PROJECTILES.

Problem:

(1) A body is projected with a velocity of 98 metre per sec. in a direction making an angle $\tan^{-1}3$. with the horizon ; S.T it rises to a vertical height of 441 metres and that its time of flight is about 19 secs. Find also horizontal range through the point of projection ($g = 9.8\ m/sec^2$).

Soln:

Given: $u = 98$

$\alpha = \tan^{-1}3.$

$\tan\alpha = 3.$

They are two components $= \sin\alpha$ & $\cos\alpha$

$$\therefore \sin\alpha = \sin\alpha \cdot \frac{\cos\alpha}{\cos\alpha}$$

$$= \frac{\sin\alpha}{\cos\alpha} \cdot \cos\alpha.$$

$$= \frac{\sin\alpha}{\cos\alpha} \cdot \frac{\cos\alpha}{1}$$

$$= \tan\alpha \cdot \frac{1}{\sec\alpha}:$$

$$= \frac{\tan\alpha}{\sec\alpha} = \frac{\tan\alpha}{\sqrt{1+\tan^2\alpha}} = \frac{3}{\sqrt{1+(3)^2}} = \frac{3}{\sqrt{10}}$$

$$\cos\alpha = \frac{\cos\alpha \cdot \sin\alpha}{\sin\alpha}$$

$$= \frac{\cos\alpha}{\sin\alpha} \cdot \sin\alpha$$

$$= \frac{1}{\tan\alpha} \cdot \sin\alpha$$

$$= \frac{\sin\alpha}{\tan\alpha} = \frac{\cancel{3}}{\sqrt{10}} \times \frac{1}{\cancel{3}} = \frac{1}{\sqrt{10}}$$

Greatest height.

$$h = \frac{u^2 \sin^2\alpha}{2g}$$

$$= \frac{(98)^2 (3)^2}{2 \times 9.8 \times 10}$$

$$= \frac{98 \times 98 \times 9}{2 \times 9.8 \times 10}$$

$$\boxed{h = 441 \text{ metres}}$$

Time of flight.

$$T = \frac{2u \sin\alpha}{g}$$

$$= \frac{2 \times 98 \times \frac{3}{\sqrt{10}}}{9.8}$$

$$= \frac{2 \times 98 \times 3}{\sqrt{10} \times 9.8} = \frac{6 \times \cancel{98} \times \sqrt{10}}{\cancel{\sqrt{10}} \times \cancel{98}} = 6 \times 3.162$$

$$= 18.972$$

$$\boxed{T = 19 \text{ secs.}}$$

Horizontal range.

$$R = \frac{2u^2 \sin\alpha \cos\alpha}{g}$$

$$= \frac{2 \times (98)^2}{9.8} \times \frac{3}{\sqrt{10}} \times \frac{1}{\sqrt{10}}$$

$$= \frac{2 \times 98 \times 98 \times 3}{9.8 \times 10}$$

$$\boxed{R = 588} \text{ metres.}$$

(2). If the greatest height attained by the particle is a quarter of its range on the horizontal plane through the point of projection, find the angle of projection.

Sol.

Let $u \rightarrow$ initial velocity.

$\alpha \rightarrow$ angle of projection.

$$h = \frac{u^2 \sin^2\alpha}{2g}, \quad R = \frac{2u^2 \sin\alpha \cos\alpha}{g}$$

Given:

$$\frac{u^2 \sin^2\alpha}{2g} = \frac{1}{\cancel{4}_2} \times \frac{\cancel{2}u^2 \sin\alpha \cos\alpha}{g}$$

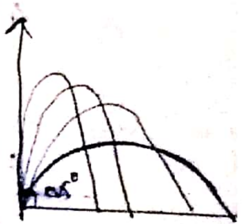
$$\frac{\cancel{u^2} \sin^{\cancel{2}}\alpha}{\cancel{2g}} = \frac{\cancel{u^2} \cancel{\sin\alpha} \cos\alpha}{\cancel{2g}}$$

$$\sin\alpha = \cos\alpha$$

$$\frac{\sin\alpha}{\cos\alpha} = 1$$

$$\tan\alpha = 1$$

$$\alpha = \tan^{-1}(1) = 45^\circ \quad \therefore \boxed{\alpha = 45^\circ}$$



(5) A stone is thrown with a velocity of 39.2 m/sec at 30° to the horizontal. Find at what times it will be at a height of 14.7m ($g = 9.8\ m/sec^2$)

Sol:

given: $u = 39.2\ m/sec$

$\alpha = 30°$

$s = 14.7m.$

$a = -g$

$(\because \sin 30° = \frac{1}{2} = 0.5)$

w.k.t. initial vertical velocity $= u \sin\alpha$

$= 39.2 \times \sin 30°$

$= 19.6\ m/sec$

$\therefore \quad s = ut + \frac{1}{2}at^2$

$14.7 = 19.6 . t - \frac{1}{2} 9.8 t^2$

$14.7 = 19.6t - 4.9t^2$

$\div 4.9$

$\frac{19.7}{4.9} = \frac{19.6}{4.9} t - t^2$

$3 = 4t - t^2$

$t^2 - 4t + 3 = 0$

$\therefore (t-3)(t-1) = 0$

$\therefore \quad t = 1 \text{ (or) } t = 3.$

$\therefore$ at the end of 1 sec & again at the end of 3 secs. it will be at a height of 14.7m //.

(6) A particle is projected so as to graze the tops of two parallel walls, the first of height 'a' at a distance b from the point of projection and the second of height 'b' at a distance 'a' from the point of projection. If the path of particle lies in a plane perpendicular to the both the walls, find the range on the horizontal plane and S.T the angle of projection exceeds $\tan^{-1}3$.

Sol: w.k.t. u – initial velocity.

α – angle of projection.

eqn. to the path

$$y = x\tan\alpha - \frac{gx^2}{2u^2\cos^2\alpha} \longrightarrow (1).$$

Let $t = \frac{\sin\alpha}{\cos\alpha} = \tan\alpha.$

(1) $\Rightarrow$

$$y = xt - \frac{gx^2}{2u^2} \cdot \frac{1}{\cos^2\alpha}$$

$$= xt - \frac{gx^2}{2u^2} \sec^2\alpha.$$

$$= xt - \frac{gx^2}{2u^2}(1+\tan^2\alpha)$$

$$= xt - \frac{gx^2}{2u^2}(1+t^2) \longrightarrow (2)$$

The tops of the two walls are $\overset{x\ y}{(b,a)}$ & (a,b) respectively lie in (2)

$$\therefore \quad y = xt - \frac{gx^2}{2u^2}(1+t^2)$$

$$a = bt - \frac{gb^2}{2u^2}(1+t^2) \longrightarrow (3)$$

$$a - bt = -\frac{gb^2}{2u^2}(1+t^2) \longrightarrow (4)$$

IIIly.

$$b = at - \frac{ga^2}{2u^2}(1+t^2) \longrightarrow (5)$$

$$b - at = -\frac{ga^2}{2u^2}(1+t^2) \longrightarrow (6)$$

$$\frac{(4)}{(6)} \Rightarrow \frac{a-bt}{b-at} = \frac{-\frac{gb^2}{2u^2}(1+t^2)}{-\frac{ga^2}{2u^2}(1+t^2)}$$

$$\frac{a-bt}{b-at} = \frac{-\cancel{\frac{g}{2u^2}(1+t^2)}\cdot b^2}{-\cancel{\frac{g}{2u^2}(1+t^2)}\cdot a^2}$$

$$\frac{a-bt}{b-at} = \frac{b^2}{a^2}$$

$$a^2(a-bt) = b^2(b-at)$$

$$a^3 - a^2bt = b^3 - ab^2t$$

$$a^3 - b^3 = a^2bt - ab^2t$$

$$a^3 - b^3 = t(a^2b - ab^2)$$

$$\therefore \quad t = \frac{a^3-b^3}{a^2b - ab^2} \longrightarrow$$

$$\tan\alpha = \frac{(a-\cancel{b})(a^2+ab+b^2)}{ab(a-\cancel{b})} \qquad (\because t = \tan\alpha)$$

$$\boxed{t = \tan\alpha = \frac{a^2+ab+b^2}{ab}} \longrightarrow (A)$$

Add & Sub $(3ab)$,

$$\tan\alpha = \frac{a^2+ab+b^2+3ab-3ab}{ab}$$

$$= \frac{a^2-2ab+b^2+3ab}{ab}$$

$$= \frac{(a-b)^2+3ab}{ab}$$

$$= \frac{(a-b)^2}{ab} + \frac{3\cancel{ab}}{\cancel{ab}}$$

$$\tan\alpha = \frac{(a-b)^2}{ab} + 3 \longrightarrow (7)$$

The first term in the right side of (7) is (+ve)

$$\therefore \quad \tan\alpha > 3$$

$$\alpha > \tan^{-1}3.$$

From (4),

$$a - bt = \frac{-gb^2}{2u^2}(1+t^2)$$

$$\frac{a-bt}{-b^2} = \frac{g(1+t^2)}{2u^2}$$

$$\frac{\cancel{+}(bt-a)}{\cancel{+}b^2} = \frac{g(1+t^2)}{2u^2} \qquad \boxed{\because \text{from } (A)}$$

$$\Rightarrow \frac{\frac{\cancel{b}(a^2+ab+b^2)}{a\cancel{b}} - a}{b^2} = \frac{\cancel{a^2}+ab+b^2-\cancel{a^2}}{ab^2} = \frac{\cancel{b}(a+b)}{ab^{\cancel{2}}}$$

$$\boxed{\frac{g(1+t^2)}{2u^2} = \frac{a+b}{ab}} \longrightarrow (8)$$

Horizontal range

$$R = \frac{u^2 \sin 2\alpha}{g}$$

$$= \frac{u^2\, 2\sin\alpha\cos\alpha}{g(1)}$$

$$= \frac{2u^2 \sin\alpha\cos\alpha}{g(\cos^2\alpha + \sin^2\alpha)}$$

$$= \frac{2u^2 \sin\alpha\cos\alpha}{\cos^2\alpha \cdot g\left(1 + \frac{\sin^2\alpha}{\cos^2\alpha}\right)}$$

$$= \frac{2u^2 \frac{\sin\alpha}{\cos\alpha}}{g\left(1 + \frac{\sin^2\alpha}{\cos^2\alpha}\right)}$$

$$= \frac{2u^2 \tan\alpha}{g(1+\tan^2\alpha)}$$

$$= \frac{2u^2 t}{g(1+t^2)} \quad (\because \text{from } (8))$$

$$= t \cdot \left(\frac{2u^2}{g(1+t^2)}\right)$$

$$= b \cdot \left(\frac{ab}{a+b}\right)$$

$$= \frac{a^2+ab+b^2}{ab} \cdot \frac{ab}{a+b}$$

$$R = \frac{a^2+ab+b^2}{a+b}.$$

(5) S.T the greatest height reached by a projectile whose initial velocity is V and angle of projection is α is unaltered if V is increased to kV and α is decreased by λ when $\operatorname{cosec}\lambda = k(\cot\lambda - \cot\alpha)$

Sol

let, v – initial velocity, $\frac{\cot\lambda - \cot\alpha}{1} = \frac{\operatorname{cosec}\lambda}{k}$

α – angle of projection.

given:

V is increased to kV $\Rightarrow$ $v = kV$

α is decreased by λ $\Rightarrow$ $\alpha = (\alpha - \lambda)$.

w.k.t.

$$h = \frac{u^2 \sin^2\alpha}{2g}$$

$$= \frac{(kV)^2 \sin^2(\alpha-\lambda)}{2g}$$

$\cot(\alpha-\beta) = \frac{\cot\alpha\cot\beta+1}{\cot\beta-\cot\alpha}$

$$= \frac{k^2V^2}{2g}\left[\sin\alpha\cos\lambda - \cos\alpha\sin\lambda\right]^2$$

$$= \frac{k^2V^2}{2g}\left[\sin\alpha\cos\lambda \cdot \frac{\sin\lambda}{\sin\lambda} - \cos\alpha\sin\lambda \cdot \frac{\sin\alpha}{\sin\alpha}\right]^2$$

$$= \frac{k^2V^2}{2g}\left[\sin\alpha\sin\lambda \frac{\cos\lambda}{\sin\lambda} - \sin\alpha\sin\lambda \frac{\cos\alpha}{\sin\alpha}\right]^2$$

$$= \frac{k^2V^2}{2g}\left[\sin\alpha\sin\lambda\cot\lambda - \sin\alpha\sin\lambda\cot\alpha\right]^2$$

$$= \frac{k^2V^2}{2g} \sin^2\alpha \sin^2\lambda \cdot [\cot\lambda - \cot\alpha]^2$$

$$= \frac{k^2V^2 \sin^2\alpha}{2g} \times \frac{1}{\operatorname{cosec}^2\lambda} \times \left[\frac{\operatorname{cosec}\lambda}{k}\right]^2$$

$\because$ Given $\operatorname{cosec}\lambda = k(\cot\lambda - \cot\alpha)$

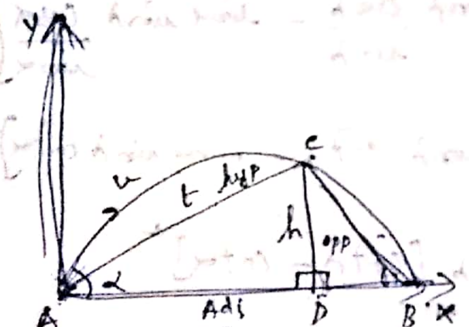
$$\boxed{\frac{\operatorname{cosec}\lambda}{k} = \cot\lambda - \cot\alpha}$$

$$\therefore h = \frac{k^2 V^2 \sin^2\alpha}{2g} \times \frac{1}{\cancel{\operatorname{cosec}\lambda}}\left(\frac{\operatorname{cosec}^{\cancel{2}}\lambda}{\cancel{k^2}}\right)$$

$$\boxed{h = \frac{V^2 \sin^2\alpha}{2g}} //.$$

(6) A particle is thrown over a triangle from one end of a horizontal base & grazing the vertex falls on the other end of the base. If A, B are the base angles, and α the angle of projection, S.T. $\tan\alpha = \tan A + \tan B$.

proof:



Let $u \to$ velocity,
$\alpha \to$ angle of projection,
$t \to$ time from A to C

Draw, $CD \perp AB$
Let $CD = h$.

Considering vertical motion

CD = Vertical distance described in time 't'.

$= u\sin\alpha . t - \frac{1}{2}gt^2$ $\longrightarrow$ (1)

AD = Horizontal distance described in time 't'.

$= u\cos\alpha . t$ $\longrightarrow$ (2)

From Δ ~~ACD~~ ΔCAD,

$$\tan A = \frac{CD}{AD}$$

$$= \frac{u\sin\alpha . t - \frac{1}{2}gt^2}{u\cos\alpha . t}$$

$$\tan A = \frac{u\sin\alpha t}{u\cos\alpha t} - \frac{\frac{1}{2}gt^2}{u\cos\alpha . t}$$

$$\tan A = \tan\alpha - \frac{gt}{2u\cos\alpha} \longrightarrow (3)$$

AB = Horizontal range

$$= \frac{2u^2\sin\alpha\cos\alpha}{g}$$

$\therefore DB = AB - AD$.

$$\boxed{DB = \frac{2u^2\sin\alpha\cos\alpha}{g} - u\cos\alpha . t}$$

From ΔCDB,

$$\tan B = \frac{CD}{DB}$$

$$= \frac{u\sin\alpha t - \frac{1}{2}gt^2}{\left(\frac{2u^2\sin\alpha\cos\alpha}{g} - u\cos\alpha . t\right)}$$

$$= \frac{(2u\sin\alpha t - gt^2)}{2\left(\frac{2u^2\sin\alpha\cos\alpha - gu\cos\alpha . t}{g}\right)}$$

$$= \frac{gt\,(\cancel{2u\sin\alpha - gt})}{2u\cos\alpha\,(\cancel{2u\sin\alpha - gt})}$$

$$= \frac{gt}{2u\cos\alpha} \longrightarrow (4)$$

Adding (3) + (4),

$\tan A + \tan B$

$$= \tan\alpha - \cancel{\frac{gt}{2u\cos\alpha}} + \cancel{\frac{gt}{2u\cos\alpha}}$$

$= \tan\alpha$

$$\therefore \boxed{\tan A + \tan B = \tan\alpha} //$$

(7) S.T. the greatest height which a particle with initial velocity v can reach on a vertical wall at a distance 'a' from the pt. of projection is $\boxed{\frac{V^2}{2g} - \frac{ga^2}{2V^2}}$. Prove also that the greatest height above the pt. of projection attained by the particle in its flight is $\boxed{V^6/2g(V^4+g^2a^2)}$.

Soln.

$V \to$ initial velocity
$a \to$ distance from the pt. of projection

path of projection.

$$y = x\tan\alpha - \frac{gx^2}{2V^2\cos^2\alpha} \longrightarrow (1)$$

put $x = a$;

$$y = a\tan\alpha - \frac{ga^2}{2V^2\cos^2\alpha}$$

$$= a\tan\alpha - \frac{ga^2}{2V^2}\sec^2\alpha$$

$$= a\tan\alpha - \frac{ga^2}{2V^2}(1+\tan^2\alpha)$$

$$y = at - \frac{ga^2}{2V^2}(1+t^2) \longrightarrow (2)$$

where $t = \tan\alpha$.

y is a fn. of t.

$\therefore$ y is maximum when $\frac{dy}{dt} = 0$

$\frac{d^2y}{dt^2}$ is negative (-ve)

Diff w.r.to t in (2),

$$\frac{dy}{dt} = a - \frac{ga^2}{2V^2}(2t)$$

$$= a - \frac{ga^2 t}{V^2}$$

$$\frac{d^2y}{dt^2} = -\frac{ga^2}{V^2}$$

$$= \text{negative}$$

$$\therefore a - \frac{ga^2 t}{V^2} = 0$$

$$a = \frac{ga^2 t}{V^2}$$

$$\boxed{t = \frac{V^2}{ga}} \longrightarrow (3)$$

Sub in (2),

$$\max. y = a\left(\frac{V^2}{ga}\right) - \frac{ga^2}{2V^2}\left(1 + \frac{V^4}{g^2a^2}\right)$$

$$= \frac{V^2}{g} - \frac{ga^2}{2V^2} - \frac{ga^2V^4}{2V^2g^2a^2}$$

$$= \frac{V^2}{g} - \frac{ga^2}{2V^2} - \frac{V^2}{2g}$$

$$= \frac{2V^2 - V^2}{2g} - \frac{ga^2}{2V^2}$$

$$\boxed{y = \frac{V^2}{2g} - \frac{ga^2}{2V^2}}$$

The greatest height reached on the wall.

Greatest height attained during the flight.

$$= \frac{V^2\sin^2\alpha}{2g}$$

$$= \frac{V^2}{2g}\cdot\frac{1}{\operatorname{cosec}^2\alpha}$$

$$= \frac{V^2}{2g}\cdot\frac{1}{(1+\cot^2\alpha)}$$

$$= \frac{V^2}{2g\left(1+\frac{1}{\tan^2\alpha}\right)}$$

$$= \frac{V^2}{2g\left(1+\frac{1}{t^2}\right)} \quad (\because \text{by } (3))$$

$$= \frac{V^2}{2g\left(1+\frac{1}{\frac{V^4}{g^2a^2}}\right)}$$

$$= \frac{V^2}{2g\left(1+\frac{g^2a^2}{V^4}\right)}$$

$$= \frac{V^2}{2g\left(\frac{V^4+g^2a^2}{V^4}\right)}$$

$$= \frac{V^2\cdot V^4}{2g(V^4+g^2a^2)}$$

$$\boxed{H = \frac{V^6}{2g(V^4+g^2a^2)}}$$

// The horizontal range of a projectile is maximum, given the magnitude u of the velocity of projection.

If $u \to$ initial velocity.

$\alpha \to$ angle of projection.

$R \to$ Hrzl. plane through the pt. of projection

$$\therefore R = \frac{2u^2\sin\alpha\cos\alpha}{g}$$

$$= \frac{u^2\,2\sin\alpha\cos\alpha}{g}$$

$$R = \frac{u^2\sin 2\alpha}{g} \longrightarrow (1)$$

$g \div = R$

Now, g being a constant for a given value u,

$R \to$ greatest when

$\sin 2\alpha \to$ greatest.

ie, $\sin 2\alpha = 1$

$2\alpha = \sin^{-1}(1)$

$2\alpha = 90^\circ$

$\boxed{\alpha = 45^\circ}$

Hence,

The given velocity of projection, the horizontal range is a maximum when the particle is projected at an angle of 45° to the horizontal.

i.e,

Maximum horizontal range bisects the angle b/w the horizontal and the vertical.

Sub $\boxed{\alpha = 45^\circ}$ in (1),

$$R = \frac{u^2 \sin 2\alpha}{g}$$

$$= \frac{u^2 \sin 2(45)}{g}$$

$$= \frac{u^2 \sin 90^\circ}{g}$$

(1) $\boxed{R = \frac{u^2}{g}}$

i.e, The maximum horizontal range is $\frac{u^2}{g}$.

TO SHOW THAT, FOR A GIVEN INITIAL VELOCITY OF PROJECTION THERE ARE, IN GENERAL TWO POSSIBLE DIRECTIONS OF PROJECTIONS SO AS TO OBTAIN A GIVEN HORIZONTAL RANGE.

Let $u \rightarrow$ velocity of projection.

$\alpha \rightarrow$ Angle of projection.

Given: $R = k$.

Then,

$$\frac{u^2 \sin 2\alpha}{g} = k$$

$$\sin 2\alpha = \frac{gk}{u^2} \longrightarrow (1)$$

$\because u$ & k given & constant

R.H.S. of (1) is (+ve).

If $\frac{gk}{u^2} < 1$

$gk < u^2$

Determine an acute angle θ whose sine is exactly equal to $\frac{gk}{u^2}$

From (1),

$$\sin\theta = \sin 2\alpha \longrightarrow (2)$$

$$\theta = 2\alpha$$

$$\alpha = \frac{\theta}{2} \longrightarrow (3)$$

$\because \sin(180^\circ - \theta) = \sin\theta \longrightarrow (4)$

Compare (2) & (4),

$$\sin 2\alpha = \sin(180^\circ - \theta)$$

$$2\alpha = 180^\circ - \theta$$

$$\alpha = \frac{180^\circ}{2} - \frac{\theta}{2}, \text{H}$$

$$\alpha = 90^\circ - \frac{\theta}{2} \longrightarrow (5)$$

From (3) & (5),

There are two values of α and so two directions of projection each giving the same range k.

Let α_1 & α_2 be two values of α.

Then.

$$\alpha_1 = \frac{\theta}{2}, \quad \alpha_2 = 90^\circ - \frac{\theta}{2}$$

$$\alpha_1 + \alpha_2 = \frac{\theta}{2} + 90^\circ - \frac{\theta}{2}$$

$$\therefore \alpha_1 + \alpha_2 = 90^\circ$$

As $\theta < 90^\circ$, $\alpha_1 < 45^\circ$

$\alpha_2 > 45^\circ$

Now,

$$45^\circ - \alpha_1 = 45^\circ - \frac{\theta}{2} \longrightarrow (6)$$

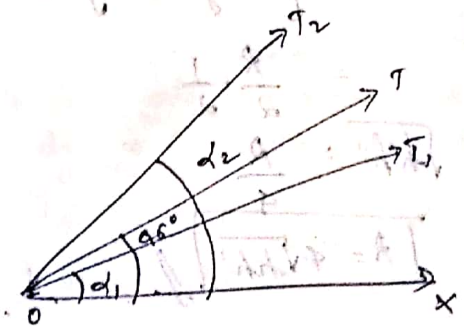
and

$$\alpha_2 - 45^\circ = 90^\circ - \frac{\theta}{2} - 45^\circ$$

$$= 45^\circ - \frac{\theta}{2} \longrightarrow (7)$$

Comparing (6) & (7),

$$45^\circ - \alpha_1 = \alpha_2 - 45^\circ \longrightarrow (8)$$



45° is the angle of projection to get maximum horizontal range with the same initial velocity.

get (8) S.T the two directions α_1 & α_2 are equally inclined to the direction of maximum range.

In fig. OT_1 & OT_2 are the directions α_1 & α_2 necessary to get a given range k.

OT is the direction giving maximum horizontal range.

$$\angle T_1OT = \angle XOT - \angle XOT_1$$

$$= 45^\circ - \alpha_1$$

$$\angle T_2OT = \angle XOT_2 - \angle XOT$$

$$= \alpha_2 - 45^\circ$$

$$\therefore \angle T_1OT = \angle T_2OT$$

$\therefore OT$ bisects the angle b/w OT_1 & OT_2.

From (1),

If $u^2 = gk$

$\because \sin 2\alpha = 1$

Then $2\alpha = \sin^{-1}(1)$

$2\alpha = 90^\circ$

$\alpha = 45^\circ$

$\therefore \alpha$ is possible & corresponds to the case of maximum range.

If $u^2 < gk$, R.H.S (1) > 1

we cannot get a real value for α.

$\therefore$ no angle of projection to get a range greater

$$R > \frac{u^2}{g}.$$

Which is really maximum range possible.

Note:

Horizontal range k, we find that $u^2 \geq gk$. So the minimum value of $u = \sqrt{gk}$.

(1) If h & h' be the greatest heights in the two paths of a projectile with a given velocity for a given range R. P.T $R = 4\sqrt{hh'}$.

Proof.

Let α & α' be the two angles of projection with given velocity u

$$\alpha + \alpha' = 90$$

$$\alpha' = 90 - \alpha \longrightarrow (1)$$

$$R = \frac{2u^2 \sin\alpha \cos\alpha}{g} \longrightarrow (2)$$

$$h = \frac{u^2 \sin^2\alpha}{2g} \longrightarrow (3)$$

$$h' = \frac{u^2 \sin^2\alpha'}{2g} \longrightarrow (4)$$

$$\therefore hh' = \frac{u^2 \sin^2\alpha}{2g} \cdot \frac{u^2 \sin^2\alpha'}{2g}$$

$$= \frac{u^4 \sin^2\alpha \sin^2\alpha'}{4g^2}$$

$$(hh')^{1/2} = \left(\frac{u^4 \sin^2\alpha \sin^2\alpha'}{4g^2}\right)^{1/2}$$

$$\sqrt{hh'} = \frac{u^2 \sin\alpha \sin\alpha'}{2g}$$

$$= \frac{u^2 \sin\alpha \sin(90^\circ - \alpha)}{2g}$$

$$= \frac{u^2 \sin\alpha \cos\alpha}{2g}$$

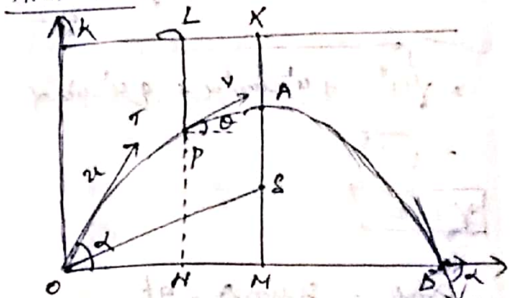
$$= \left[\frac{u^2 \sin\alpha \cos\alpha}{g}\right] \cdot \frac{1}{2}$$

$$= \frac{R}{2} \cdot \frac{1}{2}$$

$$\sqrt{hh'} = \frac{R}{4}$$

$$\boxed{R = 4\sqrt{hh'}}$$ //.

§ TO FIND THE VELOCITY OF THE PROJECTILE IN MAGNITUDE & DIRECTION AT THE END OF TIME t:



Let a particle be projected from 'O' with velocity 'u' at an angle 'α' to the horizon.

After time 't',

Let it be at P,

$v \longrightarrow$ its velocity.

$\theta \longrightarrow$ inclined angle to hor.

w.k.t,

Horizontal velocity $= u\cos\alpha$.

Horizontal velocity at $P = V\cos\theta$.

$$\therefore V\cos\theta = u\cos\alpha. \longrightarrow (1)$$

Vertical velocity $= u\sin\alpha$ and this is subject to a retardation 'g'.

$$= u\sin\alpha - gt.$$

Vertical velocity at $P = V\sin\theta$

$$\therefore V\sin\theta = u\sin\alpha - gt \longrightarrow (2)$$

Squaring & Adding (1) & (2),

$$V^2\cos^2\theta + V^2\sin^2\theta = u^2\cos^2\alpha + (u\sin\alpha - gt)^2$$

$$V^2(\cos^2\theta + \sin^2\theta) = u^2\cos^2\alpha + u^2\sin^2\alpha + g^2t^2 - 2u\sin\alpha gt$$

$$V^2(1) = u^2(\cos^2\alpha + \sin^2\alpha) - 2u\sin\alpha gt + g^2t^2$$

$$V^2 = u^2 - 2u\sin\alpha gt + g^2t^2$$

$$\therefore V = \sqrt{u^2 - 2u\sin\alpha gt + g^2t^2} \longrightarrow (3)$$

Dividing (2) & (1),

$$\frac{V\sin\theta}{V\cos\theta} = \frac{u\sin\alpha - gt}{u\cos\alpha}$$

$$\tan\theta = \frac{u\sin\alpha - gt}{u\cos\alpha} \longrightarrow (4)$$

Eqn (3) & (4) give 'V' & 'θ' i.e, the velocity at P in magnitude & direction.

Note:

Case (i):

(i) If $t < \frac{u\sin\alpha}{g}$

i.e, time taken to reach the highest pt. A.

$u\sin\alpha - gt \longrightarrow (+ve)$

$\tan\theta \longrightarrow (+ve)$

$\theta \longrightarrow (+ve)$

After time taken to reach A,

$t > \frac{u\sin\alpha}{g}$

Then $\theta = (-ve)$.

If $t = \frac{u\sin\alpha}{g}$

$\therefore \tan\theta = 0$

$\theta = 0$

Hence, the highest pt. A, the direction of the velocity is horizontal.

Case (ii):

(ii) Put $t = \frac{2u\sin\alpha}{g}$ in (3) & (4).

(3) ⇒

$v = \sqrt{u^2 - 2u\sin\alpha gt + g^2t^2}$

$= \sqrt{u^2 - 2u\sin\alpha g \cdot \frac{2u\sin\alpha}{g} + g^2 \frac{4u^2\sin^2\alpha}{g^2}}$

$= \sqrt{u^2 - \cancel{4u^2\sin^2\alpha} + \cancel{4u^2\sin^2\alpha}}$

$= \sqrt{u^2}$

$\boxed{v = u}$

$\tan\theta = \frac{u\sin\alpha - gt}{u\cos\alpha}$

$= \frac{u\sin\alpha - g \cdot \frac{2u\sin\alpha}{g}}{u\cos\alpha}$

$= \frac{u\sin\alpha - 2u\sin\alpha}{u\cos\alpha}$

$= -\frac{u\sin\alpha}{u\cos\alpha}$

$\tan\theta = -\tan\alpha$

$\boxed{\theta = -\alpha}$

Hence the particle strikes the horizontal plane downwards at B.

Case (iii):

Form (3), deduced from the principle of Energy.

Change in kinetic energy from O to P

$$\boxed{K.E = \frac{1}{2}mv^2 - \frac{1}{2}mu^2.}$$

Work done by the external force (gravity).

$$\boxed{\text{motion from O to P} = mg \cdot y}$$

where $y \to$ vertical height P above O

but $y =$ Vertical distance in time t.

$= u\sin\alpha t - \frac{1}{2}gt^2$

Hence,

Principle of Work Energy.

$\frac{1}{2}mv^2 - \frac{1}{2}mu^2 = -mg \cdot y$

$\therefore mv^2 - mu^2 = -2mgy$

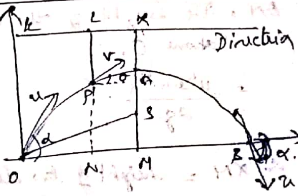
$\cancel{m}(v^2 - u^2) = -2\cancel{m}gy$

$v^2 - u^2 = -2g(u\sin\alpha t - \frac{1}{2}gt^2)$

$v^2 - u^2 = -2u\sin\alpha \cdot gt + g^2t^2$

$$\therefore \boxed{v^2 = u^2 - 2u\sin\alpha \cdot gt + g^2t^2}$$

§ To S.T the velocity at any pt. P of a projectile is equal in magnitude to the velocity acquired in falling freely from the directrix to the pt.



If $v \longrightarrow$ velocity at angle θ to the horz. when particle P at the end of t secs.

$v^2 = u^2 - 2u\sin\alpha gt + g^2t^2 \longrightarrow (1)$

Let L be a pt. vertically above P on the directrix.

If v is the velocity acquired by a particle which falls freely under gravity from L to P, Then

$v^2 = 2gLP \longrightarrow (2)$

Let $S \to$ focus.

$A \to$ vertex.

$X \to$ foot of the direc

$AX = AS = \frac{1}{4} \times$ latus rect

$= \frac{1}{4} \times \frac{2u^2\cos^2\alpha}{g}$

$Ax = \frac{u^2\cos^2\alpha}{2g}$

AM = The height of the vertex above O.

$= \frac{u^2\sin^2\alpha}{2g}$.

$\therefore XM$ = height of the directrix

$= XA + AM$

$= \frac{u^2\cos^2\alpha}{2g} + \frac{u^2\sin^2\alpha}{2g}$.

$= \frac{u^2(\sin^2\alpha + \cos^2\alpha)}{2g}$.

$XM = \frac{u^2}{2g}$.

PM = Vertical distance travelled in time 't' secs.

$= u\sin\alpha . t - \frac{1}{2}gt^2$.

$LP = LN - PN$

$= XM - PN$

$LP = \frac{u^2}{2g} - (u\sin\alpha . t - \frac{1}{2}gt^2)$.

From (4),

$v^2 = 2g\left(\frac{u^2}{2g} - (u\sin\alpha . t - \frac{gt^2}{2})\right)$

$v^2 = u^2 - 2u\sin\alpha gt + g^2t^2 \longrightarrow (5)$

From (1) & (5)

$v^2 = V^2$

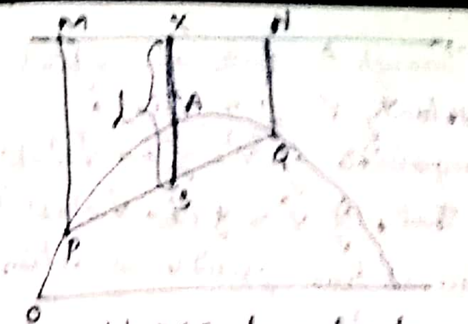
$\boxed{V = V.}$ in magnitude

Hence, the velocity at any pt. of the projectile is equal to its velocity acquired by a particle in falling freely from the directrix to that pt..

(1) If v_1 & v_2 be the velocities of a projectile at the ends of a focal chord of its path and U is the velocity at the vertex prove that

$v_1^{-2} + v_2^{-1} = U^{-2}$

Sol.



Let PSQ be a focal chord and let the velocities at P and Q be v_1 and v_2.

Draw, PM, SX & $QN \perp$ to the directrix.

A is the vertex.

$SX = l$, length of the semi-latus rectum

As P, A, Q are pts. on the parabola.

$SP = PM$

$SA = AX$ &

$SQ = QN$.

as PSQ is focal chord,

$\frac{1}{SP} + \frac{1}{SQ} = \frac{2}{l}$

w.k.t

$v_1^2 = 2g . PM = 2g . SP$

$v_2^2 = 2g . QN = 2g . SQ$.

$U^2 = 2g . AX = 2g . SA$.

$v_1^{-2} + v_2^{-2} = \frac{1}{v_1^2} + \frac{1}{v_2^2}$

$= \frac{1}{2g . SP} + \frac{1}{2g . SQ}$

$= \frac{1}{2g}\left(\frac{1}{SP} + \frac{1}{SQ}\right)$

$= \frac{1}{2g} \cdot \frac{2}{l}$

$= \frac{1}{2g} \cdot \frac{2}{SX}$

$= \frac{1}{2g} \cdot \frac{2}{2SA}$

$= \frac{1}{2g . SA}$

$= \frac{1}{U^2}$

$\boxed{v_1^{-2} + v_2^{-2} = U^{-2}}$ //

§ Given the magnitude of the velocity of projection, to show that there are two directions of projection for the particle so as to reach a given point.

Let $u \longrightarrow$ velocity,

$\alpha \longrightarrow$ angle of projection,

$$y = x\tan\alpha - \frac{gx^2}{2V^2\cos^2\alpha} \longrightarrow (1)$$

At the pt. (a, b). apply in (1)

$$b = a\tan\alpha - \frac{ga^2}{2V^2\cos^2\alpha}$$

$$= a\tan\alpha - \frac{ga^2}{2V^2}\sec^2\alpha$$

$$= a\tan\alpha - \frac{ga^2}{2V^2}(1+\tan^2\alpha)$$

$$2V^2 b = 2V^2 a\tan\alpha - ga^2 - ga^2\tan^2\alpha$$

$$ga^2\tan^2\alpha - 2aV^2\tan\alpha + (ga^2 + 2V^2 b) = 0 \longrightarrow (1)$$

$\because$ a, b, u are given, eqn (2) is a quadratic in $\tan\alpha$ & Hence, has two roots.

Corresponding values of α are the two possible directions of projection to hit the pt. (a, b).

(1). P is a pt. at a horizontal distance 'a' & a vertical distance 'b' from the pt. of projection. It is required to project a particle to pass velocity V. S.T This is impossible if $V^2 < g(b + \sqrt{a^2+b^2})$ & that, if $V^2 > g(b+\sqrt{a^2+b^2})$ There are two possible directions of projection.

Proof:

To prove: If α & β are the inclinations of the two directions

$$\boxed{b\tan(\alpha+\beta) + a = 0}$$

W.K.T,

If α is the angle of projection the quadratic in $\tan\alpha$ is

$$ga^2\tan^2\alpha - 2aV^2\tan\alpha + (ga^2 + 2V^2 b) = 0 \longrightarrow (1)$$

$$a = ga^2,\quad b = 2aV^2,\quad c = ga^2 + 2V^2 b$$

The two roots of (1) will be real

$$b^2 - 4ac = (2aV^2)^2 - 4ga^2(ga^2 + 2V^2 b)$$

$$= 4a^2V^4 - 4g^2a^4 - 8V^2ga^2b$$

$$= 4a^2(V^4 - g^2a^2 - 2V^2gb).$$

$$\therefore\ V^4 - g^2a^2 - 2V^2gb > 0.$$

$$V^4 - g^2a^2 - 2V^2gb + g^2b^2 - g^2b^2 > 0$$

$$(V^4 + g^2b^2 - 2V^2gb) + g^2b^2 - g^2a^2 > 0$$

$$(V^2 - gb)^2 - g^2b^2 - g^2a^2 > 0$$

$$(V^2 - gb)^2 - g^2(a^2+b^2) > 0$$

$$(V^2 - gb)^2 > g^2(a^2+b^2)$$

$$V^2 - gb > g\sqrt{a^2+b^2}.$$

$$V^2 > gb + g\sqrt{a^2+b^2}$$

$$\boxed{V^2 > g(b + \sqrt{a^2+b^2})}$$

This condition is satisfied, the quadratic (1), will give two real values of $\tan\alpha$. also two angles of projection.

If $V^2 < g(b + \sqrt{a^2+b^2})$.

The quadratic will be (-ve) cannot get real values for α. Impossible to hit the particular pt.

If $V^2 = g(b + \sqrt{a^2+b^2})$, the two roots are equal giving only one direction of projection.

If α & β are the two angles of projection to hit the pt. $\tan\alpha$ & $\tan\beta$.

$$\boxed{\text{Sum of roots} = \frac{-b}{a}}$$

$$\tan\alpha + \tan\beta = \frac{-(-2aV^2)}{ga^2}$$

$$= \frac{2aV^2}{ga^2} = \frac{2V^2}{ga}$$

$$\boxed{\text{product of root} = \frac{c}{a}}$$

$$\tan\alpha \cdot \tan\beta = \frac{ga^2 + 2V^2 b}{ga^2}$$

Now, $$\tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$$

$$= \frac{\frac{2V^2}{ga}}{1 - \left(\frac{ga^2 + 2V^2 b}{ga^2}\right)}$$

$$= \frac{2V^2 \cdot ga^2}{ga(ga^2 - ga^2 - 2V^2 b)}$$

$$= \frac{2V^2 a}{-2V^2 b} = -\frac{a}{b}$$

$\tan(\alpha+\beta) = -\frac{a}{b}$

$b\tan(\alpha+\beta) = -a$

$\boxed{b(\tan(\alpha+\beta)) + a = 0}$ //

$1600\tan^2\alpha$

(Q). A revolver can fire a bullet with a muzzle velocity of 63 m/sec. Is it possible to hit the top of a tower 400 m. away, its height being 30 m? $(g = 9.8\ m/sec^2)$

Let α - Angle of projection.

here $u = 63$ m/sec

$y = x\tan\alpha - \frac{gx^2}{2u^2\cos^2\alpha}$

$= x\tan\alpha - \frac{gx^2}{2u^2}(1+\tan^2\alpha)$

The top of a tower $(400, 30)$.

$30 = 400\tan\alpha - \frac{9.8 \times 400^2}{2 \times 63^2}(1+\tan^2\alpha)$.

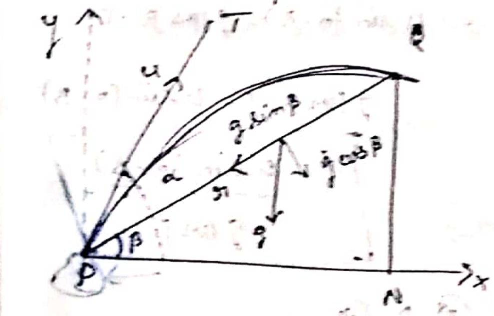
$30 = 400\tan\alpha - \frac{0.2 \times 400 \times 400}{162}(1+\tan^2\alpha)$

$3 = 40\tan\alpha - \frac{1600}{81}(1+\tan^2\alpha)$

§. Range on an Inclined plane:

State:

From a pt. on a plane, which is inclined at an angle 'β' to the horizon, a particle is projected with a velocity 'u' at an angle 'α' with the horizontal, in a plane passing through the normal to the inclined plane & the line of greatest slope. To find the range on the inclined plane.



Let P be the pt of projection & the particle strike the inclined plane at Q.

PQ → Range on the inclined plane.

Let $PQ = r$.

P → origin & the horizontal & the vertical through P.

The path is

$y = x\tan\alpha - \frac{gx^2}{2u^2\cos^2\alpha}$ → (1)

Draw QN ⊥ horizontal plane

Co-ordinates of Q are $(r\cos\beta, r\sin\beta)$ sub in (1).

$r\sin\beta = r\cos\beta \cdot \tan\alpha - \frac{gr^2\cos^2\beta}{2u^2\cos^2\alpha}$

× $2u^2\cos^2\alpha$ & cancelling r throughout.

$2u^2\cos^2\alpha \, r\sin\beta = r$ ($2u^2\cos^2\alpha\cos\beta \cdot \tan\alpha - gr\cos^2\beta$)

$2u^2\cos^2\alpha \sin\beta = 2u^2\cos^2\alpha\cos\beta\tan\alpha - gr\cos^2\beta$

$g\cos^2\beta \cdot r = 2u^2\cos^2\alpha\cos\beta\tan\alpha - 2u^2\cos^2\alpha\sin\beta$

$g\cos^2\beta \cdot r = 2u^2\cos^2\alpha\cos\beta\frac{\sin\alpha}{\cos\alpha} - 2u^2\cos^2\alpha\sin\beta$

$g\cos^2\beta \, r = 2u^2\cos\alpha$

$(\sin\alpha\cos\beta - \cos\alpha\sin\beta)$

$g\cos^2\beta \, r = 2u^2\cos\alpha \sin(\alpha-\beta)$

$$\boxed{r = \frac{2u^2\cos\alpha \sin(\alpha-\beta)}{g\cos^2\beta}}$$

§ To find the greatest distance of the projectile from the inclined plane & S.T it is attained in half the total time of flight.

Let us consider the motion ⊥^{r} to the inclined plane.

The initial velocity in this direction is $u\sin(\alpha-\beta)$ & this is subject to an acceleration $g\cos\beta$ in the same direction but acting downwards.

Let y be the distance travelled by the particle in this direction in time t.

$$y = u\sin(\alpha-\beta).t - \frac{1}{2}g\cos\beta . t^2 \longrightarrow (1)$$

Diff w.r.to t:

$$\frac{dy}{dt} = u\sin(\alpha-\beta) - \frac{2}{2}g\cos\beta . t \longrightarrow (2)$$

$$\frac{d^2y}{dt^2} = -g\cos\beta = (-ve)$$

$\therefore y$ is maximum when $\frac{dy}{dt} = 0$

$u\sin(\alpha-\beta) - g\cos\beta . t = 0$

$g\cos\beta . t = u\sin(\alpha-\beta)$

$$\boxed{t = \frac{u\sin(\alpha-\beta)}{g\cos\beta}} \longrightarrow (3)$$

Sub (3) in (1),

$$y = u\sin(\alpha-\beta) . \frac{u\sin(\alpha-\beta)}{g\cos\beta} - \frac{1}{2}g\cos\beta . \left(\frac{u\sin(\alpha-\beta)}{g\cos\beta}\right)^2$$

$$= \frac{u^2\sin^2(\alpha-\beta)}{g\cos\beta} - \frac{g\cos\beta}{2} \frac{u^2\sin^2(\alpha-\beta)}{g^2\cos^2\beta}$$

$$= \frac{u^2\sin^2(\alpha-\beta)}{g\cos\beta} - \frac{u^2\sin^2(\alpha-\beta)}{2g\cos\beta}$$

$$= \frac{2u^2\sin^2(\alpha-\beta) - u^2\sin^2(\alpha-\beta)}{g\cos\beta}$$

$$y = \frac{u^2\sin^2(\alpha-\beta)}{g\cos\beta} \longrightarrow (4)$$

(4) is the greatest distance of the projectile from the inclined plane.

From (3),

Time to this greatest distance $= \frac{u\sin(\alpha-\beta)}{g\cos\beta}$

This is clearly half of the time of flight.

§ To find determine when the range on the inclined plane is maximum, given the magnitude u of the velocity of projection:

The range R on the inclined plane is

$$R = \frac{2u^2\cos\alpha \sin(\alpha-\beta)}{g\cos^2\beta}$$

$$= \frac{u^2}{g\cos^2\beta} \, 2\cos\alpha \sin(\alpha-\beta)$$

$$= \frac{u^2}{g\cos^2\beta}\left[\sin(\alpha+\alpha-\beta) - \sin(\alpha-\alpha+\beta)\right]$$

$$\therefore R = \frac{u^2}{g\cos^2\beta}\left(\sin(2\alpha-\beta) - \sin\beta\right) \longrightarrow (1)$$

Now, u & β are given.

From (1), $\frac{u^2}{g\cos^2\beta}$ is constant.

So, R is maximum when $\sin(2\alpha-\beta) - \sin\beta$ is max.

i.e, $\sin(2\alpha-\beta)$ is greatest

when $2\alpha-\beta = \frac{\pi}{2}$

$2\alpha = \frac{\pi}{2} + \beta$

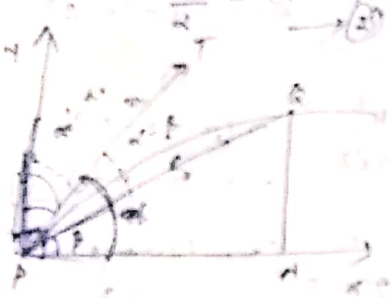
$$\boxed{\alpha = \frac{\pi}{4} + \frac{\beta}{2}}$$

for maximum range.

When 'α' take this value

$\alpha - \beta = (2\alpha - \beta) - \alpha$

$= \frac{\pi}{2} - \alpha \longrightarrow (3)$



$\alpha - \beta = \angle TPA - \angle QPA$

$= \angle TPQ \longrightarrow (4)$

$90° - \alpha = \angle YPT \longrightarrow (5)$

From (4) & (5),

$\angle TPQ = \angle YPT$

PT: The direction of projection for maximum range bisects the angle b/w the vertical & the inclined plane.

From (1), Maximum range

$R = \frac{u^2}{g\cos^2\beta}(\sin 90° - \sin\beta)$

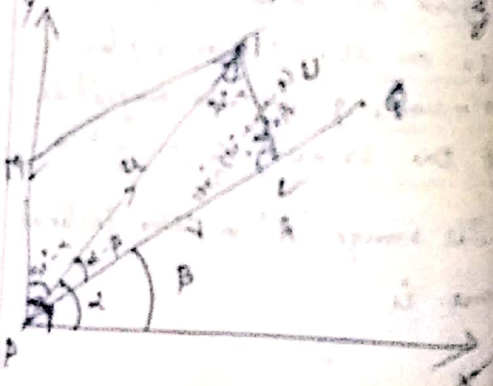
$R = \frac{u^2}{g\cos^2\beta}(1 - \sin\beta)$

$= \frac{u^2(1-\sin\beta)}{g(1 - \sin^2\beta)}$

$= \frac{u^2(1-\sin\beta)}{g(1-\sin\beta)(1+\sin\beta)}$

$$\boxed{R = \frac{u^2}{g(1+\sin\beta)}}$$

(1) If U & V be the oblique components of the initial velocity in the vertical direction & in the direction of the line of greatest slope, ST the range on the inclined plane is $\frac{2UV}{g}$



P is the pt of projection & PX, PY are the horizontal & the vertical through P.

PQ is greatest slope of the inclined plane with β.

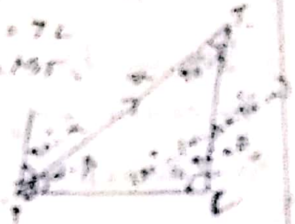
PT is the direction of projection at α.

Let PT = u velocity

From T, draw TL ∥ PY & TM ∥ PQ

Then PM = TL

PL = MT



From △PTL

we have

$\frac{TL}{\sin\angle TPL} = \frac{PL}{\sin\angle LPT} = \frac{PT}{\sin\angle PLT} \longrightarrow (1)$

$\angle TPL = \angle TPX - \angle LPX$

$= \alpha - \beta$

$\angle PTL =$ alternate $\angle MPT = 90° - \alpha$

$\angle PLT = 180° - (\angle PTL + \angle TPL)$

$= 180° - (90° - \alpha + \alpha - \beta)$

$= 180° - 90° + \alpha - \alpha + \beta$

$= 90° + \beta$

Also $PT = u$, $TL = U$, $PL = V$

$\frac{U}{\sin(\alpha-\beta)} = \frac{V}{\sin(90°-\alpha)} = \frac{u}{\sin(90°+\beta)}$

$\frac{U}{\sin(\alpha-\beta)} = \frac{V}{\cos\alpha} = \frac{u}{\cos\beta}$

compare 1st & 3rd term

$\frac{U}{\sin(\alpha-\beta)} = \frac{u}{\cos\beta}$

$$\boxed{U = \frac{u\sin(\alpha-\beta)}{\cos\beta}}$$

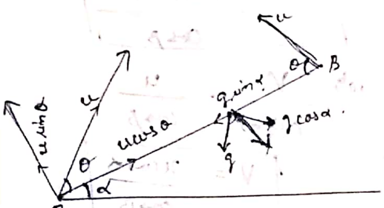
$\frac{V}{\cos\alpha} = \frac{u}{\cos\beta}$

$$\boxed{V = \frac{u\cos\alpha}{\cos\beta}}$$

$\therefore \frac{2UV}{g} = \frac{2}{g}\frac{u\sin(\alpha-\beta)}{\cos\beta}\frac{u\cos\alpha}{\cos\beta}$

$$\boxed{\frac{2UV}{g} = \frac{2u^2\sin(\alpha-\beta)\cos\alpha}{g\cos^2\beta}}$$

(2) S.T, for a given velocity of projection the maximum range down an inclined plane of inclination α bears to the maximum range up the inclined plane the ratio $\frac{1+\sin\alpha}{1-\sin\alpha}$



Let $u \rightarrow$ velocity of projection
$\theta \rightarrow$ inclination of the dir.

Velocity 'u' resolved into two components.

$u\cos\theta \rightarrow$ upward inclined plane
$u\sin\theta \rightarrow \perp^r$ to inclined plane

The acceleration 'g' can be resolved into two components.

$g\sin\alpha \rightarrow$ downward inclined plane

$g\cos\alpha \rightarrow \perp^r$ to inclined plane & downwards.

Let $T \rightarrow$ Time of flight

Distance travelled $\perp^r$ to inclined plane in time $T = 0$

$\therefore\ y = u\sin\theta \cdot T - \frac{1}{2} g\cos\alpha \cdot T^2$

$0 = u\sin\theta \cdot T - \frac{1}{2} g\cos\alpha \cdot T^2$

$\frac{1}{2} g\cos\alpha \cdot \cancel{T^2} = u\sin\theta\,\cancel{T}$

$$\boxed{T = \frac{2u\sin\theta}{g\cos\alpha}}$$

During time, the distance travelled along the plane

$R_1 = u\cos\theta \cdot T - \frac{1}{2} g\sin\alpha \cdot T^2$

$= u\cos\theta \cdot \frac{2u\sin\theta}{g\cos\alpha} - \frac{1}{\cancel{2}} \cancel{g}\sin\alpha \cdot \frac{\cancel{4}u^2\sin^2\theta}{g^{\cancel{2}}\cos^2\alpha}$

$= \frac{2u^2\cos\theta\sin\theta}{g\cos\alpha} - \frac{2u^2\sin\alpha\sin^2\theta}{g^2\cos^2\alpha}$

$= \frac{2u^2\sin\theta}{g\cos^2\alpha}\left[\cos\theta\cos\alpha - \sin\alpha\sin\theta\right]$

$= \frac{2u^2\sin\theta}{g\cos^2\alpha} \cos(\theta+\alpha)$

$= \frac{u^2}{g\cos^2\alpha}\, 2\cos(\theta+\alpha)\sin\theta$

$= \frac{u^2}{g\cos^2\alpha}\left[\sin(\theta+\alpha+\theta) - \sin(\theta+\alpha-\theta)\right]$

$R_1 = \frac{u^2}{g\cos^2\alpha}\left[\sin(2\theta+\alpha) - \sin\alpha\right]$

This is the range R_1 up the inclined plane.

R_1 is maximum, when $\sin(2\theta+\alpha) = 1$

$\therefore$ Maximum range up the plane $= \frac{u^2}{g\cos^2\alpha}\left[1-\sin\alpha\right]$

$= \frac{u^2(1-\sin\alpha)}{g(1^2-\sin^2\alpha)}$

$= \frac{u^2\cancel{(1-\sin\alpha)}}{g\cancel{(1-\sin\alpha)}(1+\sin\alpha)}$

$= \frac{u^2}{g(1+\sin\alpha)} \rightarrow (1)$

The particle is projected down the plane from B at the same angle to the plane, the time of flight has the same value $\frac{2u\sin\theta}{g\cos\alpha}$.

But the component of the initial velocity along the inclined plane is $u\cos\theta$ downwards and the component acceleration $g\sin\alpha$ is also downwards.

R_2 = distance travelled along the plane in time T.

$= u\cos\theta \cdot T + \frac{1}{2} g\sin\alpha \cdot T^2$

$= u\cos\theta \cdot \frac{2u\sin\theta}{g\cos\alpha} + \frac{\cancel{g}}{2} \frac{\sin\alpha \cdot 4u^2\sin^2\theta}{g^{\cancel{2}}\cos^2\alpha}$

$= \frac{2u^2\sin\theta}{g\cos^2\alpha}\left[\cos\theta\cos\alpha + \sin\alpha\sin\theta\right]$

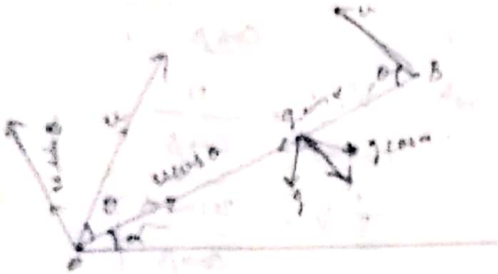
$= \frac{2u^2\sin\theta}{g\cos^2\alpha} \cos(\theta-\alpha)$

$= \frac{u^2}{g\cos^2\alpha} \sin\theta\cos(\theta-\alpha)$

$= \frac{u^2}{g\cos^2\alpha}\left[\sin(\theta+\theta-\alpha) + \sin(\theta-\theta+\alpha)\right]$

$R_2 = \frac{u^2}{g\cos^2\alpha}\left[\sin(2\theta-\alpha) + \sin\alpha\right]$

(2) S.T. for a given velocity of projection the maximum range down an inclined plane of inclination α bears to the maximum range up the inclined plane the ratio $\frac{1+\sin\alpha}{1-\sin\alpha}$



Let $u \to$ velocity of projection
$\theta \to$ inclination of the dir.

Velocity "u" resolved into two components

$u\cos\theta \to$ upward inclined plane
$u\sin\theta \to \perp^r$ to inclined plane

The acceleration g can be resolved into two components.

$g\sin\alpha \to$ downward inclined plane

$g\cos\alpha \to \perp^r$ to inclined plane & downwards.

Let $T \to$ time of flight

Distance travelled $\perp^r$ to inclined plane in time $T = 0$

$\therefore y = u\sin\theta\, T - \frac{1}{2} g\cos\alpha\, T^2$

$0 = u\sin\theta \cdot T - \frac{1}{2} g\cos\alpha\, T^2$

$\frac{1}{2} g\cos\alpha \cdot T^{\cancel{2}} = u\sin\theta\, \cancel{T}$

$$\boxed{T = \frac{2u\sin\theta}{g\cos\alpha}}$$

During time, the distance travelled along the plane

$R_1 = u\cos\theta\, T - \frac{1}{2} g\sin\alpha\, T^2$

$= u\cos\theta \cdot \frac{2u\sin\theta}{g\cos\alpha} - \frac{1}{2} g\sin\alpha \cdot \frac{4u^2\sin^2\theta}{g^2\cos^2\alpha}$

$= \frac{2u^2\cos\theta\sin\theta}{g\cos\alpha} - \frac{2u^2\sin\alpha\sin^2\theta}{g\cos^2\alpha}$

$= \frac{2u^2\sin\theta}{g\cos^2\alpha}\left[\cos\theta\cos\alpha - \sin\alpha\sin\theta\right]$

$= \frac{2u^2\sin\theta}{g\cos^2\alpha}\cos(\theta+\alpha)$

$= \frac{u^2}{g\cos^2\alpha}\, 2\cos(\theta+\alpha)\sin\theta$

$= \frac{u^2}{g\cos^2\alpha}\left[\sin(\theta+\alpha+\theta) - \sin(\theta+\alpha-\theta)\right]$

$R_1 = \frac{u^2}{g\cos^2\alpha}\left[\sin(2\theta+\alpha) - \sin\alpha\right]$

This is the range R_1 up the inclined plane

R_1 is maximum, when $\sin(2\theta+\alpha) = 1$

$\therefore$ Maximum range up the plane $= \frac{u^2}{g\cos^2\alpha}\left[1-\sin\alpha\right]$

$= \frac{u^2(1-\sin\alpha)}{g(1^2-\sin^2\alpha)}$

$= \frac{u^2\cancel{(1-\sin\alpha)}}{g\cancel{(1-\sin\alpha)}(1+\sin\alpha)}$

$= \frac{u^2}{g(1+\sin\alpha)} \longrightarrow (1)$

The particle is projected down the plane from B at the same angle to the plane, the time of flight has the same value $\frac{2u\sin\theta}{g\cos\alpha}$.

But the component of the initial velocity along the inclined plane is $u\cos\theta$ downwards and the component acceleration $g\sin\alpha$ is also downwards.

R_2 = distance travelled along the plane in time T.

$= u\cos\theta \cdot T + \frac{1}{2} g\sin\alpha \cdot T^2$.

$= u\cos\theta \cdot \frac{2u\sin\theta}{g\cos\alpha} + \frac{g}{2}\sin\alpha \cdot \frac{4u^2\sin^2\theta}{g^2\cos^2\alpha}$

$= \frac{2u^2\sin\theta}{g\cos^2\alpha}\left[\cos\theta\cos\alpha + \sin\alpha\sin\theta\right]$

$= \frac{2u^2\sin\theta}{g\cos^2\alpha}\cos(\theta-\alpha)$

$= \frac{u^2}{g\cos^2\alpha}\, 2\sin\theta\cos(\theta-\alpha)$

$= \frac{u^2}{g\cos^2\alpha}\left[\sin(\theta+\theta-\alpha) + \sin(\theta-\theta+\alpha)\right]$

$R_2 = \frac{u^2}{g\cos^2\alpha}\left[\sin(2\theta-\alpha) + \sin\alpha\right]$

R_2 is maximum when $\sin(2\theta - \alpha) = 1$

So maximum range down the plane $= \dfrac{u^2}{g\cos^2\alpha}(1+\sin\alpha)$

$= \dfrac{u^2(1+\sin\alpha)}{g(1^2-\sin^2\alpha)}$

$= \dfrac{u^2(1+\sin\alpha)}{g(1-\sin\alpha)(1+\sin\alpha)}$

$= \dfrac{u^2}{g(1-\sin\alpha)} \rightarrow (2)$

$\dfrac{(2)}{(1)}$: $\dfrac{\text{Max. range down the plane}}{\text{Max. range up the plane}}$

$= \dfrac{u^2}{g(1-\sin\alpha)} \cdot \dfrac{g(1+\sin\alpha)}{u^2}$

$= \dfrac{1+\sin\alpha}{1-\sin\alpha}$ //.

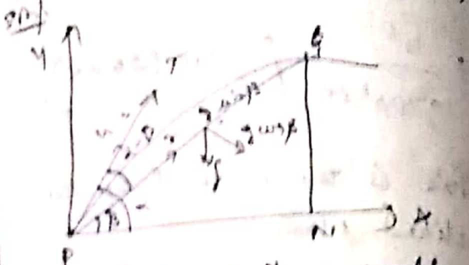
(8) A particle is projected at an angle α with a velocity u & it strikes up on inclined plane of inclination β at right angles to the plane.

P.T (i) $\cot\beta = 2\tan(\alpha-\beta)$

(ii) $\cot\beta = \tan\alpha - 2\tan\beta$

If the plane is struck horizontally, S.T $\tan\alpha = 2\tan\beta$.

Sol:



The initial velocity & accel. are split into components along the plane & ⊥r to the plane as explained.

$T = \dfrac{2u\sin(\alpha-\beta)}{g\cos\beta} \rightarrow (1)$

∵ the particle strikes the inclined plane normally, its velocity ‖ to the inclined plane at the end of time T is $= 0$

$0 = u\cos(\alpha-\beta) - g\sin\beta\, T$

$g\sin\beta\, T = u\cos(\alpha-\beta)$

$T = \dfrac{u\cos(\alpha-\beta)}{g\sin\beta} \rightarrow (2)$

Eqn (1) & (2), we have,

(1) = (2),

$\dfrac{2u\sin(\alpha-\beta)}{g\cos\beta} = \dfrac{u\cos(\alpha-\beta)}{g\sin\beta}$

$\dfrac{2\cancel{u}\sin(\alpha-\beta)}{\cancel{u}\cos(\alpha-\beta)} = \dfrac{\cancel{g}\cos\beta}{\cancel{g}\sin\beta}$

$\boxed{2\tan(\alpha-\beta) = \cot\beta} \rightarrow (A)$

$\cot\beta = 2\tan(\alpha-\beta)$

$= 2\left[\dfrac{\tan\alpha - \tan\beta}{1+\tan\alpha\tan\beta}\right]$

$\cot\beta(1+\tan\alpha\tan\beta) = 2(\tan\alpha - \tan\beta)$

$\cot\beta + \cancel{\cot\beta}\tan\alpha \dfrac{1}{\cancel{\cot\beta}} = 2(\tan\alpha - \tan\beta)$

$\cot\beta + \tan\alpha = 2\tan\alpha - 2\tan\beta$

$\cot\beta = 2\tan\alpha - 2\tan\beta - \tan\alpha$

$\boxed{\cot\beta = \tan\alpha - 2\tan\beta} \rightarrow (B)$

If the plane is struck horizontally, the vertical velocity of the projectile at the end of time T is $= 0$.

Initial vertical velocity $= u\sin\alpha$

Acceleration $= g$ downwards

Vertical velocity,

$v_T = u\sin\alpha - gT$

$0 = u\sin\alpha - gT$

$gT = u\sin\alpha$

$T = \dfrac{u\sin\alpha}{g} \rightarrow (3)$

Equating (1) & (3),

$\dfrac{2u\sin(\alpha-\beta)}{g\cos\beta} = \dfrac{u\sin\alpha}{g}$

$2\cancel{u}\sin(\alpha-\beta) = \cancel{u}\sin\alpha\cos\beta$

$2\sin(\alpha-\beta) = \sin\alpha\cos\beta$

$2(\sin\alpha\cos\beta - \cos\alpha\sin\beta) = \sin\alpha\cos\beta$

$2\sin\alpha\cos\beta - 2\cos\alpha\sin\beta = \sin\alpha\cos\beta$

$2\sin\alpha\cos\beta - \sin\alpha\cos\beta = 2\cos\alpha\sin\beta$

$\sin\alpha\cos\beta = 2\cos\alpha\sin\beta$

$\dfrac{\sin\alpha}{\cos\alpha} = 2\dfrac{\sin\beta}{\cos\beta}$

$\boxed{\tan\alpha = 2\tan\beta}$ //

Enveloping Parabola

(or)

To S.T. the envelope of all the paths described by particles projected from a pt. with a given velocity in all possible directions in the same vertical plane is another parabola.

Let $u \to$ velocity

$\alpha \to$ angle of projection

The eqn to the path:

$y = x\tan\alpha - \dfrac{gx^2}{2u^2\cos^2\alpha}$

$= x\tan\alpha - \dfrac{gx^2}{2u^2}(1+\tan^2\alpha)$

$y = xt - \dfrac{gx^2}{2u^2}(1+t^2) \longrightarrow (1)$

when $t = \tan\alpha$

Suppose a pt. (x_1, y_1) is reached with the above velocity of projection.

Then (x_1, y_1) lies on (1)

$y_1 = x_1 t - \dfrac{gx_1^2}{2u^2}(1+t^2)$

$2u^2 y_1 = 2u^2 x_1 t - gx_1^2 - gx_1^2 t^2$

$gx_1^2 t^2 - 2u^2 x_1 t + gx_1^2 + 2u^2 y_1 = 0$

$(gx_1^2)t^2 - (2u^2 x_1)t + (gx_1^2 + 2u^2 y_1) = 0 \longrightarrow (2)$

is a quadratic eqn. in $\tan\alpha$

$a = gx_1^2, \quad b = -2u^2 x_1,$

$c = gx_1^2 + 2u^2 y_1.$

(2) must be (+ve) (or) Zero

$b^2 - 4ac \geq 0$

$= 4u^4 x_1^2 - 4(gx_1^2)(gx_1^2 + 2u^2 y_1)$

$= 4u^4 x_1^2 - 4g x_1^2 (gx_1^2 + 2u^2 y_1)$

≥ 0

$4u^4 x_1^2 - 4g^2 x_1^4 - 8u^2 g x_1^2 y_1 \geq 0$

$4x_1^2(u^4 - g^2 x_1^2 - 2u^2 g y_1) \geq 0$

$u^4 - g^2 x_1^2 - 2u^2 g y_1 \geq 0$

$-g^2 x_1^2 \geq 2u^2 g y_1 - u^4$

$x_1^2 \leq -\dfrac{2u^2 g y_1}{g^2} + \dfrac{u^4}{g^2}$

is $x_1^2 \leq -\dfrac{2u^2}{g}\left(y_1 - \dfrac{u^2}{2g}\right)$

If $x_1^2 > -\dfrac{2u^2}{g}\left(y_1 - \dfrac{u^2}{2g}\right)$

(2) will be imaginary & the pt (x_1, y_1) cannot be hit with the given velocity.

Hence the pt (x, y) should be reached with the velocity

$x^2 \leq -\dfrac{2u^2}{g}\left(y - \dfrac{u^2}{2g}\right)$

Now,

$x^2 = -\dfrac{2u^2}{g}\left(y - \dfrac{u^2}{2g}\right) \longrightarrow (3)$

This means the pt. (x, y) lies on parabola.

The condition

$x^2 < -\dfrac{2u^2}{g}\left(y - \dfrac{u^2}{2g}\right)$

the pt (x, y) lies inside the parabola (3)

If $x^2 > -\dfrac{2u^2}{g}\left(y - \dfrac{u^2}{2g}\right)$

lies outside the parabola (3).

Hence, a given velocity lie within (or) on a parabola. This parabola is called the bounding parabola.

eqn (3) corresponding to the condition that the roots of the quadratic eqn (2) are equal.

Sum of the roots of (2) $= -\dfrac{b}{a}$

$\tan\alpha + \tan\alpha = \dfrac{-(-2u^2 x_1)}{gx_1^2}$,

$2\tan\alpha = \dfrac{2u^2}{gx_1}$

$\tan\alpha = \dfrac{u^2}{gx_1} \longrightarrow (4)$

From ①,

$$y = x\tan\alpha - \frac{gx^2}{2u^2}(1+\tan^2\alpha)$$

Diff ① w.r.to x,

$$\frac{dy}{dx} = \tan\alpha - \frac{2gx}{2u^2}(1+\tan^2\alpha)$$

$$\left(\frac{dy}{dx}\right)_{(x_1,y_1)} = \tan\alpha - \frac{gx_1}{u^2}(1+\tan^2\alpha)$$

$$= \frac{u^2}{gx_1} - \frac{gx_1}{u^2}\left(1+\frac{u^4}{g^2x_1^2}\right) \quad (\because ④)$$

$$= \frac{u^2}{gx_1} - \frac{gx_1}{u^2} - \frac{u^2}{gx_1}$$

$$\left(\frac{dy}{dx}\right)_{(x_1,y_1)} = -\frac{gx_1}{u^2} \longrightarrow ⑤.$$

Diff ③ w.r.to x,

$$x^2 = -\frac{2u^2}{g}\left(y - \frac{u^2}{2g}\right)$$

$$2x = -\frac{2u^2}{g}\left(\frac{dy}{dx}\right)$$

$$\frac{dy}{dx} = -\frac{gx}{u^2}$$

$$\left(\frac{dy}{dx}\right)_{(x_1,y_1)} = -\frac{gx_1}{u^2} \longrightarrow ⑥$$

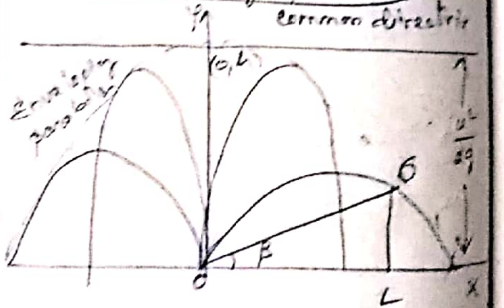
From ⑤ & ⑥.

we find that the parabolas ① & ③ have the same slope at their pt. of intersection.

So they have a common tgt at that pt. ie, They touch each other.

① is the eq. to any typical path, it follows that the bounding parabola.

③ touches all the trajectories. It is known as the enveloping parabola.



Let $h = \frac{u^2}{2g}$

Then $\frac{u^2}{g} = 2h$.

So eq. ③

$$x^2 = -\frac{2u^2}{g}\left(y - \frac{u^2}{2g}\right)$$

$$x^2 = -4h(y-h)$$

The vertex of the parabola is at $(0,h)$; latus rectum is $4h$

Distance b/w the vertex & the focus

$= \frac{1}{4} \times$ lactus rectum.

$= \frac{1}{4} \times 4h$

$= h$ //