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Class : I M.sc Mathematics

Subject : Linear Algebra

## Unit - II

**Corolloary:** Let V be a finite-dimensional vector space over the field F. If L is a linear functional on the dual space $V^*$ of V, then there is a unique vector $\alpha$ in V such that $L(f) = f(\alpha)$, for every f in $V^*$.

proof:

Theorem - 17 write.

**Corollory:**

Let V be a finite dimensional vector space over the field F. Each basis for $V^*$ is the dual of some basis for V.

**Proof:**

Let $B^* = \{f_1, f_2, \ldots, f_n\}$ be a basis for $V^*$.

(Theo: 15) WKT, Let V be a finite-dimensional vector space over the field F, and let $B = \{\alpha_1, \alpha_2 \ldots \alpha_n\}$ be a basis for V. Then there is a unique dual basis $B^* = \{f_1, f_2 \ldots f_n\}$ for $V^*$ such that $f_i(\alpha_j) = \delta_{ij}$. For each linear functional f on V, we have $f = \sum_{i=1}^{n} f(\alpha_i) f_i$ and

2 for each vector $\alpha$ in $V$ we have

$$\alpha = \sum_{i=1}^{n} f_i(\alpha)\alpha_i$$

It's follows that, there is a basis $\{L_1, L_2 \ldots L_n\}$ for $V^{**}$ such that $L_i(f_j) = \delta_{ij}$

WKT, Let $V$ be a finite-dimensional vector space over the Field $F$. If $L$ is a lineal funtional on the dual space $V^*$ of $V$, then there is a unique vector $\alpha$ in $V$ such that $L(f) = f(\alpha)$, for every $f$ in $V^*$.

It's follows that, for each $i$ there is a vector $\alpha_i$ in $V$ such that $L_i(f) = f(\alpha_i)$ for every $f$ in $V^*$.

i.e, such that $L_i = L_{\alpha_i}$

It follows immediately, that $\{\alpha_1, \alpha_2 \ldots \alpha_n\}$ is a basis for $V$ and that $B^*$ is the dual of this basis.

Hence the proof//.

Theorem : 18

If $S$ is any subset of a finite-dimensional vector space $V$, then $(S^0)^0$ is the subspace spanned by $S$.

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proof:

Let $W$ be the subspace spanned by $S$. clearly $W^0 = S^0$.

Therefore, we want to prove is that

$W = W^{00}$

(Theo: 16) WKT, Let $V$ be a finite-dimensional vector space over the field $F$, and let $W$ be a subspace of $V$. Then

$\dim W + \dim W^0 = \dim V$.

$\Rightarrow \dim W^0 + \dim W^{00} = \dim V^*$

Since $\dim V = \dim V^*$

we have, $\dim W = \dim W^{00}$

Since $W$ is a subspace of $W^{00}$,

we seethat $W = W^{00}$

Hence the proof //

Theorem : 19

If $f$ is a non-zero linear functional on the vector space $V$, then the null space of $f$ is a hyperspace in $V$. conversely, every hyperspace in $V$ is the null space of a (not unique) non-zero linear functional on $V$.

4 Proof:

Let $f$ be a non-zero linear functional on $V$ and $N_f$ it's null space.

Let $\alpha$ be a vector in $V$ which is not in $N_f$.

ie, a vector such that $f(\alpha) \neq 0$.

To prove that, every vector in $V$ is in the subspace spanned by $N_f$ and $\alpha$. That subspace consists of all vectors

$$\gamma + c\alpha, \ \gamma \text{ in } N_f, \ c \text{ in } F.$$

Let $\beta$ be in $V$. define $c = \dfrac{f(\beta)}{f(\alpha)}$ which — ①

makes sense because $f(\alpha) \neq 0$.

Then the vector $\gamma = \beta - c\alpha$ is in $N_f$

$$\text{since } f(\gamma) = f(\beta) - f(c\alpha)$$

$$= f(\beta) - c f(\alpha)$$

$$= f(\beta) - f(\beta) \text{ using ①}$$

$$f(\gamma) = 0$$

so $\beta$ is in the subspace spanned by $N_f$ and $\alpha$.

Now, let $N$ be a hyperspace in $V$.

5 Fix some vector $\alpha$ which is not in N.

Since N is a maximal proper subspace, the subspace spanned by N and $\alpha$ is the entire space V.

$\therefore$ each vector $\beta$ in V has the form

$$\beta = \gamma + c\alpha, \quad \gamma \text{ in } N, \; c \text{ in } F.$$

The vector $\gamma$ and the scalar $c$, are uniquely determined by $\beta$. If we have also $\beta = \gamma' + c'\alpha$, $\gamma'$ in N, $c'$ in F.

then $$0 = (\gamma - \gamma') + (c - c')\alpha$$

$$0 = (\gamma - \gamma') - (c' - c)\alpha$$

$$(c' - c)\alpha = (\gamma - \gamma').$$

If $c' - c \neq 0$, then $\alpha$ would be in N; hence $c' = c$ and $\gamma' = \gamma$.

In other words,

If $\beta$ is in V, there is a unique scalar $c$ such that $\beta - c\alpha$ is in N.

Call that scalar $g(\beta)$. It is easy to see that $g$ is a lineal functional on V and that N is the null space of $g$.

Hence the proof.

6 **Lemma:** If $f$ and $g$ are lineal funtionals on a vector space $V$, then $g$ is a scalar multiple of $f \Leftrightarrow$ the null space of $g$ contains the null space of $f$.

ie, if and only if $f(\alpha)=0$ implies $g(\alpha)=0$.

**proof:**

If $f=0$ then $g=0$ as well and $g$ is trivially a scalar multiple of $f$.

If $f \neq 0$ so that the null space $N_f$ is a hyperspace in $V$.

choose some vector $\alpha$ in $V$ with $f(\alpha) \neq 0$ and let $c = \dfrac{g(\alpha)}{f(\alpha)}$

The lineal funtional $h = g - cf$ is $0$ on $N_f$, since both $f$ and $g$ are $0$ there, and $h(\alpha) = g(\alpha) - c f(\alpha) = 0$.

Thus $h$ is $0$ on the subspace spanned by $N_f$ and $\alpha$ and that subspace is $V$.

we conclude that $h=0$

ie, $g = cf$

Hence the proof//

## The transpose of a Linear transformation

**def:** If $A$ is an $m \times n$ matrix over the field $F$, the transpose of $A$ is the $n \times m$ matrix $A^t$ defined by $A^t_{ij} = A_{ji}$.

**Theorem : 21**

Let $V$ and $W$ be vector spaces over the field $F$. For each linear transformation $T$ from $V$ into $W$, there is a unique linear transformation $T^t$ from $W^*$ into $V^*$ such that $(T^t g)(\alpha) = g(T\alpha)$, for every $g$ in $W^*$ and $\alpha$ in $V$.

**proof:**

Let $T^t$ is the transpose of $T$. This transformation $T^t$ is often called adjoint of $T$.

Let $V$ and $W$ be the vector spaces over the field $F$ and the linear transformation from $V$ into $W$, there is a unique linear transformation $T^t$ from $W^*$ into $V^*$ as follows.

Suppose $g$ is a linear function on $W$. Let $f(\alpha) = g(T\alpha)$ —(1) for each $\alpha$ in $V$,

8. define the funtion f from $V$ into $F$
namely the composition of $T$,
a function from $V$ into $W$, with $g$,
a funtion from $W$ into $F$.

Since both $T$ and $g$ are lineal.

WKT, (Theo-6) Let $V, W$ and $Z$ be vector spaces over the field $F$. Let $T$ be a linear transformation from $V$ into $W$ and $U$ be a linear transformation $W$ into $Z$. Then the composed funtion $UT$ defined by $(UT)(\alpha) = U(T(\alpha))$ is a lineal transformation from $V$ into $Z$.

It's follows that, $f$ is also lineal.
i.e, $f$ is lineal funtional on $V$.

Thus $T$ provides us with a rule $T^t$ which associates with each lineal funtional $g$ on $W$, a lineal funtional $f = T^t g$ on $V$ defined by (1).

Also note that $T^t$ is actually a lineal transformation from $W^*$ into $V^*$;

For, if $g_1$ and $g_2$ are in $W^*$ and $c$ is a scalar

$$[T^t(cg_1+g_2)](\alpha) = (cg_1+g_2)(T\alpha)$$
$$= cg_1(T\alpha)+g_2(T\alpha)$$
$$= c(T^t g_1)(\alpha)+(T^t g_2)(\alpha)$$

$$\Rightarrow T^t(cg_1+g_2) = cT^t g_1 + T^t g_2 \; //$$

Theorem : 22.

Let $V$ and $W$ be vector spaces over the field $F$, and let $T$ be a lineal transformation from $V$ into $W$. The null space of $T^t$ is the annihilator of the range of $T$. If $V$ and $W$ are finite dimensional, then

(i) $\text{rank}(T^t) = \text{rank}(T)$

(ii) the range of $T^t$ is the annihilator of the null space of $T$.

proof:

If $g$ is in $W^*$, then by definition

$$(T^t g)(\alpha) = g(T\alpha) \text{ for each } \alpha \text{ in } V.$$

The statement that $g$ is in the null space of $T^t$ is precisely the annihilator of the range of T.

Suppose that V and W are finite dimensional, say dim V = n, and dim W = m.

For (i) Let r be the rank of T.

ie, the dimension of the range of T.

WKT (Theo-16), Let V be a finite dimensional vector space over the field F, and let W be a subspace of V. Then

$$\dim W + \dim W^0 = \dim V.$$

It's follows that, the annihilator of the range of T then has dimension (m-r). By the first statement of this theorem, the nullity of $T^t$ must be (m-r).

But then since $T^t$ is a lineal transformation on an m-dimensional space, the rank of $T^t$ is $m-(m-r) = r$ and so T and $T^t$ have the same rank.

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For (ii) Let $N$ be the null space of $T$.

Every functional in the range of $T^t$ is in the annihilator of $N$; for suppose $f = T^t g$ for some $g$ in $W^*$; then, if $\alpha$ is in $N$,

$$f(\alpha) = (T^t g)(\alpha)$$
$$= g(T\alpha)$$
$$= g(0)$$
$$= 0.$$

Now, the range of $T^t$ is a subspace of the space $N^0$ and

$$\dim N^0 = n - \dim N$$
$$= \text{rank}(T)$$
$$\dim N^0 = \text{rank}(T^t)$$

So that the range of $T^t$ must be exactly $N^0$.

Hence the proof.

12 **Theorem : 23** Let $V$ and $W$ be finite dimensional vector spaces over the field $F$. Let $B$ be an ordered basis for $V$ with dual basis $B^*$, and let $B'$ be an ordered basis for $W$ with dual basis $B'^*$. Let $T$ be a linear transformation from $V$ into $W$; let $A$ be the matrix of $T$ relative to $B, B'$ and let $B$ be the matrix of $T^t$ relative to $B'^*, B^*$.

Then $B_{ij} = A_{ji}$

**proof:**

Let $B = \{\alpha_1, \alpha_2 \cdots \alpha_n\}$, $B' = \{\beta_1, \beta_2 \cdots \beta_m\}$

$B^* = \{f_1, f_2 \cdots f_n\}$, $B'^* = \{g_1, g_2 \cdots g_m\}$.

By definition,

$$T\alpha_j = \sum_{i=1}^{m} A_{ij}\beta_i, \quad j = 1, 2 \cdots n.$$

$$T^t g_j = \sum_{i=1}^{n} B_{ij} f_i, \quad j = 1, 2 \cdots m.$$

In other words,

$$(T^t g_j)(\alpha_i) = g_j(T\alpha_i)$$

$$= g_j\left(\sum_{k=1}^{m} A_{ki}\beta_k\right)$$

$$= \sum_{k=1}^{m} A_{ki}\, g_j(\beta_k)$$

$$= \sum_{k=1}^{m} A_{ki}\, \delta_{jk}$$

$$= A_{ji}$$

For any linear funtional $f$ on $V$.

$$f = \sum_{i=1}^{m} f(\alpha_i) f_i$$

If we apply this formula to the funtional $f = T^t g_j$ and use the fact that $(T^t g_j)(\alpha_i) = A_{ji}$

We have $T^t g_j = \sum_{i=1}^{n} A_{ji} f_i$

from which it immediately follows that $B_{ij} = A_{ji}$

Hence the proof//.

**Theorem : 24**

Let $A$ be any $m \times n$ matrix over the field $F$. Then the row rank of $A$ is equal to the column rank of $A$.

**proof:**

Let $B$ be the standard ordered basis for $F^n$ and $B'$ be the standard ordered basis for $F^m$.

14 Let T be the linear transformation from $F^n$ into $F^m$ such that the matrix of T relative to the pair $B, B'$ is A.

ie, $T(x_1, x_2 \cdots x_n) = (y_1, y_2 \cdots y_m)$

where $y_i = \sum_{j=1}^{n} A_{ij} x_j$

The column rank of A is the rank of the transformation T, because the range of T consists of all m-tuples which are linear combinations of the column vectors of A.

Relative to the dual bases $B'^*$ and $B^*$, the transpose mapping $T^t$ is represented by the matrix $A^t$.

Since the columns of $A^t$ are the rows of A, we see by the same reasoning that the row rank of A (the column rank of $A^t$) is equal to the rank of $T^t$.

WKT, (Theo-22) Let V and W be vector spaces over the field F, and let T be a linear transformation from

15 V into W. The null space of $T^t$ is the annihilator of the range of T.

If V and W are finite-dimensional, then (i) rank$(T^t)$ = rank$(T)$

(ii) the range of $T^t$ is the annihilator of the null space of T.

It's follows that T and $T^t$ have the same rank and hence the row rank of A is equal to the column rank of A.

Hence the proof//