28·1·2020

Unit I - Vector Differentiation

$\vec{i}, \vec{j}, \vec{k}$ are unit vectors in the posible directions of x, y, z axis

* if $\vec{r}(t) = f(t)\vec{i} + g(t)\vec{j} + h(t)\vec{k}$

Then $\lim_{t \to 0} \vec{r}(t) = \vec{i} \lim_{t \to 0} f(t) + \vec{j} \lim_{n \to a} g(t) + \vec{k} \lim_{n \to 0}$

* if $\vec{A} = A_1 \vec{i} + A_2 \vec{j} + A_3 \vec{k}$ magnitude of $\vec{A} = \sqrt{A_1^2 + A_2^2 + A_3^2}$
* $i \cdot i = j \cdot j = k \cdot k = 1$
* $i \cdot j = j \cdot k \cdot \; k \cdot i = 0$
* unit vector $\hat{a} = \dfrac{\vec{a}}{|\vec{a}|}$
* $\vec{A} \cdot \vec{B} = \vec{B} \cdot \vec{A}$
* $A \cdot B = |A| \cdot |B| \cos\theta$
* $A \cdot B = 0$ if A and B are $\perp^{le}$
* Let $\vec{r}$ be a position vector $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$

Captor - 1 Velocity and Acceleration

Velocity

Let the scalar variable t represented by the time and $\vec{r}$ be the position vector of a moving point P of a particle then $\vec{v} = \dfrac{d\vec{r}}{dt}$ is the velocity at P and it's direction in along the tangent plane at P

Accelaration : ②

If $\vec{v}$ is the velocity at time t of a moving point along the curve c than the accelaration of the particle at time t is than $\vec{a} = \frac{d}{dt}\left(\frac{d\vec{r}}{dt}\right)$

Problem-;

a particle noves along the curve $x = e^{-t}$ $y = 2\cos 3t$ $z = 2\sin 3t$ determine the velocity and the accelaration at any time and there magnitude of $t = 0$

Let $\vec{r}$ be the potion vector

$$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$$

$x = e^{-t}$ $y = 2\cos 3t$ $z = 2\sin 3t$

$$\vec{v} = \frac{d\vec{r}}{dt}$$

$$\vec{r} = e^{-t}i + 2\cos 3t\,\vec{j} + 2\sin 3t\,\vec{k}$$

$$\frac{d\vec{r}}{dt} = -e^{-t}\vec{i} - 2\sin 3t \cdot 3\,\vec{j} + 2\cos 3t \cdot 3\,\vec{k}$$

velocity at time $t=0$

$$\vec{v} = -\vec{i} + 6\vec{k}$$

$$\vec{a} = \frac{d}{dt}\left(\frac{d\vec{r}}{dt}\right)$$

$$= \frac{d}{dt}\left(e^{-t}\vec{i} - 6\sin 3t\,\vec{j} + 6\cos 3t\,\vec{k}\right)$$

$$= e^{-t}\vec{i} - 18\cos 3t\,\vec{j} - 18\sin 3t\,\vec{k}$$

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(3) Acceleration at $\vec{a}_1 = \frac{d}{dt}\left(\frac{d\vec{r}}{dt}\right)$ time $t=0$

$\vec{a} = \vec{i} - 18\vec{j}$

$\vec{V} = -\vec{i} + 6\vec{K}$

magnitude of the velocity $= \sqrt{A_1^2 + A_2^2}$

$= \sqrt{(-1)^2 + (6)^2}$

$= \sqrt{37}$

maginitude of acceleration $= \sqrt{A_1^2 + A_2^2}$

$= \sqrt{1^2 + (-18)^2}$

$= \sqrt{1+324}$

$= \sqrt{325}$

A particle whouse along the curve $x = e^t \cos t$ $y = e^t \sin t$ $Z = e^t$ determine the velocity and acceleration of the parti cle moving on the curve at $t=0$

Let $\vec{r}$ be a pohition vector

$\vec{r} = x\vec{i} + y\vec{j} + z\vec{K}$

$= e^t \cos t\, \vec{i} + e^t \sin t\, \vec{J} + e^t \vec{K}$

$\vec{V} = \frac{d\vec{r}}{dt} = \frac{d}{dt}\left(e^t \cos t\, \vec{i} + e^t \sin t\, \vec{j} + e^t \vec{K}\right)$

$= (-e^t \sin t + \cos t\, e^t)\vec{i} + (e^t \cos t + \sin t\, e^t)\vec{j} + e^t \vec{K}$

$\vec{V} = \vec{i} + \vec{j} + \vec{k}$ at $t=0$

$\vec{a} = \frac{d}{dt}\left(\frac{d\vec{r}}{dt}\right)$

$= \frac{d}{dt}\left(-e^t \sin t + \cos t\, e^t\right)\vec{i} + (e^t \cos t + \sin t\, e^t)\vec{j}$

$= (-e^t \cos t \mp \sin t\, e^t - e^t \sin t + \cos t\, e^t)\vec{i} +$

$(-e^t \sin t + \cos t\, e^t + e^t \cos t + \sin t\, e^t)\vec{j} + e^t \vec{k}$

At time $t = 0$

$\vec{a} = 2\vec{j} + \vec{k}$

A particle moves along the curve $x = 3t^2$ $y = t^2 - 2t$ $z = t^3$ find the velocity and accelaration of the particle moving on the curve at time $t = 1$

$\vec{r}$ is a position vector

$\vec{r} = 3t^2\vec{i} + (t^2 - 2t)\vec{j} + t^3\vec{k}$

$\vec{v} = \frac{d\vec{r}}{dt} = \frac{d}{dt}\left(3t^2\vec{i} + (t^2 - 2t)\vec{j} + t^3\vec{k}\right)$

$= 6t\vec{i} + (2t - 2)\vec{j} + 3t^2\vec{k}$

At time $t = 1$

$\vec{v} = 6\vec{i} + 3\vec{k}$

$\vec{a} = \frac{d}{dt}\left(\frac{d\vec{r}}{dt}\right) = \frac{d}{dt}\left(6t\vec{i} + (2t-2)\vec{j} + 3t^2\vec{k}\right)$

$= 6\vec{i} + 2\vec{j} + 6t\vec{k}$

At tim $t = 1$

$\vec{a} = 6\vec{i} + 2\vec{j} + 6\vec{k}$

For the curve $\vec{r} = e^{t}\vec{i} + \log(t^2+1)\vec{j} - \tan t\,\vec{k}$ find the velocity and acceleration at time $t=0$

velocity $\vec{v} = \frac{d}{dt}\left(e^{t}\vec{i} + \log(t^2+1)\vec{j} - \tan t\,\vec{k}\right)$

$= -e^{-t}\vec{i} + \frac{2t}{t^2+1}\vec{j} - \sec^2 t\,\vec{k}$

At time $t=0$

$\vec{v} = -\vec{i}$

Acceleration

$\vec{a} = \frac{d}{dt}(\vec{v}) = \frac{d}{dt}\left(-e^{-t}\vec{i} + \frac{2t}{t^2+1}\vec{j} - \sec^2 t\,\vec{k}\right)$

$= e^{-t}\vec{i} + \frac{(t^2+1)2 - 2t\cdot 2t}{(t^2+1)^2}\vec{j} - \frac{d}{dt}\sec^2 t\,\vec{k}$

$= e^{-t}\vec{i} + \frac{(-2t^2+2)}{(t^2+1)^2}\vec{j} \cancel{+}\frac{d}{dt}\sec^2 t\,\vec{k}$

$= e^{-t}\vec{i} + \frac{2-2t^2}{(t^2+1)^2}\vec{j} \cancel{+}\frac{d}{dt}\sec^2 t\,\vec{k})$

$\sec^2 t = \frac{1}{\cos^2 t}$

$= \frac{1}{\frac{1+\cos 2t}{2}}$

$= \frac{2}{1+\cos 2t}$

$\frac{d(\sec^2 t)}{dt} = \frac{(1+\cos 2t)(0) - 2(-\sin 2t\cdot 2)}{(1+\cos 2t)^2}$

$= \frac{4\sin 2t}{(1+\cos 2t)^2}$

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$$\vec{a} = e^{-t}\vec{i} + \frac{2(-t^2+1)}{(t^2+1)^2}\vec{j} - \frac{4\sin 2t}{(1+\cos 2t)^2}\vec{k}$$

at time $t=0$

$$\vec{a} = \vec{i} + 2\vec{j}$$

31-01-2020 Chapter -2 Gradient, Divergence, curl

Gradient = $\nabla \phi$

$$= \vec{i}\frac{d\phi}{dx} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z}$$

divergence = $\nabla \cdot \vec{F}$

$$\nabla = \vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}$$

curl = $\nabla \times \vec{F}$

$$= \begin{vmatrix} \vec{i} & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_2 \end{vmatrix}$$

01-02-2020 If $\phi(x,y,z) = x^2y + y^2x + z^2$ find $\nabla\phi$ at the point (1,1,1)

Solution

$$\phi = x^2y + y^2x + z^2$$

$$\nabla\phi = \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z}$$

$$\frac{\partial\phi}{\partial x} = 2xy + y^2$$

$$\frac{\partial\phi}{\partial y} = x^2 + 2yx$$

$$\frac{\partial\phi}{\partial z} = 2z$$

$$\nabla \phi = \vec{i}\,(2xy + y^2) + \vec{j}\,(x^2 + 2xy) + \vec{k}\,(2z)$$

$$\nabla \phi_{(1,1,1)} = \vec{i}\,(2(1)(1) + (1)) + \vec{j}\,(1 + 2(1)(1)) + \vec{k}\,(2(1))$$

$$= 3\vec{i} + 3\vec{j} + 2\vec{k}$$

If $\phi = x^2y - 2y^2z^3$ find $\nabla\phi$ at the point $(1,-1,2)$

$$\phi = x^2y - 2y^2z3$$

$$\nabla\phi = \vec{i}\,\frac{\partial\phi}{\partial x} + \vec{j}\,\frac{\partial\phi}{\partial y} + \vec{k}\,\frac{\partial\phi}{\partial z}$$

$$\frac{\partial\phi}{\partial x} = 2xy$$

$$\frac{\partial\phi}{\partial y} = \cancel{4}\,x^2 - 4yz3$$

$$\frac{\partial\phi}{\partial z} = -6\,z^2y^2$$

$$\nabla\phi = \vec{i}\,(2xy) + \vec{j}\,(x^2 - 4yz^3) + \vec{k}\,(-6z^2y^2)$$

$$\nabla\phi_{(1,-1,2)} = \vec{i}\,(2)(1)(-1) + \vec{j}\,((1) - (4)(-1)(8)) + \vec{k}\,((-6)(4)(1))$$

$$= -2\vec{i}\ \overset{33}{\cancel{+30}}\vec{j} - 24\,\vec{k}$$

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If $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$ and $r = |\vec{r}|$ prove that

(i) $\nabla r = \frac{\vec{r}}{r}$

(ii) $\nabla\left(\frac{1}{r}\right) = \frac{-\vec{r}}{r^3}$

(iii) $\nabla r^n = n r^{n-2} \vec{r}$

(iv) $\nabla f(x) = f'(x) \frac{\vec{r}}{r}$

(v) $\nabla(\log r) = \frac{\vec{r}}{r^2}$

(vi) $\nabla f(x) \times \vec{r} = 0$

Given $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$

$$\vec{r}\cdot\vec{r} = (x\vec{i} + y\vec{j} + z\vec{k}) \cdot (x\vec{i} + y\vec{j} + z\vec{k})$$

$$r^2 = x^2 + y^2 + z^2$$

$$r = \sqrt{x^2 + y^2 + z^2}$$

$$r = |\vec{r}|$$

we know that $r^2 = x^2 + y^2 + z^2$

diff with respect to x, y, z

$$2r\frac{\partial r}{\partial x} = 2x$$

$$\frac{\partial r}{\partial x} = \frac{x}{r}$$

$$2r\frac{\partial r}{\partial y} = 2y$$

$$\frac{\partial r}{\partial y} = \frac{y}{r}$$

$$\frac{\partial r}{\partial z} = \frac{z}{r}$$

(i) $\nabla r = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right) r$

$$= \vec{i}\frac{\partial r}{\partial x} + \vec{j}\frac{\partial r}{\partial y} + \vec{k}\frac{\partial r}{\partial z}$$

$$= \vec{i}\left(\frac{x}{r}\right) + \vec{j}\left(\frac{y}{r}\right) + \vec{k}\left(\frac{z}{r}\right)$$

$$= \frac{x\vec{i} + y\vec{j} + z\vec{k}}{r}$$

$$\nabla r = \frac{\vec{r}}{r}$$

(ii) $\nabla\left(\frac{1}{r}\right) = \frac{-\vec{r}}{r^3}$

W.K.T

$$\frac{\partial r}{\partial x} = \frac{x}{r} \quad \frac{\partial r}{\partial y} = \frac{y}{r} \quad \frac{\partial r}{\partial z} = \frac{z}{r}$$

$$\nabla\left(\frac{1}{r}\right) = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)\frac{1}{r}$$

$$= \vec{i}\frac{\partial}{\partial x}r^{-1} + \vec{j}\frac{\partial}{\partial y}r^{-1} + \vec{k}\frac{\partial}{\partial y}r^{-1}$$

$$= \vec{i}\left(-r^{-2}\frac{\partial r}{\partial x}\right) + \vec{j}\left(-r^{-2}\frac{\partial r}{\partial y}\right) + \vec{k}\left(-r^{-2}\frac{\partial r}{\partial z}\right)$$

$$= -r^{-2}\left(\vec{i}\frac{x}{r} + \vec{j}\frac{y}{r} + \vec{k}\frac{z}{r}\right)$$

$$= \frac{-r^{-2}}{r}\left(x\vec{i} + y\vec{j} + z\vec{k}\right)$$

$$\nabla\left(\frac{1}{r}\right) = -\frac{\vec{r}}{r^3}$$

(iii) $\nabla(r^n) = n r^{n-2} \vec{r}$

$$\nabla(r^n) = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right) r^n$$

$$= \left(\vec{i}\frac{\partial}{\partial x} r^n + \vec{j}\frac{\partial}{\partial y} r^n + \vec{k}\frac{\partial}{\partial z} r^n\right)$$

$$= \vec{i}\left(n r^{n-1}\frac{\partial r}{\partial x}\right) + \vec{j}\left(n r^{n-1}\frac{\partial r}{\partial y}\right) + \vec{k}\left(n r^{n-1}\frac{\partial r}{\partial z}\right)$$

$$= n r^{n-1}\left(\vec{i}\frac{\partial r}{\partial x} + \vec{j}\frac{\partial r}{\partial y} + \vec{k}\frac{\partial r}{\partial z}\right)$$

$$= n r^{n-1}\left(\vec{i}\frac{x}{r} + \vec{j}\frac{y}{r} + \vec{k}\frac{z}{r}\right)$$

$$= \frac{n r^{n-1}}{r}\left(x\vec{i} + y\vec{j} + z\vec{k}\right)$$

$$= n r^{n-1-1}\,\vec{r}$$

$$= n r^{n-2}\,\vec{r}$$

(iv) $\nabla f(r) = f'(r)\frac{\vec{r}}{r}$

$$\nabla f(r) = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right) f(r)$$

$$= \vec{i}\frac{\partial}{\partial x} f(r) + \vec{j}\frac{\partial}{\partial y} f(r) + \vec{k}\frac{\partial}{\partial z} f(r)$$

$$= \vec{i}\, f'(r)\frac{\partial r}{\partial x} + \vec{j}\, f(r)\frac{\partial r}{\partial y} + \vec{k}\, f'(r)\frac{\partial r}{\partial y}$$

$$= f'(r)\left(\vec{i}\frac{x}{r} + \vec{j}\frac{y}{r} + \vec{k}\frac{z}{r}\right)$$

$$= f'(r)\,\frac{\vec{r}}{r}$$

(v) $\nabla(\log r) = \frac{\vec{r}}{r^2}$

$$\nabla(\log r) = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)\log r$$

$$= \vec{i}\frac{\partial}{\partial x}\log r + \vec{j}\frac{\partial}{\partial y}\log r + \vec{k}\frac{\partial}{\partial z}\log r$$

$$= \vec{i}\frac{1}{r}\frac{\partial r}{\partial x} + \vec{j}\frac{1}{r}\frac{\partial r}{\partial y} + \vec{k}\frac{\partial r}{\partial z}\frac{1}{r}$$

$$= \frac{1}{r}\left(\vec{i}\frac{x}{r} + \vec{j}\frac{y}{r} + \vec{k}\frac{z}{r}\right)$$

$$= \frac{\vec{r}}{r^2}$$

(vi) $\nabla f(r) \times \vec{r} = 0$

$$\nabla f(r) = f'(r)\frac{\vec{r}}{r}$$

$$\nabla f(r) \times \vec{r} = f'(r)\frac{\vec{r}}{r} \times \vec{r}$$

$$= 0$$

$(\vec{r} \times \vec{r}) = 0$

$\vec{r} \cdot \vec{r} = 1$

(12) directional derivate, unit vector, angle b/w to the surface.

tangent plane:

Directional derivative

formula $= \nabla\phi \cdot \vec{n}$

$\vec{n} = \frac{n_1}{|n_1|}$

unit vector (n vectors equal to $\frac{n_1}{|n_1|}$)

Angle b/w normal to the surface $\cos\theta = n_1 \cdot n_2$

$\theta = \cos^{-1}(\vec{n_1} \cdot \vec{n_2})$

Orthogonal surface $\nabla\phi_1 \cdot \nabla\phi_2 = 0$

Find the direction derivative of $xyz - xy^2z^3$ at the Point $(1,2,-1)$ in the direction of the vector $\vec{i} - \vec{j} - 3\vec{k}$

soln

let $\phi = xyz - xy^2z^3$

$\nabla\phi = \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z}$

$\frac{\partial\phi}{\partial x} = \cancel{\vec{i}}\ yz - y^2z^3$

$\frac{\partial\phi}{\partial y} = xz - 2xyz^3$

$\frac{\partial\phi}{\partial z} = xy - \cancel{2}\,3xy^2z^2$

$\nabla\phi = \vec{i}(yz - y^2z^3) + \vec{j}(xz - 2xyz^3) + \vec{k}(xy - 3xy^2z^2)$

$\nabla\phi_{(1,2,-1)} = \vec{i}(-2+4) + \vec{j}(-1+4) + \vec{k}(2-12)$

$= 2\vec{i} + 3\vec{j} - 10\vec{k}$

direction der $= \nabla\phi \cdot \vec{n}$

unit vector $\vec{n} = \dfrac{\vec{n_1}}{|\vec{n_1}|}$

$$n_1 = \frac{\vec{i} - \vec{j} - 3k}{\sqrt{(-1)^2 + (-1^2) + (-3)^2}}$$

$$= \frac{\vec{i} - \vec{j} - 3\vec{k}}{\sqrt{11}}$$

2. T20 directional derivative ~~at the point (1, 2, -1)~~ $= \nabla\phi \cdot \vec{n}$

$$= 2\vec{i} + 3\vec{j} - 10\vec{k} \cdot \left(\frac{\vec{i} - \vec{j} - 3\vec{k}}{\sqrt{11}} \right)$$

$$= \frac{2 - 3 + 30}{\sqrt{11}}$$

$$= \frac{29}{\sqrt{11}}$$

Find the directional derivative of $x^3+y^3+z^3$ at the point $(1,-1,2)$ in the direction of the vector $\vec{i} + 2\vec{j} + \vec{k}$

Soln

Let $\phi = x^3 + y^3 + z^3$

$$\nabla\phi = \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z}$$

$$\frac{\partial\phi}{\partial x} = 3x^2$$

$$\frac{\partial\phi}{\partial y} = 3y^2$$

$$\frac{\partial\phi}{\partial z} = 3z^2$$

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$$\nabla\phi = \vec{i}\,3x^2 + \vec{j}\,3y^2 + \vec{k}\,3z^2$$

$$= 3\,(x^2\vec{i} + y^2\vec{j} + z^3\vec{k})$$

$$\nabla\phi_{(1,-1,2)} = 3\,(\vec{i} + \vec{j} + 4\vec{k})$$

unit vector $\vec{n} = \dfrac{\vec{n_1}}{|\vec{n_1}|}$

$$= \frac{\vec{i} + 2\vec{j} + \vec{k}}{\sqrt{1+4+1}}$$

$$= \frac{\vec{i} + 2\vec{j} + \vec{k}}{\sqrt{6}}$$

directional derivative $= \nabla\phi \cdot \vec{n}$

$$= 3(\vec{i} + \vec{j} + 4\vec{k})\left(\frac{\vec{i} + 2\vec{j} + \vec{k}}{\sqrt{6}}\right)$$

$$= 3\left(\frac{1+2+4}{\sqrt{6}}\right)$$

$$= \frac{21}{\sqrt{6}}$$

Find the maximum value of the directional derivate of the function $\phi = 2x^2 + 3y^2 + 5z^2$ at the point $(1,1,4)$

Soln

Given $\phi = 2x^2 + 3y^2 + 5z^2$

$$\nabla\phi = \vec{i}\,\frac{\partial\phi}{\partial x} + \vec{j}\,\frac{\partial\phi}{\partial y} + \vec{k}\,\frac{\partial\phi}{\partial z}$$

$$\frac{\partial\phi}{\partial x} = 4x$$

$$\frac{\partial\phi}{\partial y} = 6y$$

$$\frac{\partial\phi}{\partial z} = 10z$$

$$\nabla\phi = \cancel{10}4x\vec{i} + 6y\vec{j} + 10z\vec{k}$$

$$\nabla\phi_{(1,1,4)} = 4\vec{i} + 6\vec{j} + 40\vec{k}$$

Maximum of the directional derivative at the point $(1,1,4)$

$$\nabla\phi_{(1,1,4)} = |4\vec{i}+6\vec{j}+40\vec{k}|$$

$$= \sqrt{16+36+1600}$$

$$= \sqrt{1652}$$

Find the unit vector normal to the surface $x^2+3y^2+2z^2=6$ at the point $(2,0,1)$

Given

$$\phi = x^2+3y^2+2z^2-6$$

$$\nabla\phi = \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z}$$

$$= \vec{i}\,2x + \vec{j}\,6y + \vec{k}\,4z$$

$$\nabla\phi_{(2,0,1)} = 4\vec{i} + 0 + 4\vec{k}$$

unit vector normal to the surface $\vec{n} = \frac{\nabla\phi}{|\nabla\phi|}$

$$= \frac{4\vec{i}+4\vec{k}}{\sqrt{16+16}}$$

$$= \frac{4(\vec{i}+\vec{k})}{\sqrt{16\cdot 2}}$$

$$= \frac{\vec{i}+\vec{k}}{\sqrt{2}}$$

16 Angle b/w normal to the surface

Find the angle b/w normal to the surface $xy - z^2 = 0$ at the point $(1, 4, -2)$ and $(-3, -3, 3)$

Soln

Let $\phi = xy - z^2$

$$\nabla\phi = \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z}$$

$$= \vec{i}y + \vec{j}x + -2z\vec{k}$$

$$\nabla\phi_{(1,4,-2)} = 4\vec{i} + \vec{j} + 4\vec{k} = \nabla\phi_1$$

$$\nabla\phi_{(-3,-3,3)} = -3\vec{i} - 3\vec{j} - 6\vec{k} = \nabla\phi_2$$

The angle b/w normal to the surface $\theta = \cos^{-1}(n_1 \cdot n_2)$

$$n_1 = \frac{\nabla\phi_1}{|\nabla\phi_1|}$$

$$= \frac{4\vec{i} + \vec{j} + 4\vec{k}}{\sqrt{33}}$$

$$n_2 = \frac{\nabla\phi_2}{|\nabla\phi_2|}$$

$$= \frac{-3\vec{i} - 3\vec{j} - 6\vec{k}}{\sqrt{9+9+36}}$$

$$= \frac{-3\vec{i} - 3\vec{j} - 6\vec{k}}{\sqrt{54}}$$

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$$\vec{n_1}\cdot\vec{n_2} = \frac{4\vec{i}+\vec{j}+4\vec{k}}{\sqrt{33}} \cdot \frac{-3\vec{i}-3\vec{j}-6\vec{k}}{\sqrt{54}}$$

$$= \frac{-12-3-24}{\sqrt{33}\sqrt{54}}$$

$$= \frac{-39}{\sqrt{1782}}$$

$$\theta = \cos^{-1}\left(\frac{-39}{\sqrt{1782}}\right)$$

Show that the surface $5x^2-2yz-9x=0$ and $4x^2y+z^3-4=0$ are orthogonal at the point $(1,-1,2)$

Soln

$$\nabla\phi_1 \cdot \nabla\phi_2 = 0$$

given

$$\phi_1 = 5x^2-2yz-9x$$

$$\nabla\phi_1 = \vec{i}\frac{\partial\phi_1}{\partial x} + \vec{j}\frac{\partial\phi_1}{\partial y} + \vec{k}\frac{\partial\phi_1}{\partial z}$$

$$= \vec{i}(10x-9) + \vec{j}(-2z) + \vec{k}(-2y)$$

$$\nabla\phi_{1\,(1,-1,2)} = \vec{i} + 4\vec{j} + 2\vec{k}$$

$$\phi_2 = 4x^2y+z^3-4$$

$$\nabla\phi_2 = \vec{i}\frac{\partial\phi_2}{\partial x} + \vec{j}\frac{\partial\phi_2}{\partial y} + \vec{k}\frac{\partial\phi_2}{\partial z}$$

$$= \vec{i}(8xy) + \vec{j}(4x^2) + \vec{k}(3z^2)$$

$$\nabla\phi_{2\,(1,-1,2)} = -8\vec{i} + 4\vec{j} + 12\vec{k}$$

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$$\nabla\phi_1 \cdot \nabla\phi_2 = \cancel{((10x-9)\vec{i} - 2z\vec{j} - 2y\vec{k})} \cdot$$

$(2xy$

$$= (\vec{i} - 4\vec{j} + 2\vec{k}) \cdot (-8\vec{i} + 4\vec{j} + 12\vec{k})$$

$$= -8 - 16 + 24$$

$$= 0$$

Show that the surface $x^2+y^2+z^2=9$ and $z = x^2+y^2-3$ are orthogonal at the point $(2,-1,2)$.

we know that

$$\nabla\phi_1 \cdot \nabla\phi_2 = 0$$

$$\phi_1 = x^2+y^2+z^2-9$$

$$\nabla\phi_1 = \vec{i}\frac{\partial\phi_1}{\partial x} + \vec{j}\frac{\partial\phi_1}{\partial y} + \vec{k}\frac{\partial\phi_1}{\partial z}$$

$$= 2x\vec{i} + 2y\vec{j} + 2z\vec{k}$$

$$\nabla\phi_1{}_{(2,-1,2)} = 4\vec{i} - 2\vec{j} + 4\vec{k}$$

$$\phi_2 = x^2+y^2-z-3$$

$$\nabla\phi_2 = \vec{i}\frac{\partial\phi_2}{\partial x} + \vec{j}\frac{\partial\phi_2}{\partial y} + \vec{k}\frac{\partial\phi_2}{\partial z}$$

$$= 2x\vec{i} + 2y\vec{j} - \vec{k}$$

$$\nabla\phi_2{}_{(2,-1,2)} = 4\vec{i} - 2\vec{j} - \vec{k}$$

$$\nabla\phi_1 \cdot \nabla\phi_2 = (4\vec{i} - 2\vec{j} + 4\vec{k}) \cdot (4\vec{i} - 2\vec{j} - \vec{k})$$

$$= 16 + 4 - 4$$

$$= 16$$

Find ϕ if $\nabla\phi = (y+\sin z)\vec{i} + x\vec{j} + x\cos z\vec{k}$

Soln

$$\nabla\phi = (y+\sin z)\vec{i} + x\vec{j} + x\cos z\vec{k}$$

we know that

$$\nabla\phi = \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z}$$

$$\frac{\partial\phi}{\partial x} = y+\sin z$$

$$\int\frac{\partial\phi}{\partial x} = \int y+\sin z \,\partial x$$

$$= xy + x\sin z + f_1(y,z)$$

$$\frac{\partial\phi}{\partial y} = x$$

$$\int\frac{\partial\phi}{\partial y} = \int x\,\partial y$$

$$\phi = xy + f_2(x,z)$$

$$\frac{\partial\phi}{\partial z} = x\cos z$$

$$\int\frac{\partial\phi}{\partial z} = \int x\cos z\,\partial z$$

$$= \int x\sin z + f_3(x,y)$$

$$\phi = xy + x\sin z + c$$

were c is a orbitary constant

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(20) Find ϕ if $\nabla\phi = (6xy+z^3)\vec{i} + (3x^2-z)\vec{j} + (3xz^2-y)\vec{k}$

$$\nabla\phi = (6xy+z^3)\vec{i} + (3x^2-z)\vec{j} + (3xz^2-y)\vec{k}$$

$$\frac{\partial\phi}{\partial x} = 6xy+z^3$$

$$\int \frac{d\phi}{\cancel{\partial x}} = \int (6xy+z^3)\,dx$$

$$\phi = 3x^2y + z^3x + f_1(y,z)$$

$$\int \frac{\partial\phi}{\cancel{\partial y}} = \int (3x^2-z)\,\partial y$$

$$= 3x^2y - zy + f_2(x,z)$$

$$\int \frac{\partial\phi}{\cancel{\partial z}} = \int (3xz^2-y)\,dz$$

$$= xz^3 - zy + f_3(y,x)$$

$$\phi = 3x^2y + z^3x - zy + c$$

were c is a orbitrary constant

Solinoidal and irrotational

If $\vec{F}$ is solinoidal then $\nabla \cdot \vec{F} = 0$

irrotational

If $\vec{F}$ is irrotational then $\nabla \times \vec{F} = 0$

Show that $\vec{f} = 3y^4z^2\vec{i} + 4x^3z^2\vec{j} - 3x^2y^2\vec{k}$ is ~~irrotational~~ Solinoidal

Soln

$$\vec{f} = 3y^4z^2\vec{i} + 4x^3z^2\vec{j} - 3x^2y^2\vec{k}$$

$$\nabla \cdot \vec{F} = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right) \cdot \left(3y^4z^2\vec{i} + 4x^3z^2\vec{j} - 3x^2y^2\vec{k}\right)$$

$$= \frac{\partial}{\partial x}(3y^4z^2) + \frac{\partial}{\partial y}4x^3z^2 + \frac{\partial}{\partial z}(-3x^2y^2)$$

$$= 0 + 0 + 0$$

$$= 0$$

$\vec{F}$ is solinoidal

Find the value of a so that the vector $\vec{F} = (z+3y)\vec{i} + (x-2z)\vec{j} + (x+az)\vec{k}$ is solinoidal

$$\nabla \cdot \vec{F} = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right) \cdot \left((z+3y)i + (x-2z)\vec{j} + (x+az)\right)$$

$$= \frac{\partial}{\partial x}(z+3y) + \frac{\partial}{\partial y}(x+2z) + \frac{\partial}{\partial z}(x+az)$$

$$= 0 + 0 + a$$

$\vec{F}$ is solinoidal

$$\nabla \cdot \vec{F} = a$$

$$a = 0$$

10.2.T20 Find the value of 'a' so that the $\vec{F} = (ax+3y+4z)\vec{i} + (x-2y+3z)\vec{j} + (x-2y+3z)\vec{k}$ is solenoidal

(Q2)

$$\vec{F} = (ax+3y+4z)\vec{i} + (x-2y+3z)\vec{j} + (x-2y+3z)\vec{k}$$

$$\nabla\cdot\vec{F} = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)\cdot \vec{F}$$

$$= \frac{\partial}{\partial x}(ax+3y+4z) + \frac{\partial}{\partial y}(x-2y+3z) + \frac{\partial}{\partial z}(x-2y+3z)$$

$$= a - 2 + 3$$

$$\nabla\cdot\vec{F} = a+1$$

$\vec{F}$ is solenoidal

$$a+1=0$$

$$\boxed{a=-1}$$

Show that $\vec{F} = (y^2-z^2+3yz-2x)\vec{i} + (3xz+2xy)\vec{j} + (3xy-2xz+2z)\vec{k}$

is irrotational and solenoidal

$$\nabla\cdot\vec{F} = \left(i\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)\cdot \vec{F}$$

$$= \frac{\partial}{\partial x}(y^2-z^2+3yz-2x) + \frac{\partial}{\partial y}(3xz+2xy) + \frac{\partial}{\partial z}(3xy-2xz+2z)$$

$$= -2 + 2x - 2x + 2$$

$$= 0$$

$$\nabla\cdot\vec{F} = 0$$

$\vec{F}$ is solenoidal

irrotational

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 - z^2 + 3yz - 2x & 3xz + 2xy & 3xy - 2xz + 2z \end{vmatrix}$$

$$= \vec{i}\left(\frac{\partial}{\partial y}(3xy - 2xz + 2z) - \frac{\partial}{\partial z}(3xz + 2xy)\right)$$

$$- \vec{j}\left(\frac{\partial}{\partial x}(3xy - 2xz + 2z) - \frac{\partial}{\partial z}(y^2 - z^2 + 3yz - 2x)\right)$$

$$+ \vec{k}\left(\frac{\partial}{\partial x}(3xz + 2xy) - \frac{\partial}{\partial y}(y^2 - z^2 + 3yz - 2x)\right)$$

$$= \vec{i}\left(\cancel{3y + 2x - 3x}\ 3x - 3x\right) \cancel{+}\ \vec{j}(3y - 2z + 2z - 3y)$$

$$+ \vec{k}(3z + 2y - 2y - 3z)$$

$$= 0$$

$$\nabla \times \vec{F} = 0$$

$\vec{F}$ is irrotational

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Find value of 'a' such that $\vec{F} = (axy - z^2)\vec{i} + (x^2 + 2yz)\vec{j} + (y^2 - axz)\vec{k}$ is irrotational

$$\vec{F} = (axy - z^2)\vec{i} + (x^2 + 2yz)\vec{j} + (y^2 - axz)\vec{k}$$

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (axy - z^2) & (x^2 + 2yz) & (y^2 - axz) \end{vmatrix}$$

$$= \vec{i}(2y - 2y) - \vec{j}(-az + 2z) + \vec{k}(2x - ax)$$

$$= -\vec{j}(2-a)z + \vec{k}(2-a)x \quad \ldots \text{(1)}$$

$\vec{F}$ is irrotational

$\nabla \times \vec{F} = 0 \quad \ldots \ldots \text{(2)}$ from (1) & (2)

coeff $\vec{j}, \vec{k}$ are equal so that $-\vec{j}(2-a)z + \vec{k}(2-a)x = 0$

$$2 - a = 0$$

$$-a = -2$$

$$\boxed{a = 2}$$

11.02.'20 If $\vec{F} = xz\vec{i} - 2xyz\vec{j} + xz\vec{k}$ find div $\vec{F}$ and curl $\vec{F}$ at the point (1, 2, 0)

Soln $\vec{F} = xz\vec{i} - 2xyz\vec{j} + xz\vec{k}$

div $\vec{F} = \nabla \cdot \vec{F}$

$$\nabla \cdot \vec{F} = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right) \cdot \left(xz\vec{i} - 2xyz\vec{j} + xz\vec{k}\right)$$

$$= z - 2xz + x$$

$$\nabla \cdot \vec{F}_{(1,2,0)} = 1$$

curl $\vec{F} = \nabla \times \vec{F}$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xz & -2xyz & xz \end{vmatrix}$$

$$= \vec{i}(0 + 2xy) - \vec{j}(z - x) + \vec{k}(-2yz - 0)$$

$$\nabla \times \vec{F}_{(1,2,0)} = 4\vec{i} + \vec{j}$$

If $\vec{F} = xy^2\vec{i} + 2x^2yz\vec{j} - 3yz^2\vec{k}$ find div $\vec{F}$ and curl $\vec{F}$ at the point $(1,-1,1)$

$$\vec{F} = xy^2\vec{i} + 2x^2yz\vec{j} - 3yz^2\vec{k}$$

div $\vec{F} = \nabla \cdot \vec{F} = \left(\frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}\right) \cdot \vec{F}$

$$= y^2 + 2x^2z - 6yz$$

$$\nabla \cdot \vec{F}_{(1,-1,1)} = 1 + 2 + 6$$

$$= 9$$

$$\nabla \cdot \vec{F} = 9$$

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(26)

$$\text{curl} = \nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy^2 & 2x^2yz & -3yz^2 \end{vmatrix}$$

$$= \vec{i}(-3z^2 - 2x^2y) - \vec{j}(0-0) + \vec{k}(4xyz - 2xy)$$

$$\nabla \cdot \vec{F}_{(1,-1,1)} = \vec{i}(-3+2) + \vec{k}(-4+2)$$

$$\nabla \cdot \vec{F} = -\vec{i} - 2\vec{k}$$

If $\vec{F} = \cancel{xz^3}\ xz^3\vec{i} - 2x^2yz\vec{j} + yz\vec{k}$ find div $\vec{F}$ and curl $\vec{F}$ at the point $(1,-2,1)$

$$\vec{F} = xz^3\vec{i} - 2x^2yz\vec{j} + yz\vec{k}$$

$$\text{div}\,\vec{F} = \nabla \cdot \vec{F} = \left(\frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}\right) \cdot \vec{F}$$

$$= z^3 - 2x^2z + y$$

$$\nabla \cdot \vec{F}_{(1,-2,1)} = 1 - 2 - 2$$

$$\boxed{\nabla \cdot \vec{F} = -3}$$

$$\text{Curl}\,\vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xz^3 & -2x^2yz & yz \end{vmatrix}$$

$$= \vec{i}(z + 2x^2y) - \vec{j}(0 - 3xz^2) + \vec{k}(4xyz - 0)$$

$$\nabla \times \vec{F}_{(1,-2,1)} = -3\vec{i} + 3\vec{j} + 8\vec{k}$$

2) If $\vec{a}$ is a constant vector and $\vec{r} = x\vec{i}+y\vec{j}+z\vec{k}$

Show that $\nabla \times (\vec{a} \times \vec{r}) = 2\vec{a}$

given

$$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$$

Let $\vec{a} = a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$

$$\vec{a} \times \vec{r} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ x & y & z \end{vmatrix}$$

$$= \vec{i}(a_2z - a_3y) - \vec{j}(a_1z - a_3x) + \vec{k}(a_1y - a_2x)$$

$$\nabla \times (\vec{a} \times \vec{r}) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ a_2z - a_3y & -a_1z + a_3x & a_1y - a_2x \end{vmatrix}$$

$$= \vec{i}(a_1 + a_1) - \vec{j}(-a_2 - a_2) + \vec{k}(a_3 + a_3)$$

$$= 2a_1\vec{i} + 2a_2\vec{j} + 2a_3\vec{k}$$

$$= 2(a_1\vec{i} + a_2\vec{j} + a_3\vec{k})$$

$$= 2(\vec{a})$$

If $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$ Show $\operatorname{div}(\vec{r}/r) = \frac{2}{r}$

Given

$$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$$

$$\operatorname{div}\left(\frac{\vec{r}}{r}\right) \Rightarrow \frac{\vec{r}}{r} = \frac{x\vec{i} + y\vec{j} + z\vec{k}}{r}$$

$$\operatorname{div}\left(\frac{\vec{r}}{r}\right) = \nabla\cdot\left(\frac{\vec{r}}{r}\right) = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)\cdot\left(\frac{x\vec{i} + y\vec{j} + \ldots}{r}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{x}{r}\right) + \frac{\partial}{\partial y}\left(\frac{y}{r}\right) + \frac{\partial}{\partial z}\left(\frac{z}{r}\right)$$

$$\frac{\partial}{\partial x}\left(\frac{x}{r}\right) = \frac{r(1) - x\frac{\partial r}{\partial x}}{r^2}$$

$$\boxed{\begin{array}{l} r^2 = x^2 + y^2 + z^2 \\ \dfrac{\partial r}{\partial x} = \dfrac{x}{r} \end{array}}$$

$$= \frac{r - x\frac{x}{r}}{r^2}$$

$$= \frac{r^2 - x^2}{r^3}$$

||ly

$$\frac{\partial}{\partial y}\left(\frac{y}{r}\right) = \frac{r^2 - y^2}{r^3}$$

$$\frac{\partial}{\partial z}\left(\frac{z}{r}\right) = \frac{r^2 - z^2}{r^3}$$

$$\operatorname{div}\left(\frac{\vec{r}}{r}\right) = \frac{r^2 - x^2}{r^2} + \frac{r^2 - y^2}{r^3} + \frac{r^2 - z^2}{r^3}$$

$$= \frac{3r^2 - x^2 - y^2 - z^2}{r^3}$$

$$= \frac{3r^2 - r^2}{r^3}$$

$$\operatorname{div}\left(\frac{\vec{r}}{r}\right) = \frac{2}{r}$$

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18.2.'20 If $\vec{V} = \vec{\omega} \times \vec{r}$ prove that $\vec{\omega} = \frac{1}{2}$ curl $\vec{V}$ where

(30) $\vec{\omega}$ is a constant vector and $\vec{r}$ is a position vector

Soln

Let $\vec{\omega} = \omega_1 \vec{i} + \omega_2 \vec{j} + \omega_3 \vec{k}$

$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$

$$\vec{V} = \vec{\omega} \times \vec{r}$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \omega_1 & \omega_2 & \omega_3 \\ x & y & z \end{vmatrix}$$

$$\vec{V} = \vec{i}(\omega_2 z - \omega_3 y) - \vec{j}(\omega_1 z - \omega_3 x) + \vec{k}(\omega_1 y - \omega_2 x)$$

$$\text{curl } \vec{V} = \nabla \times \vec{V}$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \omega_2 z - \omega_3 y & \omega_3 x - \omega_1 z & \omega_1 y - \omega_2 x \end{vmatrix}$$

$$= \vec{i}(\omega_1 + \omega_1) - \vec{j}(-\omega_2 - \omega_2) + \vec{k}(\omega_3 + \omega_3)$$

$$= 2\omega_1 \vec{i} + 2\omega_2 \vec{j} + 2\omega_3 \vec{k}$$

$$\text{curl } \vec{V} = 2\vec{\omega}$$

$$\vec{\omega} = \frac{1}{2} \text{curl } \vec{V}$$

31) determine $f(r)$ Show that vector $f(r)\vec{r}$ is a both irrotational and solinoidal

Given $\vec{r}$ is a position vector

$$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$$

$$f(r)\vec{r} = f(r)x\vec{i} + f(r)y\vec{j} + f(r)z\vec{k}$$

$$\nabla \cdot f(r)\vec{r} = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right) \cdot f(r)\vec{r}$$

$$= \frac{\partial}{\partial x} f(r)x + \frac{\partial}{\partial y} f(r)y + \frac{\partial}{\partial y} f(r)z$$

$$= \left[x f'(r)\frac{\partial r}{\partial x} + f(r)\right] + \left[y f'(r)\frac{\partial r}{\partial y} + f(r)\right] + \left[z f'(r)\frac{\partial r}{\partial z} + f(r)\right]$$

$$= 3f(r) + f'(r)\left[x\frac{\partial r}{\partial x} + y\frac{\partial r}{\partial y} + z\frac{\partial r}{\partial z}\right]$$

$$= 3f(r) + f'(r)\left[\frac{x^2 + y^2 + z^2}{r}\right]$$

$$= 3f(r) + f'(r)\left(\frac{r^2}{r}\right)$$

$$= 3f(r) + f'(r)r \quad \ldots\ldots\ldots (1)$$

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(2)

$$\nabla \times f(r)\vec{r} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f(r)x & f(r)y & f(r)z \end{vmatrix}$$

$$= \vec{i}\left(z f'(r)\frac{\partial r}{\partial y} - y f'(r)\frac{\partial r}{\partial z}\right) - \vec{j}\left(z f'(r)\frac{\partial r}{\partial x} - x f'(r)\frac{\partial r}{\partial z}\right) + \vec{k}\left(y f'(r)\frac{\partial r}{\partial x} - x f'(r)\frac{\partial r}{\partial y}\right)$$

$$= \vec{i}\left(f'(r)\,(zy - zy)\right) - \vec{j}\left(f'(r)(zx - xz)\right) + \vec{k}\left(f'(r)(yx - xy)\right)$$

$$= \vec{i}\,(f'(r)(0)) - \vec{j}\,(f'(r)(0)) + \vec{k}\,(f'(r)(0))$$

$$= 0$$

In eqn (1)

$$\nabla \cdot \vec{F} = 3f(r) + f'(r)r$$

$$\nabla \cdot \vec{F} = 0$$

$$3f(r) + f'(r)r = 0$$

$$f'(r)r = -3f(r)$$

$$\frac{f'(r)}{f(r)} = -\frac{3}{r}$$

Integrating on both side

$$\int \frac{f'(r)}{f(r)}\,dr = -\int \frac{3}{r}\,dr$$

$$\log f(r) = -3\log r + \log c$$

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$$\log f(r) = -\log r^3 + \log c$$

$$\log f(r) = \log\left(\frac{c}{r^3}\right)$$

taking e on both sides

$$f(r) = \frac{c}{r^3}$$

Show that $r^n \vec{r}$ is irrotational and solinoidal for $n = -3$

$$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$$

$$r^n \vec{r} = r^n x\vec{i} + r^n y\vec{j} + r^n z\vec{k}$$

$$\nabla \times r^n \vec{r} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ r^n x & r^n y & r^n z \end{vmatrix}$$

$$= \vec{i}\left(z\, n r^{n-1} \frac{\partial r}{\partial y} - y\, n r^{n-1} \frac{\partial r}{\partial z}\right)$$

$$-\vec{j}\left(z\, n r^{n-1} \frac{\partial r}{\partial x} - x n r^{n-1} \frac{\partial r}{\partial z}\right)$$

$$+\vec{k}\left(y\, n r^{n-1} \frac{\partial r}{\partial x} - x n r^{n-1} \frac{\partial r}{\partial y}\right)$$

$$= \vec{i}\left(n r^{n-1}\left(\frac{zy}{r} - \frac{zy}{r}\right)\right) - \vec{j}\left(n r^{n-1}\left(\frac{zx}{r} - \frac{xz}{r}\right)\right)$$

$$+\vec{k}\left(n r^{n-1}\left(\frac{xy}{r} - \frac{xy}{r}\right)\right)$$

$$= 0$$

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$$\nabla \cdot r^n \vec{r} = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right) \cdot \left(r^n x\vec{i} + r^n y\vec{j} + r^n z\vec{k}\right)$$

$$= \frac{\partial}{\partial x}(r^n x) + \frac{\partial}{\partial y}(r^n y) + \frac{\partial}{\partial z}(r^n z)$$

$$= \left[x\, n r^{n-1}\frac{\partial r}{\partial x} + r^n\right] + \left[y\, n r^{n-1}\frac{\partial r}{\partial y} + r^n\right] + \left[z\, n r^{n-1}\frac{\partial r}{\partial z} + r^n\right]$$

$$= \left[\frac{x^2}{r}\, n r^{n-1} + r^n\right] + \left[\frac{y^2}{r}\, n r^{n-1} + r^n\right] + \left[\frac{z^2}{r} n r^{n-1} + r^n\right]$$

$$= 3r^n + n r^{n-1}\left[\frac{x^2+y^2+z^2}{r}\right] = 3r^n + n r^{n-1}\left[\frac{r^2}{r}\right]$$

$$= 3r^n + n r^{n-1} \cdot r^1$$

$$= 3r^n + n r^n$$

where $n = -3$

$$= 3r^n - 3r^n$$

$$= 0$$

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Vector identities

① $\nabla c = 0$ where c is a constant

② $\nabla(c\phi) = c\nabla\phi$

③ $\nabla(\phi+\psi) = \nabla\phi + \nabla\psi$

① $\nabla c = 0$

Soln

$$\nabla c = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)c$$

$$= 0 + 0 + 0$$

$$= 0$$

② $\nabla(c\phi) = \cancel{c\nabla\phi} \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)c\phi$

$$= c\left(\vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z}\right)$$

$$= c\,\nabla\phi$$

③ $\nabla(\phi+\psi) = \nabla\phi + \nabla\psi$

Soln

$$\nabla(\phi+\psi) = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)(\phi+\psi)$$

$$= \left(\vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z}\right) + \left(\vec{i}\frac{\partial\psi}{\partial x} + \vec{j}\frac{\partial\psi}{\partial y} + \vec{k}\frac{\partial\psi}{\partial z}\right)$$

$$= \nabla\phi + \nabla\psi$$

④ $\nabla(\phi\psi) = \phi\nabla\psi + \psi\nabla\phi$

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$$\nabla(\phi\psi) = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)(\phi\psi)$$

$$= \vec{i}\frac{\partial}{\partial x}(\phi\psi) + \vec{j}\frac{\partial}{\partial y}(\phi\psi) + \vec{k}\frac{\partial}{\partial z}(\phi\psi)$$

$$= \vec{i}\left(\phi\frac{\partial\psi}{\partial x} + \psi\frac{\partial\phi}{\partial x}\right) + \vec{j}\left(\phi\frac{\partial\psi}{\partial y} + \psi\frac{\partial\phi}{\partial y}\right)$$

$$+ \vec{k}\left(\phi\frac{\partial\psi}{\partial z} + \psi\frac{\partial\phi}{\partial z}\right)$$

$$= \phi\left(\vec{i}\frac{\partial\psi}{\partial x} + \vec{j}\frac{\partial\psi}{\partial y} + \vec{k}\frac{\partial\psi}{\partial z}\right) + \psi\left(\vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z}\right)$$

$$= \phi\nabla\psi + \psi\nabla\phi$$

⑤ $\nabla\left(\frac{\phi}{\psi}\right) = \frac{\psi\nabla\phi - \phi\nabla\psi}{\psi^2}$

$$\nabla\left(\frac{\phi}{\psi}\right) = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)\left(\frac{\phi}{\psi}\right)$$

$$= \vec{i}\frac{\partial}{\partial x}\left(\frac{\phi}{\psi}\right) + \vec{j}\frac{\partial}{\partial y}\left(\frac{\phi}{\psi}\right) + \vec{k}\frac{\partial}{\partial z}\left(\frac{\phi}{\psi}\right)$$

$$= \vec{i}\left(\frac{\psi\frac{\partial\phi}{\partial x} - \phi\frac{\partial\psi}{\partial x}}{\psi^2}\right) + \vec{j}\left(\frac{\psi\frac{\partial\phi}{\partial y} - \phi\frac{\partial\psi}{\partial y}}{\psi^2}\right) + \vec{k}\left(\frac{\psi\frac{\partial\phi}{\partial z} - \phi\frac{\partial\psi}{\partial z}}{\psi^2}\right)$$

$$= \frac{\psi\nabla\phi - \phi\nabla\psi}{\psi^2}$$

) $\nabla \cdot \nabla \phi = \nabla^2 \phi$

$$\nabla \cdot \nabla \phi = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right) \cdot \left(\vec{i}\frac{\partial \phi}{\partial x} + \vec{j}\frac{\partial \phi}{\partial y} + \vec{k}\frac{\partial \phi}{\partial z}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial \phi}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial \phi}{\partial y}\right) + \frac{\partial}{\partial z}\left(\frac{\partial \phi}{\partial z}\right)$$

$$= \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2}$$

$$= \nabla^2 \phi$$

7) $\nabla \times \nabla \phi = 0$

$$\nabla \times \nabla \phi = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \frac{\partial \phi}{\partial x} & \frac{\partial \phi}{\partial y} & \frac{\partial \phi}{\partial z} \end{vmatrix}$$

$$= \vec{i}\left(\frac{\partial^2 \phi}{\partial y \partial z} - \frac{\partial^2 \phi}{\partial z \partial y}\right) - \vec{j}\left(\frac{\partial^2 \phi}{\partial x \partial z} - \frac{\partial^2 \phi}{\partial z \partial x}\right) + \vec{k}\left(\frac{\partial^2 \phi}{\partial x \partial y} - \frac{\partial^2 \phi}{\partial y \partial x}\right)$$

$$= 0$$

8) $\nabla \cdot (\nabla \times \vec{F}) = 0$ where $\vec{F}$ is a constant vector.

Sol

$$\vec{F} = F_1 \vec{i} + F_2 \vec{j} + F_2 \vec{k}$$

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ F_1 & F_2 & F_3 \end{vmatrix}$$

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$$= \vec{i}\left(\frac{\partial}{\partial y}F_3 - \frac{\partial}{\partial z}F_2\right) - \vec{j}\left(\frac{\partial}{\partial x}F_3 - \frac{\partial}{\partial z}F_1\right) + \vec{k}\left(\frac{\partial}{\partial x}F_2 - \frac{\partial}{\partial y}F_1\right)$$

$$\nabla\cdot(\nabla\times\vec{F}) = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)(\nabla\times\vec{F})$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial}{\partial y}F_3 - \frac{\partial}{\partial z}F_2\right) - \frac{\partial}{\partial y}\left(\frac{\partial}{\partial x}F_3 - \frac{\partial}{\partial z}F_1\right) + \frac{\partial}{\partial z}\left(\frac{\partial F_2}{\partial x} - \frac{\partial}{\partial y}F_1\right)$$

$$= \frac{\partial^2 F_3}{\partial x\partial y} - \frac{\partial^2 F_2}{\partial x\partial z} - \frac{\partial^2 F_3}{\partial y\partial x} + \frac{\partial^2 F_1}{\partial y\partial z} + \frac{\partial^2 F_2}{\partial z\partial x} - \frac{\partial^2 F_1}{\partial y\partial z}$$

$$= 0$$

$$\nabla\times(\nabla\times\vec{F}) = \nabla(\nabla\cdot\vec{F} - \nabla^2\vec{F})$$

$$\nabla\times\vec{F} = \vec{i}\left(\frac{\partial}{\partial y}F_3 - \frac{\partial}{\partial z}F_2\right) + -\vec{J}\left(\frac{\partial}{\partial x}F_3 - \frac{\partial}{\partial z}F_1\right) + \vec{k}\left(\frac{\partial}{\partial x}F_2 - \frac{\partial}{\partial y}F_1\right)$$

$$\nabla\times(\nabla\times\vec{F}) = \begin{vmatrix} \vec{i} & \vec{J} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \left(\frac{\partial F_3}{\partial y} - \frac{\partial}{\partial z}F_2\right) & \left(\frac{\partial}{\partial z}F_1 - \frac{\partial}{\partial x}F_3\right) & \left(\frac{\partial F_2}{\partial x} - \frac{\partial}{\partial y}F_1\right) \end{vmatrix}$$

$$= \vec{i}\left(\left(\frac{\partial^2 F_2}{\partial x\partial y} - \frac{\partial^2 F_1}{\partial y^2}\right) - \left(\frac{\partial^2 F_1}{\partial z^2} - \frac{\partial^2 F_3}{\partial z\partial x}\right)\right) - \vec{J}\left(\left(\frac{\partial^2 F_2}{\partial x^2} - \frac{\partial^2 F_1}{\partial x\partial y}\right) - \left(\frac{\partial^2 F_3}{\partial y\partial z} - \frac{\partial^2 F_2}{\partial z^2}\right)\right)$$

$$+\vec{k}\left(\left(\frac{\partial^2 F_1}{\partial x\partial z} - \frac{\partial^2 F_3}{\partial x^2}\right) - \left(\frac{\partial^2 F_3}{\partial y^2} - \frac{\partial^2 F_2}{\partial y\partial z}\right)\right)$$

$$= \vec{i}\left(\frac{\partial}{\partial y}\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) - \left(\frac{\partial^2 F_1}{\partial z^2} - \frac{\partial^2 F_3}{\partial z\partial x}\right)\right)$$

$$-\vec{J}\left(\frac{\partial}{\partial x}\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial x}\right) - \left(\frac{\partial^2 F_3}{\partial z\partial y} - \frac{\partial^2 F_2}{\partial z^2}\right)\right) + \vec{k}$$

$$= \sum_i \left(\frac{\partial}{\partial y}\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) - \left(\frac{\partial^2 F_1}{\partial z^2} - \frac{\partial^2 F_3}{\partial z \partial x} \right) \right)$$

$$= \sum_i \left(\frac{\partial^2 F_2}{\partial x \partial y} - \frac{\partial^2 F_1}{\partial y^2} + \frac{\partial^2 F_3}{\partial z \partial x} - \frac{\partial^2 F_1}{\partial z^2} \right)$$

$$= \sum_i \left(\frac{\partial^2 F_2}{\partial y \partial x} + \frac{\partial^2 F_3}{\partial z \partial x} - \frac{\partial^2 F_1}{\partial z^2} - \frac{\partial^2 F_1}{\partial y^2} - \frac{\partial^2 F_1}{\partial x^2} + \frac{\partial^2 F_1}{\partial x^2} \right)$$

$$= \sum_i \left(\frac{\partial}{\partial x}\left(\frac{\partial F_2}{\partial y} + \frac{\partial F_1}{\partial x} + \frac{\partial F_3}{\partial z} \right) - \left(\frac{\partial^2 F_1}{\partial x^2} + \frac{\partial^2 F_1}{\partial y^2} + \frac{\partial^2 F_1}{\partial z^2} \right) \right)$$

$$= \sum_i \nabla (\nabla \cdot \vec{F} - \nabla^2 \vec{F})$$

$$= \nabla (\nabla \cdot \vec{F} - \nabla^2 \vec{F})$$

Prove that $\nabla^2 (r \cdot \vec{r}) = \frac{4}{r} \cdot \vec{r}$

$\cancel{\vec{r} = x}$

$\cancel{r^2 = x\vec{i}}$

$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$

$r^2 = x^2 + y^2 + z^2$

$\frac{\partial r}{\partial x} = \frac{x}{r}$

$\frac{\partial r}{\partial y} = \frac{y}{r}$

$\frac{\partial r}{\partial z} = \frac{z}{r}$

(10)

$$\nabla = i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}$$

$$r \cdot \vec{r} = xr\vec{i} + yr\vec{j} + zr\vec{k}$$

$$\nabla(r\cdot\vec{r}) = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)\cdot\left(xr\vec{i} + yr\vec{j} + zr\vec{k}\right)$$

$$= \frac{\partial}{\partial x}(xr) + \frac{\partial}{\partial y}(yr) + \frac{\partial}{\partial z}(zr)$$

$$= x\frac{dr}{\partial x} + r + y\frac{\partial r}{\partial y} + r + z\frac{\partial r}{\partial z} + r$$

$$= \frac{x^2}{r} + \frac{y^2}{r} + \frac{z^2}{r} + 3r$$

$$= \frac{r^2}{r} + 3r$$

$$= 4r$$

$$\nabla^2(r.\vec{r}) = \left(\vec{i}\frac{\partial}{\partial x}4r + \vec{j}\frac{\partial}{\partial y}4r + \vec{k}\frac{\partial}{\partial z}4r\right)$$

$$= i\,4\frac{\partial r}{\partial x} + \vec{j}\,4\frac{\partial r}{\partial y} + \vec{k}\frac{\partial r}{\partial z}$$

$$= \vec{i}\,4\frac{x}{r} + \vec{j}\,4\frac{y}{r} + \vec{k}\,4\frac{z}{r}$$

$$= 4\,\frac{(x\vec{i} + y\vec{j} + z\vec{k})}{r}$$

$$= \frac{4\vec{r}}{r}$$

(4) P.T $\nabla^2 e^r = e^r + \frac{2e^r}{r}$

Sol

$$\nabla = \vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}$$

$$\nabla e^r = \vec{i}\frac{\partial}{\partial x}(e^r) + \vec{j}\frac{\partial}{\partial y}(e^r) + \vec{k}\frac{\partial}{\partial z}(e^r)$$

$$= \vec{i}e^r\frac{\partial r}{\partial x} + \vec{j}e^r\frac{\partial r}{\partial y} + \vec{k}e^r\frac{\partial r}{\partial z}$$

$$= e^r\left(\vec{i}\frac{x}{r} + \vec{j}\frac{y}{r} + \vec{k}\frac{z}{r}\right)$$

$$= e^r\frac{\vec{r}}{r}$$

$$\nabla^2 e^r = i\frac{\partial}{\partial x}\left(\frac{e^r}{r}\vec{r}\right) + \vec{j}\frac{\partial}{\partial y}\left(\frac{e^r\vec{r}}{r}\right) + \vec{k}\frac{\partial}{\partial z}\left(\frac{e^r\vec{r}}{r}\right)$$

$$\frac{\partial}{\partial x}\left(\frac{e^r x}{r}\right) = \frac{r\left(\frac{re^r + x^2e^r}{r}\right) - (e^r x)\frac{dr}{\partial x}}{r^2}$$

$$\nabla^2 e^r = \frac{re^r + x^2e^r - e^r\frac{x^2}{r} + re^r + y^2e^r - e^r\frac{y^2}{r} + re^r + z^2e^r - \frac{z^2e^r}{r}}{r^2}$$

$$= \frac{3re^r + e^r(x^2+y^2+z^2) - \frac{e^r}{r}(x^2+y^2+z^2)}{r^2}$$

$$= \frac{3re^r + e^r r^2 - \frac{e^r}{r}(r^2)}{r^2}$$

$$= \frac{3re^r + e^r r^2 - e^r r}{r^2}$$

$$= \frac{re^r(3-1+r)}{r^2} = \frac{re^r(2+r)}{r^2}$$

$$= \frac{2e^r r}{r^2} + \frac{e^r r^2}{r^2} = \frac{2e^r}{r} + e^r$$

(42) Prove that $\phi = \frac{x}{r^3}$ then $\nabla^2 \phi = 0$

Solution

$$\nabla^2 \phi = i \frac{\partial^2 \phi}{\partial x^2} + \vec{j} \frac{\partial^2 \phi}{\partial y^2} + \vec{k} \frac{\partial^2 \phi}{\partial z^2}$$

$$\frac{\partial^2 \phi}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial \phi}{\partial x}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{r^3 (1) - x\, 3r^2 \frac{\partial r}{\partial x}}{(r^3)^2}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{r^3 - 3r^2 \frac{x^2}{r}}{r^6}\right)$$

$$= \frac{\partial}{\partial x}\left(\cancel{r^4}\ r \frac{(r^2 - 3x^2)}{r^6}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{r^2 - 3x^2}{r^5}\right)$$

$$= \frac{r^5\left(2r \frac{dr}{\partial x} - 6x\right) - (r^2 - 3x^2)\left(5r^4 \frac{\partial r}{\partial x}\right)}{r^{10}}$$

$$= \frac{r^5\left(2r \frac{x}{r} - 6x\right) - (r^2 - 3x^2)\left(5r^4 \frac{x}{r}\right)}{r^{10}}$$

$$= \frac{-4xr^5 - (r^2 - 3x^2)(5r^3 x)}{r^{10}}$$

$$= \frac{-4xr^5 - 5r^5 x + 15 r^3 x^3}{r^{10}}$$

$$= \frac{15 r^3 x^3 - \cancel{4}\, 9x \cancel{5} r^5}{r^{10}}$$

$$= \frac{15x^3 - 9xr^2}{r^7}$$

$$\frac{\partial^2 \phi}{\partial y} = \frac{\partial}{\partial y}\left(\frac{\partial \phi}{\partial y}\right)$$

$$\frac{\partial \phi}{\partial y} = \frac{r^3(0) - 3r^2 \frac{xy}{r}}{r^6}$$

$$= \frac{-3xyr}{r^6} = \frac{-3xy}{r^5}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial \phi}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{-3xy}{r^5}\right)$$

$$= \frac{r^5(-3x) - (-3xy)5r^4\frac{y}{r}}{r^{10}}$$

$$= \frac{-3xr^5 + 3xy \cdot 5r^3 y}{r^{10}}$$

$$= \frac{-3xr^2 + 15xy^2}{r^7}$$

Illy

$$\frac{\partial^2 \phi}{\partial z} = \frac{-3\cancel{x}r^2 + 15xz^2}{r^7}$$

$$\nabla^2 \phi = \frac{-9xr^2 + 15x^3 - 3xr^2 + 15xz^2 - 3xr^2 + 15xy^2}{r^7}$$

$$= \frac{-15xr^2 + 15x(x^2+y^2+z^2)}{r^7}$$

$$= \frac{-15xr^2 + 15xr^2}{r^7}$$

$$= \frac{r^2(-15x+15x)}{r^7}$$

$$= \frac{-15x+15x}{r^5}$$

$$= 0$$

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24.2.T20 Prove that $\nabla\cdot(\nabla r^n) = n(n+1)\, r^{n-2}$

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Given

$$\nabla = \vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}$$

$$\nabla r^n = i\frac{\partial}{\partial x}(r^n) + \vec{j}\frac{\partial}{\partial y}(r^n) + \vec{k}\frac{\partial}{\partial z}(r^n)$$

$$= \vec{i}\, n r^{n-1}\frac{\partial r}{\partial x} + \vec{j}\, n r^{n-1}\frac{\partial r}{\partial y} + \vec{k}\, n r^{n-1}\frac{\partial r}{\partial z}$$

$$= \vec{i}\, n r^{n-1}\frac{x}{r} + \vec{j}\, n r^{n-1}\frac{y}{r} + \vec{k}\, n r^{n-1}\frac{z}{r}$$

$$= \frac{n r^{n-1}}{r}(x\vec{i} + y\vec{j} + z\vec{k})$$

$$= n r^{n-2}\,(\vec{r})$$

$$\nabla\cdot(\nabla r^n) = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)\cdot\left[n r^{n-2}(x\vec{i} + y\vec{j} + z\vec{k})\right)$$

$$= \frac{\partial}{\partial x}\left(n r^{n-2}(x)\right) + \frac{\partial}{\partial y}\left(n r^{n-2} y\right) + \frac{\partial}{\partial z}\left(n r^{n-2} z\right)$$

$$= \sum_i \frac{\partial}{\partial x}\left(n r^{n-2} x\right)$$

$$= n r^{n-2} + x\, n(n-2) r^{n-3}\frac{\partial r}{\partial x}$$

$$= n r^{n-2} + n(n-2) r^{n-3}\frac{x^2}{r}$$

$$= n r^{n-2} + n(n-2) r^{n-4} x^2 + n r^{n-2} + n(n-2) r^{n-4} y^2$$

$$+ n(r^{n-2}) + n(n-2)\, r^{n-4} z^2$$

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$= 3r^{n-2} + n(n-2)r^{n-4}(x^2+y^2+z^2)$

$= 3r^{n-2} + n(n-2)r^{n-4} r^2$

$= 3r^{n-2} + n(n-2) r^{n-2}$

$= nr^{n-2}(3 + n-2)$

$= n r^{n-2} (n+1)$

$\nabla\cdot(\nabla r^n) = n(n+1) r^{n-2}$

Prove that $\nabla (r \nabla(\frac{1}{r^3}) = \frac{3}{r^4}$

solution

given $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$

W.K.T $= r^2 = x^2+y^2+z^2$

diff w.r.to x, y, z

$\frac{\partial r}{\partial x} = \frac{x}{r}$; $\frac{\partial r}{\partial y} = \frac{y}{r}$; $\frac{\partial r}{\partial z} = \frac{z}{r}$

$\nabla\left(\frac{1}{r^3}\right) = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)\cdot\left(\frac{1}{r^3}\right)$

$= \vec{i}\frac{\partial}{\partial x}(r^{-3}) + \vec{j}\frac{\partial}{\partial y}(r^{-3}) + \vec{k}\frac{\partial}{\partial y}(r^{-3})$

$= \sum_i i\frac{\partial}{\partial x} r^{-3}$

$= \sum -3 r^{-4} \frac{\partial r}{\partial x}\vec{i} = -3r^{-4}\frac{x}{r}\vec{i}$

$= -3r^{-4}\left(\frac{x\vec{i} + y\vec{j} + z\vec{k}}{r}\right)$

$= -3r^{-4}\frac{\vec{r}}{r} = -3r^{-5}\vec{r}$

$$r\left(\nabla\left(\frac{1}{r^3}\right)\right) = r\left(-3r^{-5}\vec{r}\right)$$

$$= -3r^{-4}\vec{r}$$

$$\nabla\left(r\left(\nabla\left(\frac{1}{r^3}\right)\right)\right) = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)\left(-3r^{-4}(x\vec{i}+y\vec{j}+z\vec{k})\right)$$

$$= \frac{\partial}{\partial x}(-3r^{-4}x) + \frac{\partial}{\partial y}(-3r^{-4}y) + \frac{\partial}{\partial z}(-3r^{-4}z)$$

$$= \sum_i \frac{\partial}{\partial x}(-3r^{-4}x)$$

$$= \sum_i -3r^{-4}(1) + x\left(12r^{-5}\frac{\partial r}{\partial x}\right)$$

$$= \sum_i -3r^{-4} + \frac{x^2}{r}12r^{-5}$$

$$= \sum_i -3r^{-4} + x^2 12 r^{-6}$$

$$= \sum_i r^{-4}\left(-3 + x^2 12 r^{-2}\right)$$

$$= r^{-4}(-3 + x^2 12 r^{-2}) + r^{-4}(-3 + y^2 12 r^{-2}) + r^{-4}(-3 + z^2 12 r^{-2})$$

$$= \frac{3}{r^4}\cancel{\left(-9 + 12(x^2 +\right.}$$

$$= \frac{1}{r^4}\left(-9 + 12r^{-2}r^2\right)$$

$$= \frac{3}{r^4}$$

(47) Prove that $\nabla^2\left(\frac{1}{r}\right)=0$

$$\nabla\left(\frac{1}{r}\right) = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)\left(\frac{1}{r}\right)$$

$$= \vec{i}\frac{\partial}{\partial x}(r^{-1}) + \vec{j}\frac{\partial}{\partial y}(r^{-1}) + \vec{k}\frac{\partial}{\partial y}(r^{-1})$$

$$= \sum_i -r^{-2}\frac{x}{r}$$

$$= \sum_i -r^{-3}x$$

$$= -r^{-3}x\vec{i} + (-r^{-3}y\vec{j}) + (-r^{-3}z\vec{k})$$

$$\nabla^2\left(\frac{1}{r}\right) = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{z}\frac{\partial}{\partial z}\right)\left(\nabla\left(\frac{1}{r}\right)\right)$$

$$= \vec{i}\frac{\partial}{\partial x}(-r^{-3}x) + \frac{\partial}{\partial y}(-r^{-3}y) + \frac{\partial}{\partial z}(-r^{-3}z)$$

$$= \sum_i -r^{-3} + \frac{x^2 3r^{-4}}{r}$$

$$= \sum -r^{-3} + x^2 3r^{-5}$$

$$= -r^{-3} + x^2 3r^{-5} - r^{-3} + y^2 3r^{-5} - r^{-3} + z^2 3r^{-5}$$

$$= -3r^{-3} + 3r^{-5}(x^2+y^2+z^2)$$

$$= -3r^{-3} + 3r^{-5} r^2$$

$$= -3r^{-3} + 3r^{-3}$$

$$= 0$$