Unit : 5

Laplace transforms - standard formulae Basic theorems & simple applications - inverse Laplace transforms - Use of Laplace transforms in solving ODE, with constant co-efficients.

T constant → a, b,

fun → f( ) φ( )

Type:

UNIT-1

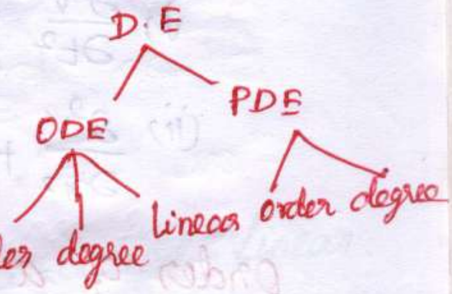
Differential Equation

An equation involved in derivatives or differentials of one or more dependent variables with respect to one or more independent variables is called the <u>differential equations</u>.

Note:

$$\frac{dy}{dx} = y' = y_1$$

$$\frac{d^2y}{dx^2} = y'' = y_2.$$



Examples:

(i) $\frac{dy}{dx} = x + \sin x$

(ii) $\frac{d^4x}{dt^4} + \frac{d^2x}{dt^2} + \left(\frac{dx}{dt}\right)^5 = e^t$

Ordinary differential equation:-

A defferential equation involving derivative with respect to a single independent variable is called the ordinary differential equation.

Example:

(i) $\frac{dy}{dx} = x + \sin x.$

(ii) $\frac{d^4x}{dt^4} + \frac{d^2x}{dt^2} + \left(\frac{dx}{dt}\right)^5 = e^t$

(iii) $k\left(\frac{d^2y}{dx^2}\right) = \left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2}$

Partial defferential equation.

A defferential equation involving partial derivatives with respect to one or more independent variables is called partial defferential equation.

Example:

(i) $\frac{\partial^2 v}{\partial t^2} = k\left(\frac{\partial^3 v}{\partial x^3}\right)^2$

(ii) $\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} = 0.$

Order of a differential equation:

The order of a defferential eqn. is the order of the highest derivatives occurring in it.

Degree of a differential equation:

The ~~g~~ degree of a differential eqn. is the degree of the highest derivati which occurs in it, after the defferential equation has been made free from radica and fractions as far as the derivatives are concerned.

Example:

Find the order and degree of the follo diffl. eqn:

(i) $\frac{dy}{dx} = x + \sin x$ (order, degree) $= (1, 1)$

(ii) $\frac{d^4x}{dt^4} + \frac{d^2x}{dt^2} + \left(\frac{dx}{dt}\right)^5 = e^t$.

(order, degree) = (4, 1)

(iii) $K\left(\frac{d^2y}{dx^2}\right) = \left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}$

Squaring on both sides.

$$K^2\left(\frac{d^2y}{dx^2}\right)^2 = \left[1+\left(\frac{dy}{dx}\right)^2\right]^3$$

(order, degree) = (2, 2)

Linear and non-linear deff/. eqn:

A differential eqn. is called linear if (i) Every dependent variable and every independent derivatives occurs in the first degree only - linear eqn. Eg: (i)

(ii) No products of dependent variable and derivatives occurs. A differential equation which is non-linear is called non-linear

Eg: (ii) to (v)

First order but higher degree :-

Type (1):

Equation solvable for p (or) $\frac{dy}{dx}$ we shall denote $\frac{dy}{dx} = p$. let the equation of the I^{st} order and of n^{th} degree be $p^n + P_1 p^{n-1} + P_2 p^{n-2} + \cdots + P_n = 0$ → (1)

Where $P_1, P_2 \cdots P_n$ are funs of x & y.

Suppose L.H.S of eqn (1) can be resolved into factors of the 1st degree of the form.

$(P-R_1)(P-R_2)\cdots(P-R_n)=0$. Therefore the general solution is $\phi(x,y,c_1),\phi(x,y,c_2)\cdots\phi(x,y,c_n)=0$.

Example:

1. Solve $p^2-5p+6=0$.

Solvable for p.

Given: $p^2-5p+6=0$.

Here $a=1, b=-5, c=6$.

$$p=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$$=\frac{5\pm\sqrt{25-4\times1\times6}}{2}$$

$$=\frac{5\pm\sqrt{25-24}}{2}$$

$$=\frac{5\pm\sqrt{1}}{2}\Rightarrow\frac{5\pm1}{2}$$

$$p=\frac{5+1}{2},\quad p=\frac{5-1}{2}$$

$$p=3,\quad p=2.$$

let us take $p=3$.

WKT $p=dy/dx$.

$$\frac{dy}{dx}=3$$

$$dy=3dx$$

Integrating on both sides

$\int dy = \int 3dx$.

$y = 3x + c_1$

$y - 3x - c_1 = 0$.

let us take $P = 2$.

WKT $P = \dfrac{dy}{dx}$.

$2 = \dfrac{dy}{dx}$.

$dy = 2dx$.

$\int$ing on both sides.

$y = 2x + c_2$.

$y - 2x - c_2 = 0$.

$\therefore$ The general solution is $(y - 3x - c_1)(y - 2x - c_2) = 0$

2) Solve $x^2p^2 + 3xyp + 2y^2 = 0$.

Solvable for P

Given :- $x^2p^2 + 3xyp + 2y^2$.

here $a = x^2$, $b = 3xy$, $c = 2y^2$.

$$P = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

$$= \frac{-3xy \pm \sqrt{9x^2y^2 - 8x^2y^2}}{2x^2}.$$

$$= \frac{-3xy \pm \sqrt{x^2y^2}}{2x^2} \Rightarrow \frac{-3xy \pm xy}{2x^2}.$$

$$P = \frac{-3xy + xy}{2x^2}, \quad P = \frac{-3xy - xy}{2x^2}.$$

$$P = -\frac{\cancel{2}xy}{\ldots} ; \quad P = -{}^{2}\!\cancel{4}xy/\cancel{2}x^2$$

$P = -y/x \; ; \; P = -2y/x.$

let us take $P = -y/x.$

WKT $P = \frac{dy}{dx}$

$$\frac{dy}{dx} = -\frac{y}{x} \Rightarrow dy = -\frac{y}{x}dx.$$

$\int$ing on both sides. $\Rightarrow \frac{dy}{y} = \frac{-dx}{x}$

$$\cancel{y = -\frac{y}{x} \cdot x + c_1}$$

$$\log y = -\log x + \log c_1$$

$$\log y + \log x = \log c_1$$

$$\log(xy) = \log c_1$$

$$xy = c_1$$

$$\boxed{xy - c_1 = 0}$$

let us take $P = \frac{-2y}{x}$

WKT $P = \frac{dy}{dx}.$

$$\frac{-2y}{x} = \frac{dy}{dx}.$$

$$\frac{dy}{y} = \frac{-2dx}{x}.$$

$\int$ing on both sides

$$\log y = -2\log x + \log c_2.$$

$$\log y + \log x^2 = \log c_2$$

$$\log(yx^2) = \log c_2$$

$$yx^2 = c_2$$

$$yx^2 - c_2 = 0$$

$\therefore$ The general solution is

$$(xy - c_1), (yx^2 - c_2) = 0$$

H.W

1) Solve $x^2p^2 + xyp - 6y^2 = 0$.

Ans: $(y - x^2c_1)(x^3y - c_2) = 0$

2) solve $xyp^2 + p(3x^2 - 2y^2) - 6xy = 0$

Ans: $(y - x^2c_1)(y^2 + 3x^2 - 2c_2) = 0$.

3) Solve $p^2 - 3p + 2 = 0$.

Ans: $(y - x - c_1), (y - 2x - c_2) = 0$.

3) Solve $p^2 + 2yp\cot x - y^2 = 0$.

solvable for p

Given $p^2 + 2y\cot x\, p - y^2 = 0$.

here $a = 1$, $b = 2y\cot x$, $c = -y^2$.

$$p = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-2y\cot x \pm \sqrt{4y^2\cot^2 x + 4y^2}}{2}$$

$$= \frac{-2y\cot x \pm \sqrt{4y^2(\cot^2 x + 1)}}{2}$$

$$= \frac{-2y\cot x \pm 2y\sqrt{\csc^2 x}}{2}$$

$$= \frac{-2y\cot x \pm 2y\,\mathrm{cosec}\,x}{2}$$

$$P = \frac{\cancel{2}(-y\cot x \pm y\,\mathrm{cosec}\,x)}{\cancel{2}}$$

$$P = -y\cot x \pm y\,\mathrm{cosec}\,x.$$

Now take $P = -y\cot x + y\,\mathrm{cosec}\,x.$

WKT $P = \frac{dy}{dx}.$

$$\frac{dy}{dx} = -y\cot x + y\,\mathrm{cosec}\,x$$

$$\frac{dy}{dx} = y(-\cot x + \mathrm{cosec}\,x)$$

$$\frac{dy}{y} = (-\cot x + \mathrm{cosec}\,x)\,dx.$$

$\int$ing on both sides.

$$\log y = -\log \sin x - \log(\mathrm{cosec}\,x + \cot x) + \log c_1.$$

$$\log y = -(\log \sin x + \log(\mathrm{cosec}\,x + \cot x)) + \log c_1$$

$$\log y = -(\log[\sin x(\mathrm{cosec}\,x + \cot x)]) + \log c_1$$

$$\log y = -\left(\log\left[\cancel{\sin x}\cdot\frac{1}{\cancel{\sin x}} + \cancel{\sin x}\cdot\frac{\cos x}{\cancel{\sin x}}\right]\right) + \log c_1$$

$$\log y = -\log(1+\cos x) + \log c_1$$

$$\log y + \log(1+\cos x) = \log c_1$$

$$\log[y(1+\cos x)] = \log c_1$$

$$y(1+\cos x) = c_1$$

$$\boxed{y(1+\cos x) - c_1 = 0}$$

let us take $P = -y\cot x - y\,\mathrm{cosec}\, x$

w.k.t $P = \frac{dy}{dx}$

$$\frac{dy}{dx} = -y(\cot x + \mathrm{cosec}\, x)$$

$$\frac{dy}{y} = -(\cot x + \mathrm{cosec}\, x)dx$$

$\int$ing on both sides.

$$\log y = -[\log \sin x - \log(\mathrm{cosec}\, x + \cot x)] + \log c_2$$

$$\log y = -\log \sin x + \log(\mathrm{cosec}\, x + \cot x) + \log c_2$$

$$\log y = \log\left[\frac{\mathrm{cosec}\, x + \cot x}{\sin x}\right] + \log c_2$$

$$\log y = \log\left[\frac{\frac{1}{\sin x} + \frac{\cos x}{\sin x}}{\sin x}\right] + \log c_2$$

$$\log y = \log\left[\frac{1+\cos x}{\sin^2 x}\right] + \log c_2$$

$$\log y = \log\left[\frac{1+\cos x}{1-\cos^2 x}\right] + \log c_2$$

$$\log y = \log\left[\frac{\cancel{1+\cos x}}{\cancel{(1+\cos x)}(1-\cos x)}\right] + \log c_2$$

$$\log y = \log\left(\frac{1}{1-\cos x}\right) + \log c_2$$

$$\log y - \log\left(\frac{1}{1-\cos x}\right) = \log c_2$$

$$\log\left(\frac{y}{\frac{1}{1-\cos x}}\right) = \log c_2$$

$$\log(y(1-\cos x)) = \log c_2$$

$$y(1-\cos x) = c_2 \Rightarrow y(1-\cos x) - c_2 = 0$$

$\therefore$ The general soln is $[y(1+\cos x) - c_1][y(1-\cos x) - c_2] = 0$

4) solve $p^2 + \left(x + y - \frac{2y}{x}\right)p + xy + \frac{y^2}{x^2} - y - \frac{y^2}{x}$

solvable for p.

here $a=1$, $b = x+y-\frac{2y}{x}$, $c = xy + \frac{y^2}{x^2} - y - \frac{y^2}{x}$

$$p = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-x-y+\frac{2y}{x} \pm \sqrt{\left(x+y-\frac{2y}{x}\right)^2 - 4\left(xy + \frac{y^2}{x^2} - y - \frac{y^2}{x}\right)}}{2}$$

$$= \frac{-x-y+\frac{2y}{x} \pm \sqrt{x^2+y^2+\cancel{\frac{4y^2}{x}}+2xy-\cancel{\frac{4y^2}{x}} - 4xy - \cancel{\frac{4y^2}{x^2}} + \cancel{4y} + \cancel{\frac{4y^2}{x}}}}{2}$$

$$= \frac{-x-y+\frac{2y}{x} \pm \sqrt{x^2+y^2-2xy}}{2}$$

$$= \frac{-x-y+\frac{2y}{x} \pm \sqrt{(x-y)^2}}{2}$$

$$p = \frac{-x-y+\frac{2y}{x} \pm (x-y)}{2}$$

$$p = \frac{\cancel{-x}-y+\frac{2y}{x}+\cancel{x}-y}{2}, \quad p = \frac{-x\cancel{-y}+\frac{2y}{x}-x+\cancel{y}}{2}$$

$$p = \frac{-2y+\frac{2y}{x}}{2} \quad ; \quad p = \frac{-2x+2y/x}{2}$$

$$= \cancel{2}\frac{\left(-y+y/x\right)}{\cancel{2}} \quad ; \quad p = +\cancel{2}\frac{\left(-x+y/x\right)}{\cancel{2}}$$

$P = \frac{-yx+y}{x}$; $P = -\frac{x^2+y}{x}$

$\boxed{P = -y + y/x}$; $\boxed{P = -x + y/x}$

let us take $P = y/x - y$.

$$P = \frac{dy}{dx}$$

$$\frac{dy}{dx} = y/x - y$$

$$\frac{dy}{dx} = y\left(\frac{1}{x} - 1\right)$$

$$\frac{dy}{y} = \left(\frac{1}{x} - 1\right)dx$$

$\int$ing on both sides.

$$\log y = \log x - x + \log c_1$$

$$\log y - \log x - \log c_1 = -x$$

$$\log\left(\frac{\frac{y}{x}}{c_1}\right) = -x$$

$$\log\left(\frac{y}{c_1 x}\right) = -x$$

Taking 'e' on both sides.

$$\frac{y}{c_1 x} = e^{-x}$$

$$y = c_1 x e^{-x}$$

$$\boxed{y - c_1 x e^{-x} = 0}$$

let us take ; $P = y/x - x$

$$P = \frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{y}{x} - x$$

$$\frac{dy}{dx} - \frac{y}{x} = -x.$$

This is linear equation in y,

$$\frac{dy}{dx} + Py = Q$$

$\therefore$ The general solution is

$$y e^{\int p\,dx} = \int Q e^{\int p\,dx} dx + c.$$

$$P = -\frac{1}{x}, \quad Q = -x.$$

$$\int p\,dx = \int -\frac{1}{x} dx \Rightarrow -\log x$$

$$e^{\int p\,dx} = e^{-\log x}.$$

$$= e^{\log x^{-1}}$$

$$= x^{-1} \Rightarrow \frac{1}{x}.$$

$$e^{\int p\,dx} = \frac{1}{x}.$$

The general solution is

$$y e^{\int p\,dx} = \int Q e^{\int p\,dx} dx + c$$

$$y \cdot \frac{1}{x} = \int -\cancel{x} \cdot \frac{1}{\cancel{x}} dx + c$$

$$y/x = -x + c$$

$$y = -x^2 + cx.$$

$$y + x^2 - cx = 0$$

$\therefore$ The general solution is

$$(y - c_1 x e^x), (y + x^2 - cx) = 0$$

H.W

1) Solve $p^2 + p(\cos x + \sec x) + 1 = 0$.

Ans: $(y + \log(\sec x + \tan x) - c_1), (y + \sin x - c_2) = 0$.

Type (b):

2m Equation solvable for y: (producer)

Suppose the equation $f(x, y, p) = 0 \rightarrow$ ①

can be solved explicity for y interms of x and p say $y = F(x, p)$.

diff. w.r to x $p = \phi(x, p, \frac{dp}{dx})$

This being first order linear differential equation is the two variables p and x can be integrated by any of the method. Hence we have

$\psi(x, y, c) \longrightarrow$ ②

Eliminating 'p' from ① & ②, we get the recurred solution.

Equation solvable for x:

2m Suppose the equation $f(x, y, p) = 0 \rightarrow$ ①

can be solved explicity for x interms of y & p

say $x = F(y, p)$

diff w.r to 'y' $\frac{dy}{dx} = \frac{1}{p}$

$\Rightarrow \phi(y, p, \frac{dp}{dy})$

Integrating, we have $\psi(y, p, c) \rightarrow$ ②

Eliminating 'p' from ① & ② we get the recurred solution.

Note :

(i) If the highest of degree of x in the differential eqn is one, then it can be solved for x.

(ii) If the highest degree of y is the dif eqn is one, then it can be solved for y.

(iii) Suppose the eqn do not contain 'x' explicity, then it can be solved for either or y.

(iv) If the highest degree of both x & y are one, then it can be solved either x or y.

(v) Suppose the eqn donot contain y explicity then it can be solved for either p or x.

1) Solve $xp^2 - 2yp + x = 0$.

Ⓐ Given $xp^2 - 2yp + x = 0$

$$xp^2 + x = 2yp$$

$$2yp = xp^2 + x$$

$$y = \frac{xp^2 + x}{2p} \rightarrow ①$$

Solvable for y.

diff. w.r. to x.

$$\frac{dy}{dx} = \frac{1}{2}\left[\frac{p\left(x \cdot 2p \cdot \frac{dp}{dx} + p^2(1) + 1\right) - (xp^2 + x)\frac{dp}{dx}}{p^2}\right]$$

WKT $p = \frac{dy}{dx}$.

$$p = \frac{1}{2p^2}\left[p\left(x \cdot 2p \cdot \frac{dp}{dx} + p^2 + 1\right) - (xp^2 + x)\frac{dp}{dx}\right]$$

$$2p^2 \cdot p = p\left(x \cdot 2p\frac{dp}{dx} + p^2+1\right) - x(p^2+1) \cdot \frac{dp}{dx}.$$

$$2p^3 = 2xp^2 \cdot \frac{dp}{dx} + p(p^2+1) - x(p^2+1)\frac{dp}{dx}.$$

$$= \frac{dp}{dx}\left[2xp^2 - x(p^2+1)\right] + p(p^2+1)$$

$$= \frac{dp}{dx}\left[2xp^2 - xp^2 - x\right] + p^3+p$$

$$= \frac{dp}{dx}\left[xp^2 - x\right] + p^3+p.$$

$$2p^3 - p^3 - p = \frac{dp}{dx}\left[xp^2 - x\right]$$

$$p^3 - p = \frac{dp}{dx}\, x(p^2-1)$$

$$p\cancel{(p^2-1)} = \frac{dp}{dx}\, x\cancel{(p^2-1)}$$

$$p = \frac{dp}{dx}\, x.$$

$$\frac{dp}{p} = \frac{dx}{x}.$$

∫ing on both sides.

$$\log p = \log x + \log c$$

$$\log p = \log(xc)$$

$$\boxed{p = xc} \longrightarrow ②$$

Using ② in ① we get.

$$① \Rightarrow y = \frac{xp^2+x}{2p}$$

$$y = \frac{x(x^2c^2)+x}{2xc} \Rightarrow \cancel{x}\frac{(x^2c^2+1)}{2\cancel{x}c}$$

2) Solve $y^2 = a^2(1+p^2)$

Given $y^2 = a^2(1+p^2)$

$$y = \pm a\sqrt{1+p^2}$$

diff. w.r to x.

$$\frac{dy}{dx} = \pm a \cdot \frac{1}{\cancel{2}\sqrt{1+p^2}} \cdot \cancel{2}p \cdot \frac{dp}{dx}.$$

WKT $\frac{dy}{dx} = p$.

$$\cancel{p} = \pm a \frac{\cancel{p}}{\sqrt{1+p^2}} \frac{dp}{dx}.$$

$$dx = \pm a \frac{dp}{\sqrt{1+p^2}}$$

$\int$ing on both sides

$$x = \pm a \log(p + \sqrt{p^2+1}) + c \rightarrow (2)$$

Given $y^2 = a^2(1+p^2)$

$$\frac{y^2}{a^2} = 1+p^2$$

$$\frac{y^2}{a^2} - 1 = p^2.$$

$$\frac{y^2-a^2}{a^2} = p^2.$$

$$p = \frac{\sqrt{y^2-a^2}}{a} \longrightarrow (3)$$

Using (3) in (2) we get

$$x = \pm a \log\left(\frac{\sqrt{y^2+a^2}}{a} + \sqrt{\frac{y^2-a^2}{a^2}+1}\right) + c$$

$$x = \pm a \log\left(\frac{\sqrt{y^2-a^2}}{a} + \sqrt{\frac{y^2-\cancel{a^2}+\cancel{a^2}}{a^2}}\right) + c$$

$$x = \pm a \log\left(\frac{\sqrt{y^2-a^2}}{} + y/a\right) + c.$$

(or) $y^2 = a^2(1+p^2)$

$$\frac{y^2}{a^2} = (1+p^2)$$

$$\frac{y^2}{a^2} - 1 = p^2 \Rightarrow p = \frac{\sqrt{y^2-a^2}}{a}$$

W.K.T $p = \frac{dy}{dx}$

$$\frac{dy}{dx} = \frac{\sqrt{y^2-a^2}}{a}$$

$$\frac{a\,dy}{\sqrt{y^2-a^2}} = dx$$

$\int$ing on both sides.

$$a \log\left(y+\sqrt{y^2-a^2}\right) = x + c$$

This is the required general soln.

3) Solve $x^2 = 1+p^2$

(So)

$$x^2 = 1+p^2$$

$$x^2-1 = p^2$$

$$p = \sqrt{x^2-1}$$

WKT $p = \frac{dy}{dx}$

$$\frac{dy}{dx} = \sqrt{x^2-1}$$

$$dy = \sqrt{x^2-1}\,dx$$

$\int$ing $\quad y = \frac{x}{2}\sqrt{x^2-1} - \frac{1}{2}\log\left(x+\sqrt{x^2-1}\right) + c$

The general soln is

$$y = \frac{1}{2}\left[x\sqrt{x^2-1} - \log\left(x+\sqrt{x^2-1}\right)\right] + c$$

4) Solve $y + xp = x^4 p^2$

(xi)

10m.

$$y + xp = x^4 p^2$$

$$y = x^4 p^2 - xp$$

$$\cancel{y = xp(x^3 p - 1)}$$

diff. w.r.to x.

$$\frac{dy}{dx} = x^4 \cdot 2p\frac{dp}{dx} + p^2 4x^3 - \left(x \cdot \frac{dp}{dx} + p(1)\right)$$

$$p = x^4 2p\frac{dp}{dx} + p^2 4x^3 - x\frac{dp}{dx} - p$$

$$\cancel{p} =$$

$$p + x\frac{dp}{dx} + p - 2px^4\frac{dp}{dx} - p^2 4x^3 = 0$$

$$x\frac{dp}{dx} + 2p - 2px^4\frac{dp}{dx} - p^2 4x^3 = 0$$

$$\left(x\frac{dp}{dx} + 2p\right) - 2px^3\left(x\frac{dp}{dx} + 2p\right) = 0$$

$$\left(x\frac{dp}{dx} + 2p\right)\left(1 - 2px^3\right) = 0$$

$$x\frac{dp}{dx} + 2p = 0$$

$$x\frac{dp}{dx} = -2p$$

$$\frac{dp}{p} = \frac{-2dx}{x}$$

Jing

$$\log p = -2\log x + \log c$$

$$\log p = -\log x^2 + \log c$$

$$\log p = \log\left(c/x^2\right)$$

$$\boxed{p = c/x^2} \rightarrow ②$$

$$1 - 2px^3 = 0$$

$$1 = 2px^3$$

$$\boxed{p = 1/2x^3} \rightarrow ③$$

Using ② in ① we get

$$y = -\cancel{x}\left(\frac{c}{x^{\cancel{2}}}\right) + x^{\cancel{4}}\left(\frac{c}{\cancel{x^2}}\right)^2$$

$$y = -c/x + c^2$$

$$\boxed{y = c^2 - c/x}$$

This is a required general soln.

If we take $p = \frac{1}{2x^3}$ it does not contain any ordinary constant c. It is called singular solution

5) Solve $y - 2px = \tan^{-1}(xp^2)$

Given $y - 2px = \tan^{-1}(xp^2)$

$$y = \tan^{-1}(xp^2) + 2px \rightarrow ①$$

diffl. w.r.to x.

$$\frac{dy}{dx} = \frac{1}{1+x^2p^4}\left(x \cdot 2p\frac{dp}{dx} + p^2(1)\right) + 2\left[p(1) + x\frac{dp}{dx}\right]$$

$$p = 2px\frac{dp}{dx} \cdot \frac{1}{1+x^2p^4} + \frac{p^2}{1+x^2p^4} + 2p + 2x\frac{dp}{dx}$$

$$p - 2p - 2xp\frac{dp}{dx} \cdot \frac{1}{1+x^2p^4} - \frac{p^2}{1+x^2p^4} - 2x\frac{dp}{dx} = 0.$$

$$-p - 2xp\frac{dp}{dx} \cdot \frac{1}{1+x^2p^4} - \frac{p^2}{1+x^2p^4} - 2x\frac{dp}{dx} = 0$$

$$p + 2xp\frac{dp}{dx} \cdot \frac{1}{1+x^2p^4} + \frac{p^2}{1+x^2p^4} + 2x\frac{dp}{dx} = 0$$

$$p\left(1 + \frac{p}{1+x^2p^4}\right) + 2x\frac{dp}{dx}\left(1 + \frac{p}{1+x^2p^4}\right) = 0$$

$$\left(1+\frac{p}{1+x^2p^4}\right)\left(p+2x\frac{dp}{dx}\right)=0$$

$$1+\frac{p}{1+x^2p^4}=0$$

$$\frac{p}{1+x^2p^4}=-1$$

$$p=-(1+x^2p^4)$$

$$p+2x\frac{dp}{dx}=0$$

$$2x\frac{dp}{dx}=-p$$

$$\frac{dp}{p}=\frac{-dx}{2x}$$

ƒing

$$\log p=-\tfrac{1}{2}\log x+\log c$$

$$2\log p=-\log x+\log c$$

$$\log p^2=\log c/x$$

$$p^2=c/x$$

$$\boxed{p=\sqrt{c/x}}$$

Using $p=\sqrt{c/x}$ in ① we get

$$① \Rightarrow y=2px+\tan^{-1}(xp^2)$$

$$=2\sqrt{c/x}\cdot x+\tan^{-1}(x\cdot c/x)$$

$$\boxed{y=2\sqrt{xc}+\tan^{-1}c}$$

b) Solve $x=y^2+\log p$

Given $x=y^2+\log p \longrightarrow ①$

diff. w.r. to y.

$$\frac{dx}{dy}=2y(1)+\frac{1}{p}\frac{dp}{dy}$$

$$\boxed{\text{wkt: } \frac{dy}{dx}=p,\ \frac{dx}{dy}=\frac{1}{p}}$$

$$\frac{1}{p}=2y+\frac{1}{p}\frac{dp}{dx}$$

$$\frac{1}{\cancel{p}} = \frac{2py + \frac{dp}{dy}}{\cancel{p}}$$

$$1 = 2py + \frac{dp}{dy}.$$

$$\frac{dp}{dy} + 2py = 1.$$

Here this eqn. is linear in P

$$P \cdot e^{\int P dy} = \int Q e^{\int P dy} dy + c \rightarrow ②.$$

Here.

$$P = 2y, \quad Q = 1.$$

$$\int P dy = \int 2y dy \Rightarrow \cancel{2} \cdot \frac{y^2}{\cancel{2}}$$

$$e^{\int P dy} = e^{y^2} \rightarrow ③.$$

Using 3 in 2 we get.

$$P \cdot e^{y^2} = \int e^{y^2} dy + c \rightarrow ④$$

Hence, we eliminates 'p' from ① & ④ we get the general solution.

7) Solve $y + xp = x - yp$.

Given $y + xp = x - yp$

$$xp + yp = x - y.$$

$$p(x+y) = x - y.$$

$$p = \frac{x-y}{x+y}.$$

W.K.T $p = \frac{dy}{dx}$.

$$\frac{dy}{dx} = \frac{x-y}{x+y}.$$

$$(x+y)\, dy = (x-y)\, dx.$$

$$x\,dy + y\,dy = x\,dx - y\,dx$$

$$x\,dy + y\,dx = x\,dx - y\,dy.$$

$$d(xy) = x\,dx - y.dy.$$

$\int$ing on both sides.

$$xy = x^2/2 - y^2/2 + c$$

$$xy = \frac{x^2 - y^2}{2} + c$$

$$\boxed{2xy = x^2 - y^2 + c}$$

Clairaut's Form:

This equation is of the form.

$$y = px + f(p) \quad \text{--- (1)}$$

diff. w.r. to x.

$$\frac{dy}{dx} = p.1 + x.\frac{dp}{dx} + f'(p).\frac{dp}{dx}.$$

$$p = p + x.\frac{dp}{dx} + f'(p)\frac{dp}{dx}$$

$$x\frac{dp}{dx} + f'(p)\frac{dp}{dx} = 0.$$

$$\frac{dp}{dx}\left(x + f'(p)\right) = 0$$

$$\frac{dp}{dx} = 0.$$

$$\boxed{dp = 0}$$

$\int$ing

$$\boxed{p = c}$$

Using $p = c$ in eqn (1).

$$y = cx + f(c)$$

This is the clairaut's equation.

① Solve $y = (x-a)p - p^2$

Given : $y = (x-a)p - p^2$

diff. w. r. to x

$$\frac{dy}{dx} = (x-a)\frac{dp}{dx} + p - 2p\frac{dp}{dx}$$

WKT $\frac{dy}{dx} = p$

$$\cancel{p} = (x-a)\frac{dp}{dx} + \cancel{p} - 2p\frac{dp}{dx}$$

$$(x-a)\frac{dp}{dx} - 2p\frac{dp}{dx} = 0$$

$$\frac{dp}{dx}\left((x-a) - 2p\right) = 0$$

$$\frac{dp}{dx} = 0$$

Sing on both sides.

$$\int dp = 0$$

$$p = c$$

Using $p = c$ in ①

$$y = (x-a)c - c^2$$

and. $(x-a) - 2p = 0$

$$x - a = 2p$$

$$p = \frac{x-a}{2} \rightarrow ②$$

$$y = (x-a)\left(\frac{x-a}{2}\right) - \left(\frac{x-a}{2}\right)^2$$

$$y = \frac{(x-a)^2}{2} - \frac{(x-a)^2}{4} \Rightarrow (x-a)^2\left[\frac{1}{2} - \frac{1}{4}\right]$$

$y = (x-a)^2\left(\frac{1}{4}\right)$. It does not contain any arbitrary constant c. So this is called singular eqn.

(x) solve $y = px + p^2$

Given: $y = px + p^2 \longrightarrow (1)$

diffl. w.r to x

$$\frac{dy}{dx} = p \cdot 1 + x \cdot \frac{dp}{dx} + 2p \cdot \frac{dp}{dx}$$

$$p = p + x\frac{dp}{dx} + 2p\frac{dp}{dx}.$$

$$\cancel{p} - \cancel{p} - x\frac{dp}{dx} - 2p\frac{dp}{dx} = 0$$

$$x\frac{dp}{dx} + 2p\frac{dp}{dx} = 0.$$

$$\frac{dp}{dx}(x + 2p) = 0.$$

$$\frac{dp}{dx} = 0$$

Sing

$$\int \frac{dp}{dx} = 0.$$

$$\boxed{p = c}$$

Using $p = c$ in eqn (1)

$$y = cx + c^2.$$

2) solve $y = px + \frac{a}{p}$

Given: $y = px + \frac{a}{p} \rightarrow (1)$

deffl. w.r to x

$$\frac{dy}{dx} = p \cdot 1 + x\frac{dp}{dx} + a\left(-\frac{1}{p^2}\right) \cdot \frac{dp}{dx}.$$

$$\cancel{p} = \cancel{p} + x\frac{dp}{dx} - \frac{a}{p^2}\frac{dp}{dx}.$$

$$-x\frac{dp}{dx} + \frac{a}{p^2}\frac{dp}{dx} = 0.$$

$$\frac{dp}{dx}\left(x - \frac{a}{p^2}\right) = 0.$$

$$\frac{dp}{dx} = 0.$$

Sing. $\int \frac{dp}{dx} = 0.$

$$\boxed{P = C}$$

$x - \frac{a}{p^2} = 0.$

$x = a/p^2$

$p^2 = a/x.$

$P = \sqrt{a/x}.$

Using $P = c$ in eqn ①

$$y = cx + \frac{a}{c}$$

4) solve: $y = Px + (1+P^2)^{1/2}$

(x)

Given: $y = Px + (1+P^2)^{1/2}$

$$y = Px + \sqrt{1+P^2} \rightarrow ①$$

diffl. w.r.t. to x

$$\frac{dy}{dx} = P(1) + x\frac{dp}{dx} + \frac{1}{\cancel{2}\sqrt{1+P^2}} \cdot \cancel{2}P\frac{dp}{dx}.$$

$$\cancel{P} = \cancel{P} + x\frac{dp}{dx} + \frac{P}{\sqrt{1+P^2}}\frac{dp}{dx}.$$

$$x\frac{dp}{dx} + \frac{P}{\sqrt{1+P^2}}\frac{dp}{dx} = 0$$

$$\frac{dp}{dx}\left(x + \frac{P}{\sqrt{1+P^2}}\right) = 0.$$

$$\frac{dp}{dx} = 0.$$

Sing

$$\int \frac{dp}{dx} = 0$$

$$\boxed{P = C}$$

Using $P = c$ in eqn ①

$$y = cx + \sqrt{1+c^2}$$ //

5) Solve: $(y-Px)(P-1)=P$

Given $(y-Px)(P-1)=P$.

$$y-Px=\frac{P}{P-1}$$

$$y=\frac{P}{P-1}+Px \rightarrow (1)$$

diff. w.r.to x.

$$\frac{dy}{dx}=\frac{(P-1)\frac{dp}{dx}-P\left(\frac{dp}{dx}\right)}{(P-1)^2}+P(1)+x\frac{dp}{dx}$$

$$\cancel{p}=\frac{(P-1)\frac{dp}{dx}-P\left(\frac{dp}{dx}\right)}{(P-1)^2}+\cancel{p}+x\frac{dp}{dx}$$

$$(P-1)\frac{dp}{dx}-P\left(\frac{dp}{dx}\right)+x(P-1)^2\frac{dp}{dx}=0.$$

$$\frac{dp}{dx}\left(\cancel{p}-1-\cancel{p}+x(P-1)^2\right)=0$$

$$\frac{dp}{dx}=0.$$

$\int$ing

$$\int\frac{dp}{dx}=0.$$

$$P=c$$

Using $P=c$ in eqn (1).

$$y=\frac{c}{c-1}+cx.$$

6) Solve: $(Px-y)(Py+x)=2P$

let $X=x^2$, $Y=y^2$

$dX=d(x^2)$, $dY=d(y^2)$

$=2x\,dx$ $\quad=2y\,dy$

$$\Rightarrow \frac{dY}{dX} = \frac{\cancel{2}y\,dy}{\cancel{2}x\,dx}$$

$$P = \frac{y}{x}\,p$$

$$p = \frac{x}{y}P$$

$$(px-y)(py+x) = 2P \longrightarrow (1)$$

$$\cancel{\left(\frac{x}{y}P - y\right)\left(\frac{x}{y}P + x\right) = 2\frac{x}{y}P}$$

$$\cancel{\left(\frac{xP-y^2}{y}\right)\left(\frac{xP+x^2}{y}\right) = \frac{2xP}{y}}$$

$$\left(\frac{x^2}{y}P - y\right)\left(\frac{x}{\cancel{y}}\cancel{y}P + x\right) = 2\frac{x}{y}P$$

$$\left(\frac{x^2P-y^2}{\cancel{y}}\right)(xP+x) = \frac{2xP}{\cancel{y}}$$

$$(Px^2-y^2)\cancel{x}(P+1) = 2\cancel{x}P$$

$$(PX-Y)(P+1) = 2P$$

$$PX - Y = \frac{2P}{P+1}$$

$$XP = \frac{2P}{P+1} + Y.$$

$$XP - \frac{2P}{P+1} = Y.$$

$$Y = XP - \frac{2P}{P+1}$$

This is clairaut form.

$\therefore$ The general soln:

$$Y = XC - \frac{2C}{C+1}$$

7) Solve : $P = \tan(y - Px)$

Given $P = \tan(y - Px)$

$\tan^{-1} P = y - Px$.

$\tan^{-1}P + Px = y$.

$\Rightarrow y = Px + \tan^{-1} P$.

$y = cx + \tan^{-1} c$.

8) Solve $y = 2Px + y^2 p^3$

$X = 2x, \quad Y = y^2$.

$dX = 2dx \quad dY = 2y\,dy$.

$\frac{dY}{dX} = \frac{\cancel{2}dx}{\cancel{2ydy}} \; \frac{\cancel{2}y\,dy}{\cancel{2}dx}$

$P = yp$

$p = P/y$

$y = 2px + y^2 p^3 \longrightarrow (1)$.

$y = 2\frac{P}{y}x + \cancel{y^2}\frac{P^3}{\cancel{y^3}}$

$y = \frac{2Px}{y} + \frac{P^3}{y}$

$y^2 = 2Px + P^3$

$\Rightarrow y^2 = 2cx + c^3$.

9) Solve: $e^{3x}(P-1) + P^3 e^{2y} = 0$.

let $X = e^x, \; Y = e^y$.

$dX = e^x dx, \; dY = e^y dy$

$$\frac{dY}{dX} = \frac{e^y dy}{e^x dx}.$$

$$P = \frac{e^y}{e^x} p \Rightarrow p = \frac{e^x}{e^y} P$$

$$e^{3x}(p-1) + p^3 e^{2y} = 0.$$

$$e^{3x}\left(\frac{e^x}{e^y} P - 1\right) + \left(\frac{e^{3x}}{e^{3y}} P^3\right) e^{2y} = 0$$

$$X^3\left(\frac{X}{Y} P - 1\right) + \left(\frac{X^3}{\cancel{Y^3}} P^3\right) \cancel{Y^2} = 0$$

$$X^3\left[\left(\frac{XP - Y}{Y}\right) + \frac{P^3}{Y}\right] = 0$$

$$XP - Y + P^3 = 0$$

$$-Y = -XP - P^3$$

$$Y = XP + P^3$$

This is clairauts form.

$$Y = cX + c^3.$$

$$e^y = ce^x + c^3.$$

10) solve: $p^3 - 4xyp + 8y^2 = 0.$

Given: $p^3 - 4xyp + 8y^2 = 0.$

Solving for 'x'.

$$4xyp = p^3 + 8y^2$$

$$x = \frac{p^3 + 8y^2}{4yp}$$

ie) $x = \frac{p^2}{4y} + \frac{2y}{p}$

diffl. w. r to 'y'

$$\frac{dx}{dy} = \frac{p^2}{4}\left(\frac{-1}{y^2}\right) + \frac{2p}{4y}\cdot\frac{dp}{dy} - \frac{2y}{p^2}\frac{dp}{dy} + \frac{2}{p}$$

$$\frac{1}{p} = \frac{-p^2}{4y^2} + \frac{\cancel{2}p}{\cancel{4}_2 y}\frac{dp}{dy} - \frac{2y}{p^2}\frac{dp}{dy} + \frac{2}{p}$$

$$\frac{1}{p} - \frac{2}{p} + \frac{p^2}{4y^2} = \frac{dp}{dy}\left(\frac{p}{2y} - \frac{2y}{p^2}\right)$$

$$\frac{p^2}{4y^2} - \frac{1}{p} = \frac{dp}{dy}\left(\frac{p^3 - 4y^2}{2p^2y}\right)$$

$$\frac{\cancel{p^3-4y^2}}{{}_2\cancel{4}\cancel{p}\cancel{y^2}} = \frac{dp}{dy}\left(\frac{\cancel{p^3-4y^2}}{\cancel{2}p^{\cancel{2}}\cancel{y}}\right)$$

$$\frac{1}{2y} = \frac{dp}{dy}\left(\frac{1}{p}\right)$$

$$\frac{dy}{y} = \frac{2}{p}dp$$

$\int$ing on both sides

$$\log y + \log c = 2\log p$$

$$\cancel{\log}(cy) = \cancel{\log} p^2$$

$$cy = p^2$$

$$p = \sqrt{cy}$$

$$4xyp = p^3 + 8y^2$$

$$4xyp = p^2 \cdot p + 8y^2$$

$$4xyp = cyp + 8y^2$$

$$4xyp - cyp = 8y^2$$

$$py(4x-c) = 8y^2$$

Squaring on both sides.

$$p^2y^2(4x-c)^2 = 64y^4$$

$$\cancel{cy}\, y^2(4x-c)^2 = 64y^4.$$

$$c(4x-c)^2 = 64y.$$

$$c(16x^2+c^2-8xc) = 64y.$$

$$16x^2c + c^3 - 8xc^2 = 64y.$$

$$c^3 - 8xc^2 + 16x^2c - 64y = 0.$$

Exact differential equation :

A differential equation $M.dx + Ndy = 0$ is said to be exact, if $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

$\therefore$ The general solution is

$\int M dx + \int N$ (terms in N not containing)$dy = c$.

Working rules :

(i) Integrate M with respect to x. Keeping y as a constant.

(ii) integrate with respect to y, only those terms of N which do not contain x.

(iii) The sum of those integrals equated to c is the solution.

1) solve: $(x^2-2xy+3y^2)dx+(y^2+6xy-x^2)d$

This soln is of the form $Mdx+Ndy$ =

Here $M = x^2-2xy+3y^2$

$N = y^2+6xy-x^2$

$$\frac{\partial M}{\partial y} = 0-2x+6y = -2x+6y \qquad \Bigg| \qquad \frac{\partial N}{\partial x} = 6y-2x = 6y-2x.$$

$$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$\therefore$ The given soln is exact.

$\therefore$ the soln is $\int M dx + \int N dy = \cancel{\phi}\, C \rightarrow (1)$

$M = x^2-2xy+3y^2.$

$$\int M dx = \int (x^2-2xy+3y^2)dx$$

$$= \frac{x^3}{3} - \frac{\cancel{2}x^2}{\cancel{2}}y + 3y^2x$$

$$\int M dx = \frac{x^3}{3} - x^2y + 3y^2x \longrightarrow (2)$$

$N = y^2+6xy-x^2.$

$$\int N.dy = \int (y^2+6xy-x^2)dy$$

$$= y^3/3 + \cancel{\frac{6xy^2}{2} - x^2y}$$

$\int N dy = y^3/3 \; (\cancel{-3xy^2 - x^2y})$ Not containing x

$\longrightarrow (3)$

Using ② & ③ in ①, we get.

$$\frac{x^3}{3} - x^2y + 3xy^2 + \frac{y^3}{3} = c$$

$$x^3 - 3x^2y + 9xy^2 + y^3 = 3c$$

$$x^3 + y^3 - 3x^2y + 9xy^2 = c.$$

② Solve: $(a^2 - 2xy - y^2)dx - (x+y)^2 dy = 0$

$$M = (a^2 - 2xy - y^2), \quad N = -(x+y)^2$$
$$= -x^2 - y^2 - 2xy$$

$$\frac{\partial M}{\partial y} = -2x - 2y = -(2x+2y) \quad \Bigg| \quad \frac{\partial N}{\partial x} = -2x - 2y = -(2x+2y)$$

$$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

$\therefore$ The given eqn is exact.

$\therefore$ The soln is $\int M dx + N dy = c \longrightarrow$ ①

$$\int M dx = \int (a^2 - 2xy - y^2) dx.$$

$$= a^2x - \cancel{2}\frac{x^2}{\cancel{2}}y - y^2x$$

$$= a^2x - x^2y - y^2x. \longrightarrow ②$$

$$\int N dx = \int -x^2 - y^2 - 2xy.$$

$$= -y^3/3 \longrightarrow ③$$

Using ② & ③ in ① we get.

$$a^2x - x^2y - y^2x - y^3/3 = c.$$

$$3a^2x - 3x^2y - 3y^2x - y^3 = 3c$$

H-W

1) Solve: $(2x^2y+4x^3-12xy^2+3y^2-xe^y+e^{2x})dy$

$+(12x^2y+2xy^2+4x^3-4y^3+2ye^{2x}-e^y)dx=$

Ans:

$4x^3y+x^2y^2+x^4-4xy^3+4ye^{2x}-xe^y+y^3=c$

2) $xdx+ydy+\frac{xdy-ydx}{x^2+y^2}=0$

Hints.

$\left(x-\frac{y}{x^2+y^2}\right)dx+\left(y+\frac{x}{x^2+y^2}\right)dy=0$

Ans: $x^2+y^2-2\tan^{-1}(x/y)=c$.

Note:

(i) If an eqn $Mdx+Ndy=0$ is not ex

It can always made exact by multiplying

some fun x & y. Such multiplier is c

integrating factor.

* $d(xy)=xdy+ydx$.
* $d(x/y)=\frac{ydx-xdy}{y^2}$
* $d(y/x)=\frac{xdy-ydx}{x^2}$
* $d(\tan^{-1}(y/x))=\frac{xdy-ydx}{x^2+y^2}$
* $d[\tan^{-1}(x/y)]=\frac{ydx-xdy}{x^2+y^2}$
* $d(\log(xy))=\frac{xdy+ydx}{xy}$.
* $d\left(\frac{-1}{xy}\right)=\frac{xdy+ydx}{x^2y^2}$

* $d(\log(y/x))=\frac{xdy-}{x}$
* $d(\log(x/y))=\frac{ydx-}{x}$
* $d(y^2/x)=\frac{2xyd}{}$
* $d(x^2/y)=\frac{2xyd}{}$
* $d(y^2/x^2)=\frac{2x^2yd}{}$

Rules for finding integrating factor:

(i) when $Mx+Ny \neq 0$ and the eqn is homogeneous, $\frac{1}{Mx+Ny}$ is an I.F of $Mdx+Ndy=0$

(ii) When $Mx-Ny \neq 0$ & the eqn is of the form $f_1(xy)\,y\,dx + f_2(xy)\,x\,dy = 0$. $\frac{1}{Mx-Ny}$ is an I.F of the eqn $Mdx+Ndy=0$

① Solve: $a(x \cdot dy + 2y\,dx) = xy\,dy$.

$$ax\,dy + 2ay\,dx - xy\,dy = 0.$$

$$(ax-xy)dy + 2ay\,dx = 0.$$

$M = 2ay$ | $N = (ax-xy)$

$\frac{\partial M}{\partial y} = 2a$ $\quad$ $\frac{\partial N}{\partial x} = a-y$

$\therefore$ The soln is not exact.

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}.$$

Now, $a(x\,dy + 2y\,dx) = xy\,dy$.

$\div xy$.

$$\frac{a\cancel{x}dy}{\cancel{x}y} + \frac{2a\cancel{y}dx}{x\cancel{y}} = \frac{\cancel{xy}dy}{\cancel{xy}}$$

$$\frac{a\,dy}{y} + \frac{2a\,dx}{x} = dy$$

$$\frac{2a\,dx}{x} + \frac{a}{y}dy - dy = 0$$

$$\left(\frac{2a}{x}\right)dx + dy\left(\frac{a}{y}-1\right) = 0.$$

$M = \frac{2a}{x}$ | $N = \frac{a}{y} - 1$

$\frac{\partial M}{\partial y} = 0$ | $\frac{\partial N}{\partial x} = 0$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

The given eqn is exact.

$$\int M dx = \int \frac{2a}{x} dx.$$

$$= 2a \log x$$

$$= a \log x^2. \longrightarrow (2)$$

$$\int N dy = \int \left(\frac{a}{y} - 1\right) dy.$$

$$= a. \log y - y. \longrightarrow (3)$$

Using (2) & (3) in (1) we get.

$$a \log x^2 + a \log y - y = c.$$

$$a(\log x^2 + \log y) - y = c.$$

$$a(\log(x^2 y)) - y = c$$

2) Solve: $(y^2 e^x + 2xy)dx - x^2 dy = 0$

Given $(y^2 e^x + 2xy)dx - x^2 dy = 0$

$M = y^2 e^x + 2xy$, $N = -x^2$

$\frac{\partial M}{\partial y} = 2y e^x + 2x$ $\quad \frac{\partial N}{\partial x} = -2x.$

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

$\therefore$ The given eqn is not exact.

Now, $(y^2 e^x + 2xy) dx - x^2 dy = 0.$

$\div y^2$

$$\left(e^x + \frac{2x}{y}\right) dx - x^2/y^2 \, dy = 0 \longrightarrow (1)$$

$M = e^x + \frac{2x}{y}$ | $N = -x^2/y^2$

$\frac{\partial M}{\partial y} = 2x\left(\frac{-1}{y^2}\right)$ | $\frac{\partial N}{\partial x} = -2x/y^2.$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$\int M dx = \int \left(e^x + \frac{2x}{y}\right) dx.$

$= e^x + \frac{\cancel{2}}{y} \cdot x^2/\cancel{2}$

$= e^x + x^2/y. \longrightarrow$ ②

$\int N dy = \int -x^2/y^2 \, dy. = 0$

$= \cancel{-x^2 \frac{y^{-1}}{-1}}$

$= \cancel{x^2/y}.$

$= 0 \longrightarrow$ ③

$e^x + x^2/y = c.$

$ye^x + x^2 = yc.$

3) $(y^2 + 2x^2 y)dx + (2x^3 - xy)dy = 0.$

$M = y^2 + 2x^2 y.$ | $N = 2x^3 - xy.$

$\frac{\partial M}{\partial y} = 2y + 2x^2$ | $\frac{\partial N}{\partial x} = 6x^2 - y.$

The given eqn is not exact

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}.$$

Now, $(y^2 + 2x^2 y)dx + (2x^3 - xy)dy = 0 \longrightarrow$ ①

Assume that $x^m y^n$ is an integrating factor

Using ①.

$$(x^m y^n y^2 + x^m y^n 2x^2y)dx + (x^m y^n 2x^3 - x^m y^n xy)dy = 0$$

$$(x^m y^{n+2} + 2x^{m+2} y^{n+1})dx + (2x^{m+3} \cdot y^n - x^{m+1} y^{n+1})dy = 0$$

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ is exact.

$M = x^m y^{n+2} + 2x^{m+2} \cdot y^{n+1}$ | $N = 2x^{m+3} y^n - x^{m+1} y^{n+1}$

$\frac{\partial M}{\partial y} = x^m (n+2) y^{n+1} + 2x^{m+2}(n+1)y^n$ | $\frac{\partial N}{\partial x} = 2(m+3)y^n x^{m+2} - (m+1)x^m y^{n+1}$

$$\Rightarrow x^m(n+2)y^{n+1} + 2x^{m+2}(n+1)y^n = 2y^n(m+3)x^{m+2} - (m+1)x^m y^{n+1}$$

$$\Rightarrow nx^m y^{n+1} + 2x^m y^{n+1} + 2nx^{m+2}y^n + 2x^{m+2}y^n =$$

$$2my^n x^{m+2} + 6y^n x^{m+2} - my^{n+1}x^m - x^m y^{n+1} = 0$$

$$\Rightarrow x^m y^{n+1}(n+m+3) + x^{m+2}y^n(2n-2m-4) = 0$$

Taking zero for R.H.S identically,

$$\Rightarrow x^m y^{n+1}(n+m+3) = 0.$$

$$n+m+3 = 0$$

$$\Rightarrow x^{m+2} \cdot y^n (2n - 2m - 4) = 0$$

$$2n - 2m - 4 = 0.$$

$$2(n-m-2) = 0.$$

$$n + m + 3 = 0$$

$$n - m - 2 = 0$$

$$2n + 1 = 0$$

$$2n = -1$$

$$m - \frac{1}{2} + 3 = 0$$

$$m = -3 + \frac{1}{2}$$

$$\boxed{m = -\frac{5}{2}}$$

$x^m y^n \rightarrow (*)$

Using $m = -5/2$ $n = -1/2$ in $(*)$

$$\left(x^{-5/2} y^{-1/2+2} + 2x^{-5/2+2} y^{-1/2+1}\right) dx + \left(2x^{-5/2+3} \cdot y^{-1/2} - x^{-5/2+1} \cdot y^{-1/2+1}\right) dy = 0.$$

$$\left(x^{-5/2} y^{3/2} + 2x^{-1/2} \cdot y^{1/2}\right) dx + \left(2x^{1/2} \cdot y^{1/2} - x^{-3/2} \cdot y^{1/2}\right) dy = 0 \rightarrow (**)$$

$M = x^{-5/2} \cdot y^{3/2} + 2x^{-1/2} y^{1/2}$.

$$\frac{\partial M}{\partial y} = 3/2 \cdot x^{-5/2} \cdot y^{1/2} + \cancel{2} x^{-1/2} \cdot \frac{1}{\cancel{2}} y^{-1/2}$$

$$= 3/2 \cdot y^{1/2} x^{-5/2} + x^{-1/2} \cdot y^{-1/2}.$$

$N = 2x^{1/2} \cdot y^{-1/2} - x^{-3/2} y^{1/2}$.

$$\frac{\partial N}{\partial x} = \cancel{2} \cdot \frac{1}{\cancel{2}} x^{-1/2} \cdot y^{-1/2} + 3/2\, x^{-5/2} \cdot y^{1/2}.$$

$$\frac{\partial N}{\partial x} = 3/2\, y^{1/2} x^{-5/2} + x^{-1/2} \cdot y^{-1/2}$$

$\therefore$ The given solution is exact.

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$$\int M\,dx = \int \left(x^{-5/2} \cdot y^{3/2} + 2x^{-1/2} y^{1/2}\right) dx.$$

$$= \frac{x^{-3/2}}{-3/2} \cdot y^{3/2} + 2\frac{x^{1/2}}{1/2} \cdot y^{1/2}$$

$$= -\frac{2}{3} x^{-3/2} y^{3/2} + 4x^{1/2} y^{1/2}.$$

$$\int N\,dy = \int \left(2x^{1/2} y^{-1/2} - x^{-3/2} y^{1/2}\right) dy.$$

$$\int M\,dx + \int N \cdot dy = c.$$

$$-\tfrac{2}{3} x^{-3/2} y^{3/2} + 4 x^{1/2} y^{1/2} = c.$$

$$-2x^{-3/2} y^{3/2} + 12 x^{1/2} y^{1/2} = c$$

$$12 x^{1/2} y^{1/2} = 2x^{-3/2} y^{3/2} + c$$

$$6\sqrt{xy} = x^{-3/2} y^{3/2} + c$$

4) Solve : $(x^2y - 2xy^2)\,dx - (x^3 - 3x^2y)\,dy = 0$

$M = x^2y - 2xy^2$ | $N = x^3 - 3x^2y.$

$\frac{\partial M}{\partial y} = x^2 - 4xy$ | $\frac{\partial N}{\partial x} = 3x^2 - 6xy$

The given eqn is not exact.

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

$$(x^2y - 2xy^2)\,dx - (x^3 - 3x^2y)\,dy = 0 \longrightarrow (1)$$

When $Mx + Ny \neq 0$ and the eqn is homogeneous

$\frac{1}{Mx+Ny}$ is an I.F of $M\,dx + N\,dy = 0.$

$$(1) \Rightarrow \left(\frac{\cancel{x^2y}}{\cancel{x^2}y^2\cancel{}} - \frac{2x\cancel{y^2}}{x^2\cancel{y^2}}\right)dx - \left(\frac{\cancel{x^3}}{\cancel{x^2}y^2} - \frac{3\cancel{x^2y}}{\cancel{x^2}y\cancel{}}\right)dy = 0$$

$$= \left(\frac{1}{y} - \frac{2}{x}\right)dx - \left(\frac{x}{y^2} - \frac{3}{y}\right)dy = 0$$

$M = \frac{1}{y} - \frac{2}{x}.$ | $N = -\frac{x}{y^2} - \frac{3}{y}$

$\frac{\partial M}{\partial y} = -\frac{1}{y^2}$ | $\frac{\partial N}{\partial x} = -\frac{1}{y^2}$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$$\int M\,dx = \int\left(\frac{1}{y} - \frac{2}{x}\right)dx.$$

$$= \frac{x}{y} - 2\log x$$

$$\int N\,dy = \int\left(-\frac{x}{y^2} + \frac{3}{y}\right)dy.$$

$$= 0 + 3\log y.$$

$$\therefore \frac{x}{y} - 2\log x + 3\log y = C$$

$$\frac{x}{y} - \log x^2 + \log y^3 = c.$$

$$\frac{x}{y} + \log\left(y^3/x^2\right) = c.$$

(5) solve : $y(xy + 2x^2y^2)dx + x(xy - x^2y^2)dy = 0$

This eqn is of the form $f_1(xy)\cdot y\,dx + f_2(xy)\,x\,dy = 0$

I.F is $\frac{1}{Mx - Ny}$

$$= \frac{1}{y(xy + 2x^2y^2)x - (xy - x^2y^2)xy}$$

$$= \frac{1}{\cancel{x^2y^2} + 2x^3y^3 - \cancel{x^2y^2} + x^3y^3}$$

$$= \frac{1}{3x^3y^3}$$

Given $\Rightarrow \left(\frac{xy^2}{3x^3y^3} + \frac{2x^2y^3}{3x^3y^3}\right)dx + \left(\frac{x^2y}{3x^2y^3} - \frac{x^3y^2}{3x^3y^3}\right)dy = 0$

$$= \left(\frac{1}{3x^2y} + \frac{2}{3x}\right)dx + \left(\frac{1}{3y^2} - \frac{1}{3y}\right)dy = 0.$$

$M = \frac{1}{3x^2y} + \frac{2}{3x}$ | $N = \frac{1}{3xy^2} - \frac{1}{3y}$.

$\frac{\partial M}{\partial y} = -\frac{1}{3x^2y^2}$ | $\frac{\partial N}{\partial x} = -\frac{1}{3x^2y^2}$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$\therefore$ The given solution is exact

$$\int M\,dx = \int\left(\frac{1}{3x^2y} + \frac{2}{3x}\right)dx.$$

$$= \frac{-1}{3yx} + \frac{2}{3}\log x$$

$$\int N\,dy = \int\left(\frac{1}{3xy^2} - \frac{1}{3y}\right)dy$$

$$= -\frac{1}{3}\log y.$$

$$\int M\,dx + \int N\,dy = c$$

$$-\frac{1}{3xy} + \frac{2}{3}\log x - \frac{1}{3}\log y = c.$$

$$-\frac{1}{xy} + 2\log x - \log y = c.$$

$$-\frac{1}{xy} + \log x^2 - \log y = c$$

$$-\frac{1}{xy} + \log(x^2/y) = c$$

$$\log(x^2/y) - \frac{1}{xy} = c$$

6) Solve : $(y-3x^2)dx - x(1-xy^2)dy = 0$.

Given: $(y-3x^2)dx - (x - x^2y^2)dy = 0$.

$M = y - 3x^2$ | $N = -x + x^2y^2$.

$\frac{\partial M}{\partial y} = 1$ | $\frac{\partial N}{\partial x} = -1 + 2xy^2$

This eqn is not exact

$Mx + Ny \neq 0$.

$(y-3x^2)x + (x^2y^2 - x)y \neq 0$

$\cancel{xy} - 3x^3 + x^2y^3 - \cancel{xy} \neq 0$

$\Rightarrow x^2y^3 - 3x^3 \neq 0$

Given: $(y-3x^2)dx - x(1-xy^2)dy = 0$.

$y.dx - 3x^2dx - xdy + x^2y^2dy = 0$

$\Rightarrow (-xdy + ydx) - 3x^2dx + x^2y^2dy = 0$.

$\Rightarrow -(xdy - ydx) - 3x^2dx + x^2y^2dy = 0$.

$= -\left(\frac{xdy - ydx}{x^2}\right) - \frac{3\cancel{x^2}dx}{\cancel{x^2}} + \frac{\cancel{x^2}y^2dy}{\cancel{x^2}} = 0$

$\Rightarrow -d(y/x) - 3dx + y^2dy = 0$

integrating we have

$= -\int d(y/x) - 3\int dx + \int y^2dy = 0$.

$= -y/x - 3x + y^3/3 = 0$.

7) ~~Solve~~: $(1+xy^2)\,dx + (1+x^2y)\,dy = 0$

$M = 1+xy^2$ | $N = 1+x^2y$

$\frac{\partial M}{\partial y} = 2xy$ | $\frac{\partial N}{\partial x} = 2xy.$

$\therefore$ The given soln is exact.

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$\int M\,dx = \int (1+xy^2)\,dx$

$= x + x^2/_2\, y^2$

$\int N\,dy = \int (1+x^2y)\,dy$

$= y + \cancel{x^2y^2/2}$

$\int M\,dx + \int N\,dy = c.$

$x + \frac{x^2y^2}{2} + y = c$

$x + y + \frac{x^2y^2}{2} = c$

8) solve: $(x^2+y^2)(x\,dx + y\,dy) = a^2(x\,dy - y\,dx)$

The above eqn can be written as.

$x^3dx + x^2y\,dy + xy^2dx + y^3dy = a^2x\,dy - a^2y\,dx$

$x^3dx + x^2y\,dy + xy^2dx + y^3dy - a^2x\,dy + a^2y\,dx =$

$(x^3 + xy^2 + a^2y)\,dx + (x^2y + y^3 - a^2x)\,dy = 0.$

$M = x^3 + xy^2 + a^2y$ | $N = x^2y + y^3 - a^2x$

$\frac{\partial M}{\partial y} = x2y + a^2$ | $\frac{\partial N}{\partial x} = 2xy - a^2$

$\therefore$ The given soln is not exact.

Given : $(x^2+y^2)(x\,dx+y\,dy) = a^2(x\,dy-y\,dx)$

$$x\,dx+y\,dy = a^2\left(\frac{x\,dy-y\,dx}{x^2+y^2}\right)$$

$$x\,dx+y\,dy = a^2\, d\left[\tan^{-1}(y/x)\right]$$

Integrating on both sides

$$\frac{x^2}{2}+y^2/2 = a^2\tan^{-1}(y/x)+c.$$

9) Solve : $\dfrac{dy}{dx} = \dfrac{2x}{x^2+y^2-2y}$

$$(x^2+y^2-2y)\,dy = 2x\,dx$$

$$x^2dy+y^2dy-2y\,dy = 2x\,dx.$$

$$(x^2+y^2)\,dy = 2x\,dx+2y\,dy.$$

$$dy = \cancel{2}\left(\frac{2x\,dx+2y\,dy}{x^2+y^2}\right)$$

$$dy = \frac{d(x^2)+d(y^2)}{x^2+y^2}.$$

$$dy = d\left(\frac{x^2+y^2}{x^2+y^2}\right)$$

integrating on both sides

$$y = \log(x^2+y^2)+c$$

10) Solve : $y\,dx - x\,dy - 3x^2y^2e^{x^3}dx = 0$

Given $y\cdot dx - x\,dy - 3x^2y^2e^{x^3}dx = 0$

$\div y^2$.
$$\frac{y\,dx-x\,dy}{y^2} - 3x^2e^{x^3}dx = 0$$

$$d(x/y) - 3x^2 e^{x^3} dx = 0$$

$$d(x/y) = 3e^{x^3} x^2 dx.$$

$\cancel{t=0}$

$\int$ing on both sides.

$t = x^3$

$dt = 3x^2 dx.$

$$x/y = \int e^t dt$$

$$x/y = e^t + c.$$

$$x/y = e^{x^3} + c.$$

11) Solve: $y\,dx + (x+y)\,dy = 0$

$$y\,dx + x\,dy + y\,dy = 0$$

$$d(xy) + y\,dy = 0.$$

$\int$ing on both sides.

$$xy + y^2/2 = c.$$

12) Solve: $(x-y)\frac{dy}{dx} = 2x - y$

$$(x-y)\,dy = (2x-y)\,dx.$$

$$x\,dy - y\,dy = 2x\,dx - y\,dx$$

$$x\,dy + y\,dx - y\,dy - 2x\,dx = 0.$$

$$d(xy) - y\,dy - 2x\,dx = 0$$

$\int$ing on both sides

$$xy - y^2/2 - \cancel{2}x^2/\cancel{2} = c$$

$$xy - y^2/2 - x^2 = c$$

UNIT - 3

Partial Differential Equation :

Definition:-

An equation involving derivatives or differentials of one or more dependent variables with respect to one or more independent variables is called partial differential equation.

Ex (i) $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$

ii) $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial t^2} = 0.$

iii) $\frac{\partial^2 v}{\partial t^2} = k\left(\frac{\partial^3 v}{\partial x^3}\right)^2$

Note :

let $z = F(x, y)$ where x and y are independent variable and z is a dependent variable, then $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$ are the first order derivatives and $\frac{\partial^2 z}{\partial x^2}, \frac{\partial^2 z}{\partial y^2}$ are the 2nd order partial derivatives.

Notation:-

(i) $P = \frac{\partial z}{\partial x}$, $q = \frac{\partial z}{\partial y}$; $r = \frac{\partial^2 z}{\partial x^2}$, $t = \frac{\partial^2 z}{\partial y^2}$

$S = \frac{\partial^2 z}{\partial x \partial y}.$

ii) Sphere:

a) let $P(x, y, z)$ be any point on the Sphere whose centre is (a, b, c) and radius is r. then the equation of sphere $(x-a)^2+(y-b)^2+(z-c)^2 = r^2$

b) The equation of the sphere whose centre is Origin $(0,0,0)$ and radius is r. then the eqn is $x^2+y^2+z^2 = r^2$

c) The parametric equation of the Sphere is $x = a\sin u \cos v$, $y = a \sin u \sin v$

$z = a\cos u$.

let the eqn of the sphere whose centre is origin and radius is a then the parametric eqn of the sphere is above values of x, y, z

and $x = a\left(\frac{1-v^2}{1+v^2}\right)\cos u$, $y = a\left(\frac{1-v^2}{1+v^2}\right)\sin v$.

$$z = \frac{2av}{1+v^2}$$

Define Jacobian:

If $u = F(x,y)$ & $v = g(x,y)$ be two continuous funs of the independent variables x & y, there exists $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ are all continuous in x & y then

$$J(u,v) = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = 0$$ is called the Jacobian of u & v with respect to x & y.

Formation of partial differential equation :-

(i) Eliminating arbitrary constant

(ii) Eliminating arbitrary ~~constant~~ function

Elimination of arbitrary constant:-

let $F(x, y, z, a, b) = 0 \rightarrow (1)$ where a, b are arbitrary constants and z be treated dependent variable fun. of x & y

$$F(x, y, z, a, b) = 0 \longrightarrow (1).$$

diff. partially w.r to x

$$\frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial x} + \frac{\partial f}{\partial z} \cdot \frac{\partial z}{\partial x} = 0.$$

$$\Rightarrow \frac{\partial F}{\partial x} + P \frac{\partial F}{\partial z} = 0 \longrightarrow (2)$$

diff partially w.r to y.

$$\frac{\partial F}{\partial y} \cdot \frac{\partial y}{\partial y} + \frac{\partial F}{\partial z} \cdot \frac{\partial z}{\partial y} = 0.$$

$$\Rightarrow \frac{\partial F}{\partial y} + q \frac{\partial F}{\partial z} = 0 \longrightarrow (3)$$

Eliminating a & b from eqn (1) (2) & (3) we get the first order partial diff eqn of the form $F[x, y, z, p, q] = 0.$

1) Eliminate the constant a or b from the following equation

(i) $z = (x+a)(y+b)$

Given $z = (x+a)(y+b)$

diff. partially w.r to 'x'.

$$\frac{\partial z}{\partial x} = (1)(y+b).$$

$$p = y+b.$$

diff. partially w.r to 'y'

$$\frac{\partial z}{\partial y} = x+a\ (1)$$

$$q = x+a.$$

$$\therefore z = pq$$

ii) $2z = (ax+y)^2 + b \rightarrow \cancel{1}$

Given $2z = (ax+y)^2 + b. \longrightarrow 1$

diff partially w.r. to 'x'.

$$\cancel{2}\frac{\partial z}{\partial x} = \cancel{2}(ax+y)(a).$$

$$\frac{\partial z}{\partial x} = a(ax+y) \longrightarrow 2$$

$$p = a(ax+y)$$

diff. partially w.r to 'y'

$$\cancel{2}\frac{\partial z}{\partial y} = \cancel{2}(ax+y)(1)$$

$$\frac{\partial z}{\partial y} = ax+y. \longrightarrow 3$$

$$q = (ax+y)$$

Sub 3 in 2

$$p = aq \longrightarrow 4$$

Using ④ in ②.

$$P = \left(\frac{P}{q}x + y\right)\frac{P}{q}$$

$$\cancel{P} = \left(\frac{Px+qy}{q}\right)\frac{\cancel{P}}{q}.$$

$$1 = \frac{Px+qy}{q^2}$$

$$q^2 = px+qy.$$

iii) $ax^2+by^2+z^2=1$

Given $ax^2+by^2+z^2=1$

$$\text{ie) } z^2 = 1-(ax^2+by^2) \longrightarrow ①$$

diff. partially w.r. to 'x'

$$\cancel{2}z\frac{\partial z}{\partial x} = -a\cancel{2}x.$$

$$zp = -ax.$$

$$\therefore a = -\frac{zp}{x} \longrightarrow ②$$

diff. partially w.r to 'y'

$$\cancel{2}z\frac{\partial z}{\partial y} = -\cancel{2}by.$$

$$zq = -by$$

$$\therefore b = -zq/y \longrightarrow ③$$

Using ② & ③ in ①

$$z^2 = 1-\left[-\frac{zp}{\cancel{x}}x^{\cancel{2}} - \frac{zq}{\cancel{y}}y^{\cancel{2}}\right]$$

$$z^2 = 1+zpx+zqy$$

$$z^2 = 1+z(px+qy).$$

H.W

(iv) $z = x + ax^2y^2 + b$ Ans: $px - x - 2y = 0$.

(v) $z = ax + by + a$ Ans: $z = p(x+1) + qy$.

(vi) $z = ax + (1-a)y + b$ Ans: $p + q - 1 = 0$.

vii) ~~$x-a^2$~~ $(x-a)^2 + (y-b)^2 + z^2 = 1$, Ans: $z^2(p^2+q^2+1) = 1$

iv) $\dfrac{x^2+y^2}{a^2} + \dfrac{z^2}{b^2} = 1$.

Given $\dfrac{x^2+y^2}{a^2} + \dfrac{z^2}{b^2} = 1. \longrightarrow ①$

$$① \quad \frac{z^2}{b^2} = 1 - \left(\frac{x^2+y^2}{a^2}\right).$$

$$z^2 = b^2\left[1 - \left(\frac{x^2+y^2}{a^2}\right)\right]$$

diff. partially w.r to 'x'

$$\cancel{2}z\frac{\partial z}{\partial x} = -\frac{b^2\cancel{2}x}{a^2}.$$

$$zp = -\frac{b^2x}{a^2}.$$

$$\frac{z}{b^2} = -\frac{x}{a^2p}. \longrightarrow ②.$$

diff. partially w.r to 'y'

$$\cancel{2}z\frac{\partial z}{\partial y} = -\frac{b^2\cancel{2}y}{a^2}$$

$$zq = -\frac{yb^2}{a^2}.$$

$$\frac{z}{b^2} = -\frac{y}{qa^2} \longrightarrow ③$$

Eqn. ② & ③

$$\frac{-x}{\cancel{a^2}p} = \frac{-y}{q\cancel{a^2}}$$

$$-xq = -yp \Rightarrow py - xq = 0.$$

5) obtain the PDE of all spheres whose c... lie on the plane $z=0$ and whose radius ... constant and equal to r.

WKT, the general eqn of sphere,

$$(x-a)^2+(y-b)^2+(z-c)^2=a^2$$

(where a is constant)

Given, radius $= r$, $z=0$, $C[a,b,0]$

$$(x-a)^2+(y-b)^2+z^2=r^2 \longrightarrow ①$$

diff partially w.r to x.

$$2(x-a)+2z\frac{\partial z}{\partial x}=0.$$

$$2[(x-a)+zp]=0.$$

$$x-a+zp=0$$

$$a = x+zp \longrightarrow ②$$

diff p. w. r to y.

$$2[(y-b)+2zq=0.$$

$$2[y-b+zq]=0.$$

$$y-b+zq=0$$

$$b=y+zq \longrightarrow ③$$

② & ③ in ①

$$(-zp)^2+(-zq)^2+z^2=r^2$$

$$z^2p^2+z^2q^2+z^2=r^2$$

$$z^2[p^2+q^2+1]=r^2.$$

ii) All sphere whose centre lie on the z-axis.

The eqn of the sphere.

$(x-a)^2+(y-b)^2+(z-c)^2=r^2 \longrightarrow (1)$

z-axis means, $C[0,0,c]$

$$x^2+y^2+(z-c)^2=r^2$$

diff. p. w. r to x.

$$2x+2(z-c)\frac{\partial z}{\partial x}=0.$$

$$\cancel{2}[x+(z-c)P]=0.$$

$$x=-(z-c)P$$

$$(z-c)=-x/P \longrightarrow (2)$$

diff. p. w. r to y.

$$2y+2(z-c)\frac{\partial z}{\partial y}=0.$$

$$2[y+(z-c)q]=0.$$

$$y=-(z-c)q.$$

$$(z-c)=-y/q \longrightarrow (3)$$

Equ (2) & (3)

$$-x/P=-y/q.$$

$$-xq=-yP.$$

$$Py-qx=0.$$

b) All the sphere the origin $(x-a)^2+(y-b)^2$ $+(z-c)^2=r^2$.

The eqn. of the sphere.

$$(x-a)^2+(y-b)^2+(z-c)^2=r^2$$

$C(0,0,0)$

$x^2+y^2+z^2=r^2 \longrightarrow (1)$

diff p. co. r to 'x'

$$2x + 2z \frac{\partial z}{\partial x} = 0.$$

$$\cancel{2} z p = -\cancel{2} x$$

$$x = -zp \longrightarrow (2).$$

diff. p. co. r to 'y'

$$2y + 2z \frac{\partial z}{\partial y} = 0.$$

$$2zq = -2y$$

$$y = -zq \longrightarrow (3)$$

$$\frac{(2)}{(3)} \Rightarrow \frac{x}{y} = \frac{-\cancel{z}p}{-\cancel{z}q}$$

$$\frac{x}{y} = p/q.$$

$$xq = yp.$$

$$xq - yp = 0.$$

7) All the spheres of radius 'c' having their centre on the x, y plane.

let the centre lie on x, y. plane, then

$$(x-a)^2 + (y-b)^2 + (z-c)^2 = c^2 \longrightarrow (1)$$

$$[x, y, 0]$$

$$(x-a)^2 + (y-b)^2 + z^2 = c^2,$$

diff p. co. r to 'x'

$$2(x-a) + 2z \frac{\partial z}{\partial x} = 0$$

$$\cancel{2}(x-a) = -\cancel{2} z p$$

$(x-a) = -zp \longrightarrow ②$.

diff. p. w. r to 'y'

$$2(y-b) + 2z\frac{\partial z}{\partial y} = 0.$$

$$\cancel{2}(y-b) = -\cancel{2}zq$$

$$(y-b) = -zq \longrightarrow ③.$$

Using ② & ③ in ①.

$$(-zp)^2 + (-zq)^2 + z^2 = c^2$$

$$z^2p^2 + z^2q^2 + z^2 = c^2$$

$$z^2(p^2+q^2+1) = c^2$$

(*) Elimination of arbitrary function:-

let u & v be any two functions of (x, y, z) and connected by an arbitrary relation $\phi(u,v)=0$.

$$\phi(u,v) = 0 \longrightarrow ①$$

diff. p. w. r to 'x'

$$\frac{\partial\phi}{\partial u}\left[\frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial x} + \frac{\partial u}{\partial z}\cdot\frac{\partial z}{\partial x}\right] + \frac{\partial\phi}{\partial v}\left[\frac{\partial v}{\partial x}\cdot\frac{\partial x}{\partial x} + \frac{\partial v}{\partial z}\cdot\frac{\partial z}{\partial x}\right] = 0 \longrightarrow ②$$

diff p. w. r to 'y'

$$\frac{\partial\phi}{\partial u}\left[\frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial y} + \frac{\partial u}{\partial z}\cdot\frac{\partial z}{\partial y}\right] + \frac{\partial\phi}{\partial v}\left[\frac{\partial v}{\partial y}\cdot\frac{\partial y}{\partial y} + \frac{\partial v}{\partial z}\cdot\frac{\partial z}{\partial y}\right] = 0 \longrightarrow ③$$

Eliminating $\frac{\partial\phi}{\partial u}$ & $\frac{\partial\phi}{\partial v}$.

$$② \Rightarrow \frac{\partial\phi}{\partial u}\left[\frac{\partial u}{\partial x} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial x}\right] = -\frac{\partial\phi}{\partial v}\left[\frac{\partial v}{\partial x} + \frac{\partial}{\partial z}\right.$$

$$③ \Rightarrow \frac{\partial\phi}{\partial u}\left[\frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial y}\right] = -\frac{\partial\phi}{\partial v}\left[\frac{\partial v}{\partial y} + \frac{\partial v}{\partial z}\right.$$

④ ÷ ⑤

$$\frac{\frac{\partial u}{\partial x} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial x}}{\frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial y}} = \frac{\cancel{-}\left[\frac{\partial v}{\partial x} + \frac{\partial v}{\partial z}\frac{\partial z}{\partial x}\right]}{\cancel{-}\left[\frac{\partial v}{\partial y} + \frac{\partial v}{\partial z}\frac{\partial z}{\partial y}\right]}$$

Notation :-

$$\frac{\partial u}{\partial x} = u_x ; \frac{\partial u}{\partial y} = u_y ; \frac{\partial u}{\partial z} = u_z$$

$$\frac{(u_x + u_z p)}{(u_y + u_z q)} = \frac{(v_x + v_z p)}{(v_y + v_z q)}$$

$$(u_x + u_z p)(v_y + v_z q) = (v_x + v_z p)(u_y + u_z$$

$$u_x v_y + u_x v_z q + u_z v_y p + u_z \cancel{v_z p q} = v_x u_y +$$

$$+ v_z u_y p + v_z \cancel{u_z p q}$$

$$0 = u_y v_x + u_y v_z p + u_z v_x q - u_x v_y - u_z v_y p$$

$$- u_x v_z q$$

$$0 = p[u_y v_z - u_z v_y] + q[u_z v_x - u_x v_z]$$

$$- [u_x v_y - u_y v_x]$$

$$u_x v_y - u_y v_x = p[u_y v_z - u_z v_y] + q[u_z v_x - u$$

where $R = u_x v_y - u_y v_x$, $P = u_y v_z - u_z v_y$

$Q = u_z v_x - u_x v_z$

$R = pP + qQ$

$Pp + Qq = R$

This eqn can be putting the form of

$R = Pp + Qq$.

1) Solve $z = F(x^2+y^2)$

Given $z = F(x^2+y^2) \longrightarrow (1)$

diff p. w. r to x.

$$\frac{\partial z}{\partial x} = F'(x^2+y^2)\, 2x.$$

$$p = 2x F'(x^2+y^2) \longrightarrow (2)$$

diff p. w. r to 'y'

$$\frac{\partial z}{\partial y} = F'(x^2+y^2)\, 2y.$$

$$q = 2y F'(x^2+y^2) \longrightarrow (3)$$

(2) ÷ (3)

$$\frac{p}{q} = \frac{\cancel{2}x\, F'\cancel{(x^2+y^2)}}{\cancel{2}y\, F'\cancel{(x^2+y^2)}}$$

$$\frac{p}{q} = x/y.$$

$$py = qx.$$

$$py - qx = 0.$$

H.W

$p = -q$

(1) Solve $z = x+y+F(x,y)$. $\quad \frac{\partial z}{\partial x} = 1 \quad \boxed{p+q=0}$

Ans: $px - qy = x - y$. $\quad p = 1$

(2) $z = F\left(\frac{xy}{z}\right)$ Ans: $xp - yq = 0$. $\quad \frac{\partial z}{\partial y} = -1$

(3) $z = x - y$ Ans: $p + q = 0$.

(4) $z = e^y F(x+y)$ Ans: $q = p + z$

5) Solve $z = F(x+ay) + \phi(x-ay)$.

Given $z = F(x+ay) + \phi(x-ay) \longrightarrow$

diff/. p. w. r to x.

$$\frac{\partial z}{\partial x} = F'(x+ay)(1) + \phi'(x-ay)(1).$$

$$p = F'(x+ay) + \phi'(x-ay) \longrightarrow ②$$

diff/. p. w r to 'y'

$$\frac{\partial z}{\partial y} = F'(x+ay)(a) + \phi'(x-ay)(-a).$$

$$\frac{\partial z}{\partial y} = aF'(x+ay) - a\phi'(x-ay)$$

$$q = aF'(x+ay) - a\phi'(x-ay) \longrightarrow ③$$

Again deff/. ② w. r to 'x'

$$\frac{\partial^2 z}{\partial x^2} = f''[x+ay] + \phi''(x-ay) \rightarrow ④$$

Again diff/. ③ w. r to 'y'

$$\frac{\partial^2 z}{\partial y^2} = F''[a+ay]a^2 + \phi''(x-ay)a^2$$

$$= a^2 [F''(x+ay) + \phi''(x-ay)]$$

$$\frac{\partial^2 z}{\partial y^2} = a^2 \frac{\partial^2 z}{\partial x^2}.$$

$$t = a^2 r$$

1) Eliminate arbitrary fun. from.

$f[x^2+y^2+z^2 \,;\, z^2 - 2xy] = 0.$

Soln:

$F(u, v) = 0.$

let $u = x^2+y^2+z^2$; $v = z^2 - 2xy$.

WKT $\dfrac{J(u,v)}{(x,y)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = 0$;

$u = x^2+y^2+z^2$.

$\dfrac{\partial u}{\partial x} = 2x + 2z\dfrac{\partial z}{\partial x}$, $\dfrac{\partial u}{\partial y} = 2y + 2z\dfrac{\partial z}{\partial y}$.

$u_x = 2x + 2zp$; $\dfrac{\partial u}{\partial y} = 2y + 2zq$.

$v = z^2 - 2xy$.

$\dfrac{\partial v}{\partial q} = 2z\dfrac{\partial z}{\partial x} - 2y$; $\dfrac{\partial v}{\partial y} = 2z\dfrac{\partial z}{\partial y} - 2x$.

$= 2zp - 2y$; $= 2zq - 2x$.

$$\begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = \begin{vmatrix} 2x+2zp & 2y+2zq \\ 2zp-2y & 2zq-2x \end{vmatrix} = 0.$$

$\Rightarrow (2x+2zp)(2zq-2x) - (2y+2zq)(2zp-2y) = 0$

$\Rightarrow 4xzq - 4x^2 + 4z^2pq - 4xzp - (4zyp - 4y^2 + 4z^2pq - 4yzq) = 0.$

$\Rightarrow 4xzq - 4x^2 + 4\cancel{z^2pq} - 4xzp - 4zyp + 4y^2 - 4\cancel{z^2pq} + 4yzq = 0.$

$\Rightarrow 4[xzq - x^2 - \cancel{4}xzp - zyp + y^2 + yzq] = 0.$

$zq(x+y) - zp(x+y) - \cancel{6}x^2 + y^2 = 0.$

$y^2 - x^2 + (x+y)(zq - zp) = 0.$

$(x+y)(zq - zp) = x^2 - y^2$

$$\cancel{(x+y)}\,(zq - zp) = \cancel{(x+y)}(x-y)$$

$$z(q-p) = (x-y)$$

$$-z(p-q) = -(y-x)$$

$$z(p-q) = y-x$$

H.W Solve $F[x^2+y^2;\ z-xy] = 0$.

Ans: $py - qx = y^2 - x^2$.

Classification of integrals:

1) let PDE be $F(x, y, z, p, q) = 0 \rightarrow (1)$ and the solution of the equation be $\phi(x, y, z, a, b) = 0 \longrightarrow (2)$

where a & b are arbitrary constant Hence eqn (2) is called complete integral of eqn (1).

Note:-

A particulary soln of eqn (1) is that giving particular values to a & b in eqn (2)

2) let PDE be $F[x, y, z, p, q] = 0 \longrightarrow (1)$

& $\phi[x, y, z, p, q] \longrightarrow (2)$ are the complete integral of the solution diff (2) p.w.r to a, b

$$\frac{\partial \phi}{\partial a}[x, y, z, a, b] = 0.$$

$$\frac{\partial \phi}{\partial a} = 0 \rightarrow (3);\quad \frac{\partial \phi}{\partial b} = 0 \rightarrow (4)$$

Eliminate a & b from eqn ②, ③ & ④ we get the solution of singular integral.

3) let PDE be $F[x, y, z, p, q] = 0 \longrightarrow ①$

and $\phi[x, y, z, a, b] = 0 \longrightarrow ②$ are the complete integral of the solution.

let us assume that an arbitrary relation between a & b of the form $b = f(a)$ in ②.

$$\therefore \phi[x, y, z, a, b] = 0.$$

$$\phi[x, y, z, a, f(a)] = 0 \longrightarrow ③$$

diff p. w. r to 'a'

$$\frac{\partial \phi}{\partial a} + \frac{\partial \phi}{\partial (F(a))} F'(a) = 0.$$

$$\frac{\partial \phi}{\partial a} + \frac{\partial \phi}{\partial b} F'(a) = 0 \longrightarrow ④$$

The elimination of 'a' between these eqn ③ & ④ is called the general integral (particular integral).

Special type of First order PDE :-

Type : 1 :-

$$F(P, q) = 0$$

(i) solution $\Rightarrow z = ax + by + c$.

(ii) No singular integral & having general integral $\{c = F(a)\}$

$p = a \;;\; q = b$

1) Solve $p^2+q^2=4$

Given $p^2+q^2=4 \longrightarrow (1)$

let us assume that $z=ax+by+c \rightarrow$

diff (2) p.w.r to x & y

$$\frac{\partial z}{\partial x}=a; \quad \frac{\partial z}{\partial y}=b.$$

$$p=a \quad ; \quad q=b$$

$(1) \Rightarrow a^2+b^2=4$

$$a^2=4-b^2.$$

$$a=\pm\sqrt{4-b^2}. \longrightarrow (3).$$

Sub. (3) in (2).

$$z=\left(\pm\sqrt{4-b^2}\right)x+by+c$$

2) Solve $p^2+q^2=npq$.

Given $p^2+q^2=npq \longrightarrow (1)$.

$$z=ax+by+c \longrightarrow (2)$$

diff. (2) p.w.r to x & y.

$$p=a, \quad q=b.$$

$(1) \Rightarrow a^2+b^2=nab.$

$$a^2-nab+b^2=0.$$

$a=1; \; b=-nb, \; c=b^2. \quad a=\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}.$

$$a=\frac{nb\pm\sqrt{n^2b^2-4b^2}}{2}$$

$$=\frac{nb\pm b\sqrt{n^2-4}}{2}$$

$$a = \frac{b\left[n \pm \sqrt{n^2-4}\right]}{2}.$$

$$(2) \Rightarrow z = b\left[\frac{n \pm \sqrt{n^2-4}}{2}\right]x + by + c.$$

$\therefore$ This is the required complete integral

To find the singular integral :-

$$(2) \Rightarrow z = b\left[\frac{n \pm \sqrt{n^2-4}}{2}\right]x + by + c.$$

$$\frac{\partial z}{\partial a} = 0 \; ; \; \frac{\partial z}{\partial b} = 0 \; ; \; \frac{\partial z}{\partial c} = 0.$$

diff p. w. r to b & c.

$$\frac{\partial z}{\partial b} = (1)\left[\frac{n \pm \sqrt{n^2-4}}{2}\right]x + y.$$

$$0 = \left(\frac{n \pm \sqrt{n^2-4}}{2}\right)x + y.$$

$$\frac{\partial z}{\partial c} = 1 \qquad \text{singular} = 0.$$

$$0 = 1$$

Here $1 = 0$ is not possible there is no ~~soln~~ singular integral

To find general integral :-

put $c = F(b)$ in eqn (2) $\quad c = F(b)$

$$(2) \Rightarrow z = b\left[\frac{n \pm \sqrt{n^2-4}}{2}\right]x + by + F(b).$$

diff p. w. r to b

$$\frac{\partial z}{\partial b} = \left(\frac{n \pm \sqrt{n^2-4}}{2}\right)x + y + F'(b)$$

We eliminate 'b' we get the required general integral

H-W

1) solve $3p^2 - 2q^2 = 4pq$

Ans: $\frac{\partial z}{\partial b} = \left[\frac{2 \pm \sqrt{10}}{3}\right] x + y + F'(b)$

2) solve $pq = k$ Ans: $z = ax + \frac{k}{a} y + c$.

3) $pq + p + q = 0$.

Ans: $z = \frac{-bx}{b+1} + by + c$.

4) $\sqrt{p} + \sqrt{q} = 0$.

Ans: $z = (1-\sqrt{b})^2 + by + c$.

5) $q^2 - 3q + p = 2$ Ans: $\frac{\partial z}{\partial b} = 2x - bx(b-3) + y + F'(b)$

3) P.T the characteristics of $q = 3p^2$ that passes through the point $(-1,0,0)$ generate the cone $(x+1)^2 + 12yz = 0$.

Given $q = 3p^2 \longrightarrow (1)$

WKT $z = ax + by + c \longrightarrow (2)$

diff p.w.r to x & y.

$\frac{\partial z}{\partial x} = a$; $\frac{\partial z}{\partial y} = b$.

$p = a$; $q = b$

$(1) \Rightarrow b = 3a^2$.

Sub in (2) $\Rightarrow z = ax + 3a^2y + c \rightarrow (3)$

The curves represented by the general integral are called the Characteristics Curves.

To find general integral:-

$$\underline{c = F(a)}$$

$$③ \Rightarrow z = ax + 3a^2y + F(a) \longrightarrow ④$$

Since ④ it passes through the pt. $\overset{x}{(-1},\overset{y}{0},\overset{z}{0})$

$$④ \Rightarrow 0 = -a + 0 + F(a)$$

$$a = F(a)$$

Sub in ④

$$z = \overset{a}{\cancel{F(a)}}\, x + 3a^2y + \cancel{F}\, a$$

$$z = a(x+1) + 3a^2y \longrightarrow ⑤$$

diffl. w. r. to 'a'

$$\frac{\partial z}{\partial a} = (1)(x+1) + 6ay \qquad \frac{\partial z}{\partial a} = 0,\ \frac{\partial z}{\partial b} = 0, \quad \frac{\partial z}{\partial a} = 0.$$

$$0 = (x+1) + 6ay$$

$$6ay = -(x+1)$$

$$a = \frac{-(x+1)}{6y} \longrightarrow ⑥$$

Using ⑥ in ⑤

$$z = \frac{-(x+1)}{6y}(x+1) + 3\left(\frac{-(x+1)}{6y}\right)^2 y.$$

$$= \frac{-[x^2+2x+1]}{6y} + \frac{\cancel{3}(x^2+2x+1)}{\cancel{36}_{12}\,\cancel{y^2}}\cancel{y}$$

$$= -\frac{(x+1)^2}{6y} + \frac{(x+1)^2}{12y}.$$

$$z = \frac{-2(x+1)^2 + (x+1)^2}{12y}$$

$$z = -\frac{(x+1)^2}{12y}.$$

$$12zy = -(x+1)^2$$

$$(x+1)^2 + 12yz = 0.$$

4) Obtain the complete integral of $p^2+q^2 =$ in the form of $z = cx\cos\alpha + cy\sin\alpha + d$ p.T $z^2 = c^2(x^2+y^2)$ is a general integral

Given $p^2+q^2 = c^2 \longrightarrow ①$

$z = ax+by+c \longrightarrow ②$.

$p = a; \quad q = b.$

$① \Rightarrow a^2+b^2 = c^2$.

let $a = c\cos\alpha; \; b = c\sin\alpha; \; c = d.$

where α is arbitrary constant

$z = ax+by+c.$

$z = cx\cos\alpha + cy\sin\alpha + d.$

$\therefore$ This is the required complete integral

To find the general integral :-

let $d = F(\alpha)$

$\therefore z = cx\cos\alpha + cy\sin\alpha + F(\alpha) \longrightarrow ③$

diffl. p. w. r to α.

$$\frac{\partial z}{\partial \alpha} = -cx\sin\alpha + cy\cos\alpha + F'(\alpha)$$

We assume, $F(\alpha)=0$; $F'(\alpha)=0$

$$0 = -cx\sin\alpha + cy\cos 6 + 0 \rightarrow ④$$

③ $\Rightarrow z = cx\cos\alpha + cy\sin\alpha$

④ $\Rightarrow 0 = -cx\sin\alpha + cy\cos\alpha$.

Squaring & adding

$$z^2+0^2 = (cx\cos\alpha + cy\sin\alpha)^2 + [-cx\sin\alpha + cy\cos\alpha]^2$$

$$z^2 = c^2x^2\cos^2\alpha + c^2y^2\sin^2\alpha + 2c^2xy\cancel{\cos\alpha\sin\alpha}$$

$$+ c^2x^2\sin^2\alpha + c^2y^2\cos^2\alpha - 2c^2\cancel{xy\sin\alpha\cos\alpha}$$

$$= c^2x^2(\cos^2\alpha+\sin^2\alpha) + c^2y^2(\sin^2\alpha+\cos^2\alpha].$$

$$= c^2x^2 + c^2y^2.$$

$$z^2 = c^2(x^2+y^2)$$

Type : 2

$$F_1(x,p) = F_2[y,q]$$

$$\int dz = \int p\,dx + \int q\,dy$$

$$F_1(x,p) = F_2(y,q) = a$$

$dz = p\,dx + q\,dy$

$\int$ing on both sides

1) Solve : $p+q = x+y$.

Given $p+q = x+y$

This eqn. can be written as

$$p-x = y-q$$

let, $p-x=a$; $y-q=a$

$$p = x+a \qquad q = y-a$$

$$\therefore dz = p\,dx + q\,dy$$

$$\int dz = \int (x+a)\,dx + \int (y-a)\,dy.$$

$$z = \frac{(x+a)^2}{2} + \frac{(y-a)^2}{2} + c$$

$\therefore$ This is the required complete integral

2) Solve : $q(p - \sin x) = \cos y$.

$$p - \sin x = \frac{\cos y}{q}$$

let $p - \sin x = a$; $\frac{\cos y}{q} = a$.

$$p = a + \sin x \; ; \quad q = \frac{\cos y}{a}.$$

$\therefore dz = p\,dx + q\,dy.$

$$\int dz = \int (a + \sin x)\,dx + \int \frac{\cos y}{a}\,dy.$$

$$z = ax - \cos x + \tfrac{1}{a} \sin y + c.$$

To find general integral :

$$c = F(a)$$

$$z = ax - \cos x - \frac{\sin y}{a} + F(a).$$

diff. p.w.r. to 'a'

$$\frac{\partial z}{\partial a} = x - 0 - \left(-\frac{\sin y}{a^2}\right) + F'(a).$$

$$0 = x + \frac{\sin y}{a^2} + F'(a)$$

$\therefore$ This is the required general integral

3) Solve $p^2+q^2=x^2+y^2$ $\quad p^2-x^2=y^2-q^2$

$$p^2-x^2=y^2-q^2$$

$$p^2-x^2=a^2 \quad ; \quad y^2-q^2=a^2$$

$$p^2=a^2+x^2 \; ; \quad q^2=y^2-a^2$$

$$p=\sqrt{a^2+x^2} \qquad q=\sqrt{y^2-a^2}$$

$$\therefore dz=p\,dx+q\,dy$$

$$\int dz=\int\sqrt{a^2+x^2}\,dx+\int\sqrt{y^2-a^2}\,dy.$$

$$z=\frac{x\sqrt{a^2+x^2}}{2}+\frac{a^2}{2}\log\left[x+\sqrt{a^2+x^2}\right]$$

$$+\frac{y\sqrt{y^2-a^2}}{2}+a^2/_2\log\left[y+\sqrt{y^2-a^2}\right]$$

$\therefore$ This is the complete integral.

H.W

1) Solve $pq=xy$ $\quad$ Ans: $z=\frac{ax^2}{2}+\frac{y^2}{2a}+c.$

2) Solve $p^2+q^2=x+y$ $\quad$ Ans: $z=\frac{2(a+x)^{3/2}}{3}+\frac{2(y-a)^{3/2}}{3}+c$

3) Solve $yp=2yx+\log q.$

Type: 3

Clairaut's form.

This is of the form $z=px+qy+F(p,q)$

$$p=a \; ; \; q=b$$

It has singular & general integral

① Solve $z = px + qy + \sqrt{(1+p^2+q^2)}$

WKT $z = px + qy + F(p,q)$ $\longrightarrow$ ①

We take $p = a$ & $q = b$

$$z = px + qy + \sqrt{1+p^2+q^2}$$

$$\therefore z = ax + by + \sqrt{1+a^2+b^2} \longrightarrow ②$$

To find Singular integral:-

~~&~~ diff. ② p.w.r.to a & b

$$\frac{\partial z}{\partial a} = x + 0 + \frac{1}{\cancel{2}\sqrt{1+a^2+b^2}}\cancel{2}a.$$

$$0 = x + \frac{a}{\sqrt{1+a^2+b^2}}$$

$$x = \frac{-a}{\sqrt{1+a^2+b^2}} \rightarrow (*) \; ; \; y = \frac{-b}{\sqrt{1+a^2+b^2}} \rightarrow (**)$$

Squaring on both sides.

$$x^2 = \frac{a^2}{1+a^2+b^2} \rightarrow ③ \; ; \; y^2 = \frac{b^2}{1+a^2+b^2} \rightarrow ④$$

③ + ④

$$x^2 + y^2 = \frac{a^2}{1+a^2+b^2} + \frac{b^2}{1+a^2+b^2}$$

$$x^2 + y^2 = \frac{a^2+b^2}{1+a^2+b^2}$$

$$1 - (x^2+y^2) = 1 - \left(\frac{a^2+b^2}{1+a^2+b^2}\right)$$

$$1 - x^2 - y^2 = \frac{1 + \cancel{a^2} + \cancel{b^2} - \cancel{a^2} - \cancel{b^2}}{}$$

$$1-x^2-y^2 = \frac{1}{1+a^2+b^2}.$$

$$1+a^2+b^2 = \frac{1}{1-x^2-y^2}$$

Taking square root on both sides.

$$\sqrt{1+a^2+b^2} = \frac{1}{\sqrt{1-x^2-y^2}} \longrightarrow (5)$$

Using (5) in (*) & (**)

$$(*) \Rightarrow x = \frac{-a}{\frac{1}{\sqrt{1-x^2-y^2}}} \;;\; (**) \Rightarrow y = \frac{-b}{\frac{1}{\sqrt{1-x^2-y^2}}}$$

$$x = -a\sqrt{1-x^2-y^2} \;;\; y = -b\sqrt{1-x^2-y^2}$$

$$a = \frac{-x}{\sqrt{1-x^2-y^2}} \;;\; b = \frac{-y}{\sqrt{1-x^2-y^2}}$$

Using (a), (b) values in (2).

$$(2) \Rightarrow z = \frac{-x^2}{\sqrt{1-x^2-y^2}} - \frac{y^2}{\sqrt{1-x^2-y^2}} + \sqrt{1+a^2+b^2}.$$

$$= \frac{-x^2-y^2}{\sqrt{1-x^2-y^2}} + \sqrt{1+\frac{x^2}{\cancel{\sqrt{1-x^2-y^2}}}+\frac{y^2}{1-x^2-y^2}}$$

$$= \frac{-(x^2+y^2)}{\sqrt{1-x^2-y^2}} + \sqrt{\frac{1-\cancel{x^2}-\cancel{y^2}+\cancel{x^2}+\cancel{y^2}}{1-x^2-y^2}}$$

$$= \frac{-(x^2+y^2)}{\sqrt{1-x^2-y^2}} + \sqrt{\frac{1}{1-x^2-y^2}}$$

$$z = \frac{-(x^2+y^2)+1}{\sqrt{1-x^2-y^2}} = \frac{1-x^2-y^2}{\sqrt{1-x^2-y^2}}$$

$$z = \sqrt{1-x^2-y^2}$$

$$z^2 = 1-x^2-y^2$$

$x^2+y^2+z^2 = 1$,,

To find general integral :-

$$② \Rightarrow z = ax+by+\sqrt{1+a^2+b^2}$$

$$b = F(a).$$

$$z = ax+F(a)y+\sqrt{1+a^2+F(a)}$$

diff P. w.r to a.

$$\frac{\partial z}{\partial a} = x+F'(a)y+\frac{1}{2\sqrt{1+a^2+F(a)}} \times (2a+F'(a))$$

$$0 = x+\frac{2a+F'(a)}{2\sqrt{1+a^2+F(a)}}$$

∴ This is the required general integral

2) Solve $z = px+qy+pq \longrightarrow ①$

$$p = a \; ; \; q = b$$

$$z = ax+by+ab \longrightarrow ②$$

To find Singular integral :-

diff. ② p. w. r to a & b

$$\frac{\partial z}{\partial a} = x+b \; : \quad \frac{\partial z}{\partial b} = a+y.$$

$$0 = x+b. \qquad 0 = a+y.$$

$$b = -x \rightarrow ③ \qquad a = -y \longrightarrow ④$$

$$② \Rightarrow z = -xy - \cancel{xy} + \cancel{xy}$$

$$z = -xy.$$

To find general integral :

② $\Rightarrow z = ax + by + ab.$

$b = F(a)$: $z = ax + F(a)y + aF(a)$

$$\frac{\partial z}{\partial a} = x + F'(a)y + aF'(a) + F(a)\cdot 1.$$

$$0 = x + F'(a)y + aF'(a) + F(a)$$

$\therefore$ This is the required general integral

H.W

⑥ $z = px + qy + p^2q^2$

Ans: $z = -2^{2/3}x^{2/3}y^{2/3}$

① $z = px + qy + 2\sqrt{\cancel{pq}}\ \sqrt{pq}$.

3) $z = px + qy + \sqrt{pq}$

② $\frac{z}{pq} = \frac{x}{q} + \frac{y}{p} + \sqrt{pq}$.

$\hookrightarrow z = 4\left(\frac{4xy}{9}\right)^{3/4}$

4) $z = px + qy + p^2 - q^2$

Ans: $4z = y^2 - x^2$

⑤ $z = px + qy + p^2 + pq + q^2$

Ans: $3z = xy - x^2 - y^2$

Type: 4

$F(x, p, q) = 0$.

(i) $dz = pdx + qdy$; let $q = a$.

(ii) $F(y, p, q) = 0$; $p = a$.

(iii) $F(z, p, q) = 0$; $q = ap$.

Type (i):-

① solve $pq = x$.

Given: $pq = x \longrightarrow$ ①.

This is $F(x, p, q) = 0$.

let $q = a$

$ap = x$.

$p = x/a$

WKT

$$dz = p\,dx + q\,dy.$$

$$\int dz = \int x/a\,dx + \int a\,dy$$

$$z = \frac{1}{a}\frac{x^2}{2} + ay + c$$

$$z = \frac{x^2}{2a} + ay + c$$

This is the required complete integral

$$c = F(a)$$

$$z = \frac{x^2}{2a} + ay + F(a) \longrightarrow (2)$$

To find general integral :-

diffl. w.r. to 'a'

$$\frac{\partial z}{\partial a} = -\frac{x^2}{2a^2} + F'(a).$$

$$0 = \frac{-x^2}{2a^2} + F'(a).$$

This is the required general integral

(2) $p = 2qx$ Ans: $z = ax^2 + ay$.

3) $\sqrt{p} + \sqrt{q} = x$ Ans:

4) $q = xp + p^2$

(ii) Type (2):

1) Solve $p = y^2q^2$

Given $p = y^2 q^2 \longrightarrow (1)$

This is $F(y, p, q) = 0.$

$$p = a$$

$P = a^2$.

① $\Rightarrow a^2 = y^2 q^2$.

$q^2 = a^2/y^2$

$q = \pm a/y$.

$dz = p\,dx + q\,dy$.

$\int dz = \int a\,dx + \int a/y\,dy$.

$z = ax + a\log y + c$.

② $q = 2yp^2$

Given $q = 2yp^2$. $\longrightarrow$ ①

$P = a$

① $\Rightarrow q = 2ya^2$

$dz = p\,dx + q\,dy$.

$\int dz = \int a\,dx + \int 2ya^2\,dy$.

$z = ax + \cancel{2}a^2 \frac{y^2}{\cancel{2}} + c$

$z = ax + a^2y^2 + c$

3) $q^2 = yp^4$

Given $q^2 = yp^4$.

$P = a$.

$q^2 = ya^4$.

$q = \sqrt{ya^4}$

$q = a^2\sqrt{y}$

$$dz = p\,dx + q\,dy.$$

$$\int dz = \int a\,dx + \int a^2\sqrt{y}\,dy.$$

$$\cancel{z = ax + a^2 \frac{1}{2\sqrt{y}}}$$

$$z = ax + a^2 \frac{y^{3/2}}{3/2} + c.$$

$$z = ax + \frac{2}{3}a^2 y^{3/2} + c.$$

Type : (iii).

(1) $pq = z$

Given $pq = z \longrightarrow (1)$

This is $F(z, p, q) = 0.$

$$q = ap \longrightarrow (2)$$

$(1) \Rightarrow p(ap) = z$

$$ap^2 = z$$

$$p^2 = z/a$$

$$p = \pm\sqrt{z/a} \longrightarrow (3)$$

$$\cancel{dz = p\,dx + q\,dy.}$$

$$\cancel{\int dz = \int}$$

Using (3) in (2) $\quad q = a\left(\pm\sqrt{z/a}\right)$

$$= \pm\sqrt{a}\sqrt{a}\,\frac{\sqrt{z}}{\sqrt{a}}$$

$$q = \pm\sqrt{az}$$

$$dz = p\,dx + q\,dy.$$

$$\int dz = \int \sqrt{z/a}\; dx + \int \sqrt{za}\; dy.$$

$$\int dz = \sqrt{\frac{z}{a}}\, x + \sqrt{za}\, y + c.$$

$$\int dz = \frac{\sqrt{z}}{\sqrt{a}}\, x + \sqrt{a}\sqrt{z}\, y + c.$$

$$\int \frac{dz}{\sqrt{z}} = \frac{x}{\sqrt{a}} + \sqrt{a}\, y + c.$$

$$2\sqrt{z} = x/\sqrt{a} + \sqrt{a}\, y + c \,//$$

(2) $p^2 + q^2 = z$

Given $p^2 + q^2 = z \longrightarrow (1)$

This is $F(z, p, q) = 0$

$$q = ap \longrightarrow (2)$$

$$p^2 + a^2p^2 = z$$

$$p^2[1 + a^2] = z$$

$$p^2 = z/1+a^2$$

$$p = \pm\sqrt{z/1+a^2} \longrightarrow (3)$$

Using (3) in (2)

$$q = a\left[\sqrt{z}/1+a^2\right]$$

WKT, $dz = p\,dx + q\,dy$

$$\int dz = \int \sqrt{\frac{z}{1+a^2}}\; dx + \int a\sqrt{\frac{z}{1+a^2}}\; dy.$$

$$\int dz = \sqrt{z}\left[\int \frac{1}{\sqrt{1+a^2}}\, dx + \int \frac{a}{\sqrt{1+a^2}}\, dy\right]$$

(9) $p^2z^2 + q^2 = 1$ Ans: $z(\sqrt{z^2+a^2}) + a^2\log(z+\sqrt{z^2+a^2}) = \pm 2(x+ay)+c$

(10) $p^3 + q^3 = 8z$ Ans: $3(1+a^3)^{1/3}\, z^{2/3} = 4(x+ay)+b.$

$$\int \frac{dz}{\sqrt{z}} = \frac{1}{\sqrt{1+a^2}}\left[\int dx + a\int dy\right]$$

$$2\sqrt{z} = \frac{1}{\sqrt{1+a^2}}[x+ay]+c.$$

$$2\sqrt{z}\sqrt{1+a^2} = x+ay+c$$

Squaring on both sides

$$4z(1+a^2) = (x+ay+c)^2.$$

H.W

3) Solve $p(1+q^2) = q(z-1)$

Ans: $2\sqrt{az-a-1} = x+ay+c$

(6) $p(1+q) = qz$ $\log(az-1) =$

(7) $z^2(p^2+q^2+1) = b^2$

Ans: $-\sqrt{b^2-z^2} = \pm\frac{(x+ay)}{\sqrt{1+a^2}}$

4) $p(1+q) = qz$ Ans: $\log(z-1/a)^{1/a} = x+ay+c$

5) $pe^y = qe^x$ Ans: $z = a(e^x+e^y)+c$

(8) $z^2(p^2z^2+q^2) = 1$ Ans: $\frac{1}{3}(z^2+a^2)^{3/2} = \pm(x+ay)+c.$

Lagrange's Method :-

A linear equation of the first order can be put in the form $Pp+Qq=R$ is called Lagrange's equation. where P, Q, R are the funs of x, y, z [which do not involve p and q]

Auxiliary Equation:

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

General Solution of Lagrange's equation :-

$$F[u,v] = 0$$