UNIT-IV

1

2 marks

1. Write down the condition for the second order homogeneous equation $ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0$ represents

(a) a cone (b) a pair of planes

The general homogeneous quadric in the 3 variables x, y, z

$$F(x,y,z) = ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0 \quad (1)$$

in which a, b, c, f, g, h are constants, represents a cone with vertex at the origin.

If quantities l, m, n satisfy the relation

$$al^2+bm^2+cn^2+2fmn+2gnl+2hlm=0$$

Then the line with direction cosines proportional to l, m, n is the generator of the cone.

If the eqn (1) splits into two linear rational factors then it represents a pair of planes.

If the eqn (1) splits into two linear factors

2

if $\Delta = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$

it represents a cone if $\Delta \neq 0$.

but a pair of planes if $\Delta = 0$.

2. Right Circular Cone (2 mark)

A right circular cone is a surface generated by a line which passes through a fixed point and makes a constant angle with the fixed line through the fixed point.

The fixed point is called the vertex, the fixed line the axis

(2 mark)

Example: 1 S.T the eqn of a right circular cone whose vertex is O, axis OZ and semi-vertical angle α is $x^2+y^2=z^2\tan^2\alpha$.

Soln.- Let the generator of the cone by $\frac{x}{l}=\frac{y}{m}=\frac{z}{n}$

The Z-axis is $\frac{x}{0}=\frac{y}{0}=\frac{z}{1}$

$$\therefore \cos\alpha = \frac{n}{\sqrt{l^2+m^2+n^2}}$$

$$\cos^2\alpha = \frac{n^2}{l^2+m^2+n^2} \Rightarrow (l^2+m^2+n^2)\cos^2\alpha = n^2$$

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$\cos^2\alpha\,(l^2+m^2)+\cos^2\alpha\, n^2 = n^2$

$\cos^2\alpha\,(l^2+m^2) = n^2 - \cos^2\alpha\, n^2$

$= n^2(1-\cos^2\alpha)$

$= n^2\sin^2\alpha$

$l^2+m^2 = n^2 \dfrac{\sin^2\alpha}{\cos^2\alpha}$

$l^2+m^2 = n^2\tan^2\alpha$

$\therefore$ The eqn of the Cone is $x^2+y^2 = z^2\tan^2\alpha$ //

Example: 2 (5 mark)

Find the eqn of the Cone with vertex O and base curve the conic in which the surface $ax^2+by^2+cz^2=1$ is cut by the plane $l_1x+m_1y+n_1z = p$.

Soln:-

Let the generator of the Cone by

$$\frac{x}{l}=\frac{y}{m}=\frac{z}{n}$$

The Co-ordinates of the point where the generator meets the base Curve is of the form $(l\lambda, m\lambda, n\lambda)$

$\lambda^2[al^2+bm^2+cn^2]=1$ —— (1)

$\lambda[ll_1+mm_1+nn_1] = p$ —— (2)

Eliminating λ from (1) & (2),

(1) $\Rightarrow$ $al^2+bm^2+cn^2 = \dfrac{1}{\lambda^2}$

(2) $\Rightarrow$ $\lambda = \dfrac{p}{ll_1+mm_1+nn_1}$

(1) $\Rightarrow$ $al^2+bm^2+cn^2 = \dfrac{1}{\left[\dfrac{p}{ll_1+mm_1+nn_1}\right]^2}$

The Eqn of the Cone is,

$$ax^2+by^2+cz^2 = \frac{[l_1x+m_1y+n_1z]^2}{p^2}$$

Example: 3 (10 mark)

Find the Condition for the eqn

$F(x,y,z) \equiv ax^2+by^2+cz^2+2fyz+2gzx+2hxy$
$+2ux+2vy+2wz+d=0.$

to represent a Cone.

(or)

Soln: Find the general Second degree Cone.

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By given,

$$F(x,y,z) \equiv ax^2+by^2+cz^2+2fyz+2gzx+2hxy$$
$$+2ux+2vy+2wz+e=0$$

If the eqn represents a Cone, let (x_1,y_1,z_1) be its vertex. Then transforming the origin to the vertex. The eqn of the cone transform into

$$F(x_1,y_1,z_1) \equiv a(x+x_1)^2+b(y+y_1)^2+c(z+z_1)^2$$
$$+2f(y+y_1)(z+z_1)+2g(z+z_1)(x+x_1)$$
$$+2h(x+x_1)(y+y_1)+2u(x+x_1)+2v(y+y_1)$$
$$+2w(z+z_1)+e=0.$$

By Equating the co-eff's of x,y,z and the Constant term to zero.

$$ax_1+hy_1+gz_1+u=0 \quad -(1)$$
$$hx_1+by_1+fz_1+v=0 \quad -(2)$$
$$gx_1+fy_1+cz_1+w=0 \quad -(3)$$
$$ux_1+vy_1+wz_1+e=0 \quad -(4)$$

Eliminating u,v,w & e we get,

$$\begin{vmatrix} a & h & g & u \\ h & b & f & v \\ g & f & c & w \\ u & v & w & e \end{vmatrix} = 0.$$

This is the Condition for the eqn to represent a Cone.

(10 mark)

Example:4: Find the Condition for the eqn $ax^2+by^2+cz^2+2fyz+2gzx+2fxy=0$ to represent a right Circular Cone. Obtain the eqn of the axis and the vertical angle of the Cone.

(or)

S.T the condition for the homogeneous eqn $ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0$ to represent a right circular cone is

$$\frac{gh-af}{f} = \frac{hf-bg}{g} = \frac{fg-ch}{h}.$$

Soln:

By given,

$$ax^2+by^2+cz^2+2fyz+2gzx+2fxy=0. \quad -(1)$$

Eqn (1) is homogeneous. So the cone has the Vertex at the origin O.

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Let the axis of the Cone be $\frac{x}{l} = \frac{y}{m} = \frac{z}{n}$. (7)

where l, m, n are its d.c's.

Let $P(x_0, y_0, z_0)$ be any point on the cone. Then d.r's of the generator OP are (x_0, y_0, z_0)

If α is the semi-vertical angle of the cone, then,

$$\cos\alpha = \frac{lx_0 + my_0 + nz_0}{\sqrt{x_0^2 + y_0^2 + z_0^2}}$$

Thus we get the eqn of the Cone as

$$x^2(\cos^2\alpha - l^2) + y^2(\cos^2\alpha - m^2) + z^2(\cos^2\alpha - n^2) - 2(mnyz + nlzx + lxym) = 0$$

Comparing with eqn ①, we get,

$$\frac{\cos^2\alpha - l^2}{a} = \frac{\cos^2\alpha - m^2}{b} = \frac{\cos^2\alpha - n^2}{c} = -\frac{mn}{f} = -\frac{nl}{g}$$

$$= -\frac{lm}{h}$$

Let these six ratios be equal to $-\frac{lmn}{k}$.

From 4th ratio,

$$-\frac{\cancel{mn}}{f} = -\frac{l\cancel{mn}}{k} \Rightarrow l = \frac{k}{f}$$

By symmetry, we have l, m, n as (8)

$$+\frac{\cancel{n}\cancel{l}}{g} = +\frac{\cancel{l}m\cancel{n}}{k} \Rightarrow m = \frac{\cancel{g}}{k}\frac{k}{g}$$

$$+\frac{\cancel{lm}}{h} = +\frac{\cancel{lm}n}{k} \Rightarrow n = \frac{k}{h}$$

Now,

$$\frac{\cos^2\alpha - l^2}{a} = -\frac{lmn}{k}$$

$$\frac{\cos^2\alpha - k^2/f^2}{a} = -\frac{k}{f}\cdot\frac{k}{g}\cdot\frac{\cancel{k}}{h}\cdot\frac{1}{\cancel{k}}$$

$$\cos^2\alpha - \frac{k^2}{f^2} = -\frac{a\cdot k^2}{fgh}$$

$$fgh\left[\cos^2\alpha - \frac{k^2}{f^2}\right] = -\frac{ak^2}{1}$$

$$fgh\cos^2\alpha - \frac{k^2gh}{f} = -ak^2$$

$$\frac{fgh\cos^2\alpha}{k^2} = \frac{gh}{f} - a$$

$$\frac{fgh\cos^2\alpha}{k^2} = \frac{gh - af}{f}$$

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III^{ly} we get,

$$\frac{\cos^2\alpha - m^2}{b} = -\frac{lmn}{k} \Rightarrow \frac{fgh\cos^2\alpha}{k^2} = \frac{hf-bg}{g}$$

$$\frac{\cos^2\alpha - n^2}{c} = -\frac{lmn}{k} \Rightarrow \frac{fgh\cos^2\alpha}{k^2} = \frac{fg-ch}{h}$$

This is the Reqd Condition.

(i) From l, m, n we get,

$$\frac{1}{f}, \frac{1}{g}, \frac{1}{h}$$

Thus the eqn of the axis are $fx = gy = hz$

(ii) The semivertical angle α may be in Particular form,

$$\frac{\cos^2\alpha - l^2}{a} = -\frac{mn}{f}$$

$$\frac{1/f}{\sqrt{1/f^2 + 1/g^2 + 1/h^2}}, \quad \frac{1/g}{\sqrt{\quad ''}}, \quad \frac{1/h}{\sqrt{\quad ''}}$$

Problems: (5 mark)

1. Find the eqn of the right circular cone whose vertex is the origin, axis is the line $\frac{x}{1} = \frac{y}{2} = \frac{z}{3}$ and semivertical angle is $30°$.

Soln:- Let $P(x_0, y_0, z_0)$ be any point on the cone. Now, the d.r's of OP and the axis of the cone are,

$$x_0, y_0, z_0 = 1, 2, 3.$$

$$\cos 30° = \frac{x_0(1) + y_0(2) + z_0(3)}{\sqrt{x_0^2+y_0^2+z_0^2}\sqrt{1^2+2^2+3^2}}$$

$$\frac{\sqrt{3}}{2} = \frac{x_0 + 2y_0 + 3z_0}{\sqrt{x_0^2+y_0^2+z_0^2}\sqrt{14}}$$

Squaring on both sides,

$$\frac{3}{4} = \frac{(x_0 + 2y_0 + 3z_0)^2}{(x_0^2+y_0^2+z_0^2)(14)}$$

$$42(x_0^2+y_0^2+z_0^2) = 4(x_0 + 2y_0 + 3z_0)^2$$

∴ The Eqn of the Cone:

$$21(x^2+y^2+z^2) = 2(x+2y+3z)^2$$

Model sum:

(2) Same problem (1), angle $60°$.

(10 mark)

(3) A right circular cone has its vertex at the origin O and its axis equally inclined to the x, y, z axes. If the eqn of the cone is $4(x^2+y^2+z^2)+9(xy+yz+zx)=0$. P.T the semi vertical angle of the cone is $\cos^{-1}\frac{1}{3\sqrt{3}}$.

soln:

Now the d.r's of the axes are 1, 1, 1.

If $P(a,b,c)$ be a point on the cone, then the d.r's of the generator OP are, a, b, c.

If α is the angle between the axis and OP, then α is the semivertical angle then,

$$\cos\alpha = \frac{1.a+1.b+1.c}{\sqrt{1^2+1^2+1^2}\sqrt{a^2+b^2+c^2}}$$

$$\cos^2\alpha = \frac{(a+b+c)^2}{3(a^2+b^2+c^2)}$$

$$= \frac{a^2+b^2+c^2+2(ab+bc+ca)}{3(a^2+b^2+c^2)}$$

$$\cos^2\alpha = \frac{(a^2+b^2+c^2)}{3(a^2+b^2+c^2)} + \frac{2(ab+bc+ca)}{3(a^2+b^2+c^2)}$$

$$= \frac{1}{3}+\frac{2}{3}\left(\frac{ab+bc+ca}{a^2+b^2+c^2}\right) \quad \text{--- (1)}$$

Now (a,b,c) satisfies the eqn of the cone.

By given, $4(a^2+b^2+c^2)+9(ab+bc+ca)=0$

$$\frac{ab+bc+ca}{a^2+b^2+c^2} = -\frac{4}{9}$$

(1) $\Rightarrow$ $$\cos^2\alpha = \frac{1}{3}+\frac{2}{3}\left(-\frac{4}{9}\right)$$

$$= \frac{1}{3}-\frac{8}{27}$$

$$= \frac{9-8}{27}$$

$$\cos^2\alpha = \frac{1}{27}$$

$$\cos\alpha = \frac{1}{3\sqrt{3}} \; //$$

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④ Find the general eqn to a Cone which touches the co-ordinate planes. (5 marks)

Soln:

If the co-ordinates plane touch a cone, the ⊥r to co-ordinate planes touch the reciprocal cone.

The direction cosines of the co-ordinates axes are $(1,0,0)$, $(0,1,0)$, $(0,0,1)$

The eqn of the cone passing through the axis is of the form

$$2fyz + 2gzx + 2hxy = 0$$

The Reqd cone is the reciprocal cone of this cone and its eqn is

$$f^2x^2 + g^2y^2 + h^2z^2 - 2ghyz - 2hfzx - 2fgxy = 0$$

This eqn can be put in the form

$$\sqrt{fx} + \sqrt{gy} + \sqrt{hz} = 0.$$

⑤ Find the eqn of the quadric cone through the x, y, z axes and the lines through the origin with d.r's $2,4,1$ & $3,3,1$.

Soln:

Since the cone passes through the x, y, z its eqn is of the form

$$fyz + gzx + hxy = 0 \quad \text{— ①}$$

D.r's are $2,4,1$ & $3,3,1$

$$① \Rightarrow \quad f(4\times1) + g(1\times2) + h(2\times4) = 0$$

$$f(3\times1) + g(1\times3) + h(3\times3) = 0$$

$$4f + 2g + 8h = 0$$

$$3f + 3g + 9h = 0$$

$$\div 3, \quad f + g + 3h = 0$$

$$\Rightarrow \frac{f}{3-4} = \frac{-g}{6-4} = \frac{h}{2-1}$$

$$\frac{f}{-1} = \frac{g}{-2} = \frac{h}{1}$$

$$f = 1, \quad g = 2, \quad h = -1$$

$$① \Rightarrow \quad yz + 2zx - xy = 0.$$

Model sum:

① $(3,5,1)$ $(1,-1,2)$ — Ans: $33yz + 25zx - 16xy = 0$

② $\frac{x}{3} = \frac{y}{5} = \frac{z}{1}$, $\frac{x}{1} = \frac{y}{-1} = \frac{z}{2}$, $\frac{x}{11} = \frac{y}{-5} = \frac{z}{-8}$

Same Ans: $33yz + 25zx - 16xy = 0$

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(10 marks)

6. Find the angle between the lines of intersection of the Cone $2xy - 2yz + zx = 0$ and the plane $x + y - z = 0$.

soln:

The cone has its vertex at the origin. If λ, μ, ν are the d.r's of the line. Then we get,

$$2\lambda\mu - 2\mu\nu + 2\nu\lambda = 0 \quad \text{--- (1)}$$

The normal to the plane are $\perp^r$. Hence,

$$\lambda + \mu - \nu = 0. \quad \text{--- (2)}$$

$\frac{(1)}{\nu^2}$, $\frac{2\lambda\mu}{\nu^2} - \frac{2\mu}{\nu} + \frac{2\lambda}{\nu} = 0$

$\frac{(2)}{\nu}$, $\frac{\lambda}{\nu} + \frac{\mu}{\nu} - 1 = 0$

Put $\frac{\lambda}{\nu} = a$, $\frac{\mu}{\nu} = b$, we get,

$$2ab - 2b + 2a = 0.$$

$$a + b - 1 = 0. \Rightarrow b = 1 - a$$

Solving we get,

$$2a(1-a) - 2(1-a) + 2a = 0$$

$$2a - 2a^2 - 2 + 2a + 2a = 0$$

$$-2a^2 + 5a - 2 = 0$$

$$2a^2 - 5a + 2 = 0$$

4

$-4 \mid -1$

$$2a^2 - 4a - a + 2 = 0$$

$$2a[a-2] - 1[a-2] = 0$$

$2a - 1 = 0$, $a - 2 = 0$

$a = \frac{1}{2}$, $a = 2$

i) $a = \frac{1}{2}$, $a + b - 1 = 0$

$\frac{1}{2} + b - 1 = 0$

$b = 1 - 1/2 \Rightarrow b = 1/2$

$\frac{\lambda}{\nu}, \frac{\mu}{\nu}, 1 = \frac{1}{2}, \frac{1}{2}, 1 \Rightarrow \lambda, \mu, \nu = 1:1:2$

ii) $a = 2$, $a + b - 1 = 0 \Rightarrow b = -1$

$\lambda : \mu : \nu = 2 : -1 : 1$

Angle btwn lines are

$$\cos\theta = \frac{1(2) + 1(-1) + 2(1)}{\sqrt{1+1+4}\sqrt{4+1+1}} = \frac{2-1+2}{\sqrt{6}\sqrt{6}}$$

$$= \frac{3}{6}$$

$\cos\theta = \frac{1}{2}$

$\theta = 60°$

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