06.12.19

P → Surface
I → Continuous
D → Limit

Unit - I

Introduction:

A partial differential equation (PDE) is an equation which contains one or more partial derivatives. Then order of partial differential equation is an order of the highest derivatives.

Let $z = f(x,y)$ where x, y are independent variables and z is the dependent variables. Then $\frac{\partial z}{\partial x}$, $\frac{\partial z}{\partial y}$ are the first order partial derivatives and $\frac{\partial^2 z}{\partial x^2}$, $\frac{\partial^2 z}{\partial y^2}$, $\frac{\partial^2 z}{\partial x \partial y}$ are the second order partial derivatives

Note:

Let θ is the function of x, y, z and t. we get a partial derivatives $\frac{\partial \theta}{\partial x}$, $\frac{\partial \theta}{\partial y}$, $\frac{\partial \theta}{\partial z}$ and $\frac{\partial \theta}{\partial t}$ are the 1^{st} order partial derivative and $\frac{\partial^2 \theta}{\partial x^2}$, $\frac{\partial^2 \theta}{\partial y^2}$, $\frac{\partial^2 \theta}{\partial z^2}$ and $\frac{\partial^2 \theta}{\partial t^2}$, are 2^{nd} order partial derivatives.

Hence we obtain a relation between the derivatives of the kind $F\left(\frac{\partial \theta}{\partial x}, \frac{\partial^2 \theta}{\partial x^2}, \ldots \frac{\partial^2 \theta}{\partial x \partial y}\right) = 0$

Notation:

Suppose that there are two independent variables x and y and z is the dependent variable (ie) $z = f(x,y)$ where, $p = \frac{\partial z}{\partial x}$, $q = \frac{\partial z}{\partial y}$, $r = \frac{\partial^2 z}{\partial x^2}$, $s = \frac{\partial^2 z}{\partial x \partial y}$, $t = \frac{\partial^2 z}{\partial y^2}$

In anothercase, there are n-independent variable that is $x_1, x_2, \ldots x_n$ and z is dependent variable where $P_1 = \frac{\partial z}{\partial x_1}$, $P_2 = \frac{\partial z}{\partial x_2} \ldots$ $\ldots P_n = \frac{\partial z}{\partial x_n}$

Also, PDE are denoted by suffixes such as $U_x = \frac{\partial u}{\partial x}$, $U_y = \frac{\partial u}{\partial y}$, $U_{xx} = \frac{\partial^2 u}{\partial x^2}$, $U_{yy} = \frac{\partial^2 u}{\partial y^2}$

Definition: (Surface)

A point x, y, z in a space is said to lies on a surface, if the co-ordinates of (x, y, z) is satisfies $F(x,y,z) = 0$ where F is the continuous differentiable function.

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Curves in a Spaces:

A curve may be specified by means of parametric equation.

Suppose f_1, f_2, f_3 are continuous function of continuous variable t which varies in an interval

$y = f_2(t)$
$z = f_3(t)$

which represent the parametric equation of a curve in the three dimensional space.

Example:

The curve in a space is a straight line with direction cosines (l, m, n) passing through points (x_0, y_0, z_0) with parametric equations

$x = x_0 + ls, \quad y = y_0 + ms, \quad z = z_0 + ns.$

Sphere:

Let the equation of the sphere whose centre is (a, b, c) and radius r.

Let $P(x, y, z)$ be any point on the sphere. Then the General equation of the sphere is

$$(x-a)^2 + (y-b)^2 + (z-c)^2 = r^2$$

Note:

Equation of the sphere whose centre origin and radius is r. Then the general equation of the sphere is $x^2 + y^2 + z^2 = r^2$

Jacobi Formula:

If $u = f(x, y)$ and $v = g(x, y)$ are two continuous function of variable x and y that $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ are all continuous in x and y.

Then,

$$\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = 0 \quad (\text{or}) \quad J\left(\frac{u,v}{x,y}\right) = 0$$

is called the Jacobian of x and y.

Parametric Equation of the Sphere:

$x = a \sin u \cos v$

$y = a \sin u \sin v$

$z = a \cos u$

(or)

$$x = a\left[\frac{1-v^2}{1+v^2}\right]\cos v$$

$$y = a\left[\frac{1-v^2}{1+v^2}\right]\sin u$$

$$z = \frac{2uv}{1+v^2}$$

Origin of first order Partial Differential Equation: (PDE)

Type : I
Elimination of arbitrary constant; $z(x, y, a, b)$

Type : II
Elimination of arbitrary function $z = f(x, y, a, b)$

Elimination of Arbitrary Constant:

Let $f(x, y, z, a, b) = 0 \longrightarrow (1)$ where, a and b are arbitrary constant and z is the dependent variable of the function x and y

Differentiating (1) partially w.r.to 'x', we have,

$$\frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial x}+\frac{\partial f}{\partial z}\cdot\frac{\partial z}{\partial x}=0 \qquad \left[\because \frac{\partial z}{\partial x}=p\right]$$

$$\frac{\partial f}{\partial x}+\frac{\partial f}{\partial z}\cdot p=0 \longrightarrow (2)$$

Diff (1) partially w.r.to y, we get,

$$\frac{\partial f}{\partial y}\frac{\partial y}{\partial y}+\frac{\partial f}{\partial z}\cdot\frac{\partial z}{\partial y}=0$$

$$\frac{\partial f}{\partial y}+\frac{\partial f}{\partial z}\cdot q=0 \qquad \left[\because \frac{\partial z}{\partial y}=q\right]$$

Eliminating a and b we get,

The P.D.E of first order of the form

$$F(x,y,z,p,q)=0$$

Problem:

1. Eliminate the constant a and b from the following equation:

i) $z=(x+a)(y+b)$

Sol Given: $z=(x+a)(y+b) \longrightarrow (1)$

Diff (1) partially w.r. to 'x'

$$\frac{\partial z}{\partial x}=(1+0)(y+b)$$

$$\frac{\partial z}{\partial x}=y+b$$

$$p=y+b \longrightarrow (2)$$

Again diff (1) w.r. to 'y'

$$\frac{\partial z}{\partial y}=(x+a)(1+0)$$

$$\frac{\partial z}{\partial y}=x+a$$

$$q=x+a \longrightarrow (3)$$

Sub ② and ③ in ①

$$\boxed{z = p \cdot q}$$

Q) $2z = (ax+y)^2 + b$

Sol) Given : $2z = (ax+y)^2 + b \rightarrow ①$

(or)

$$\left[\begin{array}{l} z = \frac{1}{2}\left[ax^2 + y^2 + 2ax + b\right] \\ z = \frac{ax^2}{2} + \frac{y^2}{2} + ax + \frac{b}{2} \end{array}\right]$$

Diff ① partially w.r to 'x', we get

$$2\frac{\partial z}{\partial x} = 2a(ax+y)$$

$$\frac{\partial z}{\partial x} = a(ax+y)$$

$$p = a(ax+y) \rightarrow ②$$

Again diff ① partially w.r to 'y' we get

$$2\frac{\partial z}{\partial y} = 2(ax+y) \cdot 1$$

$$\frac{\partial z}{\partial y} = ax+y$$

$$q = ax+y \rightarrow ③$$

$$\frac{②}{③} \Rightarrow \frac{p}{q} = \frac{a(ax+y)}{(ax+y)}$$

$$\frac{p}{q} = a \rightarrow ④$$

Sub ④ in ②

$$2z = (p/q\, x + y)^2$$

$$p = (p/q)(p/q\, x + y)$$

$$q = \frac{p}{q}x + y$$

3. Eliminate the constant a and b, $ax^2+by^2+z^2=1$

Sol) $ax^2+by^2+z^2=1$

$$z^2 = 1 - ax^2 - by^2 \rightarrow (1)$$

Diff (1) w.r.to 'x'

$$2z\frac{\partial z}{\partial x} = -2ax$$

$$z\frac{\partial z}{\partial x} = -ax$$

$$zp = -ax$$

$$\Rightarrow ax = -zp \rightarrow (2)$$

Diff (1) w.r to 'y'

$$2z\frac{\partial z}{\partial y} = -2by$$

$$z\frac{\partial z}{\partial y} = -by$$

$$zq = -by$$

$$\Rightarrow by = -zq \rightarrow (3)$$

Sub (2) and (3) in (1)

$$z^2 = 1 - ax\cdot x - by\cdot y$$

$$z^2 = 1 - x(-zp) - y(-zq)$$

$$z^2 = 1 + zxp + yzq$$

$$z^2 = z[xp + yq + \frac{1}{z}]$$

$$z - xp - yq = \frac{1}{z}$$

4. Eliminate the constant a and b $z = ax+$

Sol Given: $z = ax + yb \rightarrow (1)$

Diff (1) w.r. to 'x'

$$\frac{\partial z}{\partial x} = a$$

$$p = a \rightarrow (2)$$

Diff (1) w.r. to 'y'

$$\frac{\partial z}{\partial y} = b$$

$$q = b \rightarrow (3)$$

Sub (2) & (3) in (1)

$$z = px + qy$$

(7) Elimination of Arbitrary function:

Let u and v be any two function of x, y, z and be connected by on arbitrary relation

$$\phi(u, v) = 0 \rightarrow (1)$$

Diff partially w.r. to 'x',

$$(1) \Rightarrow \left\{\frac{\partial \phi}{\partial u}\left(\frac{\partial u}{\partial x}\frac{\partial x}{\partial x} + \frac{\partial u}{\partial z}\cdot\frac{\partial z}{\partial x}\right) + \frac{\partial \phi}{\partial v}\left(\frac{\partial v}{\partial x}\frac{\partial x}{\partial x} + \frac{\partial v}{\partial z}\frac{\partial z}{\partial x}\right)\right\} = 0 \rightarrow (2)$$

Diff partially w.r. to 'y',

$$\left\{\frac{\partial \phi}{\partial u}\left(\frac{\partial u}{\partial y}\frac{\partial y}{\partial y} + \frac{\partial u}{\partial z}\cdot\frac{\partial z}{\partial y}\right) + \frac{\partial \phi}{\partial v}\left(\frac{\partial v}{\partial y}\frac{\partial y}{\partial y} + \frac{\partial v}{\partial z}\frac{\partial z}{\partial y}\right)\right\} = 0 \rightarrow (3)$$

Eliminating $\frac{\partial \phi}{\partial u}, \frac{\partial \phi}{\partial v}$

$$\frac{\frac{\partial \phi}{\partial u}(U_x + U_z P)}{\frac{\partial \phi}{\partial u}(U_y + U_z q)} = \frac{-\frac{\partial \phi}{\partial v}(V_x + V_z P)}{-\frac{\partial \phi}{\partial v}(V_y + V_z q)}$$

$$(U_x + U_z P)(V_y + V_z q) = (V_x + V_z P)(U_y + U_z q)$$

$$U_x V_y + U_x V_z q + V_y U_z P + U_z V_z Pq = V_x U_y + V_x U_z q + V_z U_y p + U_z V_z Pq$$

$$U_x V_z q + V_y U_z P + U_x V_y = V_x U_z q + V_z U_y P + V_x U_y$$

$$U_x V_y - V_x U_y = V_x U_z q + V_z U_y P - U_x V_z q - V_y U_z P$$

$$U_x V_y - V_x U_y = P[V_z U_y - V_y U_z] + q[V_x U_z - U_x V_z]$$

$$\Rightarrow U_x V_y - V_x U_y = [V_z U_y - V_y U_z]P + [V_x U_z - U_x V_z]q$$

$$R = Pp + Qq$$

where, $P = V_z U_y - V_y U_z$

$Q = V_x U_z - U_x V_z$

$R = U_x V_y - V_x U_y$

1. Eliminate arbitrary function F for $z = xy + f(x^2+y^2)$

Sol Given: $z = xy + f(x^2+y^2) \longrightarrow (1)$

Diff partial (1) w.r to x

$$\frac{\partial z}{\partial x} = 1 \cdot y + f'(x^2+y^2) \cdot 2x$$

$$p = y + 2x f'(x^2+y^2)$$

$$P - y = 2x f'(x^2+y^2) \longrightarrow (2)$$

Diff partially ① w.r to y

$$\frac{\partial z}{\partial y} = x + f'(x^2+y^2)\,2y$$

$$q = x + f'(x^2+y^2)\,2y$$

$$q - x = 2y\,f'(x^2+y^2) \rightarrow ③$$

② ÷ ③, we get

$$\frac{p-y}{q-x} = \frac{2x\,f'(x^2+y^2)}{2y\,f'(x^2+y^2)}$$

$$\frac{p-y}{q-x} = \frac{x}{y}$$

$$y(p-y) = x(q-x)$$

$$py - y^2 - xq + x^2 = 0$$

$$x^2 - xq + py - y^2 = 0$$

$$x^2 - y^2 - xq + py = 0$$

Test

2. Eliminate the arbitrary func., F for $z = f\left(\frac{xy}{z}\right)$

Sol Given: $z = f\left(\frac{xy}{z}\right) \rightarrow ①$

diff partially ① w.r to x

$$\frac{\partial z}{\partial x} = f'\left(\frac{xy}{z}\right)\left[\frac{z(y) - xy\left(\frac{\partial z}{\partial x}\right)}{z^2}\right]$$

$$p = f'\left(\frac{xy}{z}\right)y\left[\frac{z - xp}{z^2}\right] \rightarrow ②$$

Again diff partially ① w.r. to y

$$\frac{\partial z}{\partial y} = f'\left(\frac{xy}{z}\right)\left[\frac{z(x) - xy\left(\frac{\partial z}{\partial y}\right)}{z^2}\right]$$

$$q = f'\left(\frac{xy}{z}\right) x\left[\frac{z - yq}{z^2}\right] \rightarrow (3)$$

(2) ÷ (3), we get

$$\frac{p}{q} = \frac{f'(xy/z)\, y\left[\frac{z - xp}{z^2}\right]}{f'(xy/z)\, x\left[\frac{z - yq}{z^2}\right]}$$

$$\frac{p}{q} = \frac{y(z - xp)}{x(z - yq)}$$

$$px(z - yq) = qy(z - xp)$$

$$pxz - pyqx = qyz - xpqy$$

$$pxz - qyz = 0 \Rightarrow z(px - qy) = 0$$

$$px - qy = 0$$

3. Eliminate the arbitrary func., $z = x + y + f(xy)$

Sol: Given ⇒ $z = x + y + f(xy)$ — (1)

Diff P(1) w.r to x

$$\frac{\partial z}{\partial x} = 1 + f'(xy) \cdot y$$

$$p = 1 + y\, f'(xy)$$

$$\frac{p-1}{y} = f'(xy) \rightarrow (2)$$

again diff (1) p w.r to y

$$\frac{\partial z}{\partial y} = 1 + f'(xy) \cdot x$$

$$q = 1 + x\, f'(xy)$$

$$\frac{q-1}{x} = f'(xy) \rightarrow (3)$$

② ÷ ③ , we get

$$\frac{(p-1)/y}{(q-1)/x} = \frac{f'(xy)}{f'(xy)}$$

$$\frac{(p-1)/y}{(q-1)/x} = 1$$

$$x(p-1) = y(q-1)$$

$$xp - x - yq + y = 0$$

$$xp - yq - x + y = 0$$

$$px - qy - x + y = 0$$

4. Eliminate the arbitrary func., $f(x^2+y^2+z^2, z^2-2xy) = 0$

Sol Given: $f(x^2+y^2+z^2, z^2-2xy) = 0 \rightarrow$ ①

It is of the form $f(u,v) = 0$

where $u = x^2+y^2+z^2$, $v = z^2 - 2xy$

W.k.t, by Jacobi's $J = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = 0$

$\frac{\partial u}{\partial x} = 2x+2z\frac{\partial z}{\partial x}$	$\frac{\partial u}{\partial y} = 2y+2z\frac{\partial z}{\partial y}$	$\frac{\partial v}{\partial x} = 2z \cdot \frac{\partial z}{\partial x} - 2y$
$\frac{\partial u}{\partial x} = 2x+2zp$	$\frac{\partial u}{\partial y} = 2y+2zq$	$\frac{\partial v}{\partial x} = 2zp - 2y$

$$\frac{\partial v}{\partial y} = 2z\frac{\partial z}{\partial y} - 2x$$

$$\frac{\partial v}{\partial y} = 2zq - 2x$$

Hw

$f(xy, x+y-z) = 0$

Ans: $px - qy = x - y$

$$J \Rightarrow \begin{vmatrix} 2x+2zp & 2y+2zq \\ 2zp-2y & 2zq-2x \end{vmatrix} = 0$$

$$(2x+2zp)(2zq-2x) - (2y+2zq)(2zp-2y) = 0$$

$$2\left[(xzq - x^2 + z^2pq - xzp) - (zyp - y^2 + z^2pq - zyq)\right] = 0$$

$$xzq - x^2 + z^2pq - xzp - zyp + y^2 - z^2pq + zyq = 0$$

$$xzq - x^2 - xzp - zyp + y^2 + zyq = 0$$

$$p(-xz - zy) + q(xz + zy) - x^2 + y^2 = 0$$

$$zp(-x-y) + zq(x+y) - x^2 + y^2 = 0$$

$$-zp(x+y) + zq(x+y) - x^2 + y^2 = 0$$

$$(x+y)(zq - zp) - x^2 + y^2 = 0$$

$$(x+y)z(q-p) = x^2 - y^2$$

$$(x+y)z(q-p) = (x+y)(x-y)$$

$$z(q-p) = x-y$$

$$-z(p-q) = -(y-x)$$

$$\Rightarrow \frac{y-x}{z} = p-q$$

5. Eliminate the arbitrary function $f(xy, x+y-z) = 0$

Sol Given: $f(xy, x+y-z) = 0 \rightarrow (1)$

It is of the form $f(u,v) = 0$

where $u = xy$ $\quad v = x+y-z$

w.k.t, Jacobi's $J \Rightarrow \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = 0$

$\frac{\partial u}{\partial x} = y \quad \frac{\partial u}{\partial y} = x \quad \frac{\partial v}{\partial x} = 1 - \frac{\partial z}{\partial x} = 1 - p \quad \frac{\partial v}{\partial y} = 1 - \frac{\partial z}{\partial y} = 1 - q$

$J \Rightarrow \begin{vmatrix} y & x \\ 1-p & 1-q \end{vmatrix} = 0$

$y(1-q) - x(1-p) = 0$

$y - yq - x + xp = 0$

$px - qy = x - y$

$\therefore \; px - qy = x - y$

6. Find the PDE of sphere whose centre lie on the z-axis

Sol: w.k.t, the general equation of sphere with centre (a, b, c) and radius r is

$(x-a)^2 + (y-b)^2 + (z-c)^2 = r^2$

Here, given that the centre lies on the z-axis, that implies $x^2 + y^2 + (z-c)^2 = r^2 \longrightarrow (1)$

Diff (1) partially w.r to 'x'

$2x + 2(z-c)\frac{\partial z}{\partial x} = 0$

$2x + 2(z-c)p = 0$

$x + (z-c)p = 0 \longrightarrow (2)$

Again diff ① p.w.r to 'y'

$$2y + 2(z-c)\frac{\partial z}{\partial y} = 0$$

$$2y + 2(z-c)q = 0$$

$$y+(z-c)q = 0 \rightarrow ③$$

$$②\times q \Rightarrow qx + (z-c)pq = 0$$

$$③\times p \Rightarrow py + (z-c)pq = 0$$

$$(qx-py) = 0$$

7. Find the PDE of the sphere ~~of~~ of radius 1 having the centres on the xy-plane

sol W.k.t, the general equation of the sphere with centre (a,b,c) and radius r is $(x-a)^2+(y-b)^2+(z-c)^2=r^2$

Here, given radius r, that is with centre

$(a,b,0) \Rightarrow (x-a)^2 + (y-b^2) + z^2 = 1 \rightarrow ①$

Diff ① p.w.r to 'x'

$$2(x-a)\cdot 1 + 2z\frac{\partial z}{\partial x} = 0$$

$$2(x-a) + 2zp = 0$$

$$(x-a)+zp = 0$$

$zp = a-x \Rightarrow x-a=-zp \rightarrow ②$

Diff ① p.w.r. to 'y'

$$2(y-b) + 2z\frac{\partial z}{\partial y} = 0$$

$$2(y-b) + 2zq = 0$$

$$(y-b)+zq = 0$$

$$zq = b-y$$

$$\Rightarrow y-b = -zq \rightarrow ③$$

sub ② and ③ in ①

$$(-zp)^2 + (-zq)^2 + z^2 = 1$$

$$z^2p^2 + z^2q^2 + z^2 = 1$$

$$z^2(p^2+q^2+1) = 1$$

(5)

Surface of Revolution

Find the PDE of the Surface of Revolution.

Many Surface of revolution with z-axis as the axis of revolution are of the form $z = F(r)$, $r = (x^2+y^2)^{1/2}$ where F is arbitrary continuous differentiable function.

(ie) $\boxed{z = F(x^2+y^2)^{1/2}} \rightarrow ①$

Diff ① p. w.r to 'x'

$$\frac{\partial z}{\partial x} = F'(x^2+y^2)^{1/2} \cdot \frac{1}{2}(x^2+y^2)^{-1/2} \cdot 2x$$

$$= F'(x^2+y^2)^{1/2}(x^2+y^2)^{-1/2} \cdot x$$

$$p = F'(x^2+y^2)^{1/2} \frac{x}{(x^2+y^2)^{1/2}}$$

$$p = F'(r)\frac{1}{r} \cdot x \rightarrow ②$$

Diff ② p. w.r to 'y'

$$\frac{\partial z}{\partial y} = F'(x^2+y^2)^{1/2} \cdot \frac{1}{2}(x^2+y^2)^{-1/2} \cdot 2y$$

$$q = F'(x^2+y^2)^{1/2} \frac{y}{(x^2+y^2)^{1/2}}$$

$$q = F'(r)\frac{1}{r}\cdot y \rightarrow (3)$$

$(2) \div (3) \Rightarrow$

$$\frac{p}{q} = \frac{F'(r)\ 1/r\ x}{F'(r)\ 1/r\ y}$$

$$\frac{p}{q} = \frac{x}{y}$$

$$py = xq$$

Cauchy problem for first order equations:

Statement:

(a) If $x_0(\mu)$, $y_0(\mu)$ and $z_0(\mu)$ are functions which together with their first derivatives, are continuous in the interval M defined by $\mu_1 < \mu < \mu_2$

(b) If $F(x, y, z, p, q)$ is the continuous func, of x, y, z, p and q in certain region U of the $xyzpq$ space, then it is required to established the existence of the func, $\phi(x,y)$ with the following properties

(i) $\phi(x,y)$ and its partial derivatives with respect to x and y are continuous func, of x and y in a region R of the xy-plane

(ii) For all values of x & y lying in R, the point $\{x, y, \phi(x,y), \phi_x(x,y), \phi_y(x,y)\}$ lies in U and $F(x, y, \phi(x,y), \phi_x(x,y), \phi_y(x,y)) = 0$

(iii) For all μ belonging to the interval M, the point $\{x_0(\mu), y_0(\mu)\}$ belongs to the region R and $\boxed{\phi(x_0(\mu), y_0(\mu)) = z_0}$

Classification of 1st order PDE :

Linear Equation :

The 1st order equation $F(x, y, z, p, q) = 0$ is said to be <u>linear</u> if it is linear in p, q and z is of the form $P(x,y)+Q(x,y) = R(x,y)z+S(x,y)$

$$\boxed{P(x,y)+Q(x,y) = R(x,y)\,z + S(x,y)}$$

Eg: $yx^2p + xy^2q = xyz + x^3y^3$

$$p + q = z + xy$$

Semilinear equation :

A 1st order PDE $F(x, y, z, p, q) = 0$ is said to be semi linear if it is linear in p and q and the coefficient of p and q are func. of x and y only

(ie) $P(x,y) + Q(x,y) = R(x,y,z)$

Eg: $xq + yp = \frac{z^2x^2}{y^2}$

Quasi Linear Equation :

A 1st order PDE $F(x, y, z, p, q) = 0$ is said to be Quasi Linear if it is linear in p and q (ie) $P(x,y,z)+Q(x,y,z) = R(x,y,z)$

Eg: $(x^2-y^2)p+(y^2-zx)q = x^2-xy$

Non-linear PDE:

A 1^{st} order PDE $F(x,y,z,p,q)=0$ is said to be non-linear equation which does not contain above 3 types

Eg: $\boxed{p^2+q^2=1}$

Legrange's Equation:

A 1^{st} order linear PDE is of the form $\boxed{Pp+Qq=R}$. where, P, Q and R are func, of x, y and z

Eg: $\boxed{xyp+yzq = zx}$

Test
Solve the Cauchy's problem for $zp+q=1$, when the initial curve is $x=\mu$, $y=\mu$, $z=\frac{\mu}{2}$, $0<\mu<1$

Sol Given $zp+q=1 \rightarrow (1)$

with initial condition,

$$x_0(\mu)=\mu, \quad y_0(\mu)=\mu, \quad z_0(\mu)=\frac{\mu}{2} \rightarrow (2)$$

The given equ (1) is of the form $\boxed{Pp+Qq=R}$

where $P=z$, $Q=1$, $R=1 \rightarrow (3)$

W.k.t, the transversality condition is

$$\frac{dy_0}{d\mu}\, P(x_0(\mu), y_0(\mu), z_0(\mu)) - \frac{dx_0}{d\mu}\, Q(x_0(\mu), y_0(\mu), z_0(\mu)) \neq 0 \rightarrow (4)$$

Here $y_0=\mu$, $\frac{dy_0}{d\mu}=1$, $P=z$

$x_0=\mu \Rightarrow \frac{dx_0}{d\mu}=1$, $Q=1$

$1.x - 1.1 \neq 0$

$\Rightarrow x - 1 \neq 0$ $\quad [\because z = \mu/2]$

$\cancel{\frac{\mu}{2} - 1 \neq 0}$

We know that $\frac{dx}{dt} = P$, $\frac{dy}{dt} = Q$, $\frac{dz}{dt} = R$

$\frac{dx}{dt} = x$, $\frac{dy}{dt} = 1$, $\frac{dz}{dt} = 1$

$\int dx = \int x\,dt$, $\int dy = \int dt$, $\int dz = \int dt$

$\hookrightarrow$ ⑤ $\quad y = t + y_0 \qquad z = t + z_0$

$y = t + \mu \rightarrow$ ⑥ $\qquad z = t + \mu/2 \rightarrow$ ⑦

From ⑤, $\int dx = \int x\,dt$

$= \int \left[t + \frac{\mu}{2}\right] dt$

$= \frac{t^2}{2} + \frac{\mu}{2} t + x_0$

$= \frac{t^2}{2} + \frac{\mu t}{2} + \mu$

$x = \frac{t}{2}[t + \mu] + \mu$ — ⑧

⑥ − ⑧

$y - x = t + \mu - \left[\frac{t}{2}(t+\mu) + \mu\right]$

$= t + \mu - \frac{t}{2}(t+\mu) - \mu$

$= t - \frac{t}{2}(t+\mu)$

$= t - \frac{t}{2}(y)$ $\quad [\because$ from ⑥$]$

$y - x = t\left[1 - \frac{y}{2}\right]$

$$t = \frac{(y-x)}{1-y/2} \rightarrow (9)$$

Sub (9) in (6)

$$y = \frac{(y-x)}{1-y/2} + \mu$$

$$y = \frac{2(y-x)}{2-y} + \mu$$

$$\Rightarrow \quad \mu = -\frac{2(y-x)}{(2-y)} + y$$

$$\mu = \frac{2x-2y+y(2-y)}{(2-y)}$$

$$\mu = \frac{2x-2y+2y-y^2}{(2-y)}$$

$$\boxed{\mu = \frac{2x-y^2}{2-y}}$$

Sub $\mu = \frac{2x-y^2}{2-y}$ and $t = \frac{y-x}{1-y/2}$ in (7)

$$x = \left[\frac{2(y-x)}{2-y}\right] + \left(\frac{2x-y^2}{2-y}\right)/2$$

$$= \frac{2y-2x}{2-y} + \frac{2x-y^2}{2(2-y)}$$

$$= \frac{2(2y-2x)+(2x-y^2)}{2(2-y)}$$

$$= \frac{4y-4x+2x-y^2}{2(2-y)}$$

$$= \frac{4y-2x-y^2}{2(2-y)}$$

Solve the Cauchy's problem, $2z_x + y z_y = z$ for initial condition of the curve C is

$x_0 = s,\ y_0 = s^2,\ z_0 = s,\quad 1 \le s \le 2$

Sol: Given: $2z_x + y z_y = z$

$Pp + Qq = R$

That implies $2p + yq = z \to (1)$ with initial condition, $x_0(s,0) = s,\ y_0(s,0) = s^2,\ z_0(s,0) = s$

Now, it is of the form $Pp + Qq = R$,

where $P = 2,\ Q = y,\ R = z \to (2)$

W.k.t, the transversality condition is

$$\frac{dy_0}{ds} P(x_0(s), y_0(s), z_0(s)) - \frac{dx_0}{ds} Q(x_0(s), y_0(s), z_0(s)) \neq 0$$

(ie) $2s(2) - 1 \cdot y \neq 0$

$4s - s^2 \neq 0$

W.k.t,

$$\frac{dx}{dt} = P,\quad \frac{dy}{dt} = Q,\quad \frac{dz}{dt} = R$$

$$\frac{dx}{dt} = 2,\quad \frac{dy}{dt} = y,\quad \frac{dz}{dt} = z$$

$\int dx = 2\int dt$ $\qquad \int dy = \int y\,dt$ $\qquad \int dz = \int z\,dt$

$x = 2t + x_0$ $\qquad \int \frac{dy}{y} = \int dt$ $\qquad \int \frac{dz}{z} = \int dt$

$x = 2t + s \to (3)$ $\qquad \log y = t + \log y_0$ $\qquad \log z = t + \log z_0$

$\log y = t + \log s^2$ $\qquad \log z = t + \log s$

$\log y - \log s^2 = t$ $\qquad \log z - \log s = t$

$\log(y/s^2) = t$ $\qquad \log(z/s) = t$

$y/s^2 = e^t$ $\qquad z/s = e^t$

$y = s^2 e^t \to (4)$ $\qquad z = s e^t \to (5)$

$④ \div ⑤ \Rightarrow \quad \frac{y}{z} = \frac{s^2 e^t}{s e^t}$

$$\frac{y}{z} = s \Rightarrow \boxed{s = \frac{y}{z}} \longrightarrow ⑥$$

Sub ⑥ in ③

$$x = 2t + \frac{y}{z} \Rightarrow 2t = x - y/z$$

$$\boxed{t = \frac{zx - y}{2z}}$$

Sub $t = \frac{zx-y}{2z}$, $s = \frac{y}{z}$ in ⑤

$$z = \frac{y}{z} e^{\left(\frac{zx-y}{2z}\right)}$$

Linear Equations of the 1st order:

The P.D.E of the form $Pp + Qq = R$ ⟶ ①, where P, Q, R are functions of x, y and z and which do not involve p (or) q where $p = \frac{\partial z}{\partial x}$ and $q = \frac{\partial z}{\partial y}$ and we wish to find a relation between x, y and z involving an arbitrary function.

The equ ① is refered to as Legrange's Equation.

Theorem: [Legrange's Equation]

The general solution of the Linear P.D.E $Pp + Qq = R$ is $F(u, v) = 0$ where F is an arbitrary function and $u(x,y,z) = c_1$ and $v(x,y,z) = c_2$ form a solution of the equations $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$.

Problem:

1. Find the G.S of the differential equation

$$x^2\frac{\partial z}{\partial x} + y^2\frac{\partial z}{\partial y} = (x+y)z$$

Sol Given: $x^2\frac{\partial z}{\partial x} + y^2\frac{\partial z}{\partial y} = (x+y)z \longrightarrow (1)$

$$x^2 p + y^2 q = (x+y)z \longrightarrow (2)$$

it is of the form $Pp + Qq = R$

where $P = x^2$, $Q = y^2$, $R = (x+y)z$

The auxiliary equation is

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\frac{dx}{x^2} = \frac{dy}{y^2} = \frac{dz}{(x+y)z} \longrightarrow (3)$$

Taking 1st & 2nd terms in (3)

$$\Rightarrow \frac{dx}{x^2} = \frac{dy}{y^2}$$

Integrating

$$\int \frac{dx}{x^2} = \int \frac{dy}{y^2}$$

$$\int x^{-2}dx = \int y^{-2}dy$$

$$\left[\frac{x^{-1}}{-1}\right] + c_1 = \left[\frac{y^{-1}}{-1}\right] + c_1$$

$$\frac{-1}{x} + c_1 = \frac{-1}{y} + c_1$$

$$-\frac{1}{x} + \frac{1}{y} = c_1$$

$$\frac{-y+x}{xy} = c_1$$

$$\Rightarrow c_1 = \frac{x-y}{xy} \longrightarrow (4)$$

(Some constant c_1)

Choosing $(1, -1, 0)$ has a multiplier

Subtracting 1st two term and equating with last term

$$\frac{dx - dy}{x^2 - y^2} = \frac{d(x-y)}{(x+y)(x-y)} = \frac{dz}{(x+y)z}$$

$$\int \frac{d(x-y)}{(x-y)} = \int \frac{dz}{z}$$

$$\log(x-y) = \log z + \log c_2$$

$$\log(x-y) - \log z = \log c_2$$

$$\log\left(\frac{x-y}{z}\right) = \log c_2$$

$$\frac{x-y}{z} = c_2 = V$$

$$\frac{y^2 dx - x^2 dy}{x^2 - y^2} = \frac{dz}{(x+y)z}$$

$$y^2 dx - x^2 dy = \frac{(x^2-y^2)\, dz}{(x+y)z}$$

$$y^2 dx - x^2 dy = \frac{(x+y)(x-y)\, dz}{(x+y)z}$$

$$y^2 dx - x^2 dy = \frac{(x-y)\, dz}{z}$$

G.S, $F(u, v) = 0$

$$F\left[\frac{x-y}{xy}, \frac{x-y}{z}\right] = 0$$

Test

2. Find the G.S of the partial diff. equ., $z(xp - yq) = y^2 - x^2$ ('OON')

Sol

Given: $z(xp - yq) = y^2 - x^2$ → (1)

$$zxp - zyq = y^2 - x^2$$

it is of the form $Pp + Qq = R$

where $P = zx$, $Q = -zy$, $R = y^2 - x^2$

W.k.t, the auxiliary equ., is

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\frac{dx}{zx} = \frac{dy}{-zy} = \frac{dz}{y^2 - x^2} \rightarrow (2)$$

Taking first two terms from (2)

$$\frac{dx}{zx} = \frac{dy}{-zy}$$

$$\frac{dx}{x} = -\frac{dy}{y}$$

Integrating on Both sides

$$\int \frac{dx}{x} = -\int \frac{dy}{y}$$

$$\log x = -\log y - \log c_1$$

$$\log x + \log y = \log c_1$$

$$xy = c_1 = u$$

$$\Rightarrow \boxed{u = xy}$$

choosing (x, y, z) as a multiple and adding

Multiply & divide by x in 1st term of (2)

$$\frac{x \times dx}{x \; zx} \Rightarrow \frac{x\,dx}{zx^2} \rightarrow (3)$$

$$\frac{-y \times dy}{y \times zy} \Rightarrow \frac{-y\,dy}{zy^2} \rightarrow (4)$$

$$\frac{z \times dz}{z\,(y^2 - x^2)} \Rightarrow \frac{z\,dz}{z(y^2 - x^2)} \rightarrow (5)$$

Adding (3), (4) & (5)

$$\frac{x\,dx}{zx^2} - \frac{y\,dy}{zy^2} + \frac{z\,dz}{z(y^2 - x^2)} = 0$$

$$\frac{x\,dx - y\,dy}{z(x^2 - y^2)} + \frac{z\,dz}{z(y^2 - x^2)} = 0$$

$$\frac{x\,dx - y\,dy + z\,dz}{zx^2 - zy^2 + zy^2 - zx^2} = 0 \quad \left(\frac{du}{0}\right)$$

$$x\,dx + y\,dy + z\,dz = 0$$

Integrating on B.S

$$\int x\,dx + \int y\,dy + \int z\,dz = 0 \Rightarrow \frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} = 0$$

$$\frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} = c_2 = V$$

G.S, $F(u, v) = 0$

$$F\left[xy,\ \frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} \right] = 0$$

Theorem:

If $u_i(x_1, x_2, \ldots x_n, z) = c_i$ $[i = 1, 2, \ldots n]$ are independent solutions of the equations $\frac{dx_1}{P_1} = \frac{dx_2}{P_2} = \cdots = \frac{dx_n}{P_n} = \frac{dz}{R}$ then the relation $\phi(u_1, u_2, \ldots u_n) = 0$ in which the function ϕ is arbitrary: is the G.S of the Linear PDE $P_1 \frac{\partial z}{\partial x_1} + P_2 \frac{\partial z}{\partial x_2} + \cdots + P_n \frac{\partial z}{\partial x_n} = R$

1. If u is the func. of (x,y,z) which satisfies the P.D.E $(y-z)\frac{\partial u}{\partial x} + (z-x)\frac{\partial u}{\partial y} + (x-y)\frac{\partial u}{\partial z} = 0$.

Show that U contains (x,y,z) only in combinations $x+y+z$ and $x^2+y^2+z^2$

Sol Given: $(y-z)\frac{\partial u}{\partial x} + (z-x)\frac{\partial u}{\partial y} + (x-y)\frac{\partial u}{\partial z} = 0$ → (1)

∴ the auxiliary equ.,

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} = \frac{du}{0}$$

where $P = y-z$, $Q = z-x$, $R = x-y$

$$\frac{dx}{y-z} = \frac{dy}{z-x} = \frac{dz}{x-y} = \frac{du}{0} \rightarrow (2)$$

Taking 3rd & 4th term from (2)

$$\frac{dx}{x-y} = \frac{du}{0}$$

$$\Rightarrow du = 0$$

$$\int du = 0$$

$$\boxed{u = c_1} \rightarrow (3)$$

Choosing $(1,1,1)$ as a multiplier and equating with 4th term.

$$\frac{dx + dy + dz}{(y-z)+(z-x)+(x-y)} = 0$$

$$dx + dy + dz = 0$$

$$\int dx + \int dy + \int dz + \int d\,du = 0$$

$$x+y+z+u = c_2$$

$$\Rightarrow \quad x+y+z = u + c_2$$

$$x+y+z = c_1 + c_2$$

$\longrightarrow (4)$

$$x+y+z = c_3$$

$$\Rightarrow \quad x+y+z = u$$

(x,y,z) as a multiplier and equating with 4^{th} term

$$\frac{x\,dx + y\,dy + z\,dz}{x(y-z)+y(z-x)+z(x-y)} = 0$$

$$x\,dx + y\,dy + z\,dz = 0$$

$$\int x\,dx + \int y\,dy + \int z\,dz = 0$$

$$\left[\frac{x^2}{2}\right] + \left[\frac{y^2}{2}\right] + \left[\frac{z^2}{2}\right] + c_4 = 0$$

$$\frac{x^2+y^2+z^2}{2} = v$$

$\therefore$ The general solution $\quad u = f(u,v)$

$$u = f\left[x+y+z,\ \frac{x^2+y^2+z^2}{2}\right]$$

Integral surfaces passing through a given curve.

1. Find the integral surface of the Linear PDE $x(y^2+z)p - y(x^2+z)q = (x^2-y^2)z$ which contains the st. line $x+y=0$, $z=1$

Sol: Given: $x(y^2+z)p - y(x^2+z)q = (x^2-y^2)z \rightarrow (1)$

It is of the form,

$$Pp + Qq = R$$

where $P = x(y^2+z)$ $Q = -y(x^2+z)$

and $R = (x^2-y^2)z$

W.k.t, the auxillary equation is,

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\frac{dx}{x(y^2+z)} = \frac{dy}{-y(x^2+z)} = \frac{dz}{(x^2-y^2)z}$$

Choosing $(x, y, -1)$ as a multiplier for each term

$$\frac{xdx + ydy - dz}{x^2(y^2+z) - y^2(x^2+z) - (x^2-y^2)z} = 0$$

$$\frac{xdx + ydy - dz}{x^2y^2 + x^2z - y^2x^2 - y^2z - x^2z + y^2z} = 0$$

$$xdx + ydy - dz = 0$$

$$\int x\,dx + \int y\,dy - \int dz = 0$$

$$\left[\frac{x^2}{2}\right] + \left[\frac{y^2}{2}\right] - z = c_1$$

$x^2+y^2-2z = 2C_1$

$x^2+y^2-2z = c_1 \Rightarrow U = x^2+y^2-2z$

Choosing $\left(\frac{1}{x}, \frac{1}{y}, \frac{1}{z}\right)$ as a multiplier for each term ②

$$\frac{\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz}{\frac{1}{x}x(y^2+z) - \frac{1}{y}y(x^2+z) + \frac{1}{z}z(x^2-y^2)} = 0$$

$$\frac{\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz}{y^2+z-x^2-z+x^2-y^2} = 0$$

$$\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz = 0$$

$$\int\frac{dx}{x} + \int\frac{dy}{y} + \int\frac{dz}{z} = 0$$

$$\log x + \log y + \log z = \log c_2$$

$$\log(xyz) = \log c_2$$

$$xyz = c_2 \rightarrow ③$$

$$\Rightarrow V = xyz$$

∴ The general solution $U \Rightarrow f(u,v) = 0$

$$\Rightarrow f\left[x^2+y^2-2z,\ xyz\right] = 0$$

Also the surface passes through a straight line

$x+y=0$, $z=1$

Let $x = t$

$t+y = 0$

$y = -t$

$\therefore x = t,\ y = -t,\ z = 1 \longrightarrow ④$

Sub (4) in (2)

$c_1 = t^2 + t^2 - 2$

$c_1 = 2t^2 - 2$

$\boxed{c_1 = 2(t^2-1)} \to (5)$

Sub (4) in (3)

$xyz = c_2$

$\Rightarrow \quad c_2 = (t)(-t)(1)$

$\boxed{c_2 = -t^2} \to (6)$

Sub (6) in (5)

$c_1 = 2(-c_2 - 1)$

$c_1 = -2c_2 - 2$

$c_1 + 2c_2 + 2 = 0$

$(x^2+y^2-2z) + 2(xyz) + 2 = 0$

$x^2+y^2-2z+2xyz+2 = 0$

2. Find the equation of the integral surface of the PDE $2y(z-3)p + (2x-z)q = y(2x-3)$ which passes through the circle $z=0$, $x^2+y^2=2x$

Sol Given: $2y(z-3)p + (2x-z)q = y(2x-3) \to (1)$

It is in the form of $Pp + Qq = R$

where $P = 2y(z-3)$, $Q = (2x-z)$, $R = y(2x-3)$

W.k.t, the auxiliary equ.,

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\frac{dx}{2y(z-3)} = \frac{dy}{(2x-2)} = \frac{dz}{(2x-3)y}$$

Equating 1^{st} & last term

$$\frac{dx}{2y(z-3)} = \frac{dz}{(2x-3)y}$$

$$(2x-3)\,dx = 2(z-3)\,dz$$

Integrating

$$\int (2x-3)\,dx = 2\int (z-3)\,dz$$

$$\int 2x\,dx - 3\int dx = 2\left[\int z\,dz - 3\int dz\right]$$

$$\left[\frac{2x^2}{2}\right] - 3[x] = 2\left[\left(\frac{z^2}{2}\right) - 3z\right]$$

$$x^2 - 3x = z^2 - 6z + c_1$$

$$x^2 - z^2 - 3x + 6z = c_1 = U \quad (2)$$

$$\Rightarrow U = x^2 - z^2 - 3x + 6z$$

Choosing $(1/2, y, -1)$ as a multiplier

$$\frac{\frac{1}{2}dx + y\,dy - dz}{\frac{1}{2}2y(z-3) + y(2x-2) - y(2x-3)} = 0$$

$$\frac{\frac{1}{2}dx + y\,dy - dz}{y(z-3) + 2xy - 2y - y2x + 3y} = 0$$

$$\frac{\frac{1}{2}dx + y\,dy - dz}{yz - 3y + 2xy - 2y - 2xy + 3y} = 0$$

$\frac{1}{2}dx + y\,dy - dz = 0$

$\int \frac{1}{2}dx + \int y\,dy - \int dz = 0$

$\frac{x}{2} + \frac{y^2}{2} - z = c_2$

$x + y^2 - 2z = 2c_2$

$\Rightarrow c_2 = x + y^2 - 2z \rightarrow (3)$

$v = x + y^2 - 2z$

The General solution

$f(u, v) = 0$

$f[x^2 - z^2 - 3x + 6z, \; x + y^2 - 2z] = 0$

Also the surface passes through the circle

$z = 0, \; x^2 + y^2 = 2x$

Let $x = t$

$t^2 + y^2 = 2(t)$

$y^2 = 2t - t^2$

$\therefore \; x = t, \; y^2 = 2t - t^2, \; z = 0 \rightarrow (4)$

Sub (4) in (2)

$(2) \Rightarrow t^2 - 3(t) + 6(0) = c_1$

$t^2 - 3t = c_1$

$\Rightarrow \boxed{c_1 = t^2 - 3t} \rightarrow (5)$

Sub (4) in (3)

$t + (2t - t^2) - 2(0) = c_2$

$$t + 2t - t^2 = c_2$$

$$\Rightarrow \quad c_2 = 3t - t^2$$

$$c_2 = -(t^2 - 3t)$$

$$c_2 = -c_1$$

$$\Rightarrow \quad c_2 + c_1 = 0$$

$$\Rightarrow \quad c_1 + c_2 = 0$$

$$x^2 - z^2 - 3x + 6z + x + y^2 - 2z = 0$$

$$x^2 + y^2 - z^2 - 2x + 4z = 0$$

Surface Orthogonal to given system of surface:

Step: 1

Given a system of surface $F(x,y,z) = c$, where c is the parameter obtain the auxillary equation in the form.

$$\frac{dx}{\left(\frac{\partial f}{\partial x}\right)} = \frac{dy}{\left(\frac{\partial f}{\partial y}\right)} = \frac{dz}{\left(\frac{\partial f}{\partial z}\right)} \longrightarrow (1)$$

Step: 2

Using the above equation to obtain the two independent equation $U(x,y,z) = c_1$ and $V(x,y,z) = c_2$

Step: 3

The required surface orthogonal to the given system of surface is given by, $V = f(u)$

1\. Test Find the surface which intersect the surfaces of the system $z(x+y) = c(3z+1)$ orthogonally and which passes through the circle $x^2+y^2 = 1$, $z = 1$

Sol Given $z(x+y) = c(3z+1)$

$$\Rightarrow c = \frac{z(x+y)}{(3z+1)}$$

It is of the form $F(x,y,z) = c$

where c is the parameter

$$\therefore f = \frac{z(x+y)}{(3z+1)}$$

w.k.t, the auxillary equ., is

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} \rightarrow (1)$$

where $P = \frac{\partial f}{\partial x}$, $Q = \frac{\partial f}{\partial y}$, $R = \frac{\partial f}{\partial z}$

$$P = \frac{\partial f}{\partial x} = F_x = \frac{z}{(3z+1)}$$

$$Q = \frac{\partial f}{\partial y} = F_y = \frac{z}{(3z+1)}$$

$$R = \frac{\partial f}{\partial z} = F_z = \frac{(3z+1)(x+y) - z(x+y)3}{(3z+1)^2}$$

$$F_z = \frac{x+y}{(3z+1)^2}$$

$$(1) \Rightarrow \frac{dx}{\frac{z}{(3z+1)}} = \frac{dy}{\frac{z}{(3z+1)}} = \frac{dz}{\frac{x+y}{(3z+1)^2}}$$

$$\frac{dx}{z} = \frac{dy}{z} = \frac{(3z+1)\,dz}{x+y}$$

Equating 1st two term

$$\frac{dx}{z} = \frac{dy}{z}$$

$$dx = dy$$

Integrating

$$\int dx = \int dy$$

$$x = y + c_1$$

$$x - y = c_1 \Rightarrow u = x - y$$

(2)

Choosing $(x, y, 0)$ as multiplier

$$\frac{x\,dx + y\,dy + 0}{x z + z y + 0} = 0$$

$$\frac{x\,dx + y\,dy}{xz + zy} = 0 \longrightarrow (3)$$

Equating (3) by last term

$$\frac{(3z+1)\,dz}{(x+y)} = \frac{x\,dx + y\,dy}{xz + zy}$$

$$\frac{(3z+1)\,dz}{(x+y)} = \frac{x\,dx + y\,dy}{z(x+y)}$$

$$z(3z+1)\,dz = x\,dx + y\,dy$$

$$(3z^2 + z)\,dz = x\,dx + y\,dy$$

Integrating

$$\int 3z^2\,dz + \int z\,dz = \int x\,dx + \int y\,dy$$

$$3\left(\frac{z^3}{3}\right) + \frac{z^2}{2} = \frac{x^2}{2} + \frac{y^2}{2} + c_2$$

$$z^3 + \frac{z^2}{2} = \frac{x^2}{2} + \frac{y^2}{2} + c_2$$

$z^3 + \frac{z^2}{2} - \frac{x^2}{2} - \frac{y^2}{2} = c_2$

$\Rightarrow 2z^3 + z^2 - x^2 - y^2 = 2c_2$

$\Rightarrow 2z^3 + z^2 - x^2 - y^2 = v$

$\Rightarrow c_2 = 2z^3 + z^2 - x^2 - y^2$

$\therefore$ The general solution $v = f(u)$

here $u = x - y$

$v = 2z^3 + z^2 - x^2 - y^2$

$2z^3 + z^2 - x^2 - y^2 = f(x-y)$

$2z^3 + z^2 - (x^2+y^2) = f(x-y)$

Also given $x^2+y^2 = 1$, $z = 1$

$2(1)^3 + (1)^2 - 1 = f(x-y)$

$\Rightarrow f(x-y) = 2 + 1 - 1$

$f(x-y) = 2$

$\therefore \boxed{2z^3 + z^2 - x^2 - y^2 - 2 = 0}$

2. Find the equ. of the system of the surface which cut orthogonally $x^2+y^2+z^2 = c(xy)$

Sol Given: $x^2+y^2+z^2 = c(xy)$

$\Rightarrow c = \frac{x^2+y^2+z^2}{xy}$

It is of the form $F(x,y,z) = c$

where c is parameter

$\therefore f = \frac{x^2+y^2+z^2}{xy}$

w.k.t, the auxillary eqn is

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} \rightarrow (1)$$

where $P = \frac{\partial f}{\partial x}$, $Q = \frac{\partial f}{\partial y}$, $R = \frac{\partial f}{\partial z}$

$$P = \frac{\partial f}{\partial x} = F_x = \frac{xy(2x) - (x^2+y^2+z^2)(y)}{(xy)^2}$$

$$= \frac{2x^2y - x^2y - y^3 - z^2y}{(xy)^2}$$

$$= \frac{1x^2y - y^3 - z^2y}{(xy)^2}$$

$$= \frac{x^2y - y^3 - z^2y}{(xy)^2} \Rightarrow \frac{x^2 - y^2 - z^2}{x^2y} = F_x$$

$$Q = \frac{\partial f}{\partial y} = F_y = \frac{xy(2y) - (x^2+y^2+z^2)(x)}{(xy)^2}$$

$$= \frac{2xy^2 - x^3 - xy^2 - xz^2}{(xy)^2}$$

$$= \frac{xy^2 - x^3 - xz^2}{(xy)^2}$$

$$F_y = y^2 - x^2 - z^2 / xy^2$$

$$R = \frac{\partial f}{\partial z} = F_z = \frac{2z}{xy}$$

$$(1) \Rightarrow \cancel{\frac{dx\ x^2y}{x^2 - y^2 - z^2}} = \frac{dy\ xy^2}{y^2 - x^2 - z^2} = \frac{dz\,(xy)}{2z}$$

$$\frac{dx\,(x)}{x^2 - y^2 - z^2} = \frac{dy\,(y)}{y^2 - x^2 - z^2} = \frac{dz}{2z} \rightarrow (2)$$

Choose $(1,1,0)$ as a multiplier

$$\frac{x\,dx + y\,dy}{x^2 - y^2 - z^2 + y^2 - x^2 - z^2} = 0$$

$$\frac{x\,dx + y\,dy}{-2z^2} = 0 \longrightarrow (3)$$

Equating (3) with last term in (2)

$$\frac{x\,dx + y\,dy}{-2z^2} = \frac{dz}{2z}$$

$$x\,dx + y\,dy = -z\,dz$$

$$\int x\,dx + \int y\,dy = -\int z\,dz$$

$$\frac{x^2}{2} + \frac{y^2}{2} = -(z^2/2) + c_1 \Rightarrow x^2 + y^2 + z^2 = 2c_1 = \alpha$$

$(1, -1, 0)$ as a multiplier

$$\Rightarrow \frac{x\,dx - y\,dy}{x^2 - y^2 - z^2 - y^2 + x^2 + z^2} = \frac{x\,dx - y\,dy}{2x^2 - 2y^2} \longrightarrow (4)$$

Equating (4) and last term are equal

$$\frac{x\,dx - y\,dy}{2x^2 - 2y^2} = \frac{dz}{2z}$$

$$\frac{x\,dx - y\,dy}{x^2 - y^2} = \frac{dz}{z}$$

Let $t = x^2 - y^2$

$$dt = 2x\,dx - 2y\,dy$$

$$\Rightarrow x\,dx - y\,dy = \frac{dt}{2}$$

$$\int \frac{dt}{2t} = \int \frac{dz}{z}$$

$$\frac{1}{2}\log t = \log z + \log c_2$$

$$\frac{1}{2}\log t - \log z = \log c_2$$

$$\log t - 2\log z = 2\log c_2$$

$$\log (x^2-y^2) - \log (z)^2 = 2\log c_2$$

$$\log \left[\frac{x^2-y^2}{z^2}\right] = 2\log c_2$$

$$\frac{x^2-y^2}{z^2} = c_2^2 = v$$

$$\Rightarrow c_2 = \frac{x^2-y^2}{z^2}$$

The General solution is $v = f(u)$

here $u = x^2+y^2+z^2$

$$v = \frac{x^2-y^2}{z^2}$$

$$\frac{x^2-y^2}{z^2} = f(x^2+y^2+z^2)$$

Non-linear partial differential equation of the first order equation.

Complete Integral (2M)

Let the PDE be $F(x,y,z,a,b) = 0$. Then the complete integral is $\phi(x,y,z,p,q) = 0$

General Integral :

Let us consider the relation $\phi(x,y,z,a,b) = 0$. We assume that $b = f(a)$, then $\phi(x,y,z,a,f(a)) = 0$ → ①

differential partially 'a';

$$\frac{\partial \phi}{\partial a} + \frac{\partial \phi}{\partial f(a)} F'(a) = 0 \longrightarrow (2)$$

∴ The eliminate 'a' from (1) & (2) we get the solution.

This is known as general integral.

Singular Integral:

Let $F(x, y, z, p, q) = 0$ be the PDE,

whose complete integral is,

$$\phi(x, y, z, a, b) = 0 \longrightarrow (1)$$

Differentiate partially w.r to a & b,

$$\left.\frac{\partial \phi}{\partial a} = 0, \quad \frac{\partial \phi}{\partial b} = 0 \right\} \longrightarrow (2)$$

Eliminate a and b from (1) & (2), we get the solution

This known as "Single Integral".

1. Show that $2z = (ax+y)^2 + b$ is a complete integral of $px + qy - q^2 = 0$

Sol Given: $2z = (ax+y)^2 + b \longrightarrow (1)$

diff partially w.r to x

$$2\frac{\partial z}{\partial x} = 2(ax+y) \cdot a$$

$$\frac{\partial z}{\partial x} = (ax+y) a$$

$$p = (ax+y) a \longrightarrow (2)$$

Diff partially w.r to 'y'

$$2\frac{\partial z}{\partial y} = 2(ax+y)\cdot 1$$

$$\frac{\partial z}{\partial y} = ax+y$$

$$q = ax+y \rightarrow ③$$

② ÷ ③ $\Rightarrow$ $$\frac{p}{q} = \frac{(ax+y)a}{(ax+y)}$$

$$\frac{p}{q} = a$$

$$\Rightarrow \boxed{a = p/q}$$

④ in ③ $\Rightarrow$ $$q = x(p/q)+y$$

$$\Rightarrow q = \frac{px+qy}{q}$$

$$\Rightarrow q^2 = px+qy$$

$$\Rightarrow px+qy-q^2=0$$

$\therefore$ The complete integral is $px+qy-q^2=0$

2. S.T $z = ax + \frac{y}{a} + b$ is a complete integral of $pq = 1$ then this eqn. has no singular integral and find the particular solution corresponding to $b=a$.

Sol Given: $z = ax + \frac{y}{a} + b \rightarrow ①$

diff partially w.r to 'x'

$$\frac{\partial z}{\partial x} = a$$

$$\boxed{p = a} \rightarrow ②$$

diff partially w.r to y

$$\frac{\partial z}{\partial y} = \frac{1}{a}$$

$$\boxed{q = 1/a} \rightarrow (3)$$

Sub (3) in (2)

$$p = \frac{1}{q}$$

$$\boxed{pq = 1}$$

To find the particular integral

$$(1) \Rightarrow z = ax + \frac{y}{a} + b$$

$$z = ax + \frac{y}{a} + a$$

$$z = ax + a + \frac{y}{a}$$

$$z = a(x+1) + ya^{-1} \rightarrow (4)$$

diff partially w.r to 'a' and we know that $\frac{\partial z}{\partial a} = 0$

$$\frac{\partial z}{\partial a} = (x+1) - \frac{y}{a^2}$$

$$\Rightarrow (x+1) - \frac{y}{a^2} = 0$$

$$\Rightarrow (x+1) = \frac{y}{a^2}$$

$$\Rightarrow a^2 = \frac{y}{x+1}$$

$$a = \sqrt{\frac{y}{x+1}} \rightarrow (5)$$

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Sub $a = \sqrt{\frac{y}{x+1}}$ in (4)

$$z = \sqrt{\frac{y}{x+1}}\,(x+1) + y\,\frac{1}{\sqrt{y/(x+1)}}$$

$$z = \sqrt{y}\,\sqrt{x+1} + \sqrt{y}\,\sqrt{x+1}$$

$$\boxed{z = 2\sqrt{y(x+1)}}$$

3. S.T $(x-a)^2+(y-b)^2+z^2=1$ is a complete integral of $z^2(p^2+q^2+1)=1$ by taking $b=2a$ then S.T the envelop of the sub family is $(y-2x)^2+5z^2=5$ which is a Particular integral and also S.T $z=\pm 1$ are the singular integral.

Sol Given: $(x-a)^2+(y-b)^2+z^2=1 \longrightarrow (1)$

diff partially with r to 'x'

$$2(x-a)\cdot 1 + 2z\frac{\partial z}{\partial x} = 0$$

$$2(x-a) + 2zp = 0$$

$$2zp = -2(x-a)$$

$$p = -\frac{(x-a)}{z}$$

$$p = -\frac{(x-a)}{z} \Rightarrow \boxed{(x-a) = -zp} \quad (2)$$

diff partially w.r to 'y'

$$2(y-b)\cdot 1 + 2z\frac{\partial z}{\partial y} = 0$$

$$2zq = -2(y-b)$$

$$q = -\frac{(y-b)}{z}$$

$$q = -\frac{(y-b)}{z} \Rightarrow \boxed{(y-b) = -zq} \quad (3)$$

Sub ⑤ & ③ in ①

$$(-zp)^2 + (-zq)^2 + z^2 = 1$$

$$z^2p^2 + z^2q^2 + z^2 = 1$$

$$\Rightarrow z^2(p^2+q^2+1) = 1$$

which is a complete integral

To find the particular integral,

Put $b = 2a$ in equ ①

$$① \Rightarrow (x-a)^2 + (y-2a)^2 + z^2 = 1 \longrightarrow ④$$

diff w.r to a & w.k.t $\frac{\partial z}{\partial a} = 0$

$$2(x-a)(-1) + 2(y-2a)(-2) + 2z\frac{\partial z}{\partial a} = 0$$

$$-2(x-a) - 4(y-2a) + 2z(0) = 0$$

$$-(x-a) - 2(y-2a) = 0$$

$$-x + a - 2y + 4a = 0$$

$$5a - x - 2y = 0$$

$$5a = x + 2y$$

$$\boxed{a = \frac{x+2y}{5}} \rightarrow ⑤$$

Sub ⑤ in ④

$$\left[x - \left(\frac{x+2y}{5}\right)\right]^2 + \left[y - 2\left(\frac{x+2y}{5}\right)\right]^2 + z^2 = 1$$

$$x^2 + \left(\frac{x+2y}{5}\right)^2 - 2x\left(\frac{x+2y}{5}\right) + y^2 + 4\left(\frac{x+2y}{5}\right)^2 - 4y\left(\frac{x+2y}{5}\right) + z^2 = 1$$

$$x^2 + \frac{(x+2y)^2}{25} - 2x\left(\frac{x+2y}{5}\right) + y^2 + 4\frac{(x+2y)^2}{25} - 4y\left(\frac{x+2y}{5}\right) + z^2 = 1$$

$$\frac{25x^2}{25} + \frac{(x+2y)^2}{25} - \frac{10x(x+2y)}{25} + \frac{y^2 25}{25} + \frac{4(x+2y)^2}{25} - \frac{20y(x+2y)}{25} + \frac{25z^2}{25} = 1$$

$$25x^2 + (x+2y)^2 - 10x(x+2y) + 25y^2 + 4(x+2y)^2 + 25z^2 = 25$$

$$25x^2 + x^2 + 4y^2 + 4xy - 10x^2 - 20y + 25y^2 + 4(x^2+4y^2+4xy) + 25z^2 = 25$$

$$25x^2 + x^2 + 4y^2 + 4xy - 10x^2 - 20y + 25y^2 + 4x^2 + 16y^2 + 16xy + 25z^2 = 25$$

$$20x^2 + 45y^2 + 20xy - 20y + 25z^2 = 25$$

$$5(4x^2 + 9y^2 + 4xy - 4y + 5z^2) = 25$$

$$4x^2 + 9y^2 + 4xy - 4y + 5z^2 = 5$$

(5) To find Singular integral,

$$(x-a)^2 + (y-b)^2 + z^2 = 1$$

diff partially w.r to a, $\frac{\partial z}{\partial a} = 0$

$$2(x-a) + 2z\frac{\partial z}{\partial a} = 0$$

$$2(x-a) = 0$$

$$(x-a) = 0$$

$$\boxed{x = a}$$

diff partially w.r.t 'b' & w.r.t, $\frac{\partial z}{\partial b} = 0$

$$2(y-b) + 2z\frac{\partial z}{\partial b} = 0$$

$$2(y-b) = 0$$

$$\boxed{y = b}$$

Sub $x=a$, $y=b$ in ①

$$(a-a)^2 + (b-b)^2 + z^2 = 1$$

$$z^2 = 1$$

$$z = \pm 1$$

Unit - 2

Cauchy's Method of Characteristics for solving non-linear partial differential equation.

Characteristic Strip:

A solution $\{x(t), y(t), z(t), p(t), q(t)\}$ of the system of differential equations, i.e. $f(x, y, z, p, q) = 0$ then

$$\frac{dx}{dt} = f_p, \quad \frac{dy}{dt} = f_q$$