h(z) = \dfrac{1}{z}$ and $j(z) = \dfrac{1}{z+1}$.

Taylor's expansion for $g(z)$ about $z = i$ is obviously $2(i-1) + 2(z-i)$.

Taylor's expansion for $h(z)$ about $z = i$ is given by

$$h(z) = h(i) + \sum_{n=1}^{\infty} \frac{h^{(n)}(i)}{n!}(z-i)^n.$$

Here $h(i) = \dfrac{1}{i}$; $h^{(n)}(z) = \dfrac{(-1)^n n!}{z^{n+1}}$ so that $h^{(n)}(i) = \dfrac{(-1)^n n!}{i^{n+1}}$.

$$\therefore \; h(z) = \frac{1}{i} + \sum_{n=1}^{\infty} \frac{(-1)^n n!}{i^{n+1} n!}(z-i)^n = \sum_{n=0}^{\infty} \frac{(-1)^n}{i^{n+1}}(z-i)^n.$$

Similarly we can prove that $j(z) = \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n (z-i)^n}{(1+i)^{n+1}}$.

Hence the Taylor's expansion for $f(z)$ is

$$f(z) = 2(i-1) + 2(z-i) + \sum_{n=0}^{\infty} \left[\frac{(-1)^n}{i^{n+1}} + \frac{(-1)^n}{(1+i)^{n+1}}\right](z-i)^n.$$

(ii) $f(z) = 2z - 2 + \dfrac{1}{z} + (1+z)^{-1}$ (from (1))

$$= 2z - 2 + \frac{1}{z} + (1 - z + z^2 - z^3 + \ldots) \text{ if } |z| < 1.$$

$\therefore$ In the annulus $0 < |z| < 1$ the Laurent's expansion is given by

$$f(z) = \frac{1}{z} - 1 + z + z^2 - z^3 + z^4 - \cdots.$$

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Problem 10. Expand $f(z) = \dfrac{e^{2z}}{(z-1)^3}$ about $z = 1$ as a Laurent's series. Also indicate the region of convergence of the series.

Solution.
$$f(z) = \frac{e^{2(z-1)+2}}{(z-1)^3}$$
$$= \frac{e^2 e^{2(z-1)}}{(z-1)^3}$$
$$= \frac{e^2}{(z-1)^3}\left[1 + \frac{2(z-1)}{1!} + \frac{2^2(z-1)^2}{2!} + \frac{2^3(z-1)^3}{3!} + \ldots\right]$$
$$= e^2\left[\frac{1}{(z-1)^3} + \frac{2}{(z-1)^2} + \frac{2}{(z-1)} + \frac{4}{3} + \frac{2^4}{4!}(z-1) + \ldots\right].$$

This series converges for all values of z except $z = 1$.

Exercises

1. Prove that $\dfrac{1+2z}{z^2+z^3} = \dfrac{1}{z^2} + \dfrac{1}{z} - 1 + z - z^2 + z^3 - \ldots$ where $0 < |z| < 1$.
2. Find a Laurent's series expansions in powers of z of the function $f(z) = \dfrac{1}{z(1+z^2)}$.
3. Find two different Laurent's series for $\dfrac{1}{z^2(1-z)}$ about $z = 0$ and state the regions of validity.
4. Expand $\dfrac{1}{(z-1)(z-2)}$ as a power series in z valid in

   (i) $|z| < 1$ (ii) $1 < |z| < 2$ (iii) $|z| > 2$
5. Expand $\dfrac{1}{z(z^2-3z+2)}$ as a power series valid in $1 < |z| < 2$.
6. Expand $f(z) = \dfrac{1}{(z+1)(z+3)}$ in Laurent's series valid for

   (i) $1 < |z| < 3$ (ii) $|z| > 3$

   (iii) $0 < |z+1| < 2$ (iv) $|z| < 1$
7. Expand in Laurent's series $\dfrac{1}{z(z-1)^2}$ at the point $z = 1$.

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8. Find a Laurent's series expansion in powers of z for the function $f(z) = \dfrac{1}{z(1+z^2)}$.

9. Expand $\dfrac{1}{z^2(z-3)^2}$ as a Laurent's series at $z = 3$ and state the region of validity.

10. Expand $\dfrac{1}{z(z^2-3z+2)}$ in powers of z in the regions
   (i) $0 < |z| < 1$ (ii) $1 < |z| < 2$ (iii) $|z| > 2$

11. Expand $f(z) = \dfrac{z+3}{z(z^2-z-2)}$ in powers of z
   (i) within the unit circle about the origin;
   (ii) within the annular region between the concentric circle about the origin having radii 1 and 2 respectively;
   (iii) the exterior to the circle with centre as origin and radius 2.

12. Represent $f(z) = \dfrac{z}{(z-1)(z-3)}$ by a series of powers of $z-1$ in $0 < |z-1| < 2$.

13. Give two Laurent's series expansion in powers of z for the function $f(z) = \dfrac{1}{z^2(1-z)}$ and specify the regions in which the expansions are valid.

14. Represent the function $\dfrac{z+1}{z-1}$ by
   (i) its Taylor's series in powers of z and give the region of validity;
   (ii) its Laurent's series in powers of z for the region $|z| > 1$.

15. Obtain the Laurent's series of the function $f(z) = \dfrac{1}{(z-1)(z-3)}$ valid in the region of (i) $1 < |z| < 3$ (ii) $|z| > 3$.

16. Obtain the Laurent's series expansions for $\dfrac{1}{z(1-z)^2}$ about $z = 0$ and specify the regions in which the expansions are valid.

17. Find the expansions in powers of z for $\dfrac{1}{(z^2+1)(z^2+2)}$ when
   (i) $|z| < 1$ (ii) $1 < |z| < \sqrt{2}$ (iii) $|z| > \sqrt{2}$

Answers.

2. $f(z) = \dfrac{1}{z} - z + z^3 - z^5 + \cdots;\ 0 < |z| < 1$

   $f(z) = \dfrac{1}{z^3} - \dfrac{1}{z^5} + \dfrac{1}{z^7} \cdots\cdots;\ |z| > 1.$

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3. (i) $\frac{1}{z^2}+\frac{1}{z}+1+z+z^2+\cdots;0<|z|<1$

(ii) $-\left(\frac{1}{z^3}+\frac{1}{z^4}+\frac{1}{z^5}+\cdots\right);\ |z|>1$

4. (i) $\sum_{n=0}^{\infty}\left(1-\frac{1}{2^{n+1}}\right)z^n$ (ii) $-\sum_{n=1}^{\infty}\frac{1}{z^n}-\sum_{n=0}^{\infty}\frac{z^n}{2^{n+1}}$ (iii) $\sum_{n=1}^{\infty}\frac{2^n-1}{z^{n+1}}$

5. $-\sum_{n=0}^{\infty}\left[\frac{1}{z^{n+2}}+\frac{z^{n-1}}{2^{n+1}}\right]$

6. (i) $-\frac{1}{2z^4}+\frac{1}{2z^3}-\frac{1}{2z^2}+\frac{1}{2z}-\frac{1}{6}+\frac{z}{18}-\frac{z^2}{54}+\cdots$

(ii) $\frac{1}{z^2}-\frac{4}{z^3}+\frac{13}{z^4}-\frac{40}{z^5}+\cdots$

(iii) $\frac{1}{2(z+1)}-\frac{1}{4}+\frac{1}{8}(z+1)-\frac{1}{16}(z+1)^2+\cdots$

(iv) $\frac{1}{3}-\frac{4z}{9}+\frac{13}{27}z^2-\frac{40}{81}z^3+\cdots$

7. $\sum_{n=0}^{\infty}(-1)^n(z-1)^{n-2}$

8. $\frac{1}{z}+\sum_{n=1}^{\infty}(-1)^n z^{2n-1}$ if $0<|z|<1$

9. $\frac{1}{9(z-3)^2}-\frac{2}{27(z-3)}+\frac{1}{27}-\frac{4(z-3)}{243}+\cdots;0<|z-3|<3$

10. (i) $\frac{1}{2z}+\frac{3}{4}+\frac{7z}{8}+\frac{15z^2}{16}+\frac{31z^3}{32}+\cdots$

(ii) $\left(-\frac{1}{2z}-\frac{1}{z^2}-\frac{1}{z^3}-\cdots\right)-\frac{1}{4}\left(1+\frac{z}{2}+\frac{z^2}{2^2}+\cdots\right)$

(iii) $(2-1)\frac{1}{z^3}+(2^2-1)\frac{1}{z^4}+(2^3-1)\frac{1}{z^5}+\cdots$

11. (i) $-\frac{3}{2z}+\sum_{n=0}^{\infty}\left[\frac{2}{3}(-1)^n-\frac{5}{12}\left(\frac{1}{2}\right)^n\right]z^n$

(ii) $-\frac{3}{2z}+\frac{2}{3z}\sum_{n=0}^{\infty}\frac{(-1)^n}{z^n}-\frac{5}{12}\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n$

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(iii) $-\frac{3}{2z}+\frac{2}{3z}\sum_{n=0}^{\infty}\frac{(-1)^n}{z^n}+\frac{5}{6z}\sum_{n=0}^{\infty}\left(\frac{2}{z}\right)^n$

12. $-\frac{1}{2(z-1)}-\frac{3}{4}\sum_{n=0}^{\infty}\left(\frac{z-1}{2}\right)^n$

13. (i) $\sum_{n=0}^{\infty} z^{n-2};\ 0<|z|<1$ (ii) $\sum_{n=0}^{\infty} z^{-n-3};\ |z|>1$

14. (i) $-1-2\sum_{n=0}^{\infty} z^n;\ |z|<1$ (ii) $1+2\sum_{n=0}^{\infty} z^{-n};\ |z|>1$

15. (i) $-\frac{1}{2(z-1)}-\frac{1}{4}\sum_{n=0}^{\infty}\left(\frac{z-1}{2}\right)^n$

(ii) $\frac{1}{2(z-3)}-\frac{1}{4}\sum_{n=0}^{\infty}(-1)^n\left(\frac{z-3}{2}\right)^n$

16. (i) $\sum_{n=0}^{\infty}(n+1)z^{n-1};\ |z|<1$

(ii) $\sum_{n=0}^{\infty}\left(\frac{n+1}{z^{n+3}}\right);\ |z|<1$

17. (i) $\sum_{n=0}^{\infty}(-1)^n\left(1-\frac{1}{2^{n+1}}\right)z^{2n}$

(ii) $\sum_{n=1}^{\infty}\frac{(-1)^n}{z^{2n+2}}+\sum_{n=0}^{\infty}\frac{(-1)^{n+1}z^{2n}}{2^{n+1}}$

(iii) $\sum_{n=0}^{\infty}\frac{(-1)^n(1-2^n)}{z^{2n+2}}$

7.3. Zeros of an Analytic Function

Definition. Let $f(z)$ be a function which is analytic in a region D. Let $a \in D$. Then a is said to be a **zero of order** r (where r is a positive integer) for $f(z)$ if $f(z) = (z-a)^r\varphi(z)$ where $\varphi(z)$ is analytic at a and $\varphi(a) \neq 0$.

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Example 1. Consider $f(z) = \sin z$

$$\text{we know that } \sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots\cdots\cdots$$

$$= z\left(1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \cdots\cdots\cdots\right)$$

$$= z\ \varphi(z).$$

where $\varphi(z) = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \cdots$

Obviously $\varphi(z)$ is analytic and $\varphi(0) = 1 \neq 0$.

$z = 0$ is a zero of order 1 for $\sin z$.

Example 2. Let $f(z) = (z - 2i)^2(z + 3)^3 e^z$

$2i$ is a zero of order 2 and -3 is a zero of order 3 for $f(z)$.

Example 3. Let $f(z) = z^2 \sin z$

$$\text{Then } f(z) = z^2\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots\cdots\right)$$

$$= z^3\left(1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \cdots\cdots\right)$$

$$= z^3\varphi(z)$$

where $\varphi(z) = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \cdots\cdots$

Obviously $\varphi(z)$ is analytic and $\varphi(0) \neq 0$.

$\therefore$ $z = 0$ is a zero of order 3 for $f(z) = z^2 \sin z$.

Example 4. Let $f(z) = \frac{z^3 - 1}{z^3 + 1}$.

$$f(z) = 0 \Rightarrow z^3 - 1 = 0$$

$$\Rightarrow (z - 1)(z^2 + z + 1) = 0$$

Hence the zeros of $f(z)$ are $1, \frac{-1 + i\sqrt{3}}{2}, \frac{-1 - i\sqrt{3}}{2}$ and each one is a zero of order 1.

Theorem 7.3. Suppose $f(z)$ is analytic in a region D and is not identically zero in D. Then the set of all zeros of $f(z)$ is isolated.

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Proof. Let $a \in D$ be a zero for $f(z)$. We shall prove that there exists a neighbourhood $|z-a| < \delta$ such that this neighbourhood does not contain any other zero for $f(z)$.

Suppose a is a zero of order r for $f(z)$.

Then $f(z) = (z-a)^r \varphi(z)$ ... (1)

where $\varphi(z)$ is analytic at a and $\varphi(a) \neq 0$.

Now, since φ is analytic at a, φ is continuous at a.

$\therefore$ We can find a $\delta > 0$ such that

$$|z-a| < \delta \Rightarrow |\varphi(z) - \varphi(a)| < \frac{|\varphi(a)|}{2}.$$

We claim that the neighbourhood $|z-a| < \delta$ does not contain any other zero of $f(z)$.

Suppose $b \neq a$ is another zero for $f(z)$ in this neighbourhood. Then $|b-a| < \delta$ and $f(b) = 0$.

$\therefore$ $(b-a)^r \varphi(b) = 0$ (from (1))

Now, since $b \neq a$, $(b-a)^r \neq 0$.

$\therefore$ $\varphi(b) = 0$

Further $|b-a| < \delta \Rightarrow |\varphi(b) - \varphi(a)| < \dfrac{|\varphi(a)|}{2}$

$\Rightarrow |\varphi(a)| < \dfrac{|\varphi(a)|}{2}$ which is a contradiction.

Thus the neighbourhood $|z-a| < \delta$ contains no other zero of $f(z)$ and hence the set of all zeros of $f(z)$ is isolated.

Corollary 1. Let $f(z)$ be analytic in a region D. Suppose $f(z) = 0$ on a subset of D which has a limit point in D. Then $f(z)$ is identically zero in D.

Corollary 2. Let $f(z)$ and $g(z)$ be two functions which are analytic in a region D. Suppose $f(z) = g(z)$ on a subset of D which has a limit point in D. Then $f(z) = g(z)$ in D.

(consider the function $f(z) - g(z)$ and the result follows from corollary 1)

Exercises.

1. Find all the zeros of the following functions.

   (a) $\cos z$ (b) $\dfrac{(z+1)^2(iz+2)^3}{z+7}$

2. Prove that there is no analytic functions whose zeros are precisely the points $1, \frac{1}{2}, \frac{1}{3}, \ldots\ldots, \frac{1}{n}, \ldots\ldots$

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Answers.

1. (a) $(2n+1)\pi/2; n \in Z$ (b) -1 and $-2/i$.

7.4. Singularities

Definition. A point a is called a **singular point** or a **singularity** of a function $f(z)$ if $f(z)$ is not analytic at a and f is analytic at some point of every disc $|z-a| < r$.

Example 1. Consider the function $f(z) = \frac{1}{z}$.

Then $f'(z) = -\frac{1}{z^2}$ for all $z \neq 0$.

Thus $f(z)$ is analytic except at $z = 0$.

$\therefore z = 0$ is a singular point of $f(z)$.

Example 2. Consider the function $f(z) = \frac{1}{z(z-i)}$.

0 and i are singular points for $f(z)$.

Definition. A point a is called an **isolated singularity** for $f(z)$ if

(i) $f(z)$ is not analytic at $z = a$ and

(ii) there exists $r > 0$ such that $f(z)$ is analytic in $0 < |z-a| < r$.

(i.e) the neighbourhood $|z-a| < r$ contains no singularity of $f(z)$ except a.

Example 1. $f(z) = \frac{z+1}{z^2(z^2+1)}$ has three isolated singularities $z = 0, i, -i$.

Example 2. Consider the principal branch of logarithm given by $\log re^{i\theta} = \log r + i\theta$ where $-\pi < \theta \leq \pi$.

All points on the negative real axis are singular points of this function. These singularities are not isolated.

Example 3. Consider the function $f(z) = \frac{1}{\sin z}$. The singular points are $0, \pm\pi, \pm 2\pi, \ldots\ldots$ and these are isolated singular points.

We now proceed to classify the isolated singularties of a function.

Let a be an isolated singularity for a function $f(z)$. Let $r > 0$ be such that $f(z)$ is analytic in $0 < |z-a| < r$. In this domain the function $f(z)$ can be represented as a Laurent series given by

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$$f(z) = \sum_{n=1}^{\infty} \frac{b_n}{(z-a)^n} + \sum_{n=0}^{\infty} a_n (z-a)^n \quad \text{where}$$

$$a_n = \frac{1}{2\pi i} \int_C \frac{f(\zeta)d\zeta}{(\zeta - a)^{n+1}} \text{ and } b_n = \frac{1}{2\pi i} \int_C \frac{f(\zeta)d\zeta}{(\zeta - a)^{-n+1}}.$$

The series consisting of the negative powers of $z - a$ in the above Laurent series expansion of $f(z)$ is given by $\sum_{n=1}^{\infty} \frac{b_n}{(z-a)^n}$ and is called the *principal part or singular part* of $f(z)$ at $z = a$.

The singular part of $f(z)$ at $z = a$ determines the character of the singularity. There are three types of singularities. They are

(i) ***Removable singularities***
(ii) ***Poles***
(iii) ***Essential singularities.***

Definition. Let a be an isolated singularity for $f(z)$. Then a is called a **removable singularity** if the principal part of $f(z)$ at $z = a$ has no terms.

Note. If a is a removable singularity for $f(z)$ then the Laurent's series expansion of $f(z)$ about $z = a$ is given by

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n$$
$$= a_0 + a_1(z-a) + \cdots + a_n(z-a)^n + \cdots$$

Hence $\lim_{z \to a} f(z) = a_0$

Hence by defining $f(a) = a_0$ the function $f(z)$ becomes analytic at a.

Example 1. Let $f(z) = \frac{\sin z}{z}$. Clearly 0 is an isolated singular point for $f(z)$.

$$\text{Now, } \frac{\sin z}{z} = \frac{1}{z}\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots\right)$$
$$= 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \cdots$$

Here the principal part of $f(z)$ at $z = 0$ has no terms.
Hence $z = 0$ is a removable singularity.

Also $\lim_{z \to 0} \frac{\sin z}{z} = 1$. Hence the singularity can be removed by defining $f(0) = 1$ so that the extended function becomes analytic at $z = 0$.

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Example 2. Let $f(z) = \dfrac{z - \sin z}{z^3}$.

$z = 0$ is an isolated singularity.

$$\text{Further } \frac{z - \sin z}{z^3} = \frac{1}{z^3}\left[z - \left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots\right)\right]$$

$$= \frac{1}{3!} - \frac{z^2}{5!} + \frac{z^4}{7!} - \cdots\cdots$$

$\therefore$ $z = 0$ is a removable singularity.

By defining $f(0) = \dfrac{1}{6}$ the function becomes analytic at $z = 0$.

Definition. Let a be an isolated singularity of $f(z)$. The point a is called a **pole** if the principal part of $f(z)$ at $z = a$ has a finite number of terms. If the principal part of $f(z)$ at $z = a$ is given by

$$\frac{b_1}{z-a} + \frac{b_2}{(z-a)^2} + \cdots\cdots + \frac{b_r}{(z-a)^r}.$$

where $b_r \neq 0$, we say that a is **a pole of order** r for $f(z)$.

Note. A pole of order 1 is called a **simple pole** and a pole of order 2 is called a **double pole.**

Example 1. Consider $f(z) = \dfrac{e^z}{z}$.

$$\frac{e^z}{z} = \frac{1}{z} + 1 + \frac{z}{2!} + \frac{z^2}{3!} + \cdots\cdots$$

Here the principal part of $f(z)$ at $z = 0$ has a single term $\dfrac{1}{z}$. Hence $z = 0$ is a simple pole of $f(z)$.

Example 2. Let $f(z) = \tan z = \dfrac{\sin z}{\cos z}$. The singularities of $f(z)$ are $\dfrac{\pi}{2} + n\pi$, where $n \in \mathbf{Z}$. All the singularities are poles of order 1.

Example 3. $f(z) = \dfrac{\cos z}{z^2}$ has a double pole at $z = 0$.

$$\text{For, } \frac{\cos z}{z^2} = \frac{1}{z^2}\left(1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots\cdots\right)$$

$$= \frac{1}{z^2} - \frac{1}{2!} + \frac{z^2}{4!} - \cdots\cdots$$

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Example 4. Let $f(z) = \dfrac{z^2 - 2z + 3}{z - 2}$

Then $f(z) = 2 + (z - 2) + \dfrac{3}{z - 2}$ (by partial fractions)

Here $f(z)$ has a simple pole at $z = 2$.

Definition. Let a be an isolated singularity of $f(z)$. The point a is called an **essential singularity** of $f(z)$ at $z = a$ if the principal part of $f(z)$ at $z = a$ has an infinite number of terms.

Example 1. Let $f(z) = e^{1/z}$. Obviously $z = 0$ is an isolated singularity for $f(z)$.

Further $e^{1/z} = 1 + \dfrac{1}{z} + \dfrac{1}{2!z^2} + \dfrac{1}{3!z^3} + \cdots\cdots$ The principal part of $f(z)$ has infinite number of terms. Hence $e^{1/z}$ has an essential singularity at $z = 0$.

Example 2. Let $f(z) = z^2 \sin(1/z)$. $f(z)$ has essential singularity at $z = 0$.

In the following theorem we give equivalent characterisations for an isolated singular point a of $f(z)$ to be a removable singularity.

Theorem 7.4. Let $f(z)$ be a function defined in a region D of the complex plane except possibly at a point $a \in D$ and let a be an isolated singularity for $f(z)$. Then a is a removable singularity for $f(z)$ if and only if there exists a complex number a_0 such that by defining $f(a) = a_0$ the extended function becomes analytic at a.

Proof. Suppose a is a removable singularity for $f(z)$.

$$\text{Then } f(z) = \sum_{n=0}^{\infty} a_n(z-a)^n,\ 0 < |z - a| < r$$

$$= a_0 + a_1(z - a) + a_2(z - a)^2 + \cdots\cdots$$

$\therefore$ By defining $f(a) = a_0$; f becomes analytic at a.

Conversely, suppose there exists a complex number a_0 such that by defining $f(a) = a_0$, f becomes analytic in $|z - a| < r$.

Hence f can be represented as a Taylor's series, in power of $z-a$ in this neighbourhood, given by $f(z) = \sum_0^{\infty} a_n(z - a)^n$. This shows that the principal part of $f(z)$ at $z = a$ has no terms. Hence a is a removable singularity for $f(z)$. bounded $|f(z)| \le M$

Theorem 7.5. (Riemann's theorem) Let f be a function which is bounded and analytic throughout a domain $0 < |z - z_0| < \delta$. Then either f is analytic at z_0 or else z_0 is a removable singular point of f.

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Proof. Consider the Laurent's series for the function in the given domain about z_0. The coefficient b_n of $\frac{1}{(z-z_0)^n}$ is given by $b_n = \frac{1}{2\pi i}\int\limits_C \frac{f(z)dz}{(z-z_0)^{-n+1}}$ where C is the circle $|z-z_0| = r$ where $r < \delta$.

Now, since f is bounded there exists a positive real number M such that $|f(z)| \le M$ in $0 < |z-z_0| < \delta$.

$$\therefore \quad |b_n| \le \frac{1}{2\pi}\frac{M(2\pi r)}{r^{-n+1}} \text{ (by theorem 6.2)}$$
$$= Mr^n$$

Since it is true for every r such that $0 < r < \delta$, taking limit $r \to 0$ we get $b_n = 0$. Hence the Laurent's series for $f(z)$ has no principal part. Hence the theorem follows.

Theorem 7.6. Let $f(z)$ be a function having a as an isolated singular point. The the following are equivalent.

(i) a is a pole of order r for $f(z)$.

(ii) $f(z)$ can be written in the form $f(z) = \frac{1}{(z-a)^r}\theta(z)$ where $\theta(z)$ has a removable singularity at $z = a$ and $\lim\limits_{z\to a}\theta(z) \ne 0$.

(iii) a is a zero of order r for $\frac{1}{f(z)}$.

Proof. (i) ⇒ (ii). Let a be a pole order r for $f(z)$. Then the Laurent's series expansion of $f(z)$ about a is given by $f(z) = \sum\limits_{n=1}^{r}\frac{b_n}{(z-a)^n} + \sum\limits_{n=0}^{\infty}a_n(z-a)^n$ where $b_r \ne 0$.

$$\therefore \quad f(z) = \frac{1}{(z-a)^r}[b_r + b_{r-1}(z-a) + \ldots + b_0(z-a)^{r-1} + a_0(z-a)^r + \ldots]$$
$$= \frac{1}{(z-a)^r}\theta(z) \text{ where } \theta(z) = b_r + b_{r-1}(z-a) + \ldots\ldots$$

Clearly $\lim\limits_{z\to a}\theta(z) = b_r \ne 0$ and $\theta(z)$ has a removable singularity at $z = a$.

(ii) ⇒ (iii) Let $f(z) = \frac{1}{(z-a)^r}\theta(z)$ and by suitably defining $\theta(a)$ we may assume that $\theta(z)$ is analytic at a and $\theta(a) \ne 0$.

$$\therefore \quad \frac{1}{f(z)} = (z-a)^r\frac{1}{\theta(z)} \text{ and } \frac{1}{\theta(z)} \text{ is analytic at } a \text{ and } \frac{1}{\theta(a)} \ne 0.$$

Hence a is a zero of order r for $\frac{1}{f(z)}$.

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(iii)⇒ (i) Let a be a zero of order r for $\frac{1}{f(z)}$.

Then $\frac{1}{f(z)} = (z-a)^r g(z)$ where $g(z)$ is analytic at a and $g(a) \neq 0$.

$\therefore f(z) = \frac{g_1(z)}{(z-a)^r}$ where $g_1(z)$ is analytic at a and $g_1(a) \neq 0$.

Let $g_1(z) = a_0 + a_1(z-a) + \ldots + a_n(z-a)^n + \ldots\ldots$ so that $a_0 \neq 0$.

$$\therefore \quad f(z) = \frac{a_0}{(z-a)^r} + \frac{a_1}{(z-a)^{r-1}} + \ldots + a_r + a_{r+1}(z-a) + \ldots$$

in $0 < |z-a| < r$

$\therefore$ The principal part of $f(z)$ at $z = a$ is

$$\frac{a_0}{(z-a)^r} + \frac{a_1}{(z-a)^{r-1}} + \ldots\ldots + \frac{a_{r-1}}{z-a} \text{ and } a_0 \neq 0.$$

$\therefore$ a is a pole of order r for $f(z)$.

Theorem 7.7. An isolated singularity a of $f(z)$ is a pole if and only if $\lim_{z\to a} f(z) = \infty$.

Proof. If a is a pole of order r for $f(z)$ then $f(z) = \frac{g(z)}{(z-a)^r}$ with $g(a) \neq 0$.

$$\therefore \lim_{z\to a} f(z) = \infty.$$

Conversely let a be an isolated singularity for $f(z)$ and let $\lim_{z\to a} f(z) = \infty$.

$$\text{Let } \theta(z) = \frac{1}{f(z)}.$$

$$\text{Then } \lim_{z\to a} \theta(z) = 0.$$

Hence a is a removable singularity for $\theta(z)$ and by defining $\theta(z) = 0, \theta$ becomes analytic at a. Let a be a zero of order r for the function $\theta(z)$. Then a is a pole of order r for $f(z)$.

Definition. A function $f(z)$ is said to be a **meromorphic function** if it is analytic except at a finite number of points and these finite set of points are poles.

Example 1. Let $f(z) = \frac{1}{z(z-1)^2}$.

$f(z)$ is analytic except at $z = 0$ and $z = 1$. Also 0 and 1 are poles of order 1 and 2 respectively. Hence $f(z)$ is a meromorphic function.

Example 2. $\frac{e^z}{z} = \frac{1}{z} + 1 + \frac{z}{2!} + \frac{z^2}{3!} + \cdots$ is a meromorphic function.

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Example 3. $e^{1/z}$ is not a meromorphic function since $z = 0$ is an essential singularity for $e^{1/z}$.

The following theorem due to Weierstrass describes the behaviour of a function in the neighbourhood of an essential singularity.

Theorem 7.8. Let z_0 be an essential singularity for a function $f(z)$. Let c be any complex number. Then given $\varepsilon, \delta > 0$ there exists a point z such that $|z-z_0| < \delta$ and $|f(z)-c| < \varepsilon$.

(i.e) The function $f(z)$ comes arbitrarily close to any complex number c in every neighbourhood of an essential singularity.

Proof. Suppose the theorem is false. Then there exist $\delta, \varepsilon > 0$ such that for every point z satisfying $0 < |z - z_0| < \delta$ we have $|f(z) - c| \geq \varepsilon$.

Now consider the function $g(z) = \dfrac{1}{f(z) - c}$.

$$\therefore |g(z)| = \frac{1}{|f(z) - c|} \leq \frac{1}{\varepsilon}.$$

Hence $g(z)$ is bounded and further $g(z)$ is analytic in $0 < |z - z_0| < \delta$.

Hence by Riemann's theorem $z = z_0$ is a removable singularity for $g(z)$.

Now, if $g(z_0) \neq 0$ then $\dfrac{1}{g(z)} = f(z) - c$ is analytic at z_0.

$\therefore$ By suitably defining $g(z_0)$, the function $g(z)$ becomes analytic at z_0.

If $g(z_0) = 0$ then let z_0 be a zero of order r for $g(z)$.

Then z_0 is a pole of order r for $\dfrac{1}{g(z)} = f(z) - c$.

Thus $f(z)$ is either analytic at z_0 or else z_0 is a pole of $f(z)$ which is a contradiction to the hypothesis that z_0 is an essential singularity for $f(z)$.

Hence the theorem.

Solved problems

Problem 1. Determine and classify the singular points of $f(z) = \dfrac{z}{e^z - 1}$.

Solution. The singularities of $f(z)$ are given by the values of z for which $e^z - 1 = 0$. Hence $z = 2n\pi i, n \in \mathbf{Z}$, are the singularities of $f(z)$.

$$\text{Now, } e^z - 1 = \left(1 + z + \frac{z^2}{2!} + \cdots + \frac{z^n}{n!} + \cdots\right) - 1.$$

$$= z + \frac{z^2}{2!} + \cdots + \frac{z^n}{n!} + \cdots$$

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$$\therefore \lim_{z\to 0} \frac{z}{e^z - 1} = 1$$

Hence 0 is a removable singularity for $f(z)$.

Also $\lim_{z\to 2n\pi i} \left(\frac{z}{e^z - 1}\right) = \infty$ if $n \neq 0$ and hence $2n\pi i$, $n \neq 0$, are simple poles of $f(z)$.

Problem 2. Determine and classify the singularities of $f(z) = \sin(1/z)$.

Solution. Clearly 0 is the only singularity of $f(z)$.

Also $f(z) = \frac{1}{z} - \frac{1}{3!z^3} + \frac{1}{5!z^5} - \ldots\ldots$

Thus the principal part of $f(z)$ at $z = 0$ has infinitely many terms and hence 0 is an essential singularity for $f(z)$.

Problem 3. Determine and classify the singular points of $\frac{1}{(2\sin z - 1)^2}$.

Solution. The singularities of $f(z)$ are given by the values of z for which $2\sin z - 1 = 0$.

$\therefore$ The singularities of $f(z)$ are given by $z = \frac{\pi}{6} + 2n\pi$, $n \in \mathbf{Z}$, and they are double poles.

Exercises.

1. Find the singularities of the following functions and classify the singularities.

   (i) $ze^{1/z}$ (ii) $\frac{z^2}{1+z}$ (iii) $\frac{\sin z}{z}$

   (iv) $\frac{e^{1/z}}{z-1}$ (v) $\frac{z}{e^{1/z} - 1}$ (vi) $\sin\left(\frac{1}{1-z}\right)$

   (vii) $\frac{z^2 - 2z + 3}{z - 2}$ (viii) $(z - i)\sin\left(\frac{1}{z+2i}\right)$

2. Show that the singular points of each of the following functions are poles. Determine the order of each pole.

   (i) $\frac{z+1}{z^2 - 2z}$ (ii) $\tanh z$

   (iii) $\frac{1 - e^{2z}}{z^4}$ (iv) $\frac{e^{2z}}{(z-1)^2}$

   (v) $\frac{1}{z^2+1}$ (vi) $\frac{1}{z^2(z-3)^2}$

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(vii) $\dfrac{1}{z^4 + 2z^2 + 1}$ (viii) $\dfrac{z(1+z)}{1-\cos z}$

(ix) $\dfrac{z^2 - 2z + 3}{z - 2}$ (x) $(z-i)\sin\left(\dfrac{1}{z+2i}\right)$

3. Find the order of the pole $z = 0$ for the following functions.

   (i) $\dfrac{e^z}{z}$ (ii) $\dfrac{e^z}{z^2}$ (iii) $\dfrac{1-\sin z}{z^5}$

4. Let f and g have a pole of order m and n respectively at a. What can be said about the order of pole of (i) $f+g$ (ii) fg (iii) f/g at a.

5. Show that if f has an essential singularity at a so does f^2

Answers

2. (i) 0 and 2 are simple poles (ii) 0 is a simple pole (iii) 0 is a pole of order 3. (iv) 1 is a double pole (v) i and $-i$ are simple poles (vi) 0,3 are double poles (vii) i, $-i$ are double poles (viii) 0 is a simple pole. (ix) 2 is a pole (x) $-2i$ is an essential singularity.

3. (i) 1 (ii) 2 (iii) 5.

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8. Calculus of Residues

8.0. Introduction

In this chapter we introduce the concept of the residue of a function $f(z)$ at an isolated singular point and prove Cauchy's residue theorem. Using this theorem we evaluate certain types of real definite integrals.

8.1. Residues

Definition. Let a be an isolated singularity for $f(z)$. Then the **residue** of $f(z)$ at a is defined to be the coefficient of $\frac{1}{z-a}$ in the Laurent's series expansion of $f(z)$ about a and is denoted by Res $\{f(z); a\}$.

Thus Res $\{f(z); a\} = \frac{1}{2\pi i} \int_C f(z)dz = b_1$ where C is a circle $|z-a| = r$ such that f is analytic in $0 < |z-a| < r$.

Example. Consider $f(z) = \frac{e^z}{z^2}$

$$\frac{e^z}{z^2} = \frac{1}{z^2}\left(1 + \frac{z}{1!} + \frac{z^2}{2!} + \cdots\cdots\right)$$

$$= \frac{1}{z^2} + \frac{1}{z} + \frac{1}{2!} + \frac{z}{3!} + \frac{z^2}{4!} + \cdots$$

$\therefore$ $f(z)$ has a double pole at $z = 0$.

$\therefore$ Res $\{f(z); 0\} =$ coefficient of $\frac{1}{z} = 1$.

The following lemmas provide methods for calculation of residues.

Lemma 1. If $z = a$ is a simple pole for $f(z)$ then

$$\text{Res}\,\{f(z); a\} = \lim_{z\to a}(z-a)f(z).$$

Proof. Since $z = a$ is a simple pole for $f(z)$ the Laurent's series expansion for $f(z)$ about $z = a$ is given by $f(z) = \frac{b_1}{z-a} + a_0 + a_1(z-a) + \cdots\cdots$

Now, $(z-a)f(z) = b_1 + a_0(z-a) + a_1(z-a)^2 + \cdots$

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$\therefore \quad \lim_{z \to a} (z-a) f(z) = b_1$

$$= \text{Res}\,\{f(z); a\}.$$

Lemma 2. If a is a simple pole for $f(z)$ and $f(z) = \dfrac{g(z)}{z-a}$ where $g(z)$ is analytic at a and $g(a) \neq 0$ then $\text{Res}\,\{f(z); a\} = g(a)$.

Proof. By Lemma 1, $\text{Res}\,\{f(z); a\} = \lim_{z \to a} (z-a) f(z) = \lim_{z \to a} g(z) = g(a)$.

Lemma 3. If a is a simple pole for $f(z)$ and if $f(z)$ is of the form $\dfrac{h(z)}{k(z)}$ where $h(z)$ and $k(z)$ are analytic at a and $h(a) \neq 0$ and $k(a) = 0$ then

$$\text{Res}\,\{f(z); a\} = \frac{h(a)}{k'(a)}.$$

Proof. $\text{Res}\,\{f(z); a\} = \lim_{z \to a} (z-a) f(z)$

$$= \lim_{z \to a} (z-a) \frac{h(z)}{k(z)}$$

$$= \lim_{z \to a} h(z) \lim_{z \to a} \frac{(z-a)}{k(z)}.$$

$$= \lim_{z \to a} h(z) \lim_{z \to a} \left[\frac{z-a}{k(z)-k(a)}\right] \quad \text{(since } k(a) = 0\text{)}$$

$$= h(a) \left[\frac{1}{k'(a)}\right]$$

$$= \frac{h(a)}{k'(a)}.$$

Lemma 4. Let a be a **pole or order** $m > 1$ for $f(z)$ and let $f(z) = \dfrac{g(z)}{(z-a)^m}$ where $g(z)$ is analytic at a and $g(a) \neq 0$. Then

$$\text{Res}\,\{f(z); a\} = \frac{g^{(m-1)}(a)}{(m-1)!}.$$

Proof. $g^{(m-1)}(a) = \dfrac{(m-1)!}{2\pi i} \displaystyle\int_C \frac{g(z)dz}{(z-a)^m}$ (by theorem on higher derivatives) where C is a circle $|z-a| = r$ such that $f(z)$ is analytic in $0 < |z-a| < r$.

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$$\therefore \quad \frac{g^{(m-1)}(a)}{(m-1)!} = \frac{1}{2\pi i}\int_C f(z)dz = \text{Res}\,\{f(z); a\}.$$

Solved Problems

Problem 1. Calculate the residue of $\dfrac{z+1}{z^2-2z}$ at its poles.

Solution. Let $f(z) = \dfrac{z+1}{z^2-2z} = \dfrac{z+1}{z(z-2)}$.

$z = 0$ and $z = 2$ are simple poles for $f(z)$.

$$\begin{aligned}\text{Res}\,\{f(z); 0\} &= \lim_{z\to 0}(z-0)\left[\frac{z+1}{z(z-2)}\right]\\ &= \lim_{z\to 0}\frac{z+1}{z-2} = -\frac{1}{2}\\ \text{Res}\,\{f(z); 2\} &= \lim_{z\to 2}(z-2)\left[\frac{z+1}{z(z-2)}\right]\\ &= \lim_{z\to 2}\frac{z+1}{z} = \frac{3}{2}.\end{aligned}$$

Aliter. $f(z)$ can be written as $f(z) = \dfrac{h(z)}{k(z)}$ where $h(z) = z+1$ and $k(z) = z^2 - 2z$ so that $k'(z) = 2z - 2$.

$$\begin{aligned}\therefore \quad \text{Res}\,\{f(z); 0\} &= \frac{h(0)}{k'(0)} \quad \text{(by Lemma 3)}\\ &= -\frac{1}{2}.\\ \text{Res}\,\{f(z); 2\} &= \frac{h(2)}{k'(2)} = \frac{3}{2}.\end{aligned}$$

Problem 2. Find the residue at $z = 0$ of $\dfrac{1+e^z}{z\cos z + \sin z}$.

Solution. Let $f(z) = \dfrac{1+e^z}{z\cos z + \sin z}$.

Clearly 0 is a pole of order 1 for $f(z)$.

$\therefore \quad \text{Res}\,\{f(z); 0\} = \lim\limits_{z\to 0}\dfrac{h(z)}{k'(z)}$ where $h(z) = 1 + e^z$ and $k(z) = z\cos z + \sin z$.

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Now $k'(z) = -z\sin z + \cos z + \cos z = -z\sin z + 2\cos z$.

$$\therefore \text{Res}\{f(z); 0\} = \frac{2}{2} = 1.$$

Problem 3. Find the residue of $\dfrac{1}{(z^2+a^2)^2}$ at $z = ai$.

Solution. Let $f(z) = \dfrac{1}{(z^2+a^2)^2} = \dfrac{1}{(z+ai)^2(z-ai)^2}$.

$z = ai$ and $z = -ai$ are poles of order 2 for $f(z)$.

$$\text{Let } g(z) = \frac{1}{(z+ai)^2}.$$

$$\therefore g'(z) = \frac{-2}{(z+ai)^3}.$$

$$\therefore \text{Res}\{f(z); ai\} = g'(ai) = \frac{-2}{(ai+ai)^3} = \frac{-2}{8a^3i^3} = \frac{2}{8a^3i}$$

$$= \frac{-i}{4a^3}$$

Problem 4. Find the poles of $f(z) = \dfrac{z^2+4}{z^3+2z^2+2z}$ and determine the residues at the poles.

Solution. $f(z) = \dfrac{z^2+4}{z^3+2z^2+2z} = \dfrac{z^2+4}{z(z+1-i)(z+1+i)}$.

$\therefore$ $0, i-1$ and $-1-i$ are simple poles for $f(z)$.

Here $f(z) = \dfrac{h(z)}{k(z)}$ where $h(z) = z^2+4$ and $k(z) = z^3+2z^2+2z$.

Hence $k'(z) = 3z^2+4z+2$.

$$\text{Res}\{f(z); 0\} = \frac{h(0)}{k'(0)} = \frac{4}{2} = 2.$$

$$\text{Res}\{f(z); i-1\} = \frac{h(i-1)}{k'(i-1)}$$

$$= \frac{(i-1)^2+4}{3(i-1)^2+4(i-1)+2}$$

$$= \frac{3i-1}{2} \text{ (after simplification).}$$

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Similarly Res $\{f(z); -1-i\} = \dfrac{-(1+3i)}{2}$.

Problem 5. Find the residue of $\cot z$ at $z = 0$.

Solution. $z = 0$ is a simple pole for $\cot z$. Let $f(z) = \dfrac{\cos z}{\sin z} = \dfrac{h(z)}{k(z)}$.

$$\therefore \quad \text{Res}\,\{f(z); 0\} = \frac{h(0)}{k'(0)} = \frac{\cos 0}{\cos 0} = 1.$$

Problem 6. Find the residue of $\dfrac{e^z}{z^2(z^2+9)}$ at its poles.

Solution. Let $f(z) = \dfrac{e^z}{z^2(z^2+9)}$.

Here $z = 0$ is a double pole and $z = 3i$ and $z = -3i$ are simple poles for $f(z)$.

To find the Res $\{f(z); 0\}$, let $g(z) = \dfrac{e^z}{z^2+9}$.

Clearly $g(z)$ is analytic at $z = 0$ and $g(0) \neq 0$.

$$\text{Also } g'(z) = e^z\left[\frac{(z^2+9) - 2z}{(z^2+9)^2}\right]$$

$$\therefore \quad \text{Res}\,\{f(z); 0\} = \frac{g'(0)}{1!} \text{(by Lemma 4)}$$

$$= \frac{1}{9}.$$

Now, to find Res $\{f(z); 3i\}$; let $f(z) = \dfrac{h(z)}{k(z)}$ so that $h(z) = e^z$ and $k(z) = z^2(z^2+9)$.

$$\text{Then } k'(z) = 4z^3 + 18z$$

$$\therefore \quad \text{Res}\,\{f(z); 3i\} = \frac{h(3i)}{k'(3i)}$$

$$= \frac{e^{3i}}{4(3i)^3 + 18(3i)}$$

$$= \frac{e^{3i}}{-108i + 54i}$$

$$= -\frac{e^{3i}}{54i}$$

$$= \frac{i(\cos 3 + i\sin 3)}{54}.$$

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Similarly Res $\{f(z); -3i\} = -\dfrac{(\sin 3 + i\cos 3)}{54}$.

Problem 7. Use Laurent's series to find the residue of $\dfrac{e^{2z}}{(z-1)^2}$ at $z = 1$.

Solution. Let $f(z) = \dfrac{e^{2z}}{(z-1)^2}$.

First we expand $f(z)$ as Laurent's series at $z = 1$.

$$f(z) = \frac{e^{2(z-1)+2}}{(z-1)^2}$$

$$= \frac{e^2 e^{2(z-1)}}{(z-1)^2}$$

$$= \frac{e^2}{(z-1)^2}\left[1 + \frac{2(z-1)}{1!} + \frac{2^2(z-1)^2}{2!} + \frac{2^3(z-1)^3}{3!} + \cdots\right]$$

$$= e^2\left[\frac{1}{(z-1)^2} + \frac{2}{z-1} + 2 + \frac{4}{3}(z-1) + \cdots\right]$$

This is Laurent's series expansion for $f(z)$ at $z = 1$.

$$\text{Res}\{f(z); 1\} = \text{coefficient of } \frac{1}{z-1} \text{ in Laurent's expansion}$$

$$= 2e^2.$$

Note. Without expanding in laurent's series the residue at $z = 1$ can be found as follows. Since $f(z)$ has a pole of order 2 at $z = 1$ we choose $g(z) = e^{2z}$.

$$\therefore \text{ Res}\{f(z); 1\} = \frac{g'(1)}{1!} = \left[\frac{2e^{2z}}{1!}\right]_{z=1} = 2e^2.$$

Problem 8. Find the residue of $\dfrac{ze^z}{(z-1)^3}$ at its pole.

Solution. Let $f(z) = \dfrac{ze^z}{(z-1)^3}$.

$z = 1$ is a pole of order 3 for $f(z)$.

Let $g(z) = ze^z$ so that $g'(z) = e^z(z+1)$ and $g''(z) = e^z(z+2)$.

Then Res $\{f(z); 1\} = \dfrac{g''(1)}{2!} = \dfrac{3e}{2}$.

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Problem 9. Find the residue of $\frac{1}{z - \sin z}$ at its pole.

Solution. Let $f(z) = \frac{1}{z - \sin z}$.

$$\text{Now } z - \sin z = z - \left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \dots\right)$$

$$= \frac{z^3}{3!} - \frac{z^5}{5!} + \dots$$

$$= z^3\left(\frac{1}{3!} - \frac{z^2}{5!} + \dots\right).$$

$z = 0$ is a pole of order 3 for $f(z)$ and $f(z) = \frac{1}{z^3\left(\frac{1}{3!} - \frac{z^2}{5!} + \dots\right)}$.

Now let $g(z) = \frac{1}{\left(\frac{1}{3!} - \frac{z^2}{5!} + \dots\right)}$.

Then Res $\{f(z); 0\} = \frac{g''(0)}{2!}$. Clearly $g(0) = 6$.

Now $\frac{1}{g(z)} = \frac{1}{3!} - \frac{z^2}{5!} + \frac{z^4}{7!} - \dots$. Differentiating with respect to z, we have

$$-\frac{g'(z)}{[g(z)]^2} = -\frac{2z}{5!} + \frac{4z^3}{7!} - \dots.$$

Hence $g'(0) = 0$.
Again differentiating with respect to z we have

$$\frac{[g(z)]^2[-g''(z)] + g'(z)2g(z)g'(z)}{[g(z)]^4} = \frac{-2}{5!} + \frac{12z^2}{7!} - \dots.$$

Putting $z = 0$ and using $g(0) = 6$ and $g'(0) = 0$ we get $\frac{-g''(0)}{36} = \frac{-2}{5!}$.

Hence $g''(0) = \frac{3}{5}$.

$$\therefore \text{ Res }\{f(z); 0\} = \frac{g''(0)}{2!} = \frac{3}{10}.$$

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Exercises

1. Find the order of each pole and find the residue at the poles for each of the following functions.

(i) $\dfrac{z}{z^2+1}$ (ii) $\dfrac{z+1}{z^2-2z}$

(iii) $\dfrac{2z+3}{z(z^2+1)}$ (iv) $\dfrac{z^2}{z^2+a^2}$

(v) $\dfrac{1}{z^4+2z^2+1}$ (vi) $\dfrac{1}{z^2e^z}$

(vii) $\dfrac{1}{z^3(z+4)}$ (viii) $\dfrac{z+1}{z^2(z-2)}$

(ix) $\dfrac{\cos z}{z^3}$ (x) $\dfrac{\sin z}{z^4}$

(xi) $\left(\dfrac{z+1}{z-1}\right)^2$ (xii) $\dfrac{e^{2z}}{(z-1)^2}$

(xiii) $\dfrac{e^z}{z^2+\pi^2}$ (xiv) $\dfrac{1-e^{2z}}{z^4}$

(xv) $\dfrac{e^{iz}}{z^2+a^2}$ (a real) (xvi) $\dfrac{2z}{(z+4)(z-1)^2}$

2. Find the residue of $\dfrac{z^2-2z}{(z+1)^2(z^2+4)}$ at all its poles.

3. Find the residue of $\dfrac{1+e^z}{\sin z + z\cos z}$ at the pole $z=0$.

4. Calculate the residue of $\sec^2 z$ at $z=\dfrac{\pi}{2}$.

(Hint: $\sec^2 z = \dfrac{1}{\cos^2 z} = \dfrac{2}{1+\cos 2z}$).

5. Prove that the residue of $\dfrac{z^{2n}}{(1+z)^n}$ where $n \in \mathbf{N}$ at $z=-1$ is $\dfrac{(-1)^{n+1}(2n)!}{(n-1)!(n+1)!}$.

6. Find the residue of $\dfrac{1}{(1+z^2)^n}$ at $z=i$.

7. Prove that (i) Res $\{\tan z, \pi/2\} = -1$; (ii) Res $\left\{\dfrac{1-\cos z}{z^3}, 0\right\} = \dfrac{1}{2}$.

8. Show that all the singular points of $\dfrac{1}{z(e^z-1)}$ are poles. Find the order of poles and find the radius at the poles.

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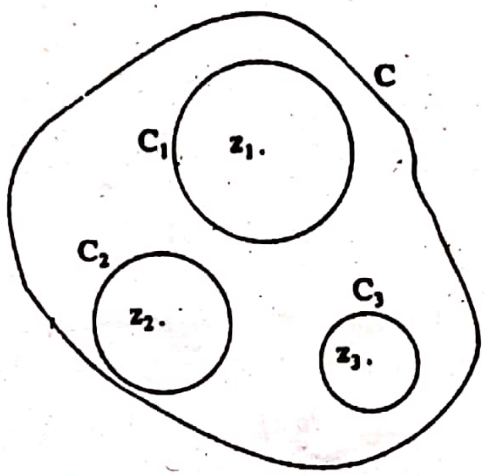
Answers.

1. (i) simple pole; Res $\frac{1}{2}$; $-i$ simple pole; Res $\frac{1}{2}$ (ii) $z = 0, 2$ simple pole; Res $-1/2$, $3/2$ (iii) $0, i, -i$ simple poles; Res 3, $(2i+3)/2$, $(2i-3)/2$ (iv) $ai, -ai$ simple poles; Res $ai/2$, $-ai/2$ (v) $i, -i$ order 2; Res $-i/4, i/4$ (vi) 0 pole of order 2; Res -1 (vii) 0 pole of order 3; Res $1/64$; -4 simple pole; Res $-1/64$ (viii) 0 pole of order 2; Res $-3/4$, 2 simple pole; Res $3/4$ (ix) 0 pole of order 3; Res $-1/2$ (x) 0 pole of order 4; Res -6 (xi) 1 simple pole; Res 4 (xii) 1 pole of order 2; Res $2e^2$ (xiii) $-\pi i$ simple pole; Res $1/2\pi i$ and πi simple pole; Res $-1/2\pi i$ (xiv) 0 pole of order 3; Res $-4/3$ (xv) ai, $-ai$ simple poles; Res $e^{-a}/2ai$, $e^{-a}/2ai$ (xvi) -4 simple pole; Res $8/25$, 1 pole of order 2; Res $8/25$ 2. $z = -1$; $-14/25$; $z = 2i$; $\frac{7+i}{25}$; $z = -2i$; $\frac{7-i}{25}$ 3. 1 4. 0 6. $\frac{-i(2n-2)!}{2^{2n-1}[(n-1)!]^2}$. 8. $z = 0$ pole of order 2; Res $-1/2$.

8.2. Cauchy's Residue Theorem

Theorem 8.1. (Cauchy's residue theorem)

Let $f(z)$ be a function which is analytic inside and on a simple closed curve C except for a finite number of singular points $z_1, z_2, \ldots\ldots, z_n$ inside C.



Then $\displaystyle\int_C f(z)dz = 2\pi i \sum_{j=1}^{n} \text{Res}\,\{f(z)\,;\,z_j\}$.

Proof. Let $C_1, C_2, \cdots, C_n$ be circles with centres $z_1, z_2, \cdots\cdots, z_n$ respectively such that all circles are interior to C and are disjoint with each other. (refer figure).

By Cauchy's theorem for multiply connected regions we have

$$\int_C f(z)dz = \int_{C_1} f(z)dz + \int_{C_2} f(z)dz + \cdots + \int_{C_n} f(z)dz.$$

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$= 2\pi i \operatorname{Res}\{f(z) ; z_1\} + 2\pi i \operatorname{Res}\{f(z); z_2\} + \cdots + 2\pi i \operatorname{Res}\{f(z); z_n\}$ (by definition of residue)

$= 2\pi i \sum_{j=1}^{n} \operatorname{Res}\{f(z) ; z_j\}$. Hence the theorem.

Example. Evaluate $\int_C \frac{z^2 dz}{(z-2)(z+3)}$ where C is the circle $|z| = 4$.

$$\text{Let } f(z) = \frac{z^2}{(z-2)(z+3)}.$$

$z = 2$ and $z = -3$ are simple poles for $f(z)$ and both of them lie inside $|z| = 4$.

$$\text{Now, } \operatorname{Res}\{f(z) ; 2\} = \lim_{z-2}(z-2)\left[\frac{z^2}{(z-2)(z+3)}\right] = \frac{4}{5}.$$

$$\operatorname{Res}\{f(z) ; -3\} = \lim_{z\to -3}(z+3)\left[\frac{z^2}{(z-2)(z+3)}\right] = -\frac{9}{5}.$$

$$\therefore \quad \text{By Residue theorem} \int_C f(z)dz = 2\pi i\left[\frac{4}{5} + \left(-\frac{9}{5}\right)\right]$$

$$= -2\pi i.$$

$$\therefore \int_C \frac{z^2 dz}{(z-2)(z+3)} = -2\pi i.$$

Theorem 8.2. (Argument theorem) Let f be a function which is analytic inside and on a simple closed curve C except for a finite number of poles inside C. Also let $f(z)$ have no zeros on C. Then $\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)} dz = N - P$ where N is the number of zeros of $f(z)$ inside C and P is the number of poles of $f(z)$ inside C. (A pole or zero of order m is counted m times).

Solution. We observe that the singularities of the function $\frac{f'(z)}{f(z)}$ inside C are the poles and zeros of $f(z)$ lying inside C.

Let z_0 be a zero of order n for $f(z)$. Let C_1 be a circle with centre z_0 such that it is the only zero of $f(z)$ inside C_1.

Then $f(z) = (z - z_0)^n g(z)$ where $g(z)$ is analytic and nonzero inside C_1. Hence $f'(z) = n(z - z_0)^{n-1} g(z) + (z - z_0)^n g'(z)$.

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$$\therefore \quad \frac{f'(z)}{f(z)} = \frac{n}{z - z_0} + \frac{g'(z)}{g(z)} \qquad \ldots (1)$$

Since $g(z)$ is analytic and non zero inside C_1, $\frac{g'(z)}{g(z)}$ is also analytic and hence can be expanded as a Taylor's series about z_0.

$$\therefore \quad \text{Res}\left\{\frac{f'(z)}{f(z)}; z_0\right\} = \text{coefficient of } \frac{1}{z - z_0} \text{ in (1)}$$

$$= n$$

$f(z) = \frac{1}{(z-z_0)^p} \cdot g(z) \Rightarrow f(z) = (z-z_0)^{-p} g(z)$

Similarly if z_1 is a pole of order p for $f(z)$, then $\text{Res}\left\{\frac{f'(z)}{f(z)}; z_1\right\} = -p$.

Hence by Cauchy's residue theorem, $\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}dz = N - P$ where N is the number of zeros and P is the number of poles of $f(z)$ within C.

Corollary. If $f(z)$ is analytic inside and on C and not zero on C, then $\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}dz = N$ where N is the number of zeros lying inside C.

Proof. Since the number of poles is zero we have $P = 0$.
Hence the result follows.

Theorem 8.3. (Rouche's theorem) If $f(z)$ and $g(z)$ are analytic inside and on a simple closed curve C and if $|g(z)| < |f(z)|$ on C then $f(z) + g(z)$ and $f(z)$ have the same number of zeros inside C.

Proof. $f(z) + g(z) = f(z)\left[1 + \frac{g(z)}{f(z)}\right] = f(z)\varphi(z)$ where $\varphi(z) = 1 + \frac{g(z)}{f(z)}$.

Hence $[f(z) + g(z)]' = f'(z) + g'(z) = f'(z)\varphi(z) + f(z)\varphi'(z)$.

$$\therefore \quad \frac{f'(z) + g'(z)}{f(z) + g(z)} = \frac{f'(z)\varphi(z) + f(z)\varphi'(z)}{f(z)\varphi(z)}$$

$$= \frac{f'(z)}{f(z)} + \frac{\varphi'(z)}{\varphi(z)}$$

$$\therefore \quad \frac{1}{2\pi i}\int_C \left[\frac{f'(z) + g'(z)}{f(z) + g(z)}\right] dz = \frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}dz + \frac{1}{2\pi i}\int_C \frac{\varphi'(z)}{\varphi(z)}dz. \qquad \ldots (1)$$

Now, by hypothesis $|g(z)| < |f(z)|$ and hence $\left|\frac{g(z)}{f(z)}\right| < 1$ on C.

$$\therefore \quad |\varphi(z) - 1| < 1 \quad \text{on } C.$$

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Hence by maximum modulus theorem, $|\varphi(z) - 1| < 1$ for every point z inside C.

$\therefore \quad \varphi(z) \neq 0$ for every point inside C.

$$\text{Hence } \int_C \frac{\varphi'(z)}{\varphi(z)} dz = \text{Number of zeros of } \varphi(z) \text{ within } C.$$

$$= 0.$$

Hence from (1), we have $\dfrac{1}{2\pi i} \displaystyle\int_C \left[\frac{f'(z) + g'(z)}{f(z) + g(z)} \right] dz = \frac{1}{2\pi i} \int_C \frac{f'(z)}{f(z)} dz$

$\therefore \quad N_1 = N_2$ where N_1 and N_2 denote respectively the number of zeros of $f(z) + g(z)$ and $f(z)$ inside C. Hence the theorem.

Remark. We can deduce the Fundamental theorem of Algebra from Rouche's theorem.

Theorem 8.4. (Fundamental theorem of algebra)
A polynomial of degree n with complex coefficients has n zeros in **C**.

Proof. Let $a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n$, where $a_n \neq 0$, be a polynomial of degree n. Let $f(z) = a_n z^n$ and $g(z) = a_0 + a_1 + \cdots + a_{n-1} z^{n-1}$.

$$\text{Clearly } \lim_{z \to \infty} \frac{g(z)}{f(z)} = 0.$$

Hence there exists a positive real number r such that $\left| \dfrac{g(z)}{f(z)} \right| < 1$ for all z with $|z| > r$.

Hence by Rouche's theorem $f(z)$ and $f(z) + g(z)$ have the same number of zeros inside the circle $|z| = r + 1$. But 0 is a zero of multiplicity n for $f(z)$. Hence the given polynomial $f(z) + g(z)$ also has n zeros.

Solved Problems

Problem 1. Evaluate $\displaystyle\int_C \frac{dz}{2z+3}$ where C is $|z| = 2$.

Solution. $z = -\dfrac{3}{2}$ is the simple pole of $f(z)$ which lies inside the circle $|z| = 2$.

$$\text{Res}\left\{ f(z); -\frac{3}{2} \right\} = \lim_{z \to -3/2} \frac{h(z)}{k'(z)} \text{ where } h(z) = 1 \text{ and } k(z) = 2z + 3.$$

$$\therefore \text{Res}\left\{ f(z); -\frac{3}{2} \right\} = \frac{1}{2}.$$

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$\therefore$ By residue theorem $\int_C f(z)dz = 2\pi i\left(\frac{1}{2}\right) = \pi i.$

Aliter. $$\int_C \frac{dz}{2z+3} = \int_C \frac{dz}{2\left(z+\frac{3}{2}\right)}$$

$$= \frac{1}{2}\int_C \frac{dz}{\left(z+\frac{3}{2}\right)}$$

$$= \frac{1}{2}(2\pi i)\left(\because \int_C \frac{dz}{z-a} = 2\pi i\right)$$

$$= \pi i.$$

Problem 2. Evaluate $\int_C \frac{dz}{z^2 e^z}$ where $C = \{z : |z| = 1\}$.

Solution. Given integral can be written as $\int_C f(z)dz$ where $f(z) = \frac{e^{-z}}{z^2}$.

$f(z)$ has pole of order 2 at $z = 0$ which lies inside the circle $|z| = 1$.

Let $g(z) = e^{-z}$. Hence $g'(z) = -e^{-z}$.

$\therefore$ By Lemma 4, Res $\{f(z); 0\} = \frac{g'(0)}{1!} = -1$.

By residue theorem $\int_C f(z)dz = 2\pi i(-1) = -2\pi i.$

Problem 3. Evaluate $\int_C \frac{2+3\sin\pi z}{z(z-1)^2}dz$ where C is the square having vertices $3+3i, 3-3i, -3+3i, -3-3i$.

Solution. Let $f(z) = \frac{2+3\sin\pi z}{z(z-1)^2}$. Here $z = 0$ is a simple pole and $z = 1$ is a double pole for $f(z)$ and both of them lie within C.

$$\text{Res}\,\{f(z); 0\} = \lim_{z\to 0} z\left(\frac{2+3\sin\pi z}{z(z-1)^2}\right) = 2.$$

$$\text{Res}\,\{f(z); 1\} = \frac{g'(1)}{1!} \text{ where } g(z) = \frac{2+3\sin\pi z}{z}.$$

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$$g'(z) = \frac{z3\pi\cos\pi z - (2 + 3\sin\pi z)}{z^2}.$$

$$\therefore\ g'(1) = -3\pi - 2.$$

$$\therefore\ \text{Res}\,\{f(z);\,1\} = -3\pi - 2.$$

$\therefore$ By residue theorem $\displaystyle\int_C f(z)dz = 2\pi i(2 - 3\pi - 2) = -6\pi^2 i.$

Problem 4. Evaluate $\displaystyle\int_C \tan z dz$ where C is $|z| = 2$.

Solution. Let $f(z) = \tan z = \dfrac{\sin z}{\cos z} = \dfrac{h(z)}{k(z)}$.

$$\cos z \text{ has zeros at } z = \frac{(2n+1)\pi}{2},\ n \in \mathbf{N}.$$

$\therefore\ f(z)$ has simple poles at $z = -\dfrac{\pi}{2}$ and $z = \dfrac{\pi}{2}$ inside the circle $|z| = 2$.

$$\text{Res}\,\{f(z);\,\pi/2\} = \frac{h(\pi/2)}{k'(\pi/2)} = \frac{\sin(\pi/2)}{-\sin(\pi/2)} = -1.$$

$$\text{Res}\,\{f(z);\,-\pi/2\} = \frac{h(-\pi/2)}{k'(-\pi/2)} = \frac{\sin(-\pi/2)}{-\sin(-\pi/2)} = -1.$$

By residue theorem $\displaystyle\int_C \tan z dz = 2\pi i[(-1) + (-1)] = -4\pi i.$

Problem 5. Prove that $\displaystyle\int_C \frac{e^{2z}}{(z+1)^3}dz = \frac{4\pi i}{e^2}$ where C is $|z| = \dfrac{3}{2}$.

Solution. Let $f(z) = \dfrac{e^{2z}}{(z+1)^3}$.

$f(z)$ has a pole of order 3 at $z = -1$.

$\text{Res}\,\{f(z);\,-1\} = \dfrac{g''(-1)}{2!}$ where $g(z) = e^{2z}$.

Now $g'(z) = 2e^{2z}$ and $g''(z) = 4e^{2z}$.

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$\therefore\ \text{Res}\{f(z); -1\} = \dfrac{4e^{-2}}{2!} = \dfrac{2}{e^2}$.

$\therefore$ By residue theorem $\displaystyle\int_C f(z)dz = 2\pi i\left(\frac{2}{e^2}\right) = \frac{4\pi i}{e^2}$.

Problem 6. Evaluate, using (i) Cauchy's integral formula (ii) residue theorem $\displaystyle\int_C \frac{z+1}{z^2+2z+4}dz$ where C is the circle $|z+1+i| = 2$.

Solution. Clearly C is a circle with centre $a = -(1+i)$ and radius 2.

$$\text{Now } \frac{z+1}{z^2+2z+4} = \frac{z+1}{(z+1)^2+(\sqrt{3})^2}$$

$$= \frac{z+1}{(z+1+i\sqrt{3})(z+1-i\sqrt{3})}$$

$$= \frac{z+1}{[z-(-1-i\sqrt{3})][(z-(-1+i\sqrt{3})]}.$$

$z_0 = -1+i\sqrt{3}$ and $z_1 = -1-i\sqrt{3}$ are the singular points of the given integrand $\dfrac{z+1}{z^2+2z+4}$.

Now $|z_0 - a| = \left|i(\sqrt{3}+1)\right| = \sqrt{3}+1 > 2$

and $|z_1 - a| = \left|-i(\sqrt{3}-1)\right| = \sqrt{3}-1 < 2$.

$\therefore\quad z_1 = -1-i\sqrt{3}$ lies inside C.

(i) *By using Cauchy integral formula.*

Consider $f(z) = \dfrac{z+1}{z-(-1-i\sqrt{3})}$.

We note that $f(z)$ is analytic at all points inside C.

$\therefore$ By Cauchy's integral formula $\displaystyle\frac{1}{2\pi i}\int_C \frac{f(z)}{z-z_1}dz = f(z_1)$;

$$\text{(i.e.) } \frac{1}{2\pi i}\int_C \frac{(z+1)dz}{[z-(1-i\sqrt{3})][z-(-1+i\sqrt{3})]} = f(-1-i\sqrt{3})$$

$$\text{(i.e.) } \frac{1}{2\pi i}\int_C \frac{(z+1)\,dz}{z^2+2z+4} = \frac{(-1-i\sqrt{3})+1}{(-1-i\sqrt{3})-(-1+i\sqrt{3})}$$

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$$= \frac{-i\sqrt{3}}{-2i\sqrt{3}}$$

$$= \frac{1}{2}.$$

$$\therefore \int_C \frac{(z+1)dz}{z^2+2z+4} = \frac{1}{2}(2\pi i) = \pi i.$$

(ii) *By using residue theorem.*

$$f(z) = \frac{z+1}{z^2+2z+4}.$$

Since $z = -1 - i\sqrt{3}$ lies inside C

Res $\{f(z); -1 - i\sqrt{3}\} = \dfrac{h(-1-i\sqrt{3})}{k'(-1-i\sqrt{3})}$ where $h(z) = z + 1$ and

$k(z) = z^2 + 2z + 4$ so that $k'(z) = 2z + 2$.

$$\therefore \text{ Res } \{f(z); -1 - i\sqrt{3}\} = \frac{-1 - i\sqrt{3} + 1}{2(-1 - i\sqrt{3}) + 2} = \frac{-i\sqrt{3}}{-i2\sqrt{3}} = \frac{1}{2}.$$

By residue theorem $\displaystyle\int_C f(z)dz = \frac{2\pi i}{2} = \pi i.$

Problem 7. Use residue calculus to evaluate $\displaystyle\int_C \frac{3\cos z}{2i - 3z}dz$ where C is the unit circle.

Solution. Let $f(z) = \dfrac{3\cos z}{2i - 3z}.$

Here $z = \dfrac{2i}{3}$ is a simple pole and lies within C.

Res $\left\{f(z); \dfrac{2i}{3}\right\} = \displaystyle\lim_{z \to 2i/3} \frac{h(z)}{k'(z)}$ where $h(z) = 3\cos z$ and $k(z) = 2i - 3z$ so that $k'(z) = -3.$

$$\therefore \text{ Res } \{f(z); 2i/3\} = \frac{3\cos(2i/3)}{-3} = -\cos(2i/3) = -\cosh(2/3).$$

By residue theorem $\displaystyle\int_C f(z)dz = 2\pi i[-\cosh(2/3)].$

(i.e.) $\displaystyle\int_C \frac{3\cos z}{2i - 3z}dz = -2\pi i\cosh(2/3).$

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Problem 8. Use residue theorem to evaluate $\int \frac{3z^2+z-1}{(z^2-1)(z-3)}dz$ around the circle $|z|=2$.

Solution. Let $f(z) = \frac{3z^2+z-1}{(z^2-1)(z-3)}$.

$f(z)$ has simple poles $1, -1, 3$ and only $1, -1$ lie inside $|z|=2$.

$\text{Res}\,\{f(z); 1\} = \frac{h(1)}{k'(1)}$ where $h(z) = 3z^2+z-1$ and $k(z) = z^3-3z^2-z+3$ so that $k'(z) = 3z^2-6z-1$.

$$\therefore\ \text{Res}\,\{f(z); 1\} = \frac{3+1-1}{3-6-1} = \frac{-3}{4}.$$

$$\text{Similarly Res}\,\{f(z); -1\} = \frac{3-1-1}{3+6-1} = \frac{1}{8}.$$

$\therefore$ By residue theorem,

$$\int \frac{3z^2-z-1}{(z^2-1)(z-3)}dz = 2\pi i\left(-\frac{3}{4}+\frac{1}{8}\right) = 2\pi i\left(\frac{-5}{8}\right) = \frac{-5\pi i}{4}$$

Problem 9. Evaluate $\int_C \frac{e^z dz}{(z+2)(z-1)}$ where C is the circle $|z-1|=1$.

Solution. $f(z) = \frac{e^z}{(z+2)(z-1)}$.

$f(z)$ has simple poles at $1, -2$; the pole 1 is inside the circle $|z-1|=1$ and $z=-2$ lies outside the circle.

$$\text{Res}\,\{f(z); 1\} = \lim_{z\to 1}(z-1)\left(\frac{e^z}{(z+2)(z-1)}\right) = \frac{e}{3}.$$

By residue theorem $\int_C f(z)dz = 2\pi i\left(\frac{e}{3}\right)$.

$$\therefore \int \frac{e^z}{(z+2)(z-1)}dz = \frac{i2\pi e}{3}.$$

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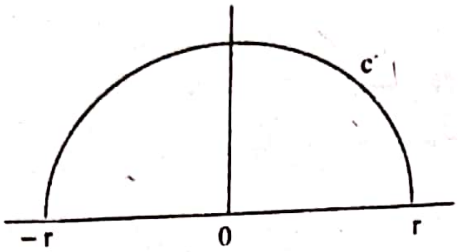
Problem 10. Show that the function $2+z^2-e^{iz}$ has precisely one zero in the open upper half plane.



Solution. Take $f(z) = 2 + z^2$ and $g(z) = -e^{iz}$. Let C be the simple closed curve consisting of the semi circle $|z| = r$ in the upper half plane together with the interval $[-r, r]$ on the real axis.

If $z \in [-r, r]$ then $|g(z)| = 1$ and $|f(z)| \geq 2$.

Hence $|f(z)| > |g(z)|$.

Now, if $z = re^{i\theta}, \theta < \theta < \pi$ then $|f(z)| = |2 + z^2| \geq |z^2| - 2 = r^2 - 2$

Also $|g(z)| = \left|-e^{ire^{i\theta}}\right| = e^{-r\sin\theta}$.

Hence for sufficiently large value of r we have $|f(z)| > |g(z)|$.

Hence by Rouche's theorem $f(z) + g(z) = 2 + z^2 - e^{iz}$ and $f(z) = 2 + z^2$ have the same number of zeros in the upper half plane. Also $2 + z^2$ has exactly one zero in the upper half of the plane namely $i\sqrt{2}$.

Hence $2 + z^2 - e^{iz}$ has exactly one root in the upper half plane.

Problem 11. Let $f(z) = \dfrac{(z^2+1)}{(z^2+2z+2)^2}$. Evaluate $\dfrac{1}{2\pi i}\displaystyle\int_C \dfrac{f'(z)}{f(z)}dz$ where C is the circle $|z| = 4$.

Solution. i and $-i$ are zeros of order 1 and $-1+i$ and $-1-i$ are poles of order 2 for $f(z)$. Also these zeros and poles lie inside C.
Hence number of zeros of $f(z) = N = 2$ and number of poles of $f(z) = P = 4$. (Poles are counted according to their multiplicity)

$\therefore$ By Argument theorem $\dfrac{1}{2\pi i}\displaystyle\int_C \dfrac{f'(z)}{f(z)}dz = N - P = 2 - 4 = -2$

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Exercises

1. Evaluate the following integrals.

(i) $\int_C \frac{3z-4}{z(z-1)}dz$ where C is the circle $|z|=2$.

(ii) $\int_C \frac{(3z-4)}{z(z-1)(z-2)}dz$ where C is the circle $|z|=\frac{3}{2}$

(iii) $\int_C \frac{2z^2+4}{z^2-1}dz$ where (a) C is the circle $|z|=2$ (b) C is the circle $|z-1|=1$.

(iv) $\int_C \frac{3dz}{z+1}$ where C is the circle $|z|=2$.

(v) $\int_C \frac{3+z}{z}dz$ where C is the circle $|z|=1$.

(vi) $\int_C \frac{z+1}{z^2-2z}dz$ where C is the circle $|z|=3$.

(vii) $\int_C \frac{3z^3+2}{(z-1)(z^2+4)}dz$ where C is (a) $|z-2|=2$ (b) $|z|=4$.

(viii) $\int_C \frac{dz}{z^3(z-1)}$ where C is the circle $|z|=3$.

(ix) $\int_C \frac{e^{-z}}{z^2}dz$ where C is the circle $|z|=1$.

(x) $\int_C \frac{dz}{z^3(z+4)}dz$ where C is (a) $|z|=2$ (b) $|z+2|=3$.

(xi) $\int_C \frac{e^z dz}{z(z-1)^2}dz$ where C is the circle $|z[=2$.

(xii) $\int_C \frac{dz}{z^2(z+1)}dz$ where C is the circle $|z|=\frac{1}{2}$.

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(xiii) $\int_C \frac{e^{zt}}{z^2(z^2+2z+2)} dz$ where C is the circle $|z| = 3$.

(xiv) $\int_C \frac{e^{zt}}{z(z^2+1)} dz (t > 0)$ where C is the square with vertices at $1+i, -1+i, -1-i, 1-i$.

(xv) $\int_C \frac{\cosh z dz}{z^3}$ where C is the square with vertices $\pm 2 \pm 2i$.

2. Prove that $\int_C \coth z dz = 0$ where C is the circle $|z| = 1$.

3. Prove that $\int_C z e^{1/z} dz = \pi i$ where C is the circle $|z| = 5$.

4. Prove that $\int_C \frac{e^z dz}{\cosh z} = 8\pi i$ where C is the circle $|z| = 5$.

Answers.

1. (i) $6\pi i$ (ii) $-2\pi i$ (iii) (a) $2\pi i$ (b) $3\pi i$ (iv) $-2\pi i$ (v) $6\pi i$ (vi) $2\pi i$ (vii) (a) πi (b) $6\pi i$ (viii) $4\pi i$ (ix) $-2\pi i$ (x) (a) $\frac{\pi i}{32}$ (b) 0 (xi) $2\pi i$ (xii) $-2\pi i$ (xiii) $2\pi i\left[\frac{t-1}{2} + \frac{e^{-t}}{4}(\sin t + \cos t)\right]$ (xiv) $2\pi i(1-\cos t)$ (xv) πi

8.3. Evaluation of Definite Integrals

We use Cauchy's residue theorem for evaluating certain types of real definite integrals.

TYPE 1. $\int_0^{2\pi} f(\cos\theta, \sin\theta) d\theta$ where $f(\cos\theta, \sin\theta)$ is a rational function of $\cos\theta$ and $\sin\theta$.

To evaluate this type of integral we substitute $z = e^{i\theta}$. As θ varies from 0 to 2π, z describes the unit circle $|z| = 1$

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Also, $\cos\theta = \dfrac{e^{i\theta}+e^{-i\theta}}{2} = \dfrac{z+z^{-1}}{2}$ and

$$\sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i} = \frac{z-z^{-1}}{2i}$$

Substituting these values in the given integrand the integral is transformed into $\int_C \theta(z)dz$ where $\theta(z) = f\left[\dfrac{z+z^{-1}}{2}, \dfrac{z-z^{-1}}{2i}\right]$ and C is the positively oriented unit circle $|z| = 1$. The integral $\int_C \theta(z)dz$ can be evaluated using the residue theorem.

Solved problems

Problem 1. Evaluate $\displaystyle\int_0^{2\pi} \frac{d\theta}{5+4\sin\theta}$

Solution. Let $I = \displaystyle\int_0^{2\pi} \frac{d\theta}{5+4\sin\theta}$.

Put $z = e^{i\theta}$.

Then $dz = izd\theta$ and $\sin\theta = \dfrac{z-z^{-1}}{2i}$.

The given integral is transformed to $I = \displaystyle\int_C \frac{dz}{iz\left[5+4\left(\frac{z-z^{-1}}{2i}\right)\right]}$ where C is the unit circle $|z| = 1$.

$$= \int_C \frac{dz}{2z^2+5iz-2}$$

Let $\quad f(z) = \dfrac{1}{2z^2+5iz-2} = \dfrac{1}{2(z+2i)(z+i/2)}$

$\therefore \quad -2i$ and $-i/2$ are simple poles of $f(z)$ and the pole $-i/2$ lies inside C.

Also Res $\{f(z)\,;\,-i/2\} = \displaystyle\lim_{z\to -i/2} \frac{1}{2(z+2i)} = \frac{1}{3i}$.

Hence by Cauchy's residue theorem $I = 2\pi i\left(\dfrac{1}{3i}\right) = \dfrac{2\pi}{3}$.

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Problem 2. Prove that $\displaystyle\int_0^{2\pi} \frac{d\theta}{1+a\sin\theta} = \frac{2\pi}{\sqrt{1-a^2}}$, $(-1 < a < 1)$

Solution. Put $z = e^{i\theta}$. Then $\sin\theta = \dfrac{z - z^{-1}}{2i}$ and $dz = iz\,d\theta$.

$$\int_0^{2\pi} \frac{d\theta}{1+a\sin\theta} = \int_C \frac{dz}{iz\left[1 + a\left(\dfrac{z - z^{-1}}{2i}\right)\right]} \text{ where } C \text{ is the unit circle.}$$

$$= \int_C \frac{2dz}{z[2i + a(z - z^{-1})]}$$

$$= \int_C \frac{2dz}{az^2 + 2iz - a}$$

Let $\quad f(z) = \dfrac{2}{az^2 + 2iz - a}$

The poles of $f(z)$ are given by $z = \dfrac{-2i \pm \sqrt{-4 + 4a^2}}{2a}$

$$= \frac{-i \pm i\sqrt{1-a^2}}{a} \text{ (since } -1 < a < 1)$$

Let $z_1 = \dfrac{-i + i\sqrt{1-a^2}}{a}$ and $z_2 = \dfrac{-i - i\sqrt{1-a^2}}{a}$

We note that $|z_2| = \dfrac{1 + \sqrt{1-a^2}}{|a|} > 1 \quad$ (since $-1 < a < 1$)

Also, since $|z_1 z_2| = 1$ it follows the $|z_1| < 1$. Hence there are no singular points on C and $z = z_1$ is the only simple pole insisde C

$$\text{Res}\{f(z)\,;\,z_1\} = \lim_{z\to z_1} (z - z_1)\left[\frac{2/a}{(z - z_1)(z - z_2)}\right]$$

$$= \frac{2/a}{z_1 - z_2}$$

$$= \frac{1}{i\sqrt{1-a^2}}$$

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By residue theorm $\displaystyle\int_0^{2\pi} \frac{d\theta}{1 + a\sin\theta} = 2\pi i\left[\frac{1}{i\sqrt{1-a^2}}\right]$

$$= \frac{2\pi}{\sqrt{1-a^2}}$$

Problem 3. Prove that $I = \displaystyle\int_0^{\pi} \frac{a\,d\theta}{a^2 + \sin^2\theta} = \frac{\pi}{\sqrt{a^2+1}} (a > 0)$

Solution. $I = \displaystyle\int_0^{\pi} \frac{a\,d\theta}{a^2 + \left(\frac{1-\cos 2\theta}{2}\right)}$

$$= \int_0^{\pi} \frac{2a\,d\theta}{2a^2 + 1 - \cos 2\theta}$$

$2\theta = \phi \quad 2d\theta = d\phi$
$\theta = 0, \quad \phi = 0$
$\theta = \pi, \quad \phi = 2\pi$

$$= \int_0^{2\pi} \frac{a\,d\varphi}{2a^2 + 1 - \cos\varphi} \quad \text{(putting } 2\theta = \varphi\text{)}$$

$$= \frac{1}{i}\int_C \frac{a\,dz}{z\left[2a^2 + 1 - \frac{(z + z^{-1})}{2}\right]} \quad \text{(putting } z = e^{i\varphi}\text{)}$$

$$= \frac{2a}{i}\int_C \frac{dz}{[2(2a^2+1)z - z^2 - 1]}$$

$$= 2ai\int_C \frac{dz}{z^2 - 2(2a^2+1)z + 1}$$

$z^2 - (4a^2+2)z + 1 = 0$
$z = \frac{4a^2 + 2 \pm \sqrt{16a^4 + 4 + 16a^2 - 4}}{2}$

$$= 2ai\int_C f(z)dz \quad \ldots (1)$$

$z = \frac{(4a^2+2) \pm 4a\sqrt{a^2+1}}{2}$

where $f(z) = \dfrac{1}{z^2 - 2(2a^2+1)z + 1}$ and C is the unit circle $|z| = 1$.

$z = (2a^2+1) \pm 2a\sqrt{a^2+1}$

Poles of $f(z)$ are the roots of $z^2 - 2(2a^2+1)z + 1 = 0$.

$\therefore \quad z = (2a^2+1) \pm 2a\sqrt{a^2+1}$

Let $z_1 = (2a^2+1) + 2a\sqrt{a^2+1}$; $z_2 = (2a^2+1) - 2a\sqrt{a^2+1}$

Clearly $|z_1| > 1$ and $|z_1 z_2| = 1$ so that $|z_2| < 1$.

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Hence the only pole inside C is $z = z_2$.

$$\text{Res}\quad \{f(z)\,;\,z_2\} = \lim_{z \to z_2} (z - z_2)\frac{1}{(z - z_1)(z - z_2)}$$

$$= \frac{1}{z_2 - z_1}$$

$$= \frac{1}{(-4a)\sqrt{a^2 + 1}}$$

From (1), $I = 2\pi i \left[\dfrac{2ai}{-4a\sqrt{a^2 + 1}}\right]$

$$= \frac{\pi}{\sqrt{a^2 + 1}}$$

Problem 4. Using Cantour integration evaluate $\displaystyle\int_0^{2\pi} \frac{d\theta}{13 + 5\sin\theta}$.

Solution. Let $I = \displaystyle\int_0^{2\pi} \frac{d\theta}{13 + 5\sin\theta}$.

Put $z = e^{i\theta}$. Then $dz = ie^{i\theta}d\theta = izd\theta$.

Also $\sin\theta = \dfrac{z - z^{-1}}{2i}$.

∴ The given integral is transformed to

$$I = \int_C \frac{dz}{iz\left[13 + 5\left(\frac{z - z^{-1}}{2i}\right)\right]} \quad \text{(where } C \text{ is the circle } |z| = 1)$$

$$= \int_C \frac{dz}{iz\left[13 + 5\left(\frac{z^2 - 1}{i2z}\right)\right]}$$

$$= \int_C \frac{2dz}{5z^2 + 26iz - 5}.$$

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Let $f(z) = \dfrac{2}{5z^2 + i26z - 5} = \dfrac{2}{(z+5i)(5z+i)}$.

$-\dfrac{i}{5}$ and $-5i$ are simple poles of $f(z)$ and the pole $-\dfrac{i}{5}$ lies inside the unit circle.

$$\text{Res}\left\{f(z); -\frac{i}{5}\right\} = \lim_{z \to -i/5} \frac{h(z)}{k'(z)} \quad (\text{where } h(z) = 2 \text{ and } k(z) = 5z^2 + i26z - 5)$$

$$= \lim_{z \to -i/5} \left(\frac{2}{10z + i26}\right)$$

$$= \frac{2}{-2i + 26i}$$

$$= \frac{1}{12i}.$$

Hence by Cauchy's residue theorem $I = 2\pi i \left(\dfrac{1}{12i}\right) = \dfrac{\pi}{6}$.

Problem 5. Use Contour integration technique to find the value of $\displaystyle\int_0^{2\pi} \frac{d\theta}{2 + \cos\theta}$.

Solution. Let $I = \displaystyle\int_0^{2\pi} \frac{d\theta}{2 + \cos\theta}$.

Put $z = e^{i\theta}$. Then $dz = ie^{i\theta} d\theta = izd\theta$.

Also $\cos\theta = \dfrac{z + z^{-1}}{2}$.

∴ The given integral is transformed to $I = \displaystyle\int_C \frac{dz}{iz\left[2 + \dfrac{z + z^{-1}}{2}\right]}$ where C is the unit circle $|z| = 1$.

$$\therefore I = \int_C \frac{dz}{iz\left[2 + \left(\dfrac{z^2+1}{2z}\right)\right]}$$

$$= \int_C \frac{2dz}{i(4z + z^2 + 1)}$$

$$= \int_C \frac{-2idz}{z^2 + 4z + 1}.$$

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$$\text{Let } f(z) = \frac{-2i}{z^2+4z+1}$$

$$= \frac{-2i}{(z+2)^2-3}$$

$$= \frac{-2i}{(z+2-\sqrt{3})(z+2+\sqrt{3})}.$$

$\therefore$ $-2+\sqrt{3}$ and $-2-\sqrt{3}$ are simple poles of $f(z)$; the pole $-2+\sqrt{3}$ lies inside C.

$$\text{Res}\left\{f(z); -2+\sqrt{3}\right\} = \lim_{z\to -2+\sqrt{3}} \left(\frac{-2i}{2z+4}\right)$$

$$= \frac{-2i}{-4+2\sqrt{3}+4}$$

$$= \frac{-i}{\sqrt{3}}.$$

Hence by Cauchy's residue theorem $I = 2\pi i\left(\frac{-i}{\sqrt{3}}\right) = \frac{2\pi}{\sqrt{3}}$.

Exercises

1. Show that $\displaystyle\int_0^{2\pi} \frac{d\theta}{5+3\cos\theta} = \frac{\pi}{2}$.

2. Show that $\displaystyle\int_0^{2\pi} \frac{\cos 2\theta d\theta}{5+4\cos\theta} = \frac{\pi}{6}$.

3. Show that $\displaystyle\int_0^{2\pi} \frac{d\theta}{2+\cos\theta} = \frac{2\pi}{\sqrt{3}}$.

4. Show that $\displaystyle\int_0^{2\pi} \frac{\cos^2 3\theta d\theta}{5-4\cos 2\theta} = \frac{3\pi}{8}$.

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5. Show that $\int_0^{2\pi} \frac{d\theta}{\cos\theta + 2\sin\theta + 4} = \frac{2\pi}{\sqrt{11}}$.

6. Show that $\int_0^{\pi} \frac{d\theta}{a + \cos\theta} = \frac{\pi}{\sqrt{a^2 - 1}} (a > 1)$.

7. Show that $\int_0^{2\pi} \frac{d\theta}{a + b\cos\theta} = \frac{2\pi}{\sqrt{a^2 - b^2}} (a > b > 0)$.

8. Show that $\int_0^{2\pi} \frac{d\theta}{1 + a\sin\theta} = \frac{2\pi}{\sqrt{1 - a^2}} (a^2 < 1)$.

9. Show that $\int_0^{\pi} \frac{d\theta}{(a + \cos\theta)^2} = \frac{2a\pi}{(a^2 - 1)^{3/2}} (a > 1)$.

10. Show that $\int_0^{\pi} \frac{1 + 2\cos\theta}{5 + 4\cos\theta} d\theta = 0$.

11. Show that $\int_0^{2\pi} \frac{d\theta}{a^2 - 2a\cos\theta + 1} = \frac{2\pi}{1 - a^2} (0 < a < 1)$.

TYPE 2. $\int_{-\infty}^{\infty} f(x)dx$ where $f(x) = \frac{g(x)}{h(x)}$ and $g(x)$, $h(x)$ are polynomials in x and the degree of $h(x)$ exceeds that of $g(x)$ by at least two. $h(x) \geq g(x) + 2$.

To evaluate this type of integral we take $f(z) = \frac{g(z)}{h(z)}$.

The poles of $f(z)$ are determined by the zeros of the equation $h(z) = 0$.

Case (i) ***No pole of f(z) lies on the real axis.***

We choose the curve C consisting of the interval $[-r, r]$ on the real axis and the semi circle $|z| = r$ lying in the upper half of the plane.

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Here r is chosen sufficiently large so that all the poles lying in the upper half of the plane are in the interior of C. Then we have

$$\int_C f(z)dz = \int_{-r}^{r} f(x)dx + \int_{C_1} f(z)dz$$

where C_1 is the semi circle.

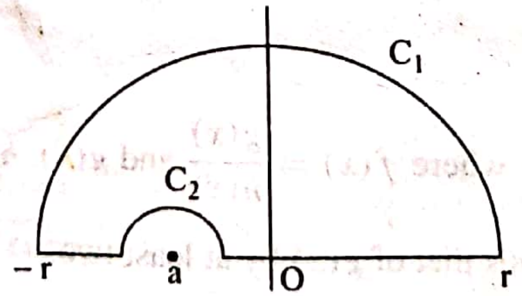
Since $\deg h(x) - \deg g(x) \geq 2$ it follows that $\int_{C_1} f(z)dz \to 0$ as $r \to \infty$ and hence

$$\int_C f(z)dz = \int_{-\infty}^{\infty} f(x)dx.$$

$\therefore \int_{-\infty}^{\infty} f(x)dx$ can be evaluated by evaluating $\int_C f(z)dz$ which in turn can be evaluated by using Cauchy's residue theorem.

Case (ii) $f(z)$ ***has poles lying on the real axis.***

Suppose a is a pole lying on the real axis. In this case we indent the real axis by a semi-circle C_2 of radius ε with centre a lying in the upper half plane where ε is chosen to be sufficiently small (refer figure).



Such an indenting must be done for every pole of $f(z)$ lying on the real axis.

It can be proved that $\int_{C_2} f(z)dz = -\pi i \text{ Res } \{f(z); a\}$. By taking limit as $r \to \infty$ and $\varepsilon \to 0$ we obtain the value of $\int_{-\infty}^{\infty} f(x)dx$.

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$z = (\cos(2n+1)\pi + i\sin(2n+1)\pi)^{1/4}$
$= \cos(2n+1)\pi/4 + i\sin(2n+1)\pi/4$
$n = 0, 1, 2, 3$

Solved Problems

Problem 1. Use Contour integration method to evaluate $\int\limits_0^\infty \frac{dx}{1+x^4}$.

Solution. Let $f(z) = \frac{1}{1+z^4}$. $\Rightarrow z^4 = -1 \Rightarrow e^{\pi i} \Rightarrow$ $z = [\cos(\pi + 2n\pi) + i\sin(\pi + 2n\pi)]^{1/4}$

The poles of $f(z)$ are given by the roots of the equation $z^4 + 1 = 0$, which are the four fourth roots of -1.

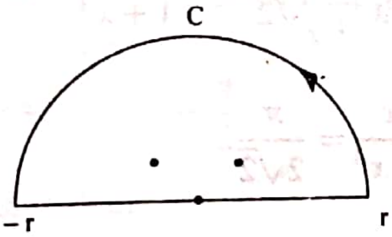
By De Moivre's theorem they are given by $e^{i\pi/4}$; $e^{i3\pi/4}$; $e^{i5\pi/4}$; $e^{i7\pi/4}$ and all are simple poles.

We choose the contour C consisting of the interval $[-r, r]$ on the real axis and the upper semi-circle $|z| = r$ which we denote by C_1.

$$\therefore \int\limits_C f(z)dz = \int\limits_{-r}^{r} f(x)dx + \int\limits_{C_1} f(z)dz \qquad \ldots (1)$$

The poles of $f(z)$ lying inside the contour C are obviously $e^{i\pi/4}$ and $e^{i3\pi/4}$ only.

We find the residues of $f(z)$ at these points.



$$\text{Res}\left\{f(z); e^{i\pi/4}\right\} = \frac{h(e^{i\pi/4})}{k'(e^{i\pi/4})}$$ where $h(z) = 1$ and $k(z) = z^4 + 1$ so that $k'(z) = 4z^3$.

$$\therefore \ \text{Res}\left\{f(z); e^{i\pi/4}\right\} = \frac{1}{4e^{i3\pi/4}} = \frac{e^{-i3\pi/4}}{4}.$$

Similarly $\text{Res}\left\{f(z); e^{i3\pi/4}\right\} = \frac{e^{-i9\pi/4}}{4}$.

By residue theorem

$$\int\limits_C f(z)dz = 2\pi i(\text{sum of the residues at the poles})$$

$$= 2\pi i\left[\frac{e^{-i3\pi/4}}{4} + \frac{e^{-i9\pi/4}}{4}\right]$$

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$$= \frac{\pi i}{2}[(\cos(3\pi/4) - i\sin(3\pi/4)) + (\cos(9\pi/4) - i\sin(9\pi/4))]$$

$$= \frac{\pi i}{2}\left[\left(-\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}\right) + \left(\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}\right)\right]$$

$$= \frac{\pi i}{2}\left(\frac{-2i}{\sqrt{2}}\right) = \frac{\pi}{\sqrt{2}}.$$

From (1) $\displaystyle\int_{-r}^{r} \frac{dx}{1+x^4} + \int_{C_1} f(z)dz = \frac{\pi}{2}.$

As $r \to \infty$, $\displaystyle\int_{C_1} f(z)dz \to 0.$

$$\therefore \int_{-\infty}^{\infty} \frac{dx}{1+x^4} = \frac{\pi}{\sqrt{2}}.$$

$$\therefore 2\int_{0}^{\infty} \frac{dx}{1+x^4} = \frac{\pi}{\sqrt{2}} \quad \left(\because \frac{1}{1+x^4} \text{ is an even function}\right).$$

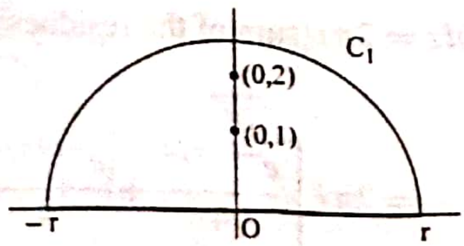
$$\therefore \int_{0}^{\infty} \frac{dx}{1+x^4} = \frac{\pi}{2\sqrt{2}}.$$

Problem 2. Using the method of contour integration evaluate $\displaystyle\int_{-\infty}^{\infty} \frac{x^2}{(x^2+1)(x^2+4)}dx.$

Solution. Let $f(z) = \dfrac{z^2}{(z^2+1)(z^2+4)}.$

The poles of $f(z)$ are $i, -i, 2i, -2i$.

Choose the contour C as shown in the figure.



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The poles i and $2i$ lie within C. By residue theorem

$$\int_C f(z)dz = 2\pi i \text{ (sum of the residues of } f(z)) \qquad \ldots (1)$$

We find the residues of $f(z)$.

$$\text{Res}\{f(z); i\} = \frac{h(i)}{k'(i)} \text{ where } h(z) = z^2 \text{ and } k(z) = (z^2+1)(z^2+4) = z^4 + 5z^2 + 4$$

so that $k'(z) = 4z^3 + 10z$.

$$\therefore \ \text{Res}\{f(z); i\} = \frac{-1}{-4i + 10i} = \frac{-1}{6i} = \frac{i}{6}.$$

$$\text{Res}\{f(z); 2i\} = -\frac{i}{3} \text{ (verify).}$$

$$\therefore \text{ From (1)} \int_C f(z)dz = 2\pi i\left(\frac{i}{6} - \frac{i}{3}\right)$$

$$= +2\pi i\left(\frac{-i}{6}\right)$$

$$= \frac{\pi}{3} \qquad \ldots (2)$$

$\therefore$ Also (1) can be written, using (2), as

$$\int_{C_1} f(z)dz + \int_{-r}^{r} \frac{x^2}{(x^2+1)(x^2+4)}dx = \frac{\pi}{3} \qquad \ldots (3)$$

Further the integral $\int_{C_1} f(z)dz \to 0$ as $r \to \infty$.

$$\therefore \int_{-\infty}^{\infty} \frac{x^2}{(x^2+1)(x^2+4)}dx = \frac{\pi}{3}.$$

Problem 3. Evaluate $\displaystyle\int_{-\infty}^{\infty} \frac{x^2}{(x^2+a^2)(x^2+b^2)}dx = \frac{\pi}{a+b}$.

Solution. Proceed as in previous problem.

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Problem 4. Evaluate $\int_{-\infty}^{\infty} \frac{x^2 - x + 2}{x^4 + 10x^2 + 9} dx$.

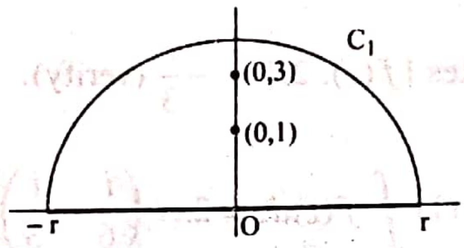
Solution. Let $f(z) = \frac{z^2 - z + 2}{z^4 + 10z^2 + 9}$.

Poles of $f(z)$ are the zeros of $z^4 + 10z^2 + 9 = 0$.

$$z^4 + 10z^2 + 9 = (z^2 + 9)(z^2 + 1).$$

$\therefore$ $z = \pm 3i; \pm i$. Hence $z = 3i, -3i, i - i$ are the simple poles of $f(z)$.

Choose the contour C as shown in the figure.



$$\int_C f(z)dz = \int_{-r}^{r} f(x)dx + \int_{C_1} f(z)dz. \quad \ldots (1)$$

The poles of $f(z)$ lying within C are i and $3i$ and both of them are simple poles.

$$\text{Res}\,\{f(z); i\} = \frac{h(i)}{k'(i)} \text{ where } h(z) = z^2 - z + 2 \text{ and}$$

$$k(z) = z^4 + 10z^2 + 9 \text{ so that } k'(z) = 4z^3 + 20z.$$

$$\therefore \quad \text{Res}\,\{f(z); i\} = \frac{-1 - i + 2}{-4i + 20i} = \frac{1 - i}{16i}.$$

Similarly $\text{Res}\,\{f(z); 3i\} = \frac{7 + 3i}{48i}$ (verify).

$$\therefore \quad \int_C dz = 2\pi i \text{ (sum of the residues at the poles)}$$

$$= 2\pi i \left(\frac{1 - i}{16i} + \frac{7 + 3i}{48i}\right)$$

$$= 2\pi i \left(\frac{10}{48i}\right) = \frac{5\pi}{12}.$$

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From (1) $\int_{-r}^{r} \frac{x^2 - x + 2}{x^4 + 10x^2 + 9} dx + \int_{C_1} \frac{z^2 - z + 2}{z^4 + 10z^2 + 9} dz = \frac{5\pi}{12}.$

Now as $r \to \infty$ the integral over $C_1 \to 0$.

$$\therefore \int_{-\infty}^{\infty} \frac{x^2 - x + 2}{x^4 + 10x^2 + 9} dx = \frac{5\pi}{12}.$$

Problem 5. Evaluate $I = \int_0^{\infty} \frac{dx}{(x^2 + a^2)^2}.$

Solution. Since $\frac{1}{(x^2 + a^2)^2}$ is an even function we have

$$\int_{-\infty}^{\infty} \frac{dx}{(x^2 + a^2)^2} = 2 \int_0^{\infty} \frac{dx}{(x^2 + a^2)^2}.$$

Let $f(z) = \frac{1}{(z^2 + a^2)^2}.$

Poles of $f(z)$ are the roots of $(z^2 + a^2)^2 = 0$.

Now, $(z^2 + a^2)^2 = (z + ai)^2 (z - ai)^2$.

$\therefore$ ai and $-ai$ are double poles of $f(z)$.

Choose the contour C consisting of the interval $[-r, r]$ on the real axis and the semi circle C_1 with centre 0 and radius r that lies in the upper half plane.

$$\therefore \int_{-r}^{r} f(x)\, dx + \int_{C_1} f(z)\, dz = \int_C f(z)\, dz. \qquad \ldots (1)$$

The poles of $f(z)$ lying within C is $z = ai$.

$\text{Res}\{f(z);\ ai\} = \frac{1}{1!} g'(ai)$ where $g(z) = \frac{1}{(z + ai)^2}.$

Now $g'(z) = -2(z + ai)^{-3}$.

$\therefore$ $g'(ai) = \frac{1}{4a^3 i}.$

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$$\therefore \ \text{Res}\{f(z); a_i\} = \frac{1}{4a^3 i}.$$

$$\therefore \int_C f(z)\,dz = 2\pi i\left(\frac{1}{4a^3 i}\right) = \frac{\pi}{2a^3}.$$

$$\therefore \int_{-r}^{r} \frac{dx}{(x^2+a^2)^2} + \int_{C_1} f(z)\,dz = \frac{\pi}{2a^3}.$$

When $r \to \infty$ the integral over $C_1 \to 0$.

$$\therefore \int_{-\infty}^{\infty} \frac{dx}{(x^2+a^2)^2} = \frac{\pi}{2a^3}.$$

$$\therefore \int_{0}^{\infty} \frac{dx}{(x^2+a^2)^2} = \frac{\pi}{4a^3}.$$

Problem 6. Prove that $\displaystyle\int_0^\infty \frac{dx}{x^6+1} = \frac{\pi}{3}$.

Solution. Since $\dfrac{1}{x^6+1}$ is an even function we have

$$\int_{-\infty}^{\infty} \frac{dx}{x^6+1} = 2\int_0^{\infty} \frac{dx}{x^6+1}.$$

Now, let $f(z) = \dfrac{1}{z^6+1}$.

The poles of $f(z)$ are given by the roots of the equation $z^6+1=0$, which are the sixth roots of -1.

By De Moivre's theorem they are given by $e^{i\pi/6}$, $e^{i3\pi/6}$, $e^{i5\pi/6}$, $e^{i7\pi/6}$ and $e^{i11\pi/6}$ and they are simple poles.

Now choose the contour C consisting of the interval $[-r, r]$ on the real axis and the upper semi circle $|z| = r$, which we denote by C_1.

The poles of $f(z)$ lying inside C are $e^{i\pi/6}$, $e^{i3\pi/6}$ and $e^{i5\pi/6}$.

$$\text{Res}\left\{f(z);\ e^{i\pi/6}\right\} = \frac{h(e^{i\pi/6})}{k'(e^{i\pi/6})} \quad \text{where } h(z) = 1 \text{ and } k(z) = z^6 + 1 \text{ so that } k'(z) = 6z^5$$

$$= \frac{1}{6e^{i5\pi/6}} = \frac{1}{6}e^{-i5\pi/6}.$$

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Similarly $\text{Res}\left\{f(z);\ e^{i3\pi/6}\right\} = \frac{1}{6}e^{-i5\pi/2}$ and

$$\text{Res}\left\{f(z);\ e^{i5\pi/6}\right\} = \frac{1}{6}e^{-i25\pi/6}.$$

$\therefore$ By residue theorem

$$\int_C f(z)\,dz = 2\pi i \text{ (sum of the residues at the 3 poles)}$$

$$= 2\pi i\left[\frac{1}{6}e^{5i\pi/6} + \frac{1}{6}e^{-5i\pi/2} + \frac{1}{6}e^{-25i\pi/6}\right]$$

$$= \frac{2\pi i}{6}\left[\left(\cos\frac{5\pi}{6} - i\sin\frac{5\pi}{6}\right) + \left(\cos\frac{5\pi}{2} - i\sin\frac{5\pi}{2}\right) + \left(\cos\frac{25\pi}{6} - i\sin\frac{25\pi}{6}\right)\right]$$

$$= \frac{\pi i}{3}\left[\left(-\frac{\sqrt{3}}{2} - \frac{i}{2}\right) + (0 - i) + \left(\frac{\sqrt{3}}{2} - \frac{i}{2}\right)\right]$$

$$= \frac{\pi i}{3}(-i - i) = \frac{2\pi}{3}.$$

$= \frac{\pi i}{3}(-2i) = -\frac{2\pi i^2}{3} = 2\pi/3$

$\therefore$ From (1), $\displaystyle\int_{-r}^{r} \frac{dx}{x^6+1} + \int_{C_1} f(z)\,dz = \frac{2\pi}{3}.$

As $r \to \infty$ the integral over $C_1 \to 0$.

$$\therefore \int_{-\infty}^{\infty} \frac{dx}{x^6+1} = \frac{2\pi}{3}.$$

$$\therefore \int_{0}^{\infty} \frac{dx}{x^6+1} = \frac{\pi}{3}.$$

Problem 7. Prove that $\displaystyle\int_0^{\infty} \frac{x^4\,dx}{x^6-1} = \frac{\pi\sqrt{3}}{6}.$

Solution. Let $f(z) = \dfrac{z^4}{z^6-1}.$

$z^6 - 1 = 0$

$z^6 = 1$

$z^6 = \cos 2n\pi + i\sin 2n\pi$

$z = (\cos 2n\pi + i\sin 2n\pi)^{1/6}$

$z = \cos 2n\pi/6 + i\sin 2n\pi/6$

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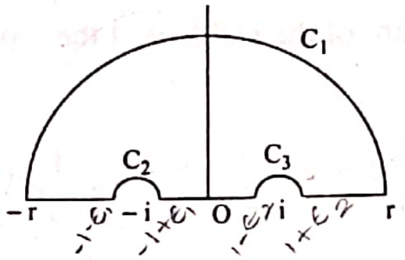
The poles of $f(z)$ are given by the sixth roots of unity, namely $e^{2n\pi i/6}$; $n = 0, 1, \ldots, 5$.

$\therefore$ $f(z)$ has 2 simple poles on the real axis, viz., 1 and -1 and the two poles $e^{\pi i/3}$ and $e^{2\pi i/3}$ lie on the upper half of the plane.

Now choose the contour C as shown in the figure.



$$\therefore \int_C f(z)\,dz = \int_{C_1} f(z)\,dz + \int_{-r}^{-1-\epsilon_1} f(x)\,dx + \int_{C_2} f(z)\,dz$$

$$+ \int_{-1+\epsilon_1}^{1-\epsilon_2} f(x)\,dx + \int_{C_3} f(z)\,dz + \int_{1+\epsilon_2}^{r} f(x)\,dx \qquad \ldots (1)$$

$$\text{Now, } \int_{C_2} f(z)dz = -\pi i \operatorname{Res}\{f(z); -1\}$$

$$= -\pi i \left(\frac{h(-1)}{k'(-1)}\right) \quad \text{where } h(z) = z^4 \text{ and } k(z) = z^6 - 1$$

$$= -\pi i(-1/6)$$

$$= \pi i/6. \qquad \ldots (2)$$

$$\text{Similarly } \int_{C_3} f(z)\,dz = -\pi i \operatorname{Res}\{f(z); 1\}$$

$$= -\pi i \left(\frac{h(1)}{k'(1)}\right)$$

$$= -\pi i(1/6)$$

$$= -\pi i/6. \qquad \ldots (3)$$

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Also $\int_C f(z)\,dz = 2\pi i\left[\text{Res}\left\{f(z); e^{\pi i/3}\right\} + \text{Res}\left\{f(z); e^{2\pi i/3}\right\}\right]$

$$= 2\pi i\left[\frac{h(e^{i\pi/3})}{6e^{i5\pi/3}} + \frac{e^{i8\pi/3}}{6e^{i10\pi/3}}\right]$$

$$= 2\pi i\left[\frac{e^{i4\pi/3}}{6e^{i5\pi/3}} + \frac{e^{i8\pi/3}}{6e^{i10\pi/3}}\right]$$

$$= \frac{\pi i}{3}\left(e^{-i\pi/3} + e^{-i2\pi/3}\right)$$

$$= \frac{\pi i}{3}\left(e^{-i\pi/3} - e^{i\pi/3}\right)$$

$$= \frac{\pi i}{3}(-2i \sin \pi/3)$$

$$= \frac{\pi\sqrt{3}}{3}. \quad \ldots (4)$$

Substituting (2), (3), (4) in (1) and taking limits as $\epsilon_1, \epsilon_2 \to 0$ and $r \to \infty$ we get

$$\int_{-\infty}^{\infty} \frac{x^4}{x^6 - 1}dx + \frac{\pi i}{6} - \frac{\pi i}{6} = \frac{\pi\sqrt{3}}{3}.$$

$$\therefore\ 2\int_0^{\infty} \frac{x^4}{x^6 - 1}dx = \frac{\pi\sqrt{3}}{3}.$$

$$\therefore\ \int_0^{\infty} \frac{x^4}{x^6 - 1}dx = \frac{\pi\sqrt{3}}{6}.$$

Exercises

1. Prove the following by using Cauchy's residue theorem.

(i) $\displaystyle\int_0^{\infty} \frac{dx}{x^2 + 1} = \frac{\pi}{2}$

(ii) $\displaystyle\int_0^{\infty} \frac{dx}{(x^2 + 1)^2} = \frac{\pi}{4}$

(iii) $\displaystyle\int_0^{\infty} \frac{x^2 dx}{(x^2 + 1)^2} = \frac{\pi}{4}$

(iv) $\displaystyle\int_0^{\infty} \frac{dx}{x^4 + x^2 + 1} = \frac{\pi\sqrt{3}}{6}$

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(v) $\int_0^\infty \frac{(2x^2-1)dx}{x^4+5x^2+4} = \frac{\pi}{4}$ (vi) $\int_0^\infty \frac{dx}{(x^2+1)^3} = \frac{\pi}{16}$

(vii) $\int_0^\infty \frac{x^2dx}{(x^2+a^2)^3} = \frac{\pi}{16a^3}$ (viii) $\int_0^\infty \frac{x^2dx}{(x^2+1)(x^2+4)} = \frac{\pi}{6}$

(ix) $\int_0^\infty \frac{x^2dx}{(x^2+4)(x^2+9)} = \frac{\pi}{200}$ (x) $\int_0^\infty \frac{dx}{x^4+1} = \frac{\pi}{2\sqrt{2}}$.

2. Show that $\int_{-\infty}^\infty \frac{dx}{(x-1)(x^2+2)} = \frac{-\sqrt{2}\pi}{6}$.

TYPE 3. $\int_{-\infty}^\infty \frac{g(x)}{h(x)}\cos ax dx$ or $\int_{-\infty}^\infty \frac{g(x)}{h(x)}\sin ax dx$ where $g(x)$ and $h(x)$ are real polynomials such that degree of $h(x)$ exceeds that of $g(x)$ by at least one and $a > 0$.

Case (i) $h(x)$ has no zeros on the real axis.

In this case take $f(z) = \frac{g(z)}{h(z)}e^{iaz}$.

$\therefore$ $f(z)$ has no poles on the real axis.

Choose the contour as in Type 2 and proceeding as in Type 2 we get the value of $\int_{-\infty}^\infty \frac{g(x)}{h(x)}e^{iax}\,dx$.

Taking the real and imaginary parts of $\frac{g(x)}{h(x)}e^{iax}dx$ we obtain the values of $\int_{-\infty}^\infty \frac{g(x)}{h(x)}\cos ax\,dx$ and $\int_{-\infty}^\infty \frac{g(x)}{h(x)}\sin ax\,dx$

Case (ii) $h(x)$ has zeros of order one on the real axis.

Take $f(z) = \frac{g(z)}{h(z)}e^{iaz}$. We notice that $f(z)$ has real poles. Suppose a is a real zero of $h(x)$ on the real axis. In this case we indent the real axis as Case (ii) of Type 2 and evaluate the integral.

To prove that the integral over the upper semicircle tends to zero as $r \to \infty$, we use the following lemma.

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Jordan's Lemma. Let $f(z)$ be a function of the complex variable z satisfying the following conditions.

(i) $f(z)$ is analytic in upper half plane except at a finite number of poles.

(ii) $f(z) \to 0$ uniformly as $|z| \to \infty$ with $0 \le \arg z \le \pi$.

(iii) a is a positive integer.

Then $\lim\limits_{r\to\infty} \int\limits_C f(z)e^{iaz}dz = 0$ where C is the semi circle with centre at the origin and radius r.

Solved Problems

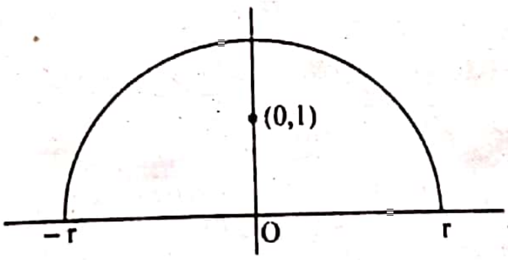
Problem 1. Prove that $\int\limits_0^\infty \frac{\cos x}{1+x^2}dx = \frac{\pi}{2e}$.

Solution. Let $f(z) = \frac{e^{iz}}{1+z^2}$.

$z^2+1=0$
$z^2=-1$
$z=\pm i$

The poles of $f(z)$ are given by i and $-i$.

Choose the contour C as shown in the figure.



The pole of $f(z)$ that lies within C is i. Hence by residue theorem

$$\int\limits_C f(z)\,dz = 2\pi i \operatorname{Res}\{f(z);\ i\}$$

$$= 2\pi i \frac{h(i)}{k'(i)} \text{ where } h(z) = e^{iz} \text{ and } k(z) = 1+z^2$$

$$= \frac{2\pi i e^{-1}}{2i} = \frac{\pi}{e}.$$

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$$\therefore \int_{-r}^{r} \frac{e^{iax}}{x^2+1}dx + \int_{C_1} \frac{e^{iaz}}{z^2+1}dz = \frac{\pi}{e}.$$

When $r \to \infty$ the integral over C_1 tends to zero.

$$\therefore \int_{-\infty}^{\infty} \frac{e^{iax}}{x^2+1}dx = \frac{\pi}{e}.$$

Equating real parts we get $\displaystyle\int_{-\infty}^{\infty} \frac{\cos x}{1+x^2}dx = \frac{\pi}{e}$.

$$\therefore 2\int_{0}^{\infty} \frac{\cos x}{1+x^2}dx = \frac{\pi}{e} \text{ (since } \frac{\cos x}{1+x^2} \text{ is an even function)}$$

$$\therefore \int_{0}^{\infty} \frac{\cos x}{1+x^2}dx = \frac{\pi}{2e}.$$

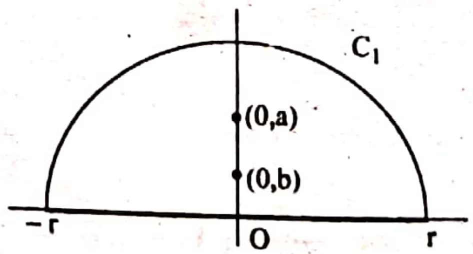
Problem 2. Using the method of contour integration evaluate

$$\int_{-\infty}^{\infty} \frac{\cos x}{(x^2+a^2)(x^2+b^2)}dx (a > b > 0).$$

Solution. Let $f(z) = \dfrac{e^{iz}}{(z^2+a^2)(z^2+b^2)}$.

The poles of $f(z)$ are $ia, -ia, ib, -ib$.

Choose the contour C as shown in the figure.



The poles of $f(z)$ which lie within C are ia and ib.

Hence by residue theorem

$$\int_C f(z)\,dz = 2\pi i(\text{sum of the residues of } f(z)). \quad \ldots (1)$$

We find the residues of $f(z)$.

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$$\text{Res}\{f(z);\ a_i\} = \frac{h(ai)}{k'(ai)} \text{ where } h(z) = e^{iz} \text{ and}$$

$k(z) = (z^2 + a^2)(z^2 + b^2) = z^4 + (a^2 + b^2)z^2 + a^2b^2$ so that

$k'(z) = 4z^3 + 2(a^2 + b^2)z.$

$$\therefore\ \text{Res}\{f(z);\ ai\} = \frac{e^{-a}}{4(ia)^3 + 2(a^2 + b^2)(ia)}$$

$$= \frac{e^{-a}}{i2a[(a^2 + b^2) - 2a^2]}$$

$$= \frac{-ie^{-a}}{2a(b^2 - a^2)}$$

$$= \frac{ie^{-a}}{2a(a^2 - b^2)}.$$

Similarly, $\text{Res}\{f(z);\ bi\} = \dfrac{ie^{-b}}{2b(b^2 - a^2)}$

$$= \frac{-ie^{-b}}{2b(a^2 - b^2)}.$$

From (1) $$\int_C f(z)dz = 2\pi i\left[\frac{i}{2(a^2 - b^2)}\left(\frac{e^{-a}}{a} - \frac{e^{-b}}{b}\right)\right]$$

$$= \frac{-\pi}{a^2 - b^2}\left(\frac{e^{-a}}{a} - \frac{e^{-b}}{b}\right)$$

$$= \frac{\pi}{a^2 - b^2}\left(\frac{e^{-b}}{b} - \frac{e^{-a}}{a}\right). \qquad \ldots (2)$$

Also (1) can be written using (2) as

$$\int_{C_1} f(z)\,dz + \int_{-r}^{r} \frac{e^{ix}}{(x^2 + a^2)(x^2 + b^2)}dx = \frac{\pi}{a^2 - b^2}\left(\frac{e^{-b}}{b} - \frac{e^{-a}}{a}\right). \qquad \ldots (3)$$

Further the integral over C_1 tends to 0 as $r \to \infty$.

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$\therefore$ (3) becomes

$$\int_{-\infty}^{\infty} \frac{e^{ix}}{(x^2+a^2)(x^2+b^2)}\,dx = \frac{\pi}{a^2-b^2}\left(\frac{e^{-b}}{b} - \frac{e^{-a}}{a}\right).$$

Equating real parts on both sides we get

$$\int_{-\infty}^{\infty} \frac{\cos x}{(x^2+a^2)(x^2+b^2)}\,dx = \frac{\pi}{a^2-b^2}\left(\frac{e^{-b}}{b} - \frac{e^{-a}}{a}\right).$$

Problem 3. Prove that $\displaystyle\int_0^{\infty} \frac{\cos ax\,dx}{(x^2+1)^2} = \frac{\pi}{4}(a+1)e^{-a}$ where $a > 0$.

Solution. Let $f(z) = \dfrac{e^{iaz}}{(z^2+1)^2}$.

The poles of $f(z)$ are given by i and $-i$ which are double poles. Now choose the contour as in Problem 1. The pole of $f(z)$ that lies within C is i.

Now, $\text{Res}\{f(z);\ i\} = \dfrac{1}{1!}g'(i)$ where $g(z) = (z-i)^2 f(z) = \dfrac{e^{iaz}}{(z+i)^2}$.

$$\therefore\ g'(z) = \frac{(z+i)^2 iae^{iaz} - e^{iaz}2(z+i)}{(z+i)^4}.$$

$$\therefore\ \text{Res}\{f(z);\ i\} = \frac{-4iae^{-a} - e^{-a}(4i)}{16} = \frac{-ie^{-a}(a+1)}{4}.$$

Hence by Cauchy's residue theorem

$$\int_C f(z)\,dz = 2\pi i\left(\frac{-ie^{-a}(a+1)}{4}\right) = \frac{\pi(a+1)e^{-a}}{2}.$$

$$\therefore \int_{-r}^{r} f(x)\,dx + \int_{C_1} f(z)\,dz = \frac{\pi(a+1)e^{-a}}{2}.$$

As $r \to \infty$, the integral over C_1 tends to zero.

$$\therefore \int_{-\infty}^{\infty} f(x)\,dx = \frac{\pi(a+1)e^{-a}}{2}.$$

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Equating real parts $\int_{-\infty}^{\infty} \frac{\cos ax}{(x^2+1)^2} dx = \frac{\pi(a+1)e^{-a}}{2}$.

$$\therefore \int_{0}^{\infty} \frac{\cos ax}{(x^2+1)^2} dx = \frac{\pi(a+1)e^{-a}}{4}.$$

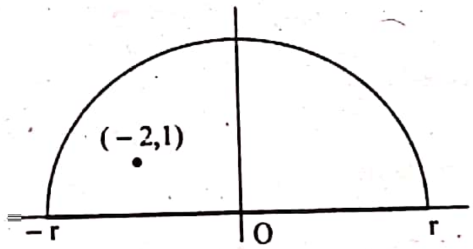
Problem 4. Prove that $\int_{-\infty}^{\infty} \frac{\sin x \, dx}{x^2+4x+5} = -\frac{\pi \sin 2}{e}$.

Solution. Let $f(z) = \frac{e^{iz}}{z^2+4z+5}$.

The poles of $f(z)$ are the roots of the equation $z^2+4z+5=0$. They are given by

$$z = \frac{-4 \pm \sqrt{16-20}}{2} = -2 \pm i.$$

Now, choose the contour C as shown in the figure.



$-2+i$ is the only pole of $f(z)$ that lies within C and it is a simple pole.

Hence by Cauchy' residue theorem,

$$\int_C f(z)\, dz = 2\pi i \, Res\{f(z); -2+i\}$$

$$= (2\pi i)\frac{h(-2+i)}{k'(-2+i)} \quad \text{where } h(z) = e^{iz} \text{ and } k(z) = z^2+4z+5$$

$$\therefore \int_{-r}^{r} f(x)\, dx + \int_{C_1} f(z)\, dz = \frac{\pi e^{-2i}}{e}.$$

Since the integral over C_1 tends to zero as $r \to \infty$, we have

$$\int_{-\infty}^{\infty} f(x)\, dx = \frac{\pi e^{-2i}}{e} = \frac{\pi}{e}(\cos 2 - i \sin 2).$$

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Equating imaginary parts we get

$$\int_{-\infty}^{\infty} \frac{\sin x dx}{x^2+4x+5} = -\frac{\pi \sin 2}{e} \qquad \ldots (4)$$

Problem 5. Evaluate $\displaystyle\int_0^{\infty} \frac{x \sin x}{x^2+a^2} dx$.

Solution. Let $f(z) = \dfrac{ze^{iz}}{z^2+a^2}$

The poles of $f(z)$ are given by ia and $-ia$ which are simple poles. Choose the contour C as in Problem 1.

Only the pole $z = ia$ lies inside C.

$\text{Res}\{f(z);\ ia\} = \dfrac{h(ia)}{k'(ia)}$ where $h(z) = ze^{iz}$ and $k(z) = z^2 + a^2$ so that $k'(z) = 2z$.

$\therefore\ \text{Res}\{f(z);\ ia\} = \dfrac{iae^{i(ia)}}{2(ia)} = \dfrac{e^{-a}}{2}$.

Hence by Cauchy's residue theorem

$$\int_C f(z)\, dz = 2\pi i \left(\frac{e^{-a}}{2}\right) = \pi i e^{-a}.$$

$$\therefore \int_{-r}^{r} f(x)\, dx + \int_{C_1} f(z)\, dz = \pi i e^{-a}.$$

When $r \to \infty$, $\displaystyle\int_{C_1} f(z)\, dz = 0$.

$$\therefore \int_{-\infty}^{\infty} f(x)\, dx = \pi i e^{-a}.$$

$$\therefore \int_{-\infty}^{\infty} \frac{xe^{ix}}{x^2+a^2} dx = \pi i e^{-a}.$$

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(i.e.) $$\int_{-\infty}^{\infty} \frac{x(\cos x + i \sin x)dx}{x^2+a^2} = \pi i e^{-a}.$$

(i.e.) $$\int_{-\infty}^{\infty} \frac{x(\cos x + i \sin x)dx}{x^2+a^2} = \pi i e^{-a}.$$

Equating imaginary parts on both sides we get

$$\int_{-\infty}^{\infty} \frac{x \sin x}{x^2+a^2} dx = \pi e^{-a}.$$

Since the above integrand is an even function we have

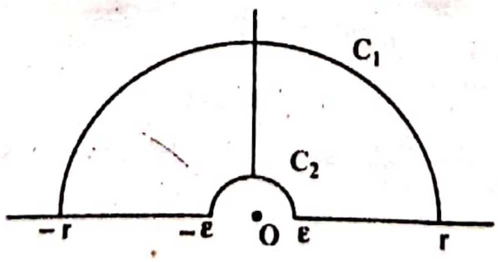
$$2\int_{0}^{\infty} \frac{x \sin x}{x^2+a^2} dx = \pi e^{-a}.$$

$$\therefore \int_{0}^{\infty} \frac{x \sin x}{x^2+a^2} dx = \frac{\pi e^{-a}}{2}.$$

Problem 6. Prove that $\displaystyle\int_0^{\infty} \frac{\sin x}{x} dx = \frac{\pi}{2}.$

Solution. Let $f(z) = \dfrac{e^{iz}}{z}$.

The only singular point of $f(z)$ is 0 which is a simple pole and it lies on the real axis. Now choose the contour C as shown in the figure.



Then $$\int_C f(z)\,dz = \int_{-r}^{-\epsilon} f(x)\,dx + \int_{C_2} f(z)\,dz$$

$$+ \int_{\epsilon}^{r} f(x)\,dx + \int_{C_1} f(z)\,dz. \qquad \ldots (1)$$

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Since $f(z)$ is analytic within C, $\int\limits_C f(z)\,dz = 0$ ... (2)

$$\text{Also} \int\limits_{C_2} f(z)\,dz = -\pi i \operatorname{Res}\{f(z);\ 0\}$$

$$= -\pi i e^0$$

$$= -\pi i. \qquad \ldots (3)$$

Further the integral over C_1 tends to 0 as $r \to \infty$.

Hence using (2) and (3) in (1) and taking limit as $r \to \infty$ we get

$$0 = \int\limits_{-\infty}^{0} f(x)\,dx - \pi i + \int\limits_{0}^{\infty} f(x)\,dx.$$

$$\therefore \int\limits_{-\infty}^{\infty} f(x)dx = \pi i.$$

Equating the imaginary parts we get $\int\limits_{-\infty}^{\infty} \frac{\sin x}{x} dx = \pi.$

$$\therefore \int\limits_{0}^{\infty} \frac{\sin x}{x} dx = \frac{\pi}{2} \text{ (since } \frac{\sin x}{x} \text{ is an even function).}$$

Exercises

1. Establish the following with the help of residues.

(i) $\displaystyle\int\limits_{-\infty}^{\infty} \frac{\sin x\,dx}{x^2 + 4x + 5} = -\frac{\pi \sin 2}{e}$

(ii) $\displaystyle\int\limits_{-\infty}^{\infty} \frac{\sin x\,dx}{x^2 - 2x + 5} = \frac{\pi \sin 1}{2e^2}$

(iii) $\displaystyle\int\limits_{0}^{\infty} \frac{\cos ax\,dx}{(x^2 + b^2)^2} = \frac{\pi}{4b^3}(1 + ab)e^{-ab}$ $(a > 0,\ b > 0)$.

(iv) $\displaystyle\int\limits_{0}^{\infty} \frac{x \sin x\,dx}{x^2 + a^2} = \frac{\pi e^{-a}}{2}$ $(a > 0)$.

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