42.1 MECHANICS OF A PARTICLE

We shall study the conservation laws for a particle in motion using Newtonian mechanics.

1. Conservation of linear momentum:

Newton's second law of motion is

$$\frac{d}{dt}(m\mathbf{v}) = \frac{d\mathbf{p}}{dt} = \mathbf{F} \quad ...(1)$$

If the total force **F** *is zero, then* $\frac{d\mathbf{p}}{dt} = 0$ *and the linear momentum is conserved.*

If $\mathbf{F}^{ext} = 0, \frac{d\mathbf{P}}{dt} = 0$. Integrating, **P** = constant.

This gives the *theorem* for conservation of linear momentum of a particle.

2. Conservation of angular momentum:

Consider a particle of mass m and linear momentum **p** at a position **r** relative to origin O of an inertial reference frame (Fig. 42.1).

- The angular momentum **L** of the particle with respect to the origin O is

$$\mathbf{L} = \mathbf{r} \times \mathbf{p}, \quad ...(1)$$

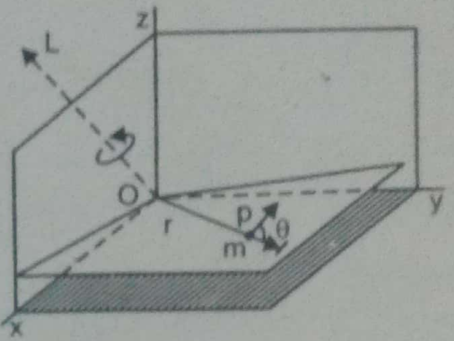


Fig. 42.1

Let F be the force acting on the particle. Then the torque $\vec{\tau}$ acting on the particle with respect to the origin O is

$$\vec{\tau} = \mathbf{r} \times \mathbf{F} \quad ...(2)$$

$$\vec{\tau} = \mathbf{r} \times \mathbf{F} = \mathbf{r} \times \frac{d\mathbf{p}}{dt}$$

$$= \frac{d}{dt}(\mathbf{r} \times \mathbf{p}) - \frac{d\mathbf{r}}{dt} \times \mathbf{p}$$

$$= \frac{d}{dt}(\mathbf{r} \times \mathbf{p}) - \mathbf{v} \times m\mathbf{v}.$$

The second term is zero, as both vectors are parallel.

$$\therefore \quad \vec{\tau} = \frac{d}{dt}(\mathbf{r} \times \mathbf{p})$$

$$\vec{\tau} = \frac{d\mathbf{L}}{dt}. \quad ...(3)$$

Thus, *time rate of change of the vector angular momentum of a particle is equal to the vector torque acting on it.*

If $\vec{\tau}_{ext} = 0$, *then* $\frac{d\mathbf{L}}{dt} = 0$. ∴ $\vec{\mathbf{L}}$ = constant.

Thus angular momentum is conserved in the absence of an external torque.

This is the principle of conservation of angular momentum.

3. Conservation of Energy:-

If the forces acting on a particle are conservative, then the total energy of the particle, which is sum of kinetic energy and potential energy, is constant or conserved.

Let the particle move from the point 1 to point 2 by the action of external force $\vec{F}$ (Fig. 42.2). The total work done in displacing the particle from point 1 to point 2 is given by

$$W_{12} = \int_1^2 dW = \int_1^2 \mathbf{F} \cdot d\mathbf{r} \qquad ...(1)$$

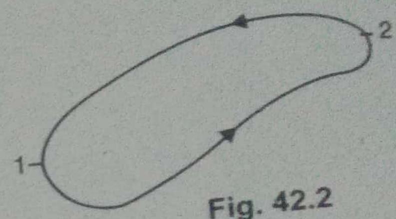


Fig. 42.2

$$= \int_1^2 \frac{d\mathbf{p}}{dt} \cdot d\mathbf{r}$$

$$= \int_1^2 \frac{d}{dt}(m\mathbf{v}) \cdot d\mathbf{r}$$

$$= m\int_1^2 \frac{d\mathbf{v}}{dt} \cdot \frac{d\mathbf{r}}{dt}\, dt \text{ (because } m \text{ is constant)}$$

$$= m\int_1^2 \frac{d\mathbf{v}}{dt} \cdot \mathbf{v}\, dt$$

$$= m\int_1^2 \frac{1}{2}\frac{d}{dt}(v^2)\, dt$$

or

$$W_{12} = \frac{1}{2}m[v^2]_1^2$$

$$= \frac{1}{2}m(v_2^2 - v_1^2) = \frac{1}{2}mv_2^2 - \frac{1}{2}mv_1^2$$

Here, v_1 and v_2 are the velocities of the particle at points 1 and 2 respectively.

$$\therefore \quad W_{12} = T_2 - T_1 \qquad ...(2)$$

Here, $T_1 = \frac{1}{2}m_1 v_1^2$ = kinetic energy of the particle at point 1,

and $T_2 = \frac{1}{2}m_2 v_2^2$ = kinetic energy of the particle at point 2.

Thus, *the total work done by a force acting on a particle is equal to the change in the kinetic energy of the particle.*

This is called the *Work - Energy theorem.*

A conservative force **F** can be expressed as the gradient of a scalar function called the potential function.

$$\mathbf{F} = -\nabla.V \qquad ...($$

Here, V is called the potential or potential energy.

We can write

$$W_{12} = \int_1^2 -\nabla V.d\mathbf{r}$$

$$= \int_1^2 -\frac{dV}{dr}dr = -\int_1^2 dV$$

$$\therefore \quad W_{12} = V_1 - V_2.$$

From Eqs. (2) and (4) we get

$$T_2 - T_1 = V_1 - V_2$$

or

$$T_2 + V_2 = T_1 + V_1 = \text{constant},$$

or in general

$$T + V = \text{constant},$$

which shows that the total energy of the particle is conserved.

This is the *energy conservation theorem*.

Conservative Forces

First Definition: *A force acting on a particle is conservative if the particle, after going t a complete round trip, returns to its initial position with the same kinetic energy as it had in*

Explanation: Suppose we throw a ball upward against gravity. The ball reaches a certain coming momentarily to rest so that its kinetic energy becomes zero. Then it returns to our under gravity with the same kinetic energy with which it was thrown. We assume the air-resistance is zero. Thus the force of gravity is conservative.

Examples of Conservative Forces. (*i*) Gravitational force (*ii*) Electrostatic force (*iii*) Elastic force.

All central forces are conservative forces.

Second Definition: *A force acting on a particle is conservative if the net work done by the force in a complete round trip of the particle is zero.*

Explanation: Suppose we throw a ball upward against gravity. When the ball is thrown up, the work done by the conservative force of gravity is negative. When the ball returns back, the work is positive. We assume that air-resistance is absent. So the negative and positive works are equal. Hence the net work done is *zero.*

If the force **F** is conservative, then the work done by it around a closed path is zero, *i.e.*,

$$\oint \mathbf{F} \cdot d\mathbf{r} = 0 \qquad \text{...(1)}$$

Physically it is clear that a system cannot be conservative if friction or other dissipative forces are present, for **F** . $d\mathbf{r}$ due to friction is always positive and the integral cannot vanish.

According to Stokes theorem,

$$\oint \mathbf{F} \cdot d\mathbf{r} = \iint_s \text{curl } \mathbf{F} \cdot d\mathbf{S} = 0 \qquad \text{[from Eq. (1)]}$$

or $$\iint_s \text{curl } \mathbf{F} \cdot d\mathbf{S} = 0$$

or $$\text{curl } \mathbf{F} = 0$$

or $$\nabla \times \mathbf{F} = 0$$

Therefore, for conservative forces $\nabla \times \mathbf{F} = 0$

But curl of a gradient is always zero.

Therefore **F** can be expressed as the gradient of a scalar function called the potential function.

$$\mathbf{F} = -\nabla V \qquad \text{...(2)}$$

Here, V is called the potential or potential energy.

42.2 MECHANICS OF A SYSTEM OF PARTICLES

When the mechanical system consists of two or more particles, we must distinguish, between the *external forces* exerted upon the particles of the system by sources not belonging to the system, and the *internal forces* arising on account of the interactions between the particles of the system themselves.

The equation of motion in terms of Newton's second law for a general system of N particles is

$$m_i \mathbf{a}_i = \dot{\mathbf{p}}_i = \mathbf{F}_i^{(e)} + \sum_{j \neq i} \mathbf{F}_i^j, \quad i = 1, 2 \,..... N. \qquad \text{...(1)}$$

From Eq. (1), we obtain equation of motion of each particle, $\boldsymbol{i}$, corresponding to each value of i. They are N equations in all. Here, $\mathbf{F}_i^{(e)}$ stands for the external force acting on i^{th} particle and $\mathbf{F}_i^j$ is the *internal force on the* i^{th} *particle due to* j^{th} *particle.*

All the particles of the system exert forces on one another. Hence the internal force on i^{th} particle must be the sum of forces due to all other particles $= \sum_{j=1}^{N} F_i^j$ excluding the term $j = i$, since by definition $\mathbf{F}_i^i$ is obviously zero.

We shall modify Eqs. (1) by assuming that Newton's third law is valid for internal forces. That is, the force $\mathbf{F}_i^j$ must be equal and opposite in direction to the force $\mathbf{F}_j^i$ that the i^{th} particle exerts on the j^{th} particle. Vectorially

$$\mathbf{F}_i^j = -\mathbf{F}_j^i \quad \text{...(2)}$$

It automatically implies that internal forces occur in pairs and act along line joining the two particles. Any combination of mutual forces must be zero then.

Summing now over all particles of the system, we obtain the equation of motion of the system as a whole:

$$\begin{aligned} \sum \dot{\mathbf{p}}_i &= \frac{d^2}{dt^2} \sum_i m_i \mathbf{r}_i \\ &= \sum_i \mathbf{F}_i^{(e)} + \sum_{i,j}{}' \mathbf{F}_i^j \\ &= \sum_i \mathbf{F}_i^{(e)}, \end{aligned} \quad \text{...(3)}$$

since

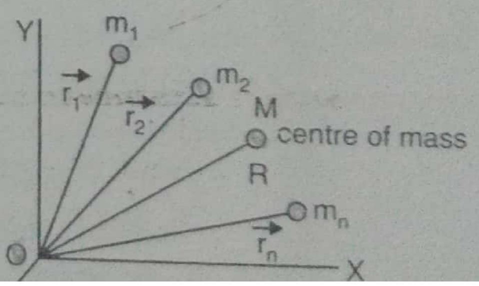
$$\begin{aligned} \sum_{i,j}{}' \mathbf{F}_i^j &= -\sum_{i,j}{}' \mathbf{F}_j^i \\ &= \frac{1}{2} \sum_{i,j}{}' \left[\mathbf{F}_i^j + \mathbf{F}_j^i\right] = 0 \end{aligned} \quad \text{[from Eq. (2)]}$$

Here, a prime on the summation symbol Σ means that the term $j = i$ is to be excluded from the sum.

We define the centre of mass $\mathbf{R}$ of the system by

$$\mathbf{R} = \frac{\sum_i m_i \mathbf{r}_i}{\sum_i m_i} = \frac{\sum_i m_i \mathbf{r}_i}{M}.$$

Here, $\sum_i m_i = M$ is the total mass of the system (Fig. 42.3). Eq. (3) becomes



$$M \frac{d^2\mathbf{R}}{dt^2} = \sum \mathbf{F}^{(e)}$$

$$\mathbf{P} = M\dot{\mathbf{R}} \qquad ...(5)$$

From Eq. (5), rate of change of total linear momentum is

$$\dot{\mathbf{P}} = M\ddot{\mathbf{R}} = \mathbf{F}^{(e)}. \qquad ...(6)$$

Eq. (6) defines two important characteristics of motion. They are:

(*i*) *Centre of mass moves as if the total external force* $\mathbf{F}^{(e)}$ *acting on the entire mass of the system were concentrated at the centre of mass.*

(*ii*) *If the total external force vanishes, the total linear momentum is conserved.*

Property (*ii*) is the *theorem of conservation of linear momentum for a system of particles*. It also implies that since $\dot{\mathbf{P}} = 0$, $\mathbf{P}$ = constant or $\dot{\mathbf{R}}$ = constant, *i.e., the centre of mass moves with constant velocity in the absence of external forces.*

Thus, we may state that *the velocity of the centre of mass of the system remains constant if there are no external forces acting on the system.*

Example. Consider the uniform motion of a radioactive nucleus undergoing disintegration (Fig. 42.4). The nucleus ejects different particles which move off in different directions in such a way that their centre of mass continues to move with constant velocity even after the disintegration.

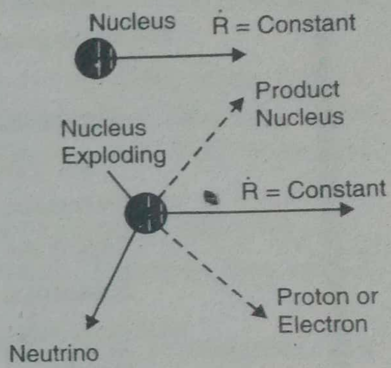


Fig. 42.4

(*a*) Conservation theorem for linear momentum:

The net linear momentum of a system of *n*-particles is

$$P = \sum_{i=1}^{n} p_i = \sum_{i=1}^{n} m_i v_i$$

From Newton's second law, $F^{ext} = \dfrac{dP}{dt}$

i.e., the rate of change of linear momentum of a system of particles is equal to the net external force acting on the system.

If $\qquad F^{ext} = 0, \quad \dfrac{dP}{dt} = 0$. Integrating, P = constant.

This gives the *theorem* for conservation of linear momentum of the system.

Statement: *"If the sum of external forces acting on the system of particles is zero, the total linear momentum of the system is constant or conserved."*

(*b*) Conservation theorem for angular momentum.

The angular momentum of i^{th} particle of the system about any point *O*, from definition is given by

$$\mathbf{L}_i = \mathbf{r}_i \times \mathbf{p}_i, \qquad ...(1)$$

Here, r_i is the radius vector of i^{th} particle from the point *O* and $\mathbf{p}_i$ its linear momentum (Fig. 42.5).

We obtain the total angular momentum of the system of particles by forming the cross product $(\mathbf{r}_i \times \mathbf{p}_i)$ for the i^{th} particle and summing over all particles.

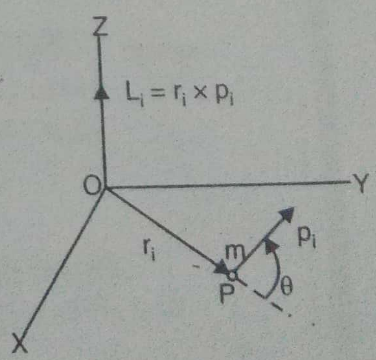


Fig. 42.5

$$\mathbf{L} = \sum_i (\mathbf{r}_i \times \mathbf{p}_i) \qquad ...(2)$$

$$\dot{\mathbf{L}} = \frac{d}{dt}\sum_i (\mathbf{r}_i \times \mathbf{p}_i) = \sum_i [\mathbf{r}_i \times \dot{\mathbf{p}}_i + \dot{\mathbf{r}}_i \times \mathbf{p}_i]$$

$$= \sum_i (\mathbf{r}_i \times \dot{\mathbf{p}}_i) \qquad \text{(because } \dot{\mathbf{r}}_i \times \mathbf{p}_i = 0)$$

$$= \sum_i \mathbf{r}_i \times \mathbf{F}_i^{(e)} + \sum_{i,j}{}' \mathbf{r}_i \times \mathbf{F}_i^j. \qquad ...(3)$$

Second term in Eq. (3) denotes the sum of internal torques which vanishes if the interacting forces are Newtonian in character. We then have the important result

$$\frac{d\mathbf{L}}{dt} = \sum_i \mathbf{r}_i \times \mathbf{F}_i^{(e)} = \tau^{(e)} = \text{sum of external torques.} \qquad ...(4)$$

The time derivative of the total angular momentum is equal to the moment of the external forces about the given point. From this we have the conservation of angular momentum of a system of particles.

*If total external torque $\tau^{(e)} = 0$, **L** is constant in time or conserved.*

Thus, *if external torque acting on a system of particles is zero, the angular momentum of the system remains constant.*

This is the conservation theorem for angular momentum of a system of particles.

EXAMPLE 1. *Express angular momentum of the system as the sum of angular momentum of motion of the centre of mass and angular momentum of the motion about the centre of mass.*

SOL. Let $\mathbf{r}_i$ be the position vector of the i^{th} particle relative to a point O fixed in an inertial frame (Fig. 42.6).

From Fig., $\quad \mathbf{r}_i = \mathbf{r}_i' + \mathbf{R} \quad$ and $\quad \mathbf{v}_i = \mathbf{v}_i' + \mathbf{v} \qquad ...(1)$

• $\mathbf{r}_i'$ and $\mathbf{v}_i'$ denote the radius vector and velocity of the i^{th} particle referred to centre of mass O' as the new origin and

$\mathbf{v} = \dot{\mathbf{R}}$ is the velocity of the centre of mass relative to O.

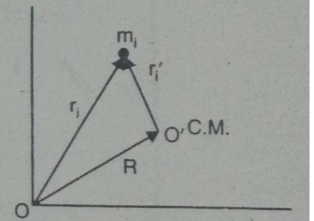


Fig. 42.6

Then $\quad \mathbf{L} = \sum m_i (\mathbf{r}_i' + \mathbf{R}) \times (\mathbf{v}_i' + \mathbf{v})$

$$= \sum_i (\mathbf{R} \times m_i \mathbf{v}) + \sum_i \mathbf{r}_i' \times m_i \mathbf{v}_i' + \left(\sum_i m_i \mathbf{r}_i'\right) \times \mathbf{v} + \mathbf{R} \times \frac{d}{dt}\left(\sum_i m_i \mathbf{r}_i'\right). \qquad ...(2)$$

But $\sum_i m_i \mathbf{r}_i' = 0$, from the definition of centre of mass.

So the last two terms in Eq. (2) vanish.

Also $\quad \sum_i (\mathbf{R} \times m_i \mathbf{v}) = \mathbf{R} \times M\mathbf{v}.$

$$\therefore \quad \mathbf{L} = \mathbf{R} \times M\mathbf{v} + \sum_i \mathbf{r}_i' \times \mathbf{p}_i' \qquad ...(3)$$

Eq. (3) shows that we can decompose angular momentum of a system of particles, with reference to a given origin O, into two distinct parts:

(*i*) angular momentum of the system about the origin as if the total mass were concentrated at the centre of mass.

(*ii*) plus the angular momentum of the motion about the centre of mass.

(c) Conservation of Energy for a System of Particles

(i) Work Energy Theorem

Consider a system of particles. Let W_i be the work done by the external force on i^{th} particle in displacing it from position 1 to 2. Then,

$$W_i = \int_1^2 \mathbf{F}_i \cdot d\mathbf{r}_i = \left[\frac{1}{2} m_i v_i^2\right]_1^2 = [T_i]_1^2$$

Here, T_i = kinetic energy of i^{th} particle.

Now, summing over all the particles of the system, we have

$$\sum W_i = \sum [T_i]_1^2$$

$$W_{12} = T_2 - T_1,$$

Here, W_{12} = total work done by the external force.

T_1 and T_2 are the kinetic energies of the system at points 1 and 2 respectively.

Thus, *the work done is equal to the change in the kinetic energy.* This is the *work-energy theorem.*

(ii) Conservation theorem for energy

For a conservative system, the force $\mathbf{F}_i$ acting on i^{th} particle is expressed as the gradient of some scalar function, *i.e.,*

$$\mathbf{F}_i = -\nabla V_i$$

Here, V_i is the potential or potential energy of i^{th} particle.

Now, $\nabla V_i = \dfrac{\partial V_i}{\partial r_i}$. $\therefore$ $\mathbf{F}_i = -\dfrac{\partial V_i}{\partial r_i}$

Work done by the external force $\mathbf{F}_i$ acting on the i^{th} particle in displacing it from position 1 to 2 is given by

$$W_i = \int_1^2 \mathbf{F}_i \cdot dr_i = -\int_1^2 \frac{\partial V_i}{\partial r_i} dr_i$$

$$= -[V_i]_1^2$$

Now summing over all the particles of the system, we have

$$\sum W_i = \sum_i [-V_i]_1^2$$

$$\therefore \quad W_{12} = V_1 - V_2 \qquad ...(2)$$

Here, W_{12} = total work done by the external forces.

V_1 and V_2 are the potential energies of the system at points 1 and 2 respectively.

Comparing Eqns. (1) and (2), $T_2 - T_1 = V_1 - V_2$

or $T_2 + V_2 = T_1 + V_1 = E$ = total energy of system.

This is the *energy conservation theorem.*

If the forces acting on the system of particles are conservative, the total energy of the system of particles, which is the sum of the total kinetic energy and the total potential energy of the system is conserved.

42.3 BASIC CONCEPTS

Degrees of Freedom. *The number of mutually independent variables required to define the state or position of a system is the number of degrees of freedom possessed by it.*

For example, the position of a simple ideal mass-point can be defined completely by the three Cartesian coordinates x, y, z. So it has three degrees of freedom. Extending this idea, for a system of N particles moving independently of each other, the number of degrees of freedom is $3N$.

Constraints. *Constraints are restrictions imposed on the position or motion of a system, because of geometrical conditions.*

Examples. (1) The beads of an abacus are constrained to one-dimensional motion by the supporting wires.

(2) Gas molecules within a container are constrained by the walls of the vessel to move only *inside* the container.

(3) The motion of rigid bodies is always such that the distance between any two particles remains unchanged.

(4) A particle placed on the surface of a solid sphere is restricted by the constraint so that it can only move on the surface or in the region exterior to the sphere.

(5) The motion of point mass of a simple pendulum is restricted since the point mass always remains at a constant distance from the point of suspension.

For a particle constrained to move on a plane, only two variables x, y or r, θ *are sufficient to describe its motion and the particle is said to have two degrees of freedom. Thus, the constraint on the motion of the particle in a plane reduces the number of degrees of freedom by one.*

Very often, we can express constraints in terms of certain equations. For example, the equation of constraint in the case of a particle moving on or outside the surface of a sphere of radius a is $x^2 + y^2 + z^2 \geq a^2$ if the origin of the coordinate system coincides with the centre of the sphere.

Types of Constraints

***(i)* Holonomic and non-holonomic constraints.**

If the constraints can be expressed as equations connecting the co-ordinates of the particles (and possibly time) in the form

$$f(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3 \ldots\ldots, \mathbf{r}_n, t) = 0 \qquad \ldots(1)$$

then the constraints are said to be *holonomic*.

Examples. (1) *The constraints involved in the motion of rigid bodies* in which the distance between any two particular points is always fixed, are *holonomic* since the conditions of constraints are expressed as

$$(\mathbf{r}_i - \mathbf{r}_j)^2 - c_{ij}^2 = 0.$$

(2) *The constraints involved* when a particle is restricted to move along a curve or surface are holonomic. Here the equation defining the curve or surface is the equation of constraint.

(3) *The constraints involved* in the motion of the point mass of a simple pendulum *are holonomic*.

Let $\mathbf{r}$ = position vector of the point mass

$\mathbf{a}$ = position vector of the point of suspension

l = length of the string.

The equation of the constraint may be written as

$$(\mathbf{r} - \mathbf{a})^2 - l^2 = 0.$$

If the constraints cannot be expressed in the form of Eq. (1), they are called *non-holonomic constraints*.

Examples. (1) The constraints involved in the motion of the particle placed on the surface of a solid sphere are non-holonomic. The conditions of constraints in this case are expressed as

$$r^2 - a^2 \geq 0,$$

where a is the radius of sphere. This is an inequality and hence not in the form of Eq. (1).

(2) The walls of the gas vessel constitute a non-holonomic constraint.

(3) Another example of a non-holonomic constraint is the condition that the velocity of the point of contact of a sphere, rolling without slipping along a stationary rough surface, is equal to zero.

(*ii*) Scleronomic and Rheonomic Constraints. If the constraints are independent of time, they are called *scleronomic*.

The constraint in the case of rigid body motion is scleronomous. If the constraints are explicitly dependent on time, they are called *rheonomic*. A bead sliding on a moving wire is an example of rheonomic constraint.

Definition of Constraint. *A constraint is defined to be some geometrical restriction on the freedom of motion of the particles of the system, which might have to be satisfied by the co-ordinates or co-ordinate differences or sometimes by velocities rather than co-ordinates.*

The effect of having 'k' equations of constraints on the system, is that 'k' out of the original $3N$ variables describing the system, become dependent rather than independent. Thus the number of independent variables is reduced to $(3N-k)$ and the system is said to possess $(3N-k)$ *'degrees of freedom'*.

In the solution of mechanical problems, the constraints introduce two types of difficulties:

(1) The co-ordinates r_i are connected by the equations of constraints. Therefore, they are not independent.

(2) The forces of constraint are not *a priori* known. In fact, they cannot be estimated till a complete solution of the problem is obtained.

The first problem can be solved by introducing *generalized co-ordinates*, whereas the second is practically an insurmountable problem. We therefore reformulate the problem such that the forces of constraint disappear.

Generalised Co-ordinates

A system consisting of N particles, free from constraints, has $3N$ independent co-ordinates or degrees of freedom. If the sum of the degrees of freedom of all the particles is k, then the system may be regarded as a collection of free particles subjected to $(3N-k)$ independent constraints. So only k co-ordinates are needed to describe the motion of the system. These new co-ordinates $q_1, q_2, q_3, \ldots\ldots q_k$ are called the *Generalised Co-ordinates of Lagrange*. Generalised co-ordinates may be lengths or angles or any other set of independent quantities which define the position of the system.

Definition. *The generalised co-ordinates of a material system are the independent parameters $q_1, q_2, q_3, \ldots\ldots, q_k$ which completely specify the configuration of the system, i.e., the position of all its particles with respect to the frame of reference.*

Example 1. Consider the simple pendulum of mass m_1 with fixed length r_1 (Fig. 42.7). The single co-ordinate θ_1 will determine uniquely the position of m_1 since the simple pendulum is a system of one degree of freedom. Since the only variable involved is θ_1, it can be chosen as the generalised co-ordinate. Thus $q = \theta_1$.

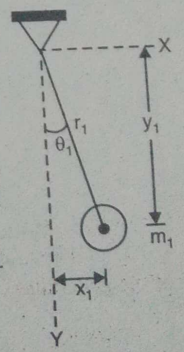


Fig. 42.7

The two coordinates x_1 and y_1 could also be used to locate m_1 but would require the inclusion of the *equation of the constraint* $x_1^2 + y_1^2 = r_1^2$. Since x_1 and y_1 are not independent, they are not generalised co-ordinates.

(2) When a particle moves in a plane, it may be described by cartesian co-ordinates x, y or the polar co-ordinates, r, θ. We can write

$$q_1 = x \quad \text{or} \quad q_1 = r = \sqrt{(x^2 + y^2)},$$

$$q_2 = y \qquad\quad q_2 = \theta = \tan^{-1}\frac{y}{x}.$$

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(3) In considering the motion of a particle in a *'spherically symmetrical force field'* or the motion of a particle constrained to move on a sphere of a fixed radius, the use of *spherical polar coordinates* (r, θ, ϕ), is more convenient. Here,

$$q_1 = r = (x^2 + y^2 + z^2)^{1/2},$$

$$q_2 = \theta = \cot^{-1} \frac{z}{(x^2 + y^2)^{1/2}}$$

$$q_3 = \phi = \tan^{-1} \frac{y}{x}.$$

(4) If it is preferred to accept a co-ordinate system moving uniformly with velocity v in x-direction, generalised co-ordinates are

$$q_1 = x - \dot{x}t.$$

$$q_2 = y \qquad \dot{x} = v = \text{constant}.$$

$$q_3 = z.$$

Transformation Equations

The rectangular cartesian co-ordinates can be expressed as the functions of generalised co-ordinates. Let x_i, y_i and z_i be the cartesian co-ordinates of i^{th} particle of the system. Let t denote the time. Then, these cartesian co-ordinates can be expressed as functions of generalised co-ordinates $q_1, q_2, q_3, \ldots, q_k$ *i.e.*,

$$\left.\begin{aligned} x_i &= x_i(q_1, q_2, \ldots, q_k, t) \\ y_i &= y_i(q_1, q_2, \ldots, q_k, t) \\ z_i &= z_i(q_1, q_2, \ldots, q_k, t) \end{aligned}\right\} \quad \ldots(1)$$

Let $\mathbf{r}_i$ be the position vector of i^{th} particle, *i.e.*, $\mathbf{r}_i = \mathbf{i}x_i + \mathbf{j}y_i + \mathbf{k}z_i$

Then $$\mathbf{r}_i = \mathbf{r}_i(q_1, q_2, q_3, \ldots, q_k, t), \quad \ldots(2)$$

Eq. (2) is the vector form of Eq. (1).

The equations like (1) and (2) are called ***transformation equations***. The functions and their derivatives in the above two equations are supposed to be continuous. The equations also contain the constraints explicitly.

42.4 GENERALISED NOTATIONS

(1) Generalised Displacement: Consider a small displacement of an N-particle system defined by changes $\delta\mathbf{r}_i$ in cartesian co-ordinates $\mathbf{r}_i$ $(i = 1, 2, \ldots, N)$ with time t held *fixed*. $\mathbf{r}_i$ are functions of generalised co-ordinates defined by equation

$$\mathbf{r}_i = \mathbf{r}_i(q_1, q_2, \ldots, q_{3N}, t).$$

An arbitrary virtual displacement $\delta\mathbf{r}_i$ is written as

$$\delta\mathbf{r}_i = \sum_{j=1}^{3N} \frac{\partial \mathbf{r}_i}{\partial q_j} \delta q_j \quad (\text{as } \delta t = 0)$$

δq_j are called the *generalised displacements* or virtual arbitrary displacements. If q_j is an angle co-ordinate, δq_j is an angular displacement.

(2) Generalised Velocity: The generalised velocities of a system are the total time derivatives of the generalised co-ordinates of the system.

Thus $$\dot{q}_i = \frac{dq_i}{dt} \quad (i = 1, 2, 3, \ldots, k).$$

(3) Generalised Momentum: The *generalised momentum* p_k *associated with* a *co-ordinate* q_k is:

$$p_k = \frac{\partial T}{\partial \dot{q}_k},$$

Configuration Space

In the case of motion of a single particle we can represent its trajectory in the three dimen space by specifying variables. For a system of N particles described by $3N$ space coordinates (3N – k) equations of constraint in the real space, it is difficult to visualise the motion of the system. It is convenient to describe the motion of a system having k coordinates in a hypothetical k dimensional space. The instantaneous configuration of the system is described by the values of the k generalised coordinates $q_1, q_2, q_3 \ldots, q_k$, and corresponds to a particular point in a cartesian hyperspace where the q's form the k coordinate axes. The point is called the system point and the k dimensional space is known as the *Configuration space*.

At some later instant, the state of the system changes and it will be represented by some other point in the configuration space. Thus, the system point moves in the configuration space tracing out a curve. This curve represents "the path of motion of the system". "The motion of the system", as used above, then refers to the motion of the system point along this path in *configuration space*. Time can be considered formally as a parameter of the curve since each point in the configuration space has one or more values of time associated with it.

Configuration space has nothing in common with the three-dimensional space which we can visualise physically. It is a purely geometric structure by means of which the laws of the variation of the state of a system can be formulated in geometrical language.

Principle of Virtual Work

Consider a system described by n generalized coordinates q_j ($j = 1, 2, 3, \ldots, n$). Suppose the system undergoes a certain displacement in the configuration space in such a way that it does not take any time and that it is consistent with the constraints on the system. Such displacements are called *virtual* because they do not represent actual displacements of the system. Since there is no actual motion of the system, the work done by the forces of constraint in such a virtual displacement is zero.

Let the virtual displacement of the i^{th} particle of the given system be $\delta \mathbf{r}_i$. If the given system is in equilibrium, the resultant force acting on the i^{th} particle of the system must be zero, *i.e.*, $\mathbf{F}_i = 0$. It is, then, obvious that *virtual work* $\mathbf{F}_i \cdot \delta \mathbf{r}_i = 0$ for the i^{th} particle and hence it is also zero for all the particles of the system.

Thus
$$dW = \sum_i \mathbf{F}_i \cdot \delta \mathbf{r}_i = 0 \qquad \ldots(1)$$

The resultant force $\mathbf{F}_i$ acting on the i^{th} particle is,

$$\mathbf{F}_i = \mathbf{F}_i^a + f_i \qquad \ldots(2)$$

Here, $\mathbf{F}_i^a$ is the applied force and f_i is the force of constraint.

Eq. (1) then becomes

$$\sum_i \mathbf{F}_i^a \cdot \delta \mathbf{r}_i + \sum_i \mathbf{f}_i \cdot \delta \mathbf{r}_i = 0 \qquad \ldots(3)$$

We now consider systems for which the virtual work done by the forces of constraints is zero, *i.e.*,

$$\sum_i \mathbf{f}_i \cdot \delta \mathbf{r}_i = 0 \qquad \ldots(4)$$

Then Eq. (3) becomes

The Lagrangian Function

The Lagrangian function L is the difference between the kinetic energy (T) and potential energy (V) of a system. It is a function of generalised coordinates, generalised velocities and time.

$$L(q, \dot{q}, t) = T(q, \dot{q}, t) - V(q, t)$$

The potential energy and Lagrangian function of a conservative system do not depend explicitly on time.

$$\therefore \quad L(q, \dot{q}) = T(q, \dot{q}) - V(q) \text{ for a conservative system.}$$

42.5 DERIVATION OF LAGRANGE'S EQUATIONS OF MOTION

(a) Lagrange's Equations from D' Alembert's Principle

Consider a system of particles whose position vectors are expressed as functions of generalized coordinates $q_1, q_2, q_3, ..., q_k .... q_f$ and the time t.

Consider any particle of the system (ith particle) of mass m_i and acted upon by an external force $\mathbf{F}_i$.

According to D' Alembert's principle,

$$\sum_i (\mathbf{F}_i - \dot{\mathbf{p}}_i) \cdot \delta \mathbf{r}_i = 0 \quad ...(1)$$

Here $\dot{\mathbf{p}}_i$ is the inertial force for ith particle and $\delta \mathbf{r}_i$ is the virtual displacement of ith particle due to action of force $\mathbf{F}_i$.

In general,

$$\mathbf{r}_i = \mathbf{r}_i(q_1, q_2, q_3, \cdots q_k, \cdots q_f, t) \quad ...(2)$$

$$\dot{\mathbf{r}}_i = \frac{\delta \mathbf{r}_i}{\delta t} = \frac{\partial \mathbf{r}_i}{\partial q_1}\dot{q}_1 + \frac{\partial \mathbf{r}_i}{\partial q_2}\dot{q}_2 + ... + \frac{\partial \mathbf{r}_i}{\partial q_k}\dot{q}_k + ... + \frac{\partial \mathbf{r}_i}{\partial q_f}\dot{q}_f + \frac{\partial \mathbf{r}_i}{\partial t}$$

$$= \sum_k \frac{\partial \mathbf{r}_i}{\partial q_k}\dot{q}_k + \frac{\partial \mathbf{r}_i}{dt} \quad ...(3)$$

The virtual displacement $\delta \mathbf{r}_i$ in terms of generalised co-ordinates is given by

$$\delta \mathbf{r}_i = \sum_k \frac{\partial \mathbf{r}_i}{\partial q_k} \delta q_k. \qquad ...(4)$$

Now, $$\dot{\mathbf{p}}_i = m_i \ddot{\mathbf{r}}_i \qquad ...(5)$$

Therefore Eq. (1) becomes $\sum_i (\mathbf{F}_i - m_i \ddot{\mathbf{r}}_i) \cdot \delta \mathbf{r}_i = 0$

or $$\sum_i m_i \ddot{\mathbf{r}}_i \cdot \delta \mathbf{r}_i = \sum_i \mathbf{F}_i \cdot \delta \mathbf{r}_i \qquad ...(6)$$

Putting the value of $\delta \mathbf{r}_i$ from Eq. (4) in (6), we get

$$\sum_i \sum_k m_i \ddot{\mathbf{r}}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_k} \delta q_k = \sum_i \sum_k \mathbf{F}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_k} \delta q_k \qquad ...(7)$$

Now, $$\frac{d}{dt}\left(\dot{\mathbf{r}}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_k} \right) = \ddot{\mathbf{r}}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_k} + \dot{\mathbf{r}}_i \cdot \frac{d}{dt}\left(\frac{\partial \mathbf{r}_i}{\partial q_k} \right),$$

$\therefore$ $$\ddot{\mathbf{r}}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_k} = \frac{d}{dt}\left(\dot{\mathbf{r}}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_k} \right) - \dot{\mathbf{r}}_i \cdot \frac{d}{dt}\left(\frac{\partial \mathbf{r}_i}{\partial q_k} \right) \qquad ...(8)$$

Putting the value of $\ddot{\mathbf{r}}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_k}$ from above equation in (7), we get

$$\sum_i \sum_k m_i \left[\frac{d}{dt}\left(\dot{\mathbf{r}}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_k} \right) - \dot{\mathbf{r}}_i \cdot \frac{d}{dt}\left(\frac{\partial \mathbf{r}_i}{\partial q_k} \right) \right] \delta q_k = \sum_i \sum_k \mathbf{F}_i \cdot \left(\frac{\partial \mathbf{r}_i}{\partial q_k} \right) \delta q_k \qquad ...(9)$$

Differentiating Eq. (3) partially with respect to $\dot{q}_k$,

$$\frac{\partial \dot{\mathbf{r}}_i}{\partial \dot{q}_k} = \frac{\partial \mathbf{r}_i}{\partial q_k} \qquad ...(10)$$

Differentiating Eq. (3) partially with respect to q_k,

...(11)

Substituting (13) and (14) in Eq. (9),

$$\sum_i\sum_k m_i\left[\frac{d}{dt}\frac{\partial}{\partial \dot{q}_k}\left(\frac{1}{2}\dot{\mathbf{r}}_i^2\right)-\dot{\mathbf{r}}_i\cdot\frac{\partial \dot{\mathbf{r}}_i}{\partial q_k}\right]\delta q_k = \sum_i\sum_k \mathbf{F}_i\cdot\frac{\partial \mathbf{r}_i}{\partial q_k}\delta q_k$$

or $$\sum_i\sum_k\left[\frac{d}{dt}\frac{\partial}{\partial \dot{q}_k}\left(\frac{1}{2}m_i\dot{\mathbf{r}}_i^2\right)-\frac{\partial}{\partial q_k}\left(\frac{1}{2}m_i\dot{\mathbf{r}}_i^2\right)\right]\delta q_k = \sum_i\sum_k \mathbf{F}_i\cdot\frac{\partial \mathbf{r}_i}{\partial q_k}\delta q_k$$

or $$\sum_k\left[\frac{d}{dt}\frac{\partial}{\partial \dot{q}_k}\left(\sum\frac{1}{2}m_i\dot{r}_i^2\right)-\frac{\partial}{\partial q_k}\left(\sum\frac{1}{2}m_i\dot{r}_i^2\right)\right]\delta q_k = \sum_i\sum_k \mathbf{F}_i\cdot\frac{\partial r_i}{\partial q_k}\delta q_k \quad ...(15)$$

$$\sum\frac{1}{2}m_i\dot{r}_i^2 = T = \text{Total kinetic energy of the system of particles} \quad ...(16)$$

and $$\sum_i \mathbf{F}_i\cdot\frac{\partial \mathbf{r}_i}{\partial q_k} = \mathbf{Q}_k, \quad ...(17)$$

Here $\mathbf{Q}_k$'s are components of *generalised force.*

Eq. (15) becomes, $$\sum_k\left\{\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_k}-\frac{\partial T}{\partial q_k}\right\}\delta q_k = \sum \mathbf{Q}_k\,\delta q_k \quad ...(18)$$

$$\therefore \quad \frac{d}{dt}\frac{\partial T}{\partial \dot{q}_k}-\frac{\partial T}{\partial q_k} = Q_k \quad ...(19)$$

This is the general form of *Lagrange's equation.* There are f such equations corresponding to f generalised co-ordinates.

When the system is wholly conservative,

$$\mathbf{F}_i = -\nabla V_i = -\frac{\partial V_i}{\partial \mathbf{r}_i}, \quad ...(20)$$

$$\mathbf{Q}_k = \sum_i \mathbf{F}_i\cdot\frac{\partial \mathbf{r}_i}{\partial q_k} = -\sum\frac{\partial V_i}{\partial \mathbf{r}_i}\cdot\frac{\partial \mathbf{r}_i}{\partial q_k} = -\sum_i\frac{\partial V_i}{\partial q_k} = -\frac{\partial}{\partial q_k}\left(\sum V_i\right) = -\frac{\partial V}{\partial q_k}$$

Here $V = \sum_i V_i$ = total potential energy of the system.

Putting this value of Q_k in Eq. (19), we get

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right)-\frac{\partial T}{\partial q_k} = -\frac{\partial V}{\partial q_k}$$

or $$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right)-\frac{\partial T}{\partial q_k}+\frac{\partial V}{\partial q_k} = 0$$

or $$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right)-\frac{\partial}{\partial q_k}(T-V) = 0 \quad ...(21)$$

The potential energy V is the function of position co-ordinates q_k and not of the *generalised velocities* $\dot{q}_k$. Therefore, Eq. (21) may be written as

$$\frac{d}{dt}\frac{\partial}{\partial \dot{q}_k}(T-V)-\frac{\partial}{\partial q_k}(T-V) = 0 \quad ...(22)$$

But $L = T - V$, where L is known as *Lagrangian function.*

Eq. (22) becomes, $$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right)-\frac{\partial L}{\partial q_k} = 0 \quad ...(23)$$

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This is Lagrange's equation for a *conservative system.*

(b) Lagrange's equations for systems containing dissipative forces.

Consider a system of particles containing dissipative or frictional forces.

The frictional force is proportional to the velocity of the particle, *i.e.,*

$$\mathbf{F}_i^{(d)} = -\lambda_i \dot{\mathbf{r}}_i,$$

Here $\mathbf{F}_i^{(d)}$ denotes the dissipative force, $\dot{\mathbf{r}}_i$ is the velocity of i^{th} particle and λ_i is the corresp... constant of proportionality.

Forces of this type are derivable from *Rayleigh's dissipation function R* defined by

$$R = \frac{1}{2}\sum_i \lambda_i \dot{\mathbf{r}}_i^2 \quad ...(2)$$

Here $i = 1, 2, ... n$ covers all the particles of the system.

$$\frac{\partial R}{\partial \dot{\mathbf{r}}_i} = \lambda_i \dot{\mathbf{r}}_i = -\mathbf{F}_i^{(d)} \quad \text{from Eq. (1)}$$

or

$$\mathbf{F}_i^{(d)} = -\frac{\partial R}{\partial \dot{\mathbf{r}}_i} \quad ...(3)$$

Here term $\frac{\partial R}{\partial \dot{\mathbf{r}}_i}$ represents the dissipative force.

The Lagrange's equation in terms of **r** is given by

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \mathbf{r}}\right) - \frac{\partial L}{\partial \dot{\mathbf{r}}} = \mathbf{F}_i^{(d)} \quad ...(4)$$

Lagrange's equation in generalised coordinates q_k is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} = \mathbf{Q}_k^{(d)}$$

where $\mathbf{Q}_k^{d}$ is component of generalised force.

In order to find the Lagrange's equation in generalised co-ordinates, we have to find the components of generalised force resulting from the dissipative force.

If $\mathbf{Q}_k^{(d)}$ is the component of generalised force along q_k, then

$$\mathbf{Q}_k^{(d)} = \sum_i \mathbf{F}_i^{(d)} \frac{\partial \mathbf{r}_i}{\partial q_k}$$

$$= -\sum \lambda_i \dot{\mathbf{r}}_i \frac{\partial \mathbf{r}_i}{\partial q_k} \quad \text{[from Eq. (1)]}$$

$$= -\sum \lambda_i \dot{\mathbf{r}}_i \frac{\partial \dot{\mathbf{r}}_i}{\partial \dot{q}_k} \quad \left[\because \frac{\partial \mathbf{r}_i}{\partial q_k} = \frac{\partial \dot{\mathbf{r}}_i}{\partial \dot{q}_k}\right]$$

or

$$\mathbf{Q}_k^{(d)} = -\sum_i \lambda_i \frac{\partial}{\partial \dot{q}_k}\left(\frac{1}{2}\dot{\mathbf{r}}_i^2\right)$$

$$= \frac{\partial}{\partial \dot{q}_k}\sum_i\left(-\frac{1}{2}\lambda_i \dot{\mathbf{r}}_i^2\right)$$

$\therefore$

$$\mathbf{Q}_k^{(d)} = -\frac{\partial R}{\partial \dot{q}_k} \quad ...(5)$$

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...containing dissipative force is given by

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} + \frac{\partial R}{\partial \dot{q}_k} = 0 \qquad ...(6)$$

Hence the term $\frac{\partial R}{\partial \dot{q}_k}$ takes into account the dissipative forces.

Thus, if dissipative forces are acting on the system, we must specify two scalar functions—the Lagrangian L and Rayleigh's dissipation function R- to derive the equations of motion.

42.6 APPLICATIONS OF LAGRANGE'S EQUATIONS

In order to use Lagrange's equations for the solution of a physical problem, one must use the following steps:

(i) Choose an appropriate coordinate system.

(ii) Write down the expressions for potential and kinetic energies.

(iii) Write down the equations of constraint, if any.

(iv) Choose the generalized coordinates.

(v) Set up the Lagrangian. $L = T - V$

(vi) Solve Lagrange's equations for each generalized coordinate using, if necessary, the equations of constraint.

(a) The Atwood's machine: (5M)

Let two small heavy particles of masses M_1 and M_2 be connected by a light inextensible rope of length l passing over a frictionless light pulley (Fig. 42.8).

It is found that the heavier particle descends while the lighter ascends, the system moving with a constant acceleration $\ddot{x}$.

The Atwood's machine is a conservative system with a holonomic constraint. There is only one independent coordinate x. The position of the other particle is determined by the constraint that the length of the rope between them is l.

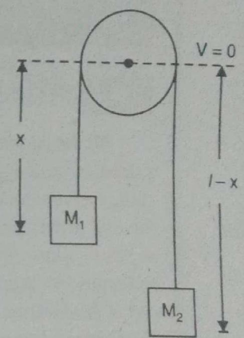

Fig. 42.8

The P.E. of the system $= V = -M_1gx - M_2g(l - x)$

The K.E. of the system $= T = \frac{1}{2}(M_1 + M_2)\dot{x}^2$

Hence, the Lagrangian function is given by

$$L = T - V = \frac{1}{2}(M_1 + M_2)\dot{x}^2 + M_1gx + M_2g(l - x)$$

The Lagrange's equation for a conservative system is

$$\frac{d}{dt}\left[\frac{\partial L}{\partial \dot{q}_i}\right] - \frac{\partial L}{\partial q_i} = 0$$

Since the system has only one degree of freedom, there is only one equation of motion, involving the derivatives.

$\therefore$ The equation of motion of the system is given by

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0$$

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From the expression for L we get, $\frac{\partial L}{\partial x} = (M_1 - M_2)g$;

$$\frac{\partial L}{\partial \dot{x}} = (M_1 + M_2)\dot{x};\ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = \frac{d}{dt}[(M_1 + M_2)\dot{x}] = (M_1 + M_2)\ddot{x}.$$

$\therefore$ We have, $(M_1 + M_2)\ddot{x} - (M_1 - M_2)g = 0$

or
$$\ddot{x} = \frac{M_1 - M_2}{M_1 + M_2} g.$$

(b) Simple pendulum: *Let the mass of the pendulum bob be m. The angle θ between rest position (OA) and deflected position (OB) is chosen as the generalized coordinate (Fig. 42.9). Let l be the length of the string. Then*

5M

$$\text{K.E. of the bob} = \frac{1}{2}mv^2$$

$$T = \frac{1}{2}ml^2\dot{\theta}^2$$

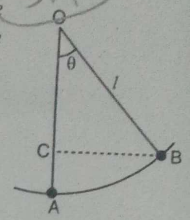

Fig. 42.9

P.E. of the bob is

$V = mg\ CA = mg\ (OA - OC) = mg\ (l - l\cos\theta) = mgl\ (1 - \cos\theta)$

The Lagrangian function L is given by

$$L = T - V = \frac{1}{2}ml^2\dot{\theta}^2 - mgl\ (1 - \cos\theta) \quad ...(1)$$

The Lagrangian equation in the generalized coordinate θ is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0 \quad ...(2)$$

From Eq. (1), $\frac{\partial L}{\partial \dot{\theta}} = ml^2\dot{\theta}$ and $\frac{\partial L}{\partial \theta} = -mgl\sin\theta$

Substituting these values in Eq. (2), we get

$$\frac{d}{dt}(ml^2\dot{\theta}) + mgl\sin\theta = 0$$

or
$$ml^2\ddot{\theta} + mgl\sin\theta = 0$$

or
$$\ddot{\theta} + \frac{g}{l}\sin\theta = 0$$

If the amplitude of motion is small, $\sin\theta \approx \theta$. Then

$$\ddot{\theta} + \frac{g}{l}\theta = 0 \quad ...(3)$$

Eq. (3) represents a S.H.M. with period $T = 2\pi\sqrt{l/g}$

(c) Compound Pendulum

A rigid body capable of oscillating in a vertical plane about a fixed horizontal axis is called a compound pendulum. Let vertical plane of oscillation be xy, O the point through which axis of rotation passes, C the centre of mass. Let mass of pendulum be m, moment of inertia about axis of rotation I and distance $OC = h$ (Fig. 42.10). Let θ be the angle through which the body is displaced.

K.E. of the system is

$$T = \frac{1}{2}I\dot{\theta}^2$$

Consider the horizontal plane passing through O as reference level.

P.E. of the system is

$$V = -mg\,(OA)$$
$$= -mgh\cos\theta$$

The Lagrangian L is written as

$$L = T - V = \frac{1}{2}I\dot{\theta}^2 + mgh\cos\theta$$

$$\therefore \quad \frac{\partial L}{\partial \dot{\theta}} = I\dot{\theta}; \quad \frac{\partial L}{\partial \theta} = -mgh\sin\theta$$

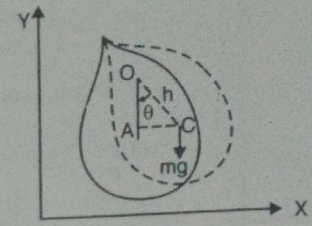

Fig. 42.10

The Lagrangian equation in the generalized coordinate θ is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0$$

or $$\frac{d}{dt}(I\dot{\theta}) + mgh\sin\theta = 0$$

or $$\ddot{\theta} + \frac{mgh}{I}\sin\theta = 0$$

If amplitude of oscillation is small, $\sin\theta \approx \theta$. Then

$$\ddot{\theta} + \frac{mgh}{I}\theta = 0$$

This is an equation for simple harmonic motion of time period

$$T = 2\pi\sqrt{\frac{I}{mgh}}$$

(*d*) Linear Harmonic Oscillator

The traditional ideal Harmonic oscillator is shown in Fig. 42.11. The displacement of the mass from its equilibrium position is x.

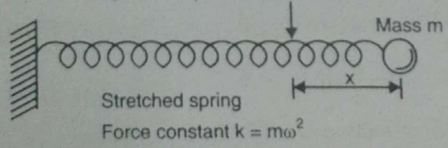

Fig. 42.11

Lagrange's equation of motion for one dimensional motion, say in x direction, is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0. \quad ...(1)$$

The kinetic energy of this system is

$$T = \frac{1}{2}m\dot{x}^2$$

Potential energy is

$$V = -\int \mathbf{F}.dx$$
$$= -\int -kx\,dx = \frac{1}{2}kx^2 + c$$

Here, c is a constant of integration and k is spring constant.

If we choose the horizontal plane passing through the position of equilibrium as the level, then $V = 0$ at $x = 0$.

$\therefore$ $c = 0$

The Lagrangian of the system is

$$L = T - V = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2$$

$$\frac{\partial L}{\partial \dot{x}} = m\dot{x} \quad \text{and} \quad \frac{\partial L}{\partial x} = -kx.$$

Eq. (1) takes the form

$$\frac{d}{dt}(m\dot{x}) - (-kx) = 0$$

or $$m\ddot{x} + kx = 0 \qquad ...(3)$$

This relation shows that the motion is simple harmonic.

$\therefore$ The frequency $$\nu = \frac{1}{2\pi}\sqrt{\frac{k}{m}} \qquad ...(4)$$

HAMILTONIAN FORMULATION OF MECHANICS

42.7 PHASE SPACE

While using the Lagrangian formulation, there are $3n$ equations of motion of the 2nd order for a system containing n particles in the absence of holonomic constraints. For general consideration it is more convenient to write $6n$ partial differential equations of the 1st order in place of $3n$ equations of the 2nd order. For the purpose there must be $6n$ degrees of freedom or $6n$ dimensional space known as *phase space.* In phase space, momenta are also regarded as independent variables like space co-ordinates. A single particle in phase space is specified by six co-ordinates, 3 position co-ordinates and 3 momentum co-ordinates. This six dimensional space is sometimes called the μ space. *The phase space is a superposition of* μ *spaces.*

In brief *phase space is a 6n dimensional space formed of co-ordinates* $q_1, q_2, ..., q_k, ..., q_{3n}, p_1, p_2, ..., p_k, ..., p_{3n}$. A single point in this phase space will fix all the position coordinates and momenta. The point thus describes the state of motion of the system besides giving its position.

42.8 THE HAMILTONIAN FUNCTION H

The Lagrangian of a system is a function of

$$q_1, q_2, \cdots q_k, \cdots \dot{q}_1, \dot{q}_2 \cdots, \dot{q}_k, \cdots \text{ and } t.$$

$$L = L(q_k, \dot{q}_k, t)$$

$$\frac{dL}{dt} = \sum_k \frac{\partial L}{\partial q_k}\dot{q}_k + \sum_k \frac{\partial L}{\partial \dot{q}_k}\ddot{q}_k + \frac{\partial L}{\partial t} \qquad ...(1)$$

The Lagrangian equation is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} = 0$$

or $$\frac{\partial L}{\partial q_k} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) \qquad ...(2)$$

Substituting this value of $\frac{\partial L}{\partial q_k}$ in Eq. (1),

$$\frac{dL}{dt} = \sum_k \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right)\dot{q}_k + \sum_k \frac{\partial L}{\partial \dot{q}_k}\ddot{q}_k + \frac{\partial L}{\partial t}$$

$$= \sum_k \frac{d}{dt}\left\{\frac{\partial L}{\partial \dot{q}_k}\dot{q}_k\right\} + \frac{\partial L}{\partial t}$$

or
$$\frac{d}{dt}\left\{L - \sum_k \frac{\partial L}{\partial \dot{q}_k}\dot{q}_k\right\} = \frac{\partial L}{\partial t} \qquad ...(3)$$

But $\frac{\partial L}{\partial \dot{q}_k} = p_k$ generalised momentum.

Eq. (3) becomes, $\frac{d}{dt}\left\{L - \sum_k p_k \dot{q}_k\right\} = \frac{\partial L}{\partial t}$

or
$$\frac{d}{dt}\left\{\sum_k p_k \dot{q}_k - L\right\} = -\frac{\partial L}{\partial t} \qquad ...(4)$$

Hamiltonian function $H = \sum p_k \dot{q}_k - L(q, \dot{q}, t)$ ..(5)

The Hamiltonian H is related to the Lagrangian L through Eq. (5).

42.9 HAMILTON'S CANONICAL EQUATIONS OF MOTION

5M

The Hamiltonian function H is function of p, q and t.

$$H = H(p, q, t)$$

$$= H(p_1, p_2, ..., p_k, ..., q_1, q_2, ..., q_k, ..., t)$$

$$dH = \sum_k \frac{\partial H}{\partial p_k} dp_k + \sum_k \frac{\partial H}{\partial q_k} dq_k + \frac{\partial H}{\partial t} dt \qquad ...(1)$$

Now,
$$H = \sum p_k \dot{q}_k - L(q, \dot{q}, t).$$

∴
$$dH = \sum_k \dot{q}_k\, dp_k + \sum_k p_k\, d\dot{q}_k - dL, \qquad ...(2)$$

Now,
$$L = L(q_1, q_2, ..., q_k, ..., \dot{q}_1, \dot{q}_2, ..., \dot{q}_k, ..., t)$$

$$dL = \sum_k \frac{\partial L}{\partial q_k} dq_k + \sum_k \frac{\partial L}{\partial \dot{q}_k} d\dot{q}_k + \frac{\partial L}{\partial t} dt$$

$$= \sum_k \frac{\partial L}{\partial q_k} dq_k + \sum_k p_k\, d\dot{q}_k + \frac{\partial L}{\partial t} dt \qquad ...(3)$$

Substituting this value of dL in Eq. (2),

$$dH = \sum_k \dot{q}_k\, dp_k + \sum_k p_k\, d\dot{q}_k - \sum_k \frac{\partial L}{\partial q_k} dq_k - \sum p_k\, d\dot{q}_k - \frac{\partial L}{\partial t} dt$$

$$= \sum_k \dot{q}_k\, dp_k - \sum_k \dot{p}_k\, dq_k - \frac{\partial L}{\partial t} dt \qquad ...(4)$$

Comparing coefficients of dp_k and dq_k in Eqs. (1) and (4),

$$\left.\begin{aligned}\dot{q}_k &= \frac{\partial H}{\partial p_k},\\ -\dot{p}_k &= \frac{\partial H}{\partial q_k},\end{aligned}\right\} \quad ...(5)$$

Eqs. (5) are called *Hamilton's equations or Hamilton's canonical equations of motion.* p's and q's are called canonical variables.

Also, $$-\frac{\partial L}{\partial t} = \frac{\partial H}{\partial t} \quad ...(6)$$

42.10 PHYSICAL SIGNIFICANCE OF THE HAMILTONIAN FUNCTION

$$H = H(p_1, p_2, ..., p_k, ..., q_1, q_2, ..., q_k, ..., t)$$

The total time derivative of H is

$$\frac{dH}{dt} = \sum_k \frac{\partial H}{\partial q_k}\dot{q}_k + \sum_k \frac{\partial H}{\partial p_k}\dot{p}_k + \frac{\partial H}{\partial t}$$

$$= -\sum_k \dot{p}_k \dot{q}_k + \sum_k \dot{q}_k \dot{p}_k + \frac{\partial H}{\partial t}$$

$$= \frac{\partial H}{\partial t}. \quad ...(1)$$

But $$\frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t} \quad ...(2)$$

$\therefore$ $$\frac{dH}{dt} = -\frac{\partial L}{\partial t} \quad ...(3)$$

If L is not an explicit function of time, $\frac{\partial L}{\partial t} = 0$. Hence

$$\frac{dH}{dt} = 0.$$

or $$H = \text{constant}. \quad ...(4)$$

Therefore H is a constant of the system.

For conservative systems the potential energy V does not depend upon generalised velocity, *i.e.*,

$$\frac{\partial V}{\partial \dot{q}_k} = 0 \quad ...(5)$$

Also we know

$$H = \sum p_k \dot{q}_k - L$$

$$= \sum_k \frac{\partial L}{\partial \dot{q}_k}\dot{q}_k - L$$

$$= \sum_k \dot{q}_k \left\{\frac{\partial}{\partial \dot{q}_k}(T - V)\right\} - L$$

$$= \sum_k \dot{q}_k \left\{\frac{\partial T}{\partial \dot{q}_k} - \frac{\partial V}{\partial \dot{q}_k}\right\} - L$$

$$= \sum_k \dot{q}_k \frac{\partial T}{\partial \dot{q}_k} - L \quad \text{from Eq. (5)}$$

$$= \sum \dot{q}_k \frac{\partial}{\partial \dot{q}_k}\left(\sum_i \frac{1}{2} m_i \dot{r}_i^2\right) - L$$
$$= \sum_k m\dot{q}_k^2 - L$$
$$= 2T - L$$
$$= 2T - (T - V)$$
$$= T + V$$
$$= \text{K.E.} + \text{P.E.}$$
$$= E = \text{total energy of the system.}$$

Thus for conservative systems where the co-ordinate transformation is independent of time, the Hamiltonian function H represents the total energy of the system.

42.11 HAMILTON'S VARIATIONAL PRINCIPLE

10M $\frac{\partial L}{\partial t} - \frac{\partial H}{\partial t}$

Statement: *The path actually traversed by a conservative, holonomic dynamical system from time t_1 to t_2 is one over which the integral of the Lagrangian between limits t_1 and t_2 is stationary i.e., the time integral of the Lagrangian is extremum.*

Explanation: The motion of the system from time t_1 to time t_2 is such that the line integral

$$I = \int_{t_1}^{t_2} L\, dt \quad ...(1)$$

where $L = T - V$, is an extremum for the path of motion.

or
$$\delta \int_{t_1}^{t_2} L\, dt = 0 \quad ...(2)$$

This principle helps us to distinguish the actual path from the neighbouring paths.

42.11.1 Derivation of Hamilton's Canonical Equations of Motion from Hamilton's Variational Principle

Hamilton's principle is

$$\delta I = \delta \int_{t_1}^{t_2} L\, dt = 0$$

The relation between Lagrangian and Hamiltonian is

$$L = \sum p_i \dot{q}_i - H(q, p, t)$$

$$\therefore \quad \delta \int_{t_1}^{t_2} \left[\sum_i p_i \dot{q}_i - H(q, p, t)\right] dt = 0$$

or
$$\delta \int_{t_1}^{t_2} \left[\sum_i p_i \frac{dq_i}{dt} - H(q_i, p_i, t)\right] dt = 0$$

or
$$\delta \sum_i \int_{q_1}^{q_2} p_i\, dq_i - \delta \int_{t_1}^{t_2} H\, dt = 0. \quad ...(1)$$

Eq. (1) is called the *modified Hamilton's principle*.

Now labelling each of the possible paths in the configuration space with a parameter α, the δ-variation can be expressed as

$$\delta = d\alpha\, (\partial/\partial\alpha).$$

Now $\delta I = \frac{\partial I}{\partial \alpha} d\alpha = d\alpha \frac{\partial}{\partial \alpha} \int_{t_1}^{t_2} \left[\sum p_i \dot{q}_i - H(q, p, t)\right] dt = 0$

The times t_1, t_2 are not varied and so they are not functions of α. Thus, the diff... integration may be interchanged.

$$d\alpha \int_{t_1}^{t_2} \frac{\partial}{\partial \alpha}\left[\sum_i p_i \dot{q}_i - H(q, p, t)\right] dt = 0$$

or $$d\alpha \int_{t_1}^{t_2} \sum_i \left[\frac{\partial p_i}{\partial \alpha}\dot{q}_i + \frac{\partial \dot{q}_i}{\partial \alpha} p_i - \frac{\partial H}{\partial q_i}\frac{\partial q_i}{\partial \alpha} - \frac{\partial H}{\partial p_i}\frac{\partial p_i}{\partial \alpha}\right] dt = 0$$

Further, we have

$$\int_{t_1}^{t_2} \frac{\partial \dot{q}_i}{\partial \alpha} p_i \, dt = \int_{t_1}^{t_2} p_i \frac{d}{dt}\left(\frac{\partial q_i}{\partial \alpha}\right) dt = p_i \left[\frac{\partial q_i}{\partial \alpha}\right]_{t_1}^{t_2} - \int_{t_1}^{t_2} \dot{p}_i \frac{\partial q_i}{\partial \alpha} dt.$$

But all the varied paths have the same end points. Hence $\frac{\partial q_i}{\partial \alpha}$ vanishes for t_1 and t_2.

$$\therefore \quad \int_{t_1}^{t_2} \frac{\partial \dot{q}_i}{\partial \alpha} p_i \, dt = -\int_{t_1}^{t_2} \dot{p}_i \frac{\partial q_i}{\partial \alpha} dt.$$

Therefore Eq. (2) becomes,

$$d\alpha \int_{t_1}^{t_2} \sum \left[\frac{\partial p_i}{\partial \alpha}\dot{q}_i - \frac{\partial H}{\partial q_i}\frac{\partial q_i}{\partial \alpha} - \frac{\partial H}{\partial p_i}\frac{\partial p_i}{\partial \alpha} - \dot{p}_i \frac{\partial q_i}{\partial \alpha}\right] dt = 0$$

$$\int_{t_1}^{t_2} \sum_i \left[\dot{q}_i \frac{\partial p_i}{\partial \alpha} d\alpha - \frac{\partial H}{\partial q_i}\frac{\partial q_i}{\partial \alpha} d\alpha - \frac{\partial H}{\partial p_i}\frac{\partial p_i}{\partial \alpha} d\alpha - \dot{p}_i \frac{\partial q_i}{\partial \alpha} d\alpha\right] dt = 0 \qquad ...(3)$$

But $\quad \delta p_i = d\alpha \cdot (\partial p_i/\partial \alpha)$ and $\delta q_i = (\partial q_i/\partial \alpha) \cdot d\alpha$

$$\therefore \quad \int_{t_1}^{t_2} \sum_i \left[\delta p_i \left(\dot{q}_i - \frac{\partial H}{\partial p_i}\right) + \delta q_i \left(-\frac{\partial H}{\partial q_i} - \dot{p}_i\right)\right] dt = 0 \qquad ...(4)$$

The variations δq_i and δp_i are independent of each other. Hence Eq. (4) holds good only when the coefficients of δp_i and δq_i vanish separately.

$$\therefore \quad \dot{q}_i = \frac{\partial H}{\partial p_i}; \qquad \dot{p}_i = -\frac{\partial H}{\partial q_i}$$

These are Hamilton's canonical equations of motion.

42.12 APPLICATIONS OF HAMILTON'S EQUATIONS OF MOTION

1. Linear Harmonic Oscillator

The system is conservative and constraint is independent of time. So Hamiltonian will represent the total energy of the system.

The kinetic energy of harmonic oscillator,

$$T = \frac{1}{2} m\dot{x}^2$$

The potential energy of harmonic oscillator,

$$V = -\int F\, dx$$

$$= \int kx\, dx = \frac{1}{2} kx^2$$

$\therefore$ **The Lagrangian of the system**

$$L = T - V = \frac{1}{2} m\dot{x}^2 - \frac{1}{2} kx^2$$

$$\frac{\partial L}{\partial \dot{x}} = \frac{\partial T}{\partial \dot{x}} = p = m\dot{x}$$

or

$$\dot{x} = p/m$$

The *Hamiltonian H* in terms of momenta is

$$H = T + V = \frac{1}{2} m(p/m)^2 + \frac{1}{2} kx^2 = \frac{p^2}{2m} + \frac{1}{2} kx^2$$

Equations of motion are

$$\dot{p} = -\frac{\partial H}{\partial x} = -kx \quad ...(1)$$

and

$$\dot{x} = \frac{\partial H}{\partial p} = p/m \quad ...(2)$$

From Eq. (2) $\ddot{x} = \dot{p}/m = -kx/m$

$$\therefore \quad m\ddot{x} + kx = 0 \quad ...(3)$$

This relation shows that the motion is simple harmonic.

$$\therefore \quad \text{The frequency } \nu = \frac{1}{2\pi}\sqrt{\frac{k}{m}} \quad ...(4)$$

2. Simple Pendulum

Let the mass of the pendulum bob be m. The angle θ between rest position (OA) and deflected position (OB) is chosen as the generalized coordinate (Fig. 42.12). Let l be the length of the string.

$$\text{K.E. of the bob} = \frac{1}{2} mv^2$$

$$T = \frac{1}{2} ml^2 \dot{\theta}^2$$

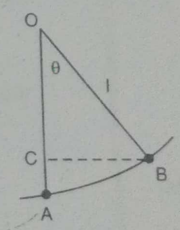

Fig. 42.12

P.E. of the bob is

$$V = mgCA = mg(OA - OC)$$

$$= mg(l - l\cos\theta) = mgl(1 - \cos\theta)$$

The Lagrangian function L is given by

$$L = T - V = \frac{1}{2} ml^2 \dot{\theta}^2 - mgl(1 - \cos\theta) \quad ...(1)$$

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = ml^2 \dot{\theta} \quad ...(2)$$

The Hamiltonian is

$$H = \sum p_j \dot{q}_j - L$$

$$= p_\theta \dot{\theta} - L$$

$$= ml^2 \dot{\theta}^2 - \left\{\frac{1}{2} ml^2 \dot{\theta}^2 - mgl(1 - \cos\theta)\right\}$$

$$= \frac{1}{2} ml^2 \dot{\theta}^2 + mgl(1 - \cos\theta) \quad ...(3)$$

$$= T + V,$$

Now putting Eq. (2) into Eq. (3), we get

$$H = \frac{1}{2} ml^2 \left(\frac{p_\theta}{ml^2}\right)^2 + mgl\,(1 - \cos\theta),$$

$$\left.\begin{aligned} \frac{\partial H}{\partial p_\theta} &= \frac{p_\theta}{ml^2} \\ \frac{\partial H}{\partial \theta} &= mgl \sin\theta. \end{aligned}\right] \qquad ...(4)$$

Hamilton's equations of motion for $\dot{\theta}$ and $\dot{p}_\theta$ are

$$\dot{\theta} = \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{ml^2} \qquad ...(5)$$

$$\dot{p}_\theta = -\frac{\partial H}{\partial \theta} = -mgl \sin\theta. \qquad ...(6)$$

Eqs. (5) and (6) represent Hamilton's equations for a simple pendulum.

From Eq. (5), we have $p_\theta = ml^2 \dot{\theta}$

or

$$\dot{p}_\theta = ml^2 \ddot{\theta} \qquad ...(7)$$

Putting the value of p_θ from Eq. (7) into Eq. (6), we get

$$ml^2 \ddot{\theta} = -mgl \sin\theta$$

or

$$\ddot{\theta} + \frac{g}{l} \sin\theta = 0$$

If the amplitude of motion is small, $\sin\theta \approx \theta$. Then

$$\ddot{\theta} + \frac{g}{l} \theta = 0 \qquad ...(8)$$

This gives the equation of motion of the simple pendulum.

Eq. (8) represents a S.H.M. with period

$$T = 2\pi \sqrt{l/g} \qquad ...(9)$$

3. Compound Pendulum

A rigid body capable of oscillating in a vertical plane about a fixed horizontal axis is called a compound pendulum. Let vertical plane of oscillation be xy, O the point through which axis of rotation passes, C the centre of mass. Let mass of pendulum be m, moment of inertia about axis of rotation I and distance $OC = h$ (Fig. 42.13). Let θ be the angle through which the body is displaced.

K.E. of the system is

$$T = \frac{1}{2} I \dot{\theta}^2$$

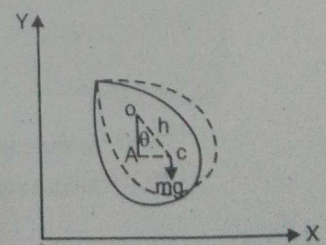

Fig. 42.13

Consider the horizontal plane passing through O as reference level.

P.E. of the system is

$$V = -mg\,(OA) = -mgh \cos\theta$$

The Lagrangian L is written as

$$L = T - V = \frac{1}{2} I\dot{\theta}^2 + mgh\cos\theta$$

Then $$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = I\dot{\theta} \qquad \text{...(1)}$$

The hamiltonian is

$$H = \sum p_j \dot{q}_j - L = p_\theta \dot{\theta} - L$$

$$= I\dot{\theta}\cdot\dot{\theta} - \frac{1}{2} I\dot{\theta}^2 - mgh\cos\theta$$

$$= \frac{1}{2} I\dot{\theta}^2 - mgh\cos\theta$$

$$= \frac{1}{2} I\left(\frac{p_\theta}{I}\right)^2 - mgh\cos\theta$$

$$= \frac{p_\theta^2}{2I} - mgh\cos\theta \qquad \text{...(2)}$$

$$= T + V$$

which implies that system is conservative.

Hamilton's equations of motion for θ and p_θ are

$$\left.\begin{aligned} \dot{\theta} &= \frac{\partial H}{\partial p_\theta} \\ \dot{p}_\theta &= -\frac{\partial H}{\partial \theta} \end{aligned}\right] \qquad \text{...(3)}$$

From Eq. (2), we find

$$\frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{I}, \quad \frac{\partial H}{\partial \theta} = mgh\sin\theta.$$

Putting these in Eq. (3), we get

$$\dot{\theta} = \frac{p_\theta}{I} \quad \text{or} \quad \dot{p}_\theta = I\ddot{\theta} \qquad \text{...(4)}$$

and $$\dot{p}_\theta = -mgh\sin\theta. \qquad \text{...(5)}$$

From Eqs. (4) and (5), we get

$$I\ddot{\theta} = -mgh\sin\theta$$

or $$\ddot{\theta} + \frac{mgh}{I}\sin\theta = 0$$

If amplitude of oscillation is small, $\sin\theta \approx \theta$. Then

$$\ddot{\theta} + \frac{mgh}{I}\theta = 0 \qquad \text{...(6)}$$

This gives the equation of motion of compound pendulum.

This is an equation for simple harmonic motion of time period

$$T = 2\pi\sqrt{\frac{I}{mgh}} \qquad \text{...(7)}$$

Angular frequency, $$\omega = \sqrt{mgh/I} \qquad \text{...(8)}$$

4. Motion of a Particle in a Central Force Field

Let P be the point mass m moving in the xy plane (Fig. 42.14).

The point P is acted upon by a central force which, is directed towards O, along the line ... (r, θ) be the polar co-ordinates of the particle. Let $V(r)$ be the potential energy due to central ...

$$\text{Lagrangian } L = T - V(r)$$
$$= \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - V(r)$$

Momenta are given as

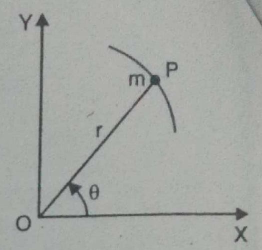

Fig. 42.14

$$\left.\begin{aligned} p_r &= \frac{\partial H}{\partial \dot{r}} = m\dot{r} \\ p_\theta &= \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta} \end{aligned}\right] \quad ...(2)$$

$$H = \sum p_k \dot{q}_k - L$$
$$= p_r \dot{r} + p_\theta \dot{\theta} - L$$
$$= (m\dot{r})\dot{r} + (mr^2\dot{\theta})\dot{\theta} - L$$
$$= m\dot{r}^2 + mr^2\dot{\theta}^2 - \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + V(r) \quad \text{from (1)}$$
$$= \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + V(r) = T + V(r) \quad ...(3)$$
$$= \text{total energy.}$$

From (2) we have

$$\dot{r} = \frac{p_r}{m}, \quad \dot{\theta} = \frac{p_\theta}{mr^2}$$

With these substitutions the Hamiltonian takes the form

$$H = \frac{1}{2}m\left\{\left(\frac{p_r}{m}\right)^2 + r^2\left(\frac{p_\theta}{mr^2}\right)^2\right\} + V(r)$$
$$= \frac{1}{2m}\left(p_r^2 + \frac{p_\theta^2}{r^2}\right) + V(r) \quad ...(4)$$

The Hamiltonian equations, corresponding to generalised co-ordinates q_k, are

$$\dot{q}_k = \frac{\partial H}{\partial p_k},$$
$$-\dot{p}_k = \frac{\partial H}{\partial q_k}$$

In this case we have two co-ordinates r and θ.

Therefore there will be four Hamilton's equations.

The two equations for $\dot{q}_k$ are:

$$\left.\begin{aligned} \dot{r} &= \frac{\partial H}{\partial p_r} = \frac{p_r}{m}, \\ \dot{\theta} &= \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{mr^2} \end{aligned}\right\} \quad ...(5)$$

The two equations for p_k are:

$$\dot{p}_r = -\frac{\partial H}{\partial r} = \frac{p_\theta^2}{mr^3} - \frac{\partial V(r)}{\partial r}$$

$$\dot{p}_\theta = -\frac{\partial H}{\partial \theta} = 0$$

The last equation shows that the time rate of change of angular momentum is zero, *i.e.*, angular momentum is conserved in planetary motion.

EXERCISE

1. Derive Lagrange's equations of motion from *D'* Alembert's principle. *(MKU, 1999)*
2. Derive Lagrange's equations of motion. Apply it in the case of (*i*) Atwood's machine (*ii*) Simple Pendulum (*iii*) Compound Pendulum (*iv*) Linear harmonic oscillator.
3. Derive Hamilton's canonical equations...