Class : I M.Sc Mathematics

Subject : Linear Algebra

UNIT - V - Diagonalization

Invariant subspaces :

Let V be a vector space and T be a linear operator on V. If W is a subspace of V, we say that W is invariant under T if for each vector α in W the vector $T\alpha$ is in W.

i.e, if T(W) is contained in W.

T-conductor

Def: Let W be an invariant subspace for T and let α be a vector in V. The T-conductor of α into W is the set $S_T(\alpha : W)$, which consists of all polynomials g (over the scalar field) such that $g(T)\alpha$ is in W.

Lemma : 1

Let W be an invariant subspace for T. The characteristic polynomial for the restriction operator T_W divides the characteristic polynomial for T. The minimal polynomial for T_W divides

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2 the minimal polynomial for T.

<u>proof</u>:

Suppose we choose an ordered basis $B = \{\alpha_1, \alpha_2 \dots \alpha_n\}$ for V such that $B' = \{\alpha_1, \alpha_2 \dots \alpha_r\}$ is an ordered basis for W ($r = \dim W$).

Let $A = [T]_B$ so that

$$T\alpha_j = \sum_{i=1}^{n} A_{ij} \alpha_i$$

Since W is invariant under T, the vector $T\alpha_j$ belongs to W for $j \leq r$.

$$\Rightarrow T\alpha_j = \sum_{i=1}^{r} A_{ij} \alpha_i \ , j \leq r$$

In other words, $A_{ij} = 0$ if $j \leq r$ and $i > r$

Schematically, A has the block form

$$A = \begin{bmatrix} B & C \\ 0 & D \end{bmatrix} \quad \text{———} \ (1)$$

where B is an $r \times r$ matrix

C is an $r \times (n-r)$ matrix

D is an $(n-r) \times (n-r)$ matrix.

and $A = [T]_B$, $B = [T_W]_{B'}$

Now, (1) $\Rightarrow \det(xI - A) = \det(xI - B)\det(xI - D)$

$\therefore$ characteristic polynomial for T_W divides the characteristic polynomial for T.

3 The k^{th} power of the matrix A has the block form

$$A^k = \begin{bmatrix} B^k & C_k \\ 0 & D^k \end{bmatrix}$$ where C_k is some $r \times (n-r)$ matrix.

$\therefore$ Any polynomial which Annihilates B (and D too). So, the minimal polynomial for B divides the minimal polynomial for A.

Hence the proof //

Lemma: 2

If W is an invariant subspace for T, then W is invariant under every polynomial in T. Thus, for each α in V, the conductor $S(\alpha; W)$ is an ideal in the polynomial algebra $F[x]$.

proof:

If β is in W, then $T\beta$ is in W.

consequently, $T(T\beta) = T^2\beta$ is in W.

By induction, $T^k\beta$ is in W for each k.

Take linear combinations to see that $f(T)\beta$ is in W for every polynomial f.

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4 The definition of $S(\alpha; W)$ makes sense if W is any subset of V. If W is a subspace, then $S(\alpha; W)$ is a subspace of $F[x]$, because

$$(cf+g)(T) = cf(T) + g(T)$$

If W is also invariant under T, let g be a polynomial in $S(\alpha; W)$

i.e, let $g(T)\alpha$ be in W.

If f is any polynomial, then $f(T)[g(T)\alpha]$ will be in W.

since $(fg)(T) = f(T)\,g(T)$

fg is in $S(\alpha; W)$. Thus the conductor absorbs multiplication by any polynomial.

Hence the proof//

Def: (Tri-angulable).

The linear operator T is called tri-angulable if there is an ordered basis in which T is represented by a triangular matrix.

Lemma: 3 Let V be a finite-dimensional vector space over the field F. Let T be a linear operator on V such that the minimal polynomial for T is a

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5 product of linear factors.

$$p = (x-c_1)^{r_1}(x-c_2)^{r_2}\cdots(x-c_k)^{r_k}, \quad c_i \text{ in } F.$$

Let W be a proper ($W \neq V$) subspace of V which is invariant under T. There exists a vector α in V such that

(i) α is not in W;

(ii) $(T-cI)\alpha$ is in W, for some characteristic value c of the operator T.

<u>proof:</u>

Let β be any vector in V which is not in W.

Let g be the T-conductor of β into W. Then g divides p, the minimal polynomial for T.

Since β is not in W, the polynomial g is not constant

$$\therefore g = (x-c_1)^{e_1}\cdots(x-c_k)^{e_k}$$

where atleast one of the integers e_i is positive.

Choose j so that $e_j > 0$. Then $(x-c_j)$ divides g:

$$g = (x-c_j)h$$

By the definition of g, the vector $\alpha = h(T)\beta$ cannot be in W. But $(T-c_jI)\alpha = (T-c_jI)h(T)\beta$

$$= g(T)\beta \text{ is in } W$$

Hence the Lemma.//

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6 **Theorem: 1**

Let V be a finite dimensional vector space over the field F and let T be a linear operator on V. Then T is diagonalizable $\Longleftrightarrow$ the minimal polynomial for T has the form

$p = (x - c_1)(x - c_2) \cdots (x - c_k)$, where

$c_1, c_2 \ldots c_k$ are distinct elements of F.

proof:

Suppose that T is diagonalizable linear operator and let $(c_1, c_2 \cdots c_k)$ be the distinct characteristic values of T. Then the minimal polynomial for T is

$T = (x - c_1)(x - c_2) \cdots (x - c_k)$

If α is a characteristic vector, then one of the operators

$(T - c_1 I)(T - c_2 I) \cdots (T - c_k I)\alpha = 0$, for every characteristic vector α.

There is a basis for the underlying space which consists of characteristic vectors of T.

Hence $p(T) = (T - c_1 I)(T - c_2 I) \cdots (T - c_k I) = 0$

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7 If T is diagonalizable linear operator, then the minimal polynomial for T is a product of distinct linear factors.

To prove the converse part:

Let W be the subspace spanned by all of the characteristic vectors of T, and suppose $W \neq V$, there is a vector α not in W and characteristic value c_j of T such that the vector $\beta = (T - c_j I)\alpha$ is in W.

Since β is in W.

$\therefore \beta = \beta_1 + \cdots + \beta_k$, where $T\beta_i = c_i \beta_i$, $1 \leq i \leq k$, and therefore the vector

$h(T)\beta = h(c_1)\beta_1 + \cdots + h(c_k)\beta_k$ is in W, for every polynomial h.

Now, $p = (x - c_j)q$, for some polynomial q. Also $q - q(c_j) = (x - c_j)h$

we have, $q(T)\alpha - q(c_j)\alpha = h(T)(T - c_j I)\alpha$
$$= h(T)\beta$$

But $h(T)\beta$ is in W and since

$$0 = p(T)\alpha = (T - c_j I)q(T)\alpha$$

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8 the vector $q(T)\alpha$ is in W.

$\therefore$ $q(c_j)\alpha$ is in W.

since α is not in W, we have $q(c_j)=0$. That contradicts the fact that p has distinct roots.

Hence the proof.

Simultaneous triangulation (or) diagonalization

Let V be a finite dimensional space and let $\mathcal{F}$ be a family of linear operators on V. The operators in $\mathcal{F}$ is said to be simultaneously triangulate (or) diagonalize operator in $\mathcal{F}$ if we find one basis B such that all of the matrices $[T]_B$, T in $\mathcal{F}$ are triangular (or) diagonal.

Lemma : 4

Let $\mathcal{F}$ be a commuting family of triangulable linear operators on V. Let W be a proper subspace of V which is invariant under $\mathcal{F}$. There exists a vector α in V such that

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9 (a) α is not in W;

(b) for each T in $\mathcal{F}$, the vector $T\alpha$ is in the subspace spanned by α and W.

proof:

with out loss of generality, to assume that $\mathcal{F}$ contains only a finite number of operators.

Let $\{T_1, T_2 \ldots T_r\}$ be a maximal linearly independent subset of $\mathcal{F}$.

i.e, a basis for the subspace spanned by $\mathcal{F}$.

If α is a vector such that (b) holds for each T_i, then (b) will hold for every operator which is a lineal combination of $T_1 T_2 \cdots T_r$.

WKT, (Lemma-3) Let V be a finite dimensional vector space over the field F. Let T be a lineal operator on V such that the minimal polynomial for T is a product of linear factors $P=(x-c_1)^{r_1} \ldots (x-c_k)^{r_k}$, c_i in F.

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10 Let W be a proper $(W \neq V)$ subspace of V which is invariant under $\mathcal{F}$. There exists a vector α in V such that

(a) α is not in W;

(b) $(T - cI)\alpha$ is in W, for some characteristic value c of the operator T.

It's follows that, we can find a vector β_1 (not in W) and scalar c_1 such that $(T_1 - c_1 I)\beta_1$ is in W.

Let V_1 be the collection of all vectors β in V such that $(T_1 - c_1 I)\beta$ is in W. Then V_1 is a subspace of V which is properly larger than W.

Furthermore, V_1 is invariant under $\mathcal{F}$.

If T commutes with T_1, then

$$(T_1 - c_1 I)(T\beta) = T(T_1 - c_1 I)\beta$$

If β is in V_1, then $(T_1 - c_1 I)\beta$ is in W. Since W is invariant under each T in $\mathcal{F}$, we have $T(T_1 - c_1 I)\beta$ is in W.

i.e, $T\beta$ is in V_1, $\forall \beta \in V_1$ and all $T \in \mathcal{F}$.

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Now; W is a proper subspace of V_1.

Let U_2 be the linear operator on V_1 obtained by restricting T_2 to the subspace V_1.

The minimal polynomial for U_2 divides the minimal polynomial for T_2. Therefore, we obtain a vector β_2 in V_1 (not in W) and a scalar c_2 such that $(T_2 - c_2 I)\beta_2$ is in W.

Note that

(a) β_2 is not in W

(b) $(T_1 - c_1 I)\beta_2$ is in W

(c) $(T_2 - c_2 I)\beta_2$ is in W

Let V_2 be the set of all vectors β in V_1 such that $(T_2 - c_2 I)\beta$ is in W. Then V_2 is invariant under $\mathcal{F}$.

Again we apply the result to U_3, the restriction of T_3 to V_2. If we continue in this way, we shall reach a vector $\alpha = \beta_r$ (not in W) such that $(T_j - c_j I)\alpha$ is in W, $j = 1, 2 \ldots r$.

Hence the Lemma.//

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Direct - Sum Decompositions

def: Let $W_1, W_2 \ldots W_K$ be subspaces of the vector space V. We say that $W_1, W_2 \ldots W_K$ are independent if $\alpha_1 + \alpha_2 + \ldots + \alpha_K = 0$, α_i in W_i implies that each α_i is 0.

For $K = 2$, the meaning of independence is $\{0\}$ intersection,

ie, W_1 and W_2 are independent $\Leftrightarrow$ $W_1 \cap W_2 = \{0\}$.

If $K > 2$, the independence of $W_1, W_2 \ldots W_K$ says much more than $W_1 \cap W_2 \cap \ldots \cap W_K = \{0\}$. It says that each W_j intersects the sum of the other subspaces W_i only in the zero vector.

The significance of independence is this, Let $W = W_1 + \ldots + W_K$ be the subspace spanned by $W_1 \ldots W_K$. Each vector α in W can be expressed as a sum

$$\alpha = \alpha_1 + \ldots + \alpha_K, \quad \alpha_i \text{ in } W_i$$

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13 If $W_1, \ldots, W_k$ are independent, then that expression for α is unique; for if $\alpha = \beta_1 + \cdots + \beta_k$, $\ldots \beta_i$ in W_i

then $0 = (\alpha_1 - \beta_1) + \cdots + (\alpha_k - \beta_k)$.

Hence $\alpha_i - \beta_i = 0$, $i = 1, 2 \ldots k$.

Thus, when $W_1, \ldots W_k$ are independent, we can operate with the vectors in W as k-tuples $(\alpha_1, \alpha_2 \ldots \alpha_k)$, α_i in W_i, in the same way as we operate with vectors in R^k as k-tuples of numbers.

Lemma : 5

Let V be a finite-dimensional vector space. Let $W_1, W_2 \ldots W_k$ be subspaces of V and let $W = W_1 + \cdots + W_k$. The following are equivalent.

(a) $W_1, \ldots W_k$ are independent

(b) For each j, $2 \leq j \leq k$ we have

$$W_j \cap (W_1 + \cdots + W_{j-1}) = \{0\}$$

(c) If B_i is an ordered basis for W_i, $1 \leq i \leq k$, then the sequence $B = (B_1 \ldots B_k)$ is an ordered basis for W.

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14 <u>proof:</u>

Assume (a)

Let α be a vector in the intersection $W_j \cap (W_1 + \cdots + W_{j-1})$. Then there are vectors $\alpha_1, \ldots \alpha_{j-1}$ with α_i in W_i such that $\alpha = \alpha_1 + \cdots + \alpha_{j-1}$.

Since $\alpha_1 + \cdots + \alpha_{j-1} + (-\alpha) + 0 + \cdots + 0 = 0$ and since $W_1, \ldots W_K$ are independent, it must be that $\alpha_1 = \alpha_2 = \cdots = \alpha_{j-1} = \alpha = 0$.

Now, let us observe that (b) $\Rightarrow$ (a)

Suppose $0 = \alpha_1 + \cdots + \alpha_K$, α_i in W_i

Let j be the largest integer i such that $\alpha_i \neq 0$. then $0 = \alpha_1 + \cdots + \alpha_j$, $\alpha_j \neq 0$

Thus $\alpha_j = -\alpha_1 - \cdots - \alpha_{j-1}$ is a non-zero vector in $W_j \cap (W_1 + \cdots + W_{j-1})$.

Now that we know (a) and (b) are the same, let us see why (a) is equivalent to (c).

Assume (a). Let B_i be a basis for W_i, $1 \leq i \leq K$, and let $B = (B_1, \ldots B_K)$.

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15 Any lineal relation between the vectors in B will have the form $\beta_1 + \cdots + \beta_K = 0$ where β_i is some lineal combination of the vectors in B_i.

Since $W_1, \ldots, W_K$ are independent, each B_i is independent, the relation we have between the vectors in B is the trivial relation.

Hence the proof//

def: projection

If V is a vector space, a projection of V is a lineal operator E on V such that $E^2 = E$.

Note: Suppose that E is a projection. Let R be the range of E and let N be the null space of E.

(i) The vector β is in the range $R \Leftrightarrow E\beta = \beta$. If $\beta = E\alpha$, then $E\beta = E^2\alpha = E\alpha = \beta$. Conversely, if $\beta = E\beta$, then β is in the range of E.

(ii) $V = R \oplus N$

(iii) The unique expression for α as a sum of vectors in R and N is $\alpha = E\alpha + (\alpha - E\alpha)$

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Theorem : 2

If $V = W_1 \oplus \cdots \oplus W_k$, Then there exist k linear operators $E_1, \cdots E_k$ on V Such that

(i) each E_i is a projection $(E_i^2 = E_i)$

(ii) $E_i E_j = 0$, if $i \neq j$

(iii) $I = E_1 + \cdots + E_k$

(iv) the range of E_i is W_i.

conversely, if $E_1, \cdots E_k$ are k linear operators on V which satisfy conditions (i), (ii), and (iii), and if we let W_i be the range of E_i, then $V = W_1 \oplus \cdots \oplus W_k$

proof:

we have only to prove the converse statement.

Suppose $E_1, \cdots E_k$ are linear operators on V which Satisfy the first three conditions, and let W_i be the range of E_i. Then certainly $V = W_1 + \cdots + W_k$;

By condition (iii) we have

$$\alpha = E_1\alpha + \cdots\cdots + E_k\alpha.$$ for each α in V, and $E_i\alpha$ is in W_i.

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17 This expression for α is unique, because if $\alpha = \alpha_1 + \cdots + \alpha_k$ with α_i in W_i, say $\alpha_i = E_i \beta_i$, then using (i) and (ii) we have

$$E_j \alpha = \sum_{i=1}^{k} E_j \alpha_i$$

$$= \sum_{i=1}^{k} E_j E_i \beta_i$$

$$= E_j^2 \beta_j$$

$$= E_j \beta_j$$

$$E_j \alpha = \alpha_j$$

This shows that V is the direct sum of the W_i.

Hence the proof//

Invariant Direct Sums

def: If α is a vector in V, we have unique vectors $\alpha_1, \ldots \alpha_k$ with α_i in W_i such that $\alpha = \alpha_1 + \cdots + \alpha_k$ and then $T\alpha = T_1\alpha_1 + \cdots + T_k\alpha_k$.

We shall describe this situation by saying that T is the direct sum

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18 of the operators $T_1, \dots T_k$. It must be remembered in using this termino-logy that the T_i are not linear operators on the space V but on the various subspaces W_i.

The fact that $V = W_1 \oplus \cdots \oplus W_k$ enables us to associate with each α in V a unique k-tuple $(\alpha_1, \alpha_2 \dots \alpha_k)$ of vectors α_i in W_i.

Theorem : 3

Let T be a linear operator on the space V, and let $W_1, \dots, W_k$ and $E_1, \dots, E_k$ are k linear operators on V. Then a necessary and sufficient condition that each subspace W_i be invariant under T is that T commute with each of the projections E_i.

i.e, $TE_i = E_i T$, $i = 1, \dots k$.

proof:

Suppose T commutes with each E_i. Let α be in W_j. Then $E_j\alpha = \alpha$ and

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19 $T\alpha = T(E_j\alpha)$

$= E_j(T\alpha)$

which shows that $T\alpha$ is in the range of E_j.

i.e, that W_j is invariant under T.

Assume that, Now each W_i is invariant under T.

we have to prove that, $TE_j = E_jT$.

Let α be any vector in V. Then

$$\alpha = E_1\alpha + \cdots + E_k\alpha$$

$$T\alpha = TE_1\alpha + \cdots + TE_k\alpha$$

Since $E_i\alpha$ is in W_i, which is invariant under T, we must have $T(E_i\alpha) = E_i\beta_i$ for some vector β_i.

Then $E_jTE_i\alpha = E_jE_i\beta_i$

$$= \begin{cases} 0, & \text{if } i \neq j \\ E_j\beta_j, & \text{if } i = j \end{cases}$$

Thus $E_jT\alpha = E_jTE_1\alpha + \cdots + E_jTE_k\alpha$

$= E_j\beta_j = TE_j\alpha$

This holds for each α in V, so $E_jT = TE_j$

Hence the proof//

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Theorem : 4

Let T be a linear operator on a finite-dimensional space V. If T is diagonalizable and if $c_1 \ldots c_k$ are the distinct characteristic value of T, Then $\exists$ a linear operators $E_1 \ldots E_k$ on V such that

(i) $T = c_1 E_1 + \cdots + c_k E_k$

(ii) $I = E_1 + \cdots + E_k$

(iii) $E_i E_j = 0, \ i \neq j$

(iv) $E_i^2 = E_i$ (E_i is a projection)

(v) the range of E_i is the characteristic space for T associated with c_i.

Conversely, if $\exists$ k distinct scalars $c_1, \ldots, c_k$ and k non-zero linear operators $E_1 \ldots E_k$ which satisfy conditions (i), (ii) and (iii), then T is diagonalizable, $c_1, \ldots, c_k$ are the distinct characteristic values of T, and conditions (iv) and (v) are satisfied also.

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21) Proof:

Suppose that T is diagonalizable, with distinct characteristic values $c_1, c_2 \cdots c_k$.
Let W_i be the space of characteristic vectors associated with the characteristic value c_i.

As we have seen $V = W_1 \oplus \cdots \oplus W_k$
Let $E_1, \cdots, E_k$ be the projections associated with this decomposition as well known theorem.

Then (ii), (iii), (iv) and (v) are satisfied.
To verify (i) proceed as follows.

For each α in V,

$$\alpha = E_1\alpha + \cdots + E_k\alpha \text{ and so}$$

$$T\alpha = TE_1\alpha + \cdots + TE_k\alpha$$

$$T\alpha = c_1E_1\alpha + \cdots + c_kE_k\alpha.$$

$$\Rightarrow T = c_1E_1 + \cdots + c_kE_k.$$

Now, suppose that we are given a linear operator T along with distinct scalars c_i and non-zero operators E_i which satisfy (i), (ii) and (iii).

Since $E_iE_j = 0$ when $i \neq j$,
we multiply both sides of $I = E_1 + \cdots + E_k$ by E_i and obtain immediately $E_i^2 = E_i$,

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Multiplying $T = c_1E_1 + \cdots + c_kE_k$ by E_i, we have $TE_i = c_iE_i$, which shows that any vector in the range of E_i is in the null space of $(T - c_iI)$.

Since we have assumed that $E_i \neq 0$, this prove that there is no non-zero vector in the null space of $(T - c_iI)$.
i.e, That c_i is a characteristic value of T.

Furthermore, the c_i are all of the Characteristic values of T;
For, if c is any scalar, then
$T - cI = (c_1 - c)E_1 + \cdots + (c_k - c)E_k$.
so if $(T - cI)\alpha = 0$, we must have
$$(c_i - c)E_i\alpha = 0.$$
If α is not the zero vector, then $E_i\alpha \neq 0$ for some i, so that for this i we have $c_i - c = 0$.

Certainly T is diagonalizable, since we have shown that every non-zero vector in the range of E_i is a characteristic

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23 vector of T, and the fact that $I = E_1 + \cdots + E_k$ shows that these characteristic vectors span V.

All that remains to be demonstrated is that the null space of $(T - c_i I)$ is exactly the range of E_i.

But this is clear, because if $T\alpha = c_i \alpha$,

then $\sum_{j=1}^{k} (c_j - c_i) E_j \alpha = 0$

$\Rightarrow (c_j - c_i) E_j \alpha = 0$, for each j

$\Rightarrow E_j \alpha = 0$, $j \neq i$

Since $\alpha = E_1\alpha + \cdots + E_k\alpha$, and $E_j\alpha = 0$ for $j \neq i$, we have $\alpha = E_i \alpha$, which proves that α is in the range of E_i.

Hence the proof //

(X) <u>Theorem : 5</u> <u>primary Decomposition theorem</u>

Let T be a linear operator on the finite-dimensional vector space V over the field F. Let p be the minimal polynomial for T,

$$p = p_1^{r_1} \cdots p_k^{r_k}$$

where the p_i are distinct irreducible monic polynomials over F and the r_i

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24 are positive integers. Let W_i be the null space of $p_i(T)^{r_i}$, $i = 1, \ldots k$. Then

(i) $V = W_1 \oplus \cdots \oplus W_k$;

(ii) each W_i is invariant under T;

(iii) if T_i is the operator induced on W_i by T, then the minimal polynomial for T_i is $p_i^{r_i}$!

<u>Proof:</u>

If the direct sum decomposition (i) is valid. The projection E_i will be the identity on W_i and zero on the other W_j.

We shall find a polynomial h_i such that $h_i(T)$ is the identity on W_i and is zero on the other W_j, and so that $h_1(T) + \cdots + h_k(T) = I$ etc.

For each i, let $f_i = \dfrac{p}{p_i^{r_i}} = \prod_{j \neq i} p_j^{r_j}$

Since $p_1, \ldots p_k$ are distinct prime polynomials, the polynomials $f_1, \ldots f_k$ are relatively prime.

Thus there are polynomials $g_1, \ldots g_k$ such that $\sum_{i=1}^{n} f_i g_i = 1$.

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25 If $i \neq j$, then $f_i f_j$ is divisible by the polynomial p, because $f_i f_j$ contains each $p_m^{r_m}$ as a factor.

We shall show that the polynomials $h_i = f_i g_i$ behave in the manner described in the first paragraph of the proof.

Let $E_i = h_i(T) = f_i(T) g_i(T)$.

Since $h_1 + \cdots + h_k = 1$ and p divides $f_i f_j$ for $i \neq j$, we have

$$E_1 + \cdots + E_k = I$$

$$E_i E_j = 0, \text{ if } i \neq j$$

Thus the E_i are projections which correspond to some direct-sum decomposition of the space V.

We wish to show that the range of E_i is exactly the subspace W_i.

It is clear that each vector in the range of E_i is in W_i.

For, if α is in the range of E_i, then $\alpha = E_i \alpha$ and so

$$\begin{aligned} p_i(T)^{r_i}\alpha &= p_i(T)^{r_i} E_i \alpha \\ &= p_i(T)^{r_i} f_i(T) g_i(T)\alpha \\ &= 0 \end{aligned}$$

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26 because $p^{r_i} f_i g_i$ is divisible by the minimal polynomial p.

Conversely,

Suppose that α is in the null space of $p_i(T)^{r_i}$.

If $j \neq i$, then $f_j g_j$ is divisible by $p_i^{r_i}$ and so $f_j(T) g_j(T) \alpha = 0$

i.e, $E_j \alpha = 0$, for $j \neq i$.

But then it is immediate that $E_i \alpha = \alpha$.

i.e, that α is in the range of E_i.

This completes the proof of statement (i).

It is certainly clear that the subspaces W_i are invariant under T.

If T_i is the operator induced on W_i by T, then evidently $p_i(T)^{r_i} = 0$, because by definition $p_i(T)^{r_i}$ is 0 on the subspace W_i.

This shows that the minimal polynomial for T_i divides $p_i^{r_i}$.

Conversely, let g be any polynomial such that $g(T_i) = 0$. Then $g(T) f_i(T) = 0$

Thus $g f_i$ is divisible by the minimal polynomial p of T.

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27 i.e, $p_i^{r_i}$ f_i divides $g f_i$.

It is easily seen that $p_i^{r_i}$ divides g.

Hence the minimal polynomial for T_i is $p_i^{r_i}$.

Hence the proof//

def: NIL POTENT

Let N be a linear operator on the vector space V. we say that N is nilpotent if there is some positive integer 'r' such that $N^r = 0$

Theorem: 6

Let T be a linear operator on the finite-dimensional vector space V over the field F. suppose that the minimal polynomial for T decomposes over F into a product of linear polynomials. Then there is a diagonalizable operator D on V and a nilpotent N on V such that (i) $T = D + N$

(ii) $DN = ND$

The diagonalizable operator D and the nilpotent operator N are uniquely

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28 determined by (i) and (ii) and each of them is a polynomial in T.

proof:

we have just observed that, we can write $T = D + N$ where D is diagonalizable and N is nilpotent, and where D and N not only commute but are polynomials in T.

Now, suppose that we also have $T = D' + N'$ where D' is diagonalizable, N' is nilpotent, and $D'N' = N'D'$.

we shall prove that $D = D'$ and $N = N'$.

Since D' and N' commute with one another and $T = D' + N'$.

we see that D' and N' commute with T. Thus D' and N' commute with any polynomial in T.

Hence they commute with D and with N. Now we have $D + N = D' + N'$

(or) $D - D' = N' - N$

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29 and all four of these operators commute with one another.

Since D and D' are both diagonali-zable and they commute, they are simultaneously diagonalizable, and $D-D'$ is diagonalizable.

Since N and N' are both nilpotent and they commute, the operator $(N'-N)$ is nilpotent;

For using the fact that N and N' commute $(N'-N)^r = \sum_{j=0}^{r} \binom{r}{j} (N')^{r-j}(-N)^j$

and so when r is sufficiently large every term in this expression for $(N'-N)$ will be 0 (zero).

Now $D-D'$ is a diagonalizable operator which is also nilpotent. Such an operator is obviously the zero operator;

For since it is nilpotent, the minimal polynomial for this operator is of the form x^r for some $r \leq m$; but then

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Since the operator is diagonalizable, the minimal polynomial cannot have a repeated root.

Hence $r=1$ and the minimal polynomial is simply x, which says the operator is zero.

Thus we see that $D = D'$ and $N = N'$

Hence the proof//

Corollary:

If $E_1, \dots E_k$ are the projections associated with the primary decomposition of T, then each E_i is a polynomial in T and accordingly if a linear operator U commutes with T then U commutes with each of the E_i.

i.e, each subspace W_i is invariant under U.

Proof:

Let us take a look at the special case in which the minimal polynomial for T is a product of first degree

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31 polynomials.

i.e, the case in which each p_i is of the form $p_i = x - c_i$.

Now, the range of E_i is the null space W_i of $(T - c_i I)^{r_i}$.

Let us put $D = c_1 E_1 + \cdots + c_k E_k$

WKT, D is diagonalizable operator which we shall call the diagonalizable part of T. Let us look at the operator

$N = T - D$.

Now, $T = TE_1 + \cdots + TE_k$

$D = c_1 E_1 + \cdots + c_k E_k$

so $N = (T - c_1 I) E_1 + \cdots + (T - c_k I) E_k$

lly $N^2 = (T - c_1 I)^2 E_1 + \cdots + (T - c_k I)^2 E_k$

and ingeneral that

$$N^r = (T - c_1 I)^r E_1 + \cdots + (T - c_k I)^r E_k.$$

when $r \geq r_i$, for each i, we shall have $N^r = 0$, because the operator $(T - c_i I)^r$ will be zero on the range of E_i.

Hence the proof//

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Theorem : 7

Let V be a finite-dimensional vector space over the field F and let T be a linear operator on V. Then T is triangulable $\Leftrightarrow$ the minimal polynomial for T is a product of polynomial over F.

Proof: Suppose that the minimal polynomial factors $P = (x-c_1)^{r_1} \cdots (x-c_k)^{r_k}$ — ①

{ wkt, Let V be a finite-dimensional vector space over the field F. Let T be a linear operator on V such that the minimal polynomial for T is a product of linear factors $P = (x-c_1)^{r_1} \cdots (x-c_k)^{r_k}$ }

We shall arrive at an ordered basis $B = \{\alpha_1, \ldots \alpha_n\}$ in which the matrix representing T is upper triangular:

$$[T]_B = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a_{22} & a_{23} & \cdots & a_{2n} \\ 0 & 0 & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & & a_{nn} \end{bmatrix} \quad \text{— ②}$$

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33

Now, equ (2) says that

$$T\alpha_j = a_{1j}\alpha_1 + \cdots\cdots + a_{jj}\alpha_j, \quad 1 \le j \le n$$

i.e, $T\alpha_j$ is in the subspace spanned by $\alpha_1, \cdots \alpha_j$.

To find $\alpha_1, \cdots, \alpha_n$, we start by applying the result to the subspace $W = \{0\}$, to obtain the vector α_1.

Then apply the result to W_1, the space spanned by α_1 and we have obtain α_2.

Next we apply the result to W_2, the space spanned by α_1 and α_2.

Continue in this way, after $\alpha_1, \cdots \alpha_i$ have been found, It is the triangular-type relations for $j = 1, \cdots i$ which ensure that the subspace spanned by $\alpha_1 \cdots \alpha_i$ is invariant under T.

If T is triangulable, It's evident that the characteristic polynomial for T has the form $f = (x-c_1)^{d_1} \cdots (x-c_k)^{d_k}$, c_i in F.

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we see that equation ②, the diagonal entries $a_{11}, \dots a_{m}$ are the charalteristic values, with c_i repeated d_i times. But, if f can be so faltered, so can the minimal polynomial p, because it divides f.

Hence the proof.//

Theorem : 8

Let T be a linear operator on an n-dimensional vector space [or, let A be an $n \times n$ matrix]. The charalteristic and minimal polynomials for T [for A] have the same roots, except for multiplicities.

proof:

Let p be the minimal polynomial for T. Let c be the scalar.

we have to prove that $p(c) = 0 \iff$ c is charalteristic value of T.

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35 First, suppose that $p(c) = 0$. Then

$p = (x-c)q$. where q is

a minimal polynomial.

Since $\deg q < \deg p$, the definition of the minimal polynomial p tells us that

$q(T) \neq 0$.

Choose a vector β such that $q(T)\beta \neq 0$.

Let $\alpha = q(T)\beta$. Then

$$0 = p(T)\beta$$

$$\Rightarrow \quad 0 = (T-cI)q(T)\beta$$

$$\Rightarrow \quad 0 = (T-cI)\alpha$$

and thus, c is a characteristic value of T.

Now, suppose that c is a characteristic value of T, say, $T\alpha = c\alpha$ with $\alpha \neq 0$.

WKT, suppose that $T\alpha = c\alpha$. If f is any polynomial, then $f(T)\alpha = f(c)\alpha$. — (*)

As we noted in a result (*), we have

$$p(T)\alpha = p(c)\alpha$$

Since $p(T) = 0$ and $\alpha \neq 0$ we have $p(c) = 0$

Hence the proof.

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