$$\frac{dp}{dt} = f_x = -pf_z$$

$$\frac{dq}{dt} = -f_y = -qf_z$$

$$\frac{dz}{dt} = pf_p + qf_q$$ can be transformed as strip.

The first three function, $x(t), y(t), z(t)$ determine a space curve $p(t), q(t)$ determine the tangent plane $(p, q, -1)$ as normal vector.

$\therefore$ The curve along with this tangent plane at each point is called a characteristic strip and the curve is called characteristic curve.

Initial strip:

Let $x = x_0(s)$, $y = y_0(s)$, $z = z_0(s)$

we take, some arbitrary initial data

Suppose that this curve we can specified this function $p_0(s)$ and $q_0(s)$ such that together with the initial data curve.

This satisfies the equation $f(x_0(s), y_0(s), z_0(s), p_0(s), q_0(s))$ and the strip condition is $\frac{dz_0}{ds} = p_0\frac{dx_0}{ds} + q_0\frac{dy_0}{ds}$

Then such an initial elements $(x_0(s), y_0(s), z_0(s), p_0(s), q_0(s))$ is said to be initial strip for the initial data curve $(x_0(s), y_0(s), z_0(s))$.

1. Find the solution of the characteristics equ $z = \frac{p^2+q^2}{2} + (p-x)(q-y)$ which passes through the x-axis.

Sol) Given: $z = \frac{p^2+q^2}{2} + (p-x)(q-y) \rightarrow (1)$

and the integral surface which passes through the x-axis

(ie) $y = 0$, $z = 0 \rightarrow (2)$

we have $x = s$, $y = 0$, $z = 0$

$\therefore$ Let the initial value of $(x_0, y_0, z_0, p_0, q_0)$ be (x, y, z, p, q) taken as

$$\left.\begin{aligned} x &= x_0(s) = s \\ y &= y_0(s) = 0 \\ z &= z_0(s) = 0 \end{aligned}\right\} \rightarrow (3)$$

Since the initial values $(x_0, y_0, z_0, p_0, q_0)$ satisfies the eqn (1) we have

$$z_0 = \frac{p_0^2+q_0^2}{2} + (p_0 - x_0)(q_0 - y_0) \rightarrow (4)$$

Sub (3) in (4)

$$0 = \frac{p_0^2+q_0^2}{2} + (p_0 - s)(q_0 - 0)$$

$$\Rightarrow \frac{p_0^2+q_0^2}{2} + q_0(p_0 - s) = 0 \rightarrow (5)$$

W.k.t.

By using strip condition

$\frac{dz_0}{ds} =$

By

$0 =$

0

$\Rightarrow p_0$

using

$$\frac{dz_0}{ds} = p_0 \frac{dx_0}{ds} + q_0 \frac{dy_0}{ds}$$

By sub $x_0, y_0\ z_0$, we have

$$0 = p_0 \frac{ds}{ds} + q_0 (0)$$

$$0 = p_0 (1).$$

$$\Rightarrow p_0 = 0$$

using $p_0 = 0$ in (5)

$$0 = \frac{0 + q_0^2}{2} + q_0 (-s)$$

$$\Rightarrow \frac{q_0^2}{2} - q_0 s = 0$$

$$q_0 \left(\frac{q_0}{2} - s\right) = 0$$

$q_0 = 0$ | $q_0 - 2s = 0$, $\boxed{q_0 = 2s}$

Therefore, the initial solution of $(x_0, y_0, z_0, p_0, q_0)$ is $(x_0 = s,\ y_0 = 0,\ z_0 = 0,\ p_0 = 0,\ q_0 = 0,\ q_0 = 2s)$

Let $F(x, y, z, p, q) = \frac{p^2 + q^2}{2} + (p - x)(q - y) - z$

∴ The characteristic equ., is

$$\frac{dx}{dt} = F_p = \frac{2p}{2} + (1)(q - y) = p + q - y \rightarrow (6)$$

$$\frac{dy}{dt} = F_q = \frac{2q}{2} + (1)(p - x) = q + p - x \rightarrow (7)$$

$$\frac{dp}{dt} = -F_x - pF_z$$

$$f_x = (-1)(q - y) \Rightarrow F_x = -1$$

$$\frac{dp}{dt} = -F_x - pF_z$$

$$= q - y + p$$

$$\frac{dp}{dt} = p + q - y \rightarrow (8)$$

$$\frac{dq}{dt} = -F_y - qF_z$$

$$F_y = (-1)(p - x)$$

$$\frac{dq}{dt} = p - x + q$$

$$= p + q - x \rightarrow (9)$$

$$\frac{dz}{dt} = pF_p + qF_q$$

$$= p(p + q - y) + q(p + q - x)$$

$$\frac{dz}{dt} = p^2 + qp - py + pq + q^2 - qx \rightarrow (10)$$

From (6) & (8)

$$\frac{dx}{dt} = p + q - y \qquad \frac{dp}{dt} = p + q - y$$

$$\frac{dx}{dt} - \frac{dp}{dt} = 0$$

$$\int \frac{d(x-p)}{dt} = 0$$

$$\int d(x-p) = \int 0$$

$$x - p = c_1 \rightarrow (7)$$

Using initial condition, we have

$$S - 0 = c_1$$

$c_1 = s$

Sub $c_1 = s$ in (*)

$x - p = s$

$\Rightarrow \boxed{x = p + s} \rightarrow$ (A)

From (7) & (9)

$$\frac{dy}{dt} = p + q - x = \frac{dq}{dt}$$

$$\frac{dy}{dt} - \frac{dq}{dt} = 0$$

$$\int \frac{d(y-q)}{dt} = 0$$

$$\int d(y-q) = 0$$

$$y - q = c_2 \rightarrow (*_1)$$

Using initial condition,

$$0 - 2s = c_2$$

$$c_2 = -2s$$

Sub $c_2 = -2s$ in $(*_1)$

$$y - q = -2s$$

$$q - 2s = y$$

$$\Rightarrow \boxed{y = q - 2s} \rightarrow (B)$$

Taking $\frac{dp}{dt} + \frac{dq}{dt} - \frac{dx}{dt} = p + q - x$

$$\frac{d(p+q-x)}{dt} = p + q - x$$

$$\int \frac{d(p+q-x)}{(p+q-x)} = \int dt$$

$\log(p+q-x) = t + \log c_3$

$e^{\log\left(\frac{p+q-x}{c_3}\right)} = e^t$

$\frac{p+q-x}{c_3} = e^t$

$\boxed{p+q-x = c_3 e^t} \longrightarrow (*_2)$

$\Rightarrow p + q_0 - x_0 = c_3 e^{t_0}$

$0 + 2s - s = c_3(1)$

$s = c_3$

Sub $c_3 = s$ in $(*_2)$

$p+q-x = s e^t \longrightarrow (11)$

Taking,

$\frac{dp}{dt} + \frac{dq}{dt} - \frac{dy}{dt} = p+q-y$

$\frac{d(p+q-y)}{dt} = p+q-y$

$\int \frac{d(p+q-y)}{(p+q-y)} = \int dt$

$\log(p+q-y) = t + \log c_4$

$e^{\log\frac{(p+q-y)}{c_4}} = e^t$

$\frac{p+q-y}{c_4} = e^t$

$p+q-y = c_4 e^t$

$\Rightarrow P_0 + q_0 - y_0 = c_4 e^t$

$0 + 2s - 0 = c_4$

$\Rightarrow \boxed{c_4 = 2s}$

$$\boxed{p + q - y = 2se^t} \longrightarrow (12)$$

Now, we have to find the values of x, y, p, q in terms of s & t

from equ (10)

$$\frac{dz}{dt} = pF_p + qF_q$$

$$= p(p+q-y) + q(p+q-x)$$

$$\frac{dz}{dt} = p(2se^t) + q(se^t)$$

$$\frac{dz}{dt} = se^t(2p+q) \longrightarrow (13)$$

from (11)

$p+q-x = se^t \quad \{\because x = p+s\}$

$p+q-p-s = se^t$

$q - s = se^t$

$$\boxed{q = s + se^t}$$

from (12)

$p+q-y = 2se^t. \quad \{\because y = q - 2s\}$

$p+q-q+2s = 2e^ts$

$p = 2se^t - 2s$

$$\boxed{p = 2s(e^t - 1)}$$

Sub p & q in (13)

$$\frac{dz}{dt} = se^t\,(2\,(2s(e^t-1) + s(1+e^t))$$

$$= s^2e^t\,(4e^t - 4 + 1 + e^t)$$

$$= s^2e^t\,(5e^t - 3)$$

$$\frac{dz}{dt} = 5s^2e^{2t} - 3s^2e^t$$

$$\Rightarrow dz = 5s^2e^{2t}\,dt - 3s^2e^t\,dt$$

$$\Rightarrow z = \frac{5s^2e^{2t}}{2} - 3s^2e^t + c_5$$

$$0 = \frac{5s^2}{2} - 3s^2 + c_5$$

$$c_5 = 3s^2 - \frac{5s^2}{2}$$

$$= \frac{6s^2 - 5s^2}{2}$$

$$\boxed{c_5 = s^2/2}$$

$$z = \frac{5s^2e^{2t}}{2} - 3s^2e^t + \frac{s^2}{2}$$

$$z = \frac{s^2}{2}\left[5e^{2t} - 6e^t + 1\right] \longrightarrow (14)$$

from (A),

$$x = p + s$$

$$x = 2s\,(e^t - 1) + s$$

$= 2se^t - 2s + s$

$= 2e^t - s$

$x = s(2e^t - 1) \rightarrow (15)$

from (13)

$y = q - 2s$

$= s(1 - e^t) - 2s$

$= s + se^t - 2s$

$= se^t - s$

$\boxed{y = s(e^t - 1)} \rightarrow (16)$

$(15) \div (16) \Rightarrow \quad \frac{x}{y} = \frac{s(2e^t - 1)}{s(e^t - 1)} = \frac{2e^t - 1}{e^t - 1}$

$x(e^t - 1) = y(2e^t - 1)$

$xe^t - x = 2ye^t - y$

$xe^t - 2ye^t = x - y$

$e^t(x - 2y) = x - y$

$e^t = \frac{x - y}{x - 2y}$

from (16)

$y = s(e^t - 1)$

$y = s\left[\frac{x - y}{x - 2y} - 1\right]$

$y = s\left[\frac{x - y - x + 2y}{x - 2y}\right]$

$y(x-2y) = s(y)$

$$\boxed{s = x-2y}$$

from (14),

$$z = \frac{s^2}{2}\left[5e^{2t} - 6t + 1\right]$$

$$= \frac{(x-2y)^2}{2}\left[5\left(\frac{x-y}{x-2y}\right)^2 - 6\left(\frac{x-y}{x-2y}\right) + 1\right]$$

$$= \frac{(x-2y)^2}{2}\left[\frac{5(x-y)^2 - 6(x-2y)(x-y) + (x-2y)^2}{(x-2y)^2}\right]$$

$$= \frac{1}{2}\left[5(x^2+y^2-2xy) - 6(x^2 - xy - 2xy + 2y^2) + x^2 + 4y^2 - 4xy\right]$$

$$= \frac{1}{2}\left[5x^2 + 5y^2 - 10xy - 6x^2 + 6xy + 12xy - 12y^2 + x^2 + 4y^2 - 4xy\right]$$

$$= \frac{1}{2}\left[-3y^2 + 4xy\right]$$

$$= \frac{y}{2}\left[-3y + 4x\right]$$

$$z = \frac{y}{2}(4x - 3y)$$

Compatible System of first order PDE:

Def:

If Every solution of the 1st order P.D.E $f(x, y, z, p, q) = 0$ is a solution of another

PDE $g(x,y,z,p,q)=0$ then the equ. is said to be Compatible.

Result:

(i) The sufficient condition of f and g are compatible if $[f,g]=0$

where $[f,g] = \dfrac{\partial[f,g]}{\partial[x,p]} + p\dfrac{\partial[f,g]}{\partial[z,p]}$

$+ \dfrac{\partial[f,g]}{\partial[y,q]} + q\dfrac{\partial[f,g]}{\partial[z,q]}$

(ii) J = Jacobian of p, q

$= \dfrac{\partial[f,g]}{\partial[p,q]}$ $= \begin{vmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial p} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial p} \end{vmatrix} = \begin{vmatrix} f_x & f_p \\ g_x & g_p \end{vmatrix}$

$\neq 0$

(iii) $dz = p\,dx + q\,dy$

1. S.T the equ. $xp = yq$ and $z(xp+yq) = 2xy$ are compatible and solve them.

Sol Given: $xp = yq \rightarrow (1)$

$z(xp+yq) = 2xy \rightarrow (2)$

Let $f = xp - yq$ and $g = zxp + zyq - 2xy$

$f_x = p$ $\qquad g_x = zp - 2y$

$f_y = -q$ $\qquad g_y = zq - 2x$

$f_z = 0$ $\qquad g_z = xp + yq$

$f_p = x$ | $g_p = zx$

$f_q = -y$ | $g_q = zy$

To prove: $[f, g] = 0$

W.k.t,

$$[f,g] = \frac{\partial[f,g]}{\partial[x,p]} + p\,\frac{\partial[f,g]}{\partial[z,p]} + \frac{\partial[f,g]}{\partial[y,q]} + q\,\frac{\partial[f,g]}{\partial[z,q]} \rightarrow (3)$$

$$\frac{\partial[f,g]}{\partial[x,p]} = \begin{vmatrix} f_x & f_p \\ g_x & g_p \end{vmatrix}$$

$$= \begin{vmatrix} p & x \\ zp-2y & zx \end{vmatrix}$$

$$= zxp - xzp + 2xy \Rightarrow 2xy \neq 0$$

$$\frac{\partial[f,g]}{\partial[z,p]} = \begin{vmatrix} f_z & f_p \\ g_z & g_p \end{vmatrix}$$

$$= \begin{vmatrix} 0 & x \\ xp+yq & z\,x \end{vmatrix}$$

$$= 0 - x^2p - xyq$$

$$= -x^2p - xyq$$

$$= -(x^2p + xyq) \neq 0$$

$$\frac{\partial[f,g]}{\partial[y,q]} = \begin{vmatrix} f_y & f_q \\ g_y & g_q \end{vmatrix}$$

$$= \begin{vmatrix} -q & -y \\ zq-2x & zy \end{vmatrix}$$

$$= -qzy + yzq - 2xy$$

$$= -2xy$$

$$\neq 0$$

$$\frac{\partial[f,g]}{\partial[z,q]} = \begin{vmatrix} f_z & f_q \\ g_z & g_q \end{vmatrix}$$

$$= \begin{vmatrix} 0 & -y \\ xp+yq & zy \end{vmatrix}$$

$$= 0 + xyp + y^2q$$

$$= y^2q + xyp$$

$$\neq 0$$

$\therefore$ (3) $\Rightarrow$ $[f,g] = 2xy - (x^2p + xyq)p - 2xy + q(y^2q + xyp)$

$$= 2xy - x^2p^2 - xypq - 2xy + y^2q^2 + xypq$$

$$= y^2q^2 - x^2p^2$$

$$= (yq)^2 - (xp)^2 \quad \text{(using (1))}$$

$$= (yq)^2 - (yq)^2$$

$$= 0$$

$\therefore$ The given system of equ., are compatiable.

W.k.t, $dz = p\,dx + q\,dy \rightarrow$ (A)

To find the value of p and q by using the equ., (1) & (2)

From equ (1) $\Rightarrow$ $xp = yq$

(2) $\Rightarrow$ $z(xp + yq) = 2xy$

Sub (1) in (2)

$$(2) \Rightarrow z(xp + xp) = 2xy$$

$$2xpz = 2xy$$

$$\boxed{p = y/z}$$

Sub $p = y/z$ in (1)

$$x(y/z) = yq$$

$$\boxed{q = x/z}$$

Sub p & q value in (A)

$$dz = \frac{y}{z}dx + \frac{x}{z}dy$$

$$z\,dz = y\,dx + x\,dy$$

$$z\,dz = d(xy)$$

Integrating

$$\int z\,dz = \int d(xy)$$

$$z^2/2 = xy + C \Rightarrow z^2 = 2xy + c$$

2. S.T the equ $xp - yq = x$ and $x^2p+q = xz$ are compatiable and solve them.

sol Given: $xp - yq = x \rightarrow (1)$

$$x^2p+q = xz \rightarrow (2)$$

Let $f = xp - yq - x$ and

$$g = x^2p+q-xz.$$

$f_x = p-1$ $\qquad$ $g_x = 2xp - z$

$f_y = -q$ $\qquad$ $g_y = 0$

$f_z = 0$ $\qquad$ $g_z = -x$

$f_p = x$ $\qquad$ $g_p = x^2$

$f_q = -y$ $\qquad$ $g_q = 1$

To prove: $[f,g]=0$

w.k.t,

$$[f,g] = \frac{\partial[f,g]}{\partial[x,p]} + p\,\frac{\partial[f,g]}{\partial[z,p]} + \frac{\partial[f,g]}{\partial[y,q]} + q\,\frac{\partial[f,g]}{\partial[z,q]} \rightarrow (3)$$

$$\frac{\partial[f,g]}{\partial[x,p]} = \begin{vmatrix} f_x & f_p \\ g_x & g_p \end{vmatrix}$$

$$= \begin{vmatrix} p-1 & x \\ 2xp-z & x^2 \end{vmatrix}$$

$$= x^2p - x^2 - 2x^2p + xz$$

$$= xz - x^2 - x^2p \Rightarrow q - x^2$$

$$\neq 0$$

$$\frac{\partial[f,g]}{\partial[z,p]} = \begin{vmatrix} f_z & f_p \\ g_z & g_p \end{vmatrix}$$

$$= \begin{vmatrix} 0 & x \\ -x & x^2 \end{vmatrix}$$

$$= 0 + x^2$$

$$= x^2$$

$$\neq 0$$

$$\frac{\partial[f,g]}{\partial[y,q]} = \begin{vmatrix} f_y & f_q \\ g_y & g_q \end{vmatrix}$$

$$= \begin{vmatrix} -q & -y \\ 0 & 1 \end{vmatrix}$$

$$= -q + 0$$

$$= -q$$

$$\neq 0$$

$$\frac{\partial[f,g]}{\partial[z,q]} = \begin{vmatrix} f_z & f_q \\ g_z & g_q \end{vmatrix}$$

$$= \begin{vmatrix} 0 & -y \\ -x & 1 \end{vmatrix}$$

$$= -xy$$

$$\neq 0$$

$[f,g] = q - x^2 + px^2 - q - qxy$

$= -x^2 + px^2 - qxy$

$= -x^2 + x(px - qy)$ $\quad [\because px - qy = x]$

$= -x^2 + x^2$

$[f,g] = 0$

$\therefore$ The given system of eq^n. are compaatiable

w.k.t, $dz = p\,dx + q\,dy \rightarrow (A)$

to find the value of p & q by using the equ (1) & (2)

From equ (1) $\Rightarrow xp - yq = x$

equ (2) $\Rightarrow x^2p + q = xz$

(1) $\Rightarrow xp = x + yq$

$p = \dfrac{x+yq}{x} \rightarrow (3)$

$x^2\left(\dfrac{x+yq}{x}\right) + q = xz$

$x(x+yq) + q = xz$

$x^2 + q + xyq = xz$

$q(1-y) = xz - x)$

$q = \dfrac{x(z-1)}{(1-y)} \rightarrow (4)$

Sub $q = \frac{x(z-1)}{(1-y)}$ in (3)

$$P = \frac{x - y\left(\frac{x(z-1)}{(1-y)}\right)}{x}$$

$$P = \frac{x(1-y) - xy(z-1)}{x(1-y)}$$

$$P = \frac{(1-y) - y(z-1)}{(1-y)}$$

Sub p & q in (A)

$$dz = \frac{(1-y) - y(z-1)}{(1-y)}\, dx + \frac{x(z-1)}{(1-y)}\, dy$$

$$(1-y)\, dz = \left[(1-y) - y(z-1)\right] dx + x(z-1)\, dy$$

(1) $\times x \Rightarrow x^2 p - xyq = x^2$

(2) $\Rightarrow x^2 p + q = xz$

$\quad (-) \quad (-) \quad (-)$

$$-xyq - q = x^2 - xz$$

$$q(-xy-1) = x^2 - xz$$

$$q = \frac{x^2 - xz}{-(xy+1)}$$

$$q = \frac{xz - x^2}{xy+1}$$

$$q = \frac{x(z-x)}{1+xy}$$

Sub $q = \dfrac{x(z-x)}{1+xy}$ in ①

$$xp - y\left(\frac{x(z-x)}{(1+xy)}\right) = x$$

$$p - y\frac{(z-x)}{(1+xy)} = 1$$

$$p = 1 + \frac{y(z-x)}{(1+xy)}$$

$$p = \frac{1+xy+zy-xy}{1+xy}$$

$$p = \frac{1+zy}{1+xy}$$

Ⓐ $\Rightarrow$ $dz = \left(\dfrac{1+zy}{1+xy}\right)dx + \left(\dfrac{x(z-x)}{(1+xy)}\right)dy$

Sub dx on both sides

$$dz - dx = \left(\frac{1+yz}{1+xy}\right)dx + \left(\frac{x(z-x)}{1+xy}\right)dy - dx$$

$$= \left[\frac{1+yz}{1+xy} - 1\right]dx + \left(\frac{x(z-x)}{1+xy}\right)dy$$

$$= \left[\frac{1+yz-1-xy}{1+xy}\right]dx + \left(\frac{x(z-x)}{1+xy}\right)dy$$

$$= \left(\frac{y(z-x)}{1+xy}\right)dx + \left(\frac{x(z-x)}{1+xy}\right)dy$$

$$dz - dx = (z-x)\left[\frac{y\,dx + x\,dy}{(1+xy)}\right]$$

$$d(z-x) = (z-x)\left[\frac{d(xy)}{1+xy}\right]$$

$$\int \frac{d(z-x)}{z-x} = \int \frac{d(xy)}{1+xy}$$

$$\log(z-x) = \log(1+xy) + \log c$$

$$\log(z-x) - \log(1+xy) = \log c$$

$$\log\left(\frac{z-x}{1+xy}\right) = \log c$$

$$e^{\log(z-x/1+xy)} = e^{\log c}$$

$$\frac{z-x}{1+xy} = c$$

3. S.T the equ., $z = px+qy$ is compaatiable with any equation $f(x,y,z,p,q)=0$ is homogeneous in x, y & z solve the equation $2xy\,(p^2+q^2) = z\,(yp+xq)$

Sol. Given: $z = px+qy \longrightarrow (1)$

$$f(x,y,z,p,q) = 0 \longrightarrow (2)$$

To prove: The above two equ., is compaatiable.

(ie) $[f,g]=0$

Let $f = f(x,y,z,p,q) \longrightarrow (3)$

$g = px+qy-z \longrightarrow (4)$

$f_x = f_x \qquad g_x = p$

$f_y = f_y \qquad g_y = q$

$f_z = f_z \qquad g_z = -1$

$f_p = f_p \qquad g_p = x$

$f_q = f_q \qquad g_q = y$

w.k.t,

$$[f,g] = \frac{\partial[f,g]}{\partial[x,p]} + p\frac{\partial[f,g]}{\partial[z,p]} + \frac{\partial[f,g]}{\partial[y,q]} + q\frac{\partial[f,g]}{\partial[z,q]}$$

$$\frac{\partial[f,g]}{\partial[x,p]} = \begin{vmatrix} f_x & f_p \\ g_x & g_p \end{vmatrix}$$

$$= \begin{vmatrix} f_x & f_p \\ p & x \end{vmatrix}$$

$$= x f_x - p f_p$$

$$\neq 0$$

$$\frac{\partial[f,g]}{\partial[z,p]} = \begin{vmatrix} f_z & f_p \\ g_z & g_p \end{vmatrix}$$

$$= \begin{vmatrix} f_z & f_p \\ -1 & x \end{vmatrix}$$

$$= x f_z + f_p$$

$$\neq 0$$

$$\frac{\partial[f,g]}{\partial[y,q]} = \begin{vmatrix} f_y & f_q \\ g_y & g_q \end{vmatrix}$$

$$= \begin{vmatrix} f_y & f_q \\ q & y \end{vmatrix}$$

$$= y f_y - q f_q$$

$$\neq 0$$

$$\frac{\partial[f,g]}{\partial[z,q]} = \begin{vmatrix} f_z & f_q \\ g_z & g_q \end{vmatrix}$$

$$= \begin{vmatrix} f_z & f_q \\ -1 & y \end{vmatrix}$$

$$= y f_z + f_q$$

$$\neq 0$$

$$[f,g] = x f_x - p f_p + xp f_z + p f_p + y f_y - q f_q + qy f_z + q f_q$$

$$= x f_x + xp f_z + y f_y + qy f_z$$

$$= x f_x + y f_y + f_z (xp + yq)$$

$$[f,g] = x f_x + y f_y + z f_z$$

Since $f(x,y,z,p,q)$ is homogeneous

(ie) $x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} + z\frac{\partial f}{\partial z} = 0$ [Euler's formula]

$\therefore [f, g] = 0$

Given: $z = px + qy \rightarrow$ (A)

and $2xy(p^2+q^2) = z(yp+xq) \rightarrow$ (B)

Sub (A) in (B)

$$2xy(p^2+q^2) = (px+qy)(yp+xq)$$

$$2xy(p^2+q^2) = p^2xy + x^2pq + y^2pq + xyq^2$$

$$2xy(p^2+q^2) = (p^2+q^2)xy + x^2pq + y^2pq$$

$$xy(p^2+q^2) = pq(x^2+y^2)$$

$$\frac{p^2+q^2}{pq} = \frac{x^2+y^2}{xy}$$

$$\frac{p}{q} + \frac{q}{p} = \frac{x}{y} + \frac{y}{x}$$

Let $\frac{p}{q} = \frac{x}{y}$

$$\Rightarrow p = \frac{qx}{y}$$

Sub p in (A)

$$z = \frac{qx^2}{y} + qy$$

$$z = q\left[\frac{x^2+y^2}{y}\right]$$

$$q = \frac{zy}{x^2+y^2}$$

2f2

Sub q in p

$$p = \frac{qx}{y}$$

$$p = \left(\frac{zy}{x^2+y^2}\right)\frac{x}{y}$$

$$p = \frac{xz}{x^2+y^2}$$

W.k.t $dz = p\,dx + q\,dy$

$$dz = \frac{xz}{x^2+y^2}\,dx + \frac{zy}{(x^2+y^2)}\,dy$$

$$\frac{dz}{z} = \frac{x\,dx}{x^2+y^2} + \frac{y\,dy}{x^2+y^2}$$

$$\int \frac{dz}{z} = \int \frac{x\,dx + y\,dy}{x^2+y^2}$$

Multiply 2 on Both sides

$$\int \frac{2\,dz}{z} = \int \frac{2(x\,dx + y\,dy)}{x^2+y^2}$$

$$2\log z = \log(x^2+y^2) + \log c$$

$$\log z^2 = \log(x^2+y^2) + \log c$$

$$\log z^2 - \log(x^2+y^2) = \log c$$

$$\log\left[\frac{z^2}{x^2+y^2}\right] = \log c$$

$$\boxed{\frac{x^2}{x^2+y^2} = c}$$

4. S.T (HW) the equ., $f(x,y,p,q)=0$ & $g(x,y,p,q)=0$ are compatiable if $\frac{\partial[f,g]}{\partial[x,p]} + \frac{\partial[f,g]}{\partial[y,q]} = 0$. Verify that the equ., $p = p(x,y)$, $q = q(x,y)$ are compatiable.

Charpit Method:

Auxillary Equation;

$$\frac{dx}{f_p} = \frac{dy}{f_q} = \frac{dz}{pf_p+qf_q} = \frac{dp}{-f_x-pf_z} = \frac{dq}{-f_y-qf_z}$$

1. Find the complete integral of the following equ. $p^2x+q^2y=z$.

Sol: Given: $p^2x+q^2y=z \rightarrow (1)$

Let $f(x,y,z,p,q) = p^2x+q^2y-z \rightarrow (2)$

W.k.t, the auxillary equ.,

$$\frac{dx}{f_p} = \frac{dy}{f_q} = \frac{dz}{pf_p+qf_q} = \frac{dp}{-f_x-pf_z} = \frac{dq}{-f_y-qf_z}$$

$f_x = p^2$ $\qquad f_p = 2px$

$f_y = q^2$ $\qquad f_q = 2qy$

$f_z = -1$

$$\frac{dx}{2px} = \frac{dy}{2qy} = \frac{dz}{p(2px)+q(2qy)} = \frac{dp}{-p^2+p} = \frac{dq}{+q}$$

Taking 1^{st} and 4^{th} term

$$\frac{dx}{2px} = \frac{dp}{p-p^2}$$

Mult & Div by p^2 in 1^{st} term, $2px$ ...

$$\frac{p^2dx \;\cancel{+}\; 2pxdp}{2p^3x + 2px(p-p^2)} = 0$$

$$\frac{p^2dx + 2pxdp}{2p^3x + 2p^2x - 2p^3x} = 0$$

$$\frac{p^2dx + 2pxdp}{2p^2x} = 0$$

$$\frac{pdx + 2xdp}{2px} = 0$$

$$\frac{d(p^2x)}{2p^2x} = 0 \longrightarrow 3$$

Taking 2^{nd} & 5^{th} term

$$dy/2qy = dq/-q^2+q$$

Multip/&div q^2 in 2^{nd} & $2qy$ in 5^{th}

$$\frac{q^2dy + 2\ldots}{2q^3y\ldots}$$

$$\frac{q^2dy}{2q^3\ldots}$$

$q^{\ldots}$

Equ 3

$d\ldots$

$\log$

$\log$

Sub

$$\frac{q^2 dy + 2qy\, dq}{2q^3 y + 2qy(q^2 - q^2)} = 0$$

$$\frac{q^2 dy + 2qy\, dq}{2q^3 y - 2q^3 y + 2q^2 y} = 0$$

$$\frac{q^2 dy + 2qy dq}{2q^2 y} = 0$$

$$\frac{d(q^2 y)}{2q^2 y} = 0 \rightarrow (4)$$

Equ (3) & (4)

$$\frac{d(p^2 x)}{2p^2 x} = \frac{d(q^2 y)}{2q^2 y}$$

$$\log(p^2 x) = \log(q^2 y) + \log c$$

$$\log\left(\frac{p^2 x}{q^2 y}\right) = \log c$$

$$e^{\log(p^2x/q^2y)} = e^{\log c}$$

$$\frac{p^2 x}{q^2 y} = c$$

$$p^2 x = c\, q^2 y \longrightarrow (5)$$

$$\Rightarrow q^2 y = p^2 x / c$$

Sub (5) in (1)

$$p^2 x + \frac{p^2 x}{c} = z$$

$$\Rightarrow cp^2 x + p^2 x = cz$$

$$\Rightarrow p^2 x (c+1) = zc$$

$$p^2 = \frac{zc}{x(c+1)}$$

$$\boxed{p = \sqrt{\frac{zc}{x(c+1)}}}$$

Sub $p = \sqrt{zc/x(c+1)}$ in

$$q^2 y = \frac{\frac{zc}{x(c+1)} x}{c}$$

$$q^2 y = \frac{z}{(c+1)}$$

$$\boxed{q = \sqrt{\frac{z}{y(c+1)}}}$$

w.k.t, $dz = p\,dx + q\,dy$

$$dz = \sqrt{\frac{cz}{x(c+1)}}\,dx + \sqrt{\frac{z}{y(c+1)}}\,dy$$

(5) $\sqrt{\frac{c+1}{z}}\,dz = \sqrt{\frac{c}{x}}\,dx + \frac{1}{\sqrt{y}}\,dy$

$$\frac{\sqrt{c+1}\,dz}{\sqrt{z}} = \frac{\sqrt{c}}{\sqrt{x}}\,dx + \frac{1}{\sqrt{y}}\,dy$$

$$\sqrt{c+1}\int \frac{dz}{\sqrt{z}} = \sqrt{c}\int \frac{dx}{\sqrt{x}} + \int \frac{dy}{\sqrt{y}}$$

$$\sqrt{c+1}\; 2\sqrt{z} = \sqrt{c}\cdot 2\sqrt{x} + 2\sqrt{y} + \ldots$$

$$\sqrt{c+1}\cdot\sqrt{z} = \sqrt{c}\sqrt{x} + \sqrt{y} + d$$

$$(c+1)\sqrt{z} = c$$

where c & d are constant.

Home Sum:

4. Sol

Given : $f(x, y, p, q) = 0 \longrightarrow (1)$

$g(x, y, p, q) = 0 \longrightarrow (2)$

To prove above two equation are compatible.

(ie) $[f, g] = 0$

Let $f = f(x, y, p, q)$

$g = g(x, y, p, q)$

$f_x = f_x$ $\qquad$ $g_x = g_x$

$f_y = f_y$ $\qquad$ $g_y = g_y$

$f_p = f_p$ $\qquad$ $g_p = g_p$

$f_q = f_q$ $\qquad$ $g_q = g_q$

w.k.t

$$[f,g] = \frac{\partial[f,g]}{\partial[x,p]} + p\frac{\partial[f,g]}{\partial[z,p]} + \frac{\partial[f,g]}{\partial[y,q]} + q\frac{\partial[f,g]}{\partial[z,q]}$$

$$\frac{\partial[f,g]}{\partial[x,p]} = \begin{vmatrix} f_x & f_p \\ g_x & g_p \end{vmatrix}$$

$$= f_x g_p - g_x f_p$$

$$\neq 0$$

$$\frac{\partial[f,g]}{\partial[y,q]} = \begin{vmatrix} f_y & f_q \\ g_y & g_q \end{vmatrix}$$

$$= f_y g_q - f_q g_y$$

$$\neq 0$$

$$\frac{\partial[f,g]}{\partial[z,p]} = 0$$

$$\frac{\partial[f,g]}{\partial[z,q]} = 0$$

$$[f,g] = f_x g_p - g_x f_p + p(0) + f_y g_q - f_q g_y + q(0)$$

$$= f_x g_p - f_p g_x + f_y g_q - f_q g_y$$

$$= \frac{\partial[f,g]}{\partial[x,p]} + \frac{\partial[f,g]}{\partial[y,q]}$$

$[f,g] = 0$

$\therefore$ Is compatible

5. Show that $p = 5x - 7y$, $q = 6x + 8y$ are not compattable.

Sol Given: $p = 5x - 7y \longrightarrow (1)$

$q = 6x + 8y \longrightarrow (2)$

Let $f = 5x - 7y - p$

$g = 6x + 8y - q$.

$f_x = 5 - p$ | $g_x = 6$

$f_y = -7$ | $g_y = 8$

$f_z = 0$ | $g_z = 0$

$f_p = -x$ | $g_p = 0$

$f_q = 0$ | $g_q = -1$

$$\frac{\partial[f,g]}{\partial[x,p]} = \begin{vmatrix} 5-p & -x \\ 6 & 0 \end{vmatrix}$$

$$= 0 + 6x$$

$$= 6x$$
$$\neq 0$$

$$\frac{\partial[f,g]}{\partial[z,p]} = \begin{vmatrix} 0 & -x \\ 0 & 0 \end{vmatrix}$$

$$= 0$$

$$\frac{\partial[f,g]}{\partial[y,q]} = \begin{vmatrix} 7 & 0 \\ 8 & -1 \end{vmatrix}$$
$$= -7$$
$$\neq 0$$

$$\frac{\partial[f,g]}{\partial[z\,q]} = \begin{vmatrix} 0 & 0 \\ 0 & -1 \end{vmatrix}$$

$= 0$

$\partial[f,g] = 6x + 0 - 7 \neq 0$

$= 6x - 7$

$p = 5x - 7y$, $q = 6x + 8y$ are not compatible

Find the complete integral of $pqxy = z^2$

Sol Given: $pqxy = z^2 \rightarrow (1)$

Let $f(x,y,z,p,q) = pqxy - z^2 \longrightarrow (2)$

W.k.t, the auxiliary eqn.,

$$\frac{dx}{f_p} = \frac{dy}{f_q} = \frac{dz}{pf_p + qf_q} = \frac{dp}{-f_x - pf_z} = \frac{dq}{-f_y - qf_z}$$

$f_x = pqy \quad f_z = -2z \quad f_q = pxy$

$f_y = pqx \quad f_p = qxy$

$$\frac{dx}{qxy} = \frac{dy}{pxy} = \frac{dz}{pqxy + pqxy} = \frac{dp}{-pqy + 2zp} = \frac{dq}{-pq\ldots}$$

$\rightarrow (3)$

Taking 1st and 4th term & choose multiplier p and x

$$\frac{pdx + xdp}{pqxy + x(-pqy + 2zp)} = 0$$

$$\frac{p\,dx + x\,dp}{pqxy - pqxy + 2xzp} = 0$$

$$\frac{p\,dx + x\,dp}{2xzp} = 0$$

$$\frac{d(xp)}{2xzp} = 0 \longrightarrow (4)$$

Taking 2nd & 5th term & choose multiplier q and y

$$\frac{q\,dy + y\,dq}{pqxy + y(-pqx + 2zq)} = 0$$

$$\frac{q\,dy + y\,dq}{pqxy - pqxy + 2zyq} = 0$$

$$\frac{d(qy)}{2zyq} = 0 \longrightarrow (5)$$

Equ (4) & (5)

$$\frac{d(xp)}{2xzp} = \frac{d(qy)}{2zyq}$$

$$\frac{d(xp)}{xp} = \frac{d(qy)}{yq}$$

$$\int \frac{d(xp)}{(xp)} = \int \frac{d(qy)}{(yq)}$$

$$\log(xp) = \log(yq) + \log c^2$$

$$\log(xp) - \log(yq) = \log c^2$$

$$\log(xp/yq) = \log c^2 \Rightarrow \boxed{\frac{xp}{yq} = c^2} \Rightarrow xp = yqc^2$$

Given $pq\,xy = z^2$

$$(yqc)\,qy = z^2$$

$$c^2y^2q^2 = z^2$$

$$q^2 = \frac{z^2}{c^2y^2}$$

$$\boxed{q = \frac{z}{cy}}$$

Sub $q = \frac{z}{cy}$ in ①

$$p\left(\frac{z}{cy}\right)xy = z^2$$

$$\boxed{p = \frac{zc}{x}}$$

w.k.t $dz = p\,dx + q\,dy$

$$dz = \left(\frac{zc}{x}\right)dx + \left(\frac{z}{cy}\right)dy$$

$$\int \frac{dz}{z} = \int c\,\frac{dx}{x} + \frac{1}{c}\int \frac{dy}{y}$$

$$\log z = c\log x + \frac{1}{c}\log y + \log d$$

$$\log z = \log x^c + \log y^{1/c} + \log d$$

$$\log z = \log(x^c \cdot y^{1/c} \cdot d)$$

$$e^{\log z} = e^{\log(x^c \cdot y^{1/c} \cdot d)}$$

$$z = x^c \cdot y^{1/c} \cdot d$$

Special
Type
1. Find
Sol
2. P+

Special type of 1st order equation :

Type: 1 $f(p,q)=0$

Charpit Auxiliary equ.,

$$\frac{dx}{f_p} = \frac{dy}{f_q} = \frac{dz}{pf_p+qf_q} = \frac{dp}{-f_x-pf_z} = \frac{dq}{-f_y-qf_z}$$

here, $f_x, f_y, f_z = 0$

$$\therefore \frac{dx}{f_p} = \frac{dy}{f_q} = \frac{dz}{pf_q+qf_q} \qquad (P=a)$$

1. Find the complete integral $pq=1$

Sol Given $pq=1$

put $P=a$

$aq=1$

$q=1/a$

W.k.t

$dz = p\,dx + q\,dy$

$dz = a\,dx + \frac{1}{a}dy$

$\int dz = \int a\,dx + \int \frac{1}{a}dy$

$z = ax + \frac{y}{a} + c$

2. $P+q=Pq$

$Pp+Qq=R$

$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

Given $p+q=pq$

Put $p=a$

$a+q=aq$

$a=aq-q$

$a=q(a-1)$

$q=\frac{a}{(a-1)}$

$\frac{a}{a-1}$

W.k.t, $dz = p\,dx + q\,dy$

$= a\,dx + \frac{a}{a-1}dy$

$\int dz = \int a\,dx + \frac{a}{a-1}\int dy$

$= ax + \frac{a}{a-1}y + c$

Type: 2

$f(z, p, q) = 0$

Note:

$P = aq$

Charpit auxillary equation

$$\frac{dx}{f_p} = \frac{dy}{f_q} = \frac{dz}{pf_p} = \frac{dp}{-pf_z} = \frac{dq}{-qf_z}$$

1. Find the complete integral of $zpq = p+q$

Sol Given: $zpq = p+q$

put $p = aq$

$zaq^2 = aq + q$

$zaq^2 = (a+1)q$

$\boxed{q = \frac{a+1}{za}}$

$P = \frac{a+1}{za}(a)$

$\boxed{P = \frac{a+1}{z}}$

W.k.t, $dz = p\,dx + q\,dy$

2. Find

Sol

$$dz = \frac{a+1}{z}\,dx + \frac{a+1}{za}\,dy$$

$$\int z\,dz = \int (a+1)\,dx + \int \frac{a+1}{a}\,dy$$

$$\frac{z^2}{2} = (a+1)x + \frac{a+1}{a}\,y + c$$

2. Find the complete integral of $z = p^2 - q^2$

Sol) Given: $z = p^2 - q^2$

Put $p = aq$

$$z = a^2q^2 - q^2$$

$$= q^2(a^2-1)$$

$$\frac{z}{a^2-1} = q^2$$

$$q = \frac{\sqrt{z}}{\sqrt{a^2-1}}$$

$$p = \frac{a\sqrt{z}}{\sqrt{a^2-1}}$$

W.k.t, $dz = p\,dx + q\,dy$

$$dz = \frac{a\sqrt{z}}{\sqrt{a^2-1}}\,dx + \frac{\sqrt{z}}{\sqrt{a^2-1}}\,dy$$

$$\int \frac{dz}{\sqrt{z}} = \int \frac{a\,dx}{\sqrt{a^2-1}} + \int \frac{dy}{\sqrt{a^2-1}}$$

$$\int \frac{dz}{z^{1/2}} = \frac{a}{\sqrt{a^2-1}}\int dx + \frac{1}{\sqrt{a^2-1}}\int dy$$

$$\frac{z^{1/2}}{1/2} = \frac{a}{\sqrt{a^2-1}}\,x + \frac{1}{\sqrt{a^2-1}}\,y + c$$

$$2\sqrt{z} = \frac{a}{\sqrt{a^2-1}}\,x + \frac{1}{\sqrt{a^2-1}}\,y + c$$

3. Find the complete integral $p^2z^2+q^2=1$

sol given: $p^2z^2+q^2=1$

put $p=aq$

$$a^2q^2z^2+q^2=1$$

$$q^2(a^2z^2+1)=1$$

$$q^2=\frac{1}{a^2z^2+1}$$

$$q=\frac{1}{\sqrt{a^2z^2+1}}$$

$$p=\frac{a}{\sqrt{a^2z^2+1}}$$

W.k.t,

$$dz = p\,dx + q\,dy$$

$$dz = \frac{a}{\sqrt{a^2z^2+1}}\,dx + \frac{1}{\sqrt{a^2z^2+1}}\,dy$$

$$\int \sqrt{a^2z^2+1}\,dz = \int a\,dx + \int dy$$

$$\because \int\sqrt{x^2+a^2}\,dz = \frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}\log\left(x+\sqrt{x^2+a^2}\right)$$

$$\frac{az}{2}\sqrt{a^2z^2+1} + \frac{1}{2}\log\left(az+\sqrt{a^2z^2+1}\right) = ax+y+c$$

Type:3

Clairauts Equation

$$z = px+qy+f(p,q)$$

$$p=a,\quad q=b$$

1. Find the complete integrals of $z - xp - yq = \frac{1}{p+q}$

Sol Given: $z - xp - yq = \frac{1}{p+q}$

$$z = xp + yq + \frac{1}{p+q}$$

It is a form of complete integral

$$z - px - qy = \frac{1}{p+q}$$

$$z = xp + yq + \frac{1}{p+q}$$

It is of the form,

$$z = px + qy + f(p,q)$$

$$p = a, \quad q = b$$

$$z = ax + by + \frac{1}{a+b}$$

2. $z = px + qy + pq$

Sol It is of the form

$$z = px + qy + f(p,q)$$

$$p = a, \quad q = b$$

$$z = ax + by + ab$$

3. $z = px + qy + \log pq$

Sol It is of the form

$$z = px + qy + f(p,q)$$

$$p = a, \quad q = b$$

$$z = ax + by + \log ab$$

4. $pqz = p^2(xq + p^2) + q^2(yp + q^2)$

$$z = \frac{p^2(xq + p^2) + q^2(yp + q^2)}{pq}$$

$$= \frac{p^2xq}{pq} + \frac{p^4}{pq} + \frac{q^2yp}{pq} + \frac{q^4}{pq}$$

$$= px + qy + \frac{p^3}{q} + \frac{q^3}{p}$$

$$= px + qy + \left(\frac{p^4 + q^4}{pq}\right)$$

It is of the form

$p = a, \quad q = b$

$$z = ax + by + \left(\frac{a^4 + b^4}{ab}\right)$$

Type: 4

$f(x, p) = g(y, q)$

$g(y, q) = a^2, \quad f(x, p) = a^2$

1. Find the complete integral of $p^2q^2 + x^2y^2 = x^2q^2(x^2+y^2)$

Sol Given: $p^2q^2 + x^2y^2 = x^2q^2(x^2+y^2)$

$$\frac{p^2q^2 + x^2y^2}{x^2q^2} = x^2 + y^2$$

$$\frac{p^2q^2}{x^2q^2} + \frac{x^2y^2}{x^2q^2} = x^2 + y^2$$

$$\frac{p^2}{x^2} + \frac{y^2}{q^2} = x^2 + y^2$$

$$\frac{p^2}{x^2} - x^2 = y^2 - \frac{y^2}{q^2}$$

It is of the form,

$$f(x, p) = g(y, q)$$

Let $g(y, q) = a^2$

$$\Rightarrow y^2 - \frac{y^2}{q^2} = a^2$$

$$\Rightarrow q^2y^2 - y^2 = a^2q^2$$

$$q^2y^2 - a^2q^2 = y^2$$

$$q^2(y^2 - a^2) = y^2$$

$$q^2 = \frac{y^2}{y^2 - a^2}$$

$$q = \frac{y}{\sqrt{y^2 - a^2}}$$

$$f(x, p) = a^2$$

$$\Rightarrow p^2/x^2 - x^2 = a^2$$

$$p^2 - x^4 = a^2x^2$$

$$p^2 = a^2x^2 + x^4$$

$$p^2 = x^2(a^2 + x^2)$$

$$p = x\sqrt{a^2 + x^2}$$

sub $dz = p\,dx + q\,dy$

$$dz = x\sqrt{a^2 + x^2}\,dx + \frac{y}{\sqrt{y^2 - a^2}}\,dy$$

Integrating we get.

$$\int dz = \int x\sqrt{a^2+x^2}\,dx + \int \frac{y}{\sqrt{y^2-a^2}}\,dy$$

$t = a^2+x^2$ $\qquad$ $t = y^2-a^2$

$dt = 2x\,dx$ $\qquad$ $dt = 2y\,dy$

$\frac{dt}{2} = x\,dx$ $\qquad$ $\frac{dt}{2} = y\,dy$

$$\int dz = \int \sqrt{t}\,\frac{dt}{2} + \int \frac{1}{\sqrt{t}}\,\frac{dt}{2}$$

$$= \frac{1}{2}\,\frac{t^{3/2}}{3/2} + \frac{1}{2}\,\frac{t^{1/2}}{1/2} + c$$

$$= \frac{t\sqrt{t}}{3} + \sqrt{t} + c$$

$$= \frac{(a^2+x^2)^{3/2}}{3} + \sqrt{y^2-a^2} + c$$

2. Find the complete integral of $p^2y(1+x^2) = qx^2$

sol

Given: $p^2y(1+x^2) = qx^2$

$$\frac{p^2}{x^2}(1+x^2) = \frac{q}{y}$$

It is of the form

$$f(x,p) = g(y,q)$$

$$\Rightarrow f(x,p) = a^2$$

$$\frac{p^2}{x^2}(1+x^2) = a^2$$

$$p^2(1+x^2) = a^2x^2$$

$$p^2 = \frac{a^2x^2}{1+x^2}$$

Sub

Ja

No

$$p = \frac{ax}{\sqrt{1+x^2}}$$

$$\Rightarrow g(y, q) = a^2$$

$$\frac{q}{y} = a^2$$

$$q = a^2 y$$

$$q = y a^2$$

Sub $dz = p\,dx + q\,dy$

$$dz = \frac{a}{\sqrt{1+x^2}}\,dx + y a^2 dy$$

$$t = 1 + x^2$$

$$dt = 2x\,dx$$

$$\frac{dt}{2} = x\,dx$$

$$\int dz = a\int \frac{dt}{2\sqrt{t}} + a^2 \int y\,dy$$

$$= \frac{a}{2}\int t^{-1/2}\,dt + a^2 \int y\,dy$$

$$= \frac{a}{2}\,\frac{t^{1/2}}{1/2} + \frac{a^2y^2}{2} + c$$

$$z = a(1+x^2)^{1/2} + \frac{1}{2}a^2y^2 + c.$$

Jacobi Method:

Note:

(i) $p = \frac{-u_1}{u_3}$, $q = \frac{-u_2}{u_3}$

(ii) $du = u_1\,dx + u_2\,dy + u_3\,dz$

(iii) Auxiliary equation (A.E)

$$\frac{dx}{f_{u_1}} = \frac{dy}{f_{u_2}} = \frac{dz}{f_{u_3}} = \frac{du_1}{-f_x} = \frac{du_2}{-f_y} = \frac{du_3}{-f_z}$$ Taking

1. Find the complete integral of $p^2x + q^2y = z$

Sol) Given: $p^2x + q^2y = z \longrightarrow (1)$

W.k.t, by using Jacobi method.

$$p = -\frac{u_1}{u_3}, \quad q = -\frac{u_2}{u_3} \longrightarrow (2)$$

Sub equ (2) in (1)

$$\left(\frac{u_1}{u_3}\right)^2 (x) + \left(\frac{-u_2}{u_3}\right)^2 y = z$$

$$\frac{u_1^2}{u_3^2}x + \frac{u_2^2}{u_3^2}y = z$$

$$u_1^2 x + u_2^2 y = z u_3^2$$

$$u_1^2 x + u_2^2 y - z u_3^2 = 0$$

Let $f(x, y, z, u_1, u_2, u_3) = u_1^2 x + u_2^2 y - z u_3^2 \rightarrow (3)$

$f_x = u_1^2 \qquad f_{u_1} = 2u_1 x$

$f_y = u_2^2 \qquad f_{u_2} = 2u_2 y$

$f_z = -u_3^2 \qquad f_{u_3} = -2z u_3$

The auxiliary equation

$$\frac{dx}{f_{u_1}} = \frac{dy}{f_{u_2}} = \frac{dz}{f_{u_3}} = \frac{du_1}{-f_x} = \frac{du_2}{-f_y} = \frac{du_3}{-f_z}$$

$$\frac{dx}{2u_1 x} = \frac{dy}{2u_2 y} = \frac{dz}{-2z u_3} = \frac{du_1}{-u_1^2} = \frac{du_2}{-u_2^2} = \frac{du_3}{-u_3^2}$$

Taking ① & ② term

$$\frac{dx}{2u_1x} = \frac{du_1}{-u_1^2}$$

$$\frac{dx}{2x} = \frac{-du_1}{u_1}$$

Integrating

$$\int \frac{dx}{2x} = -\int \frac{du_1}{u_1}$$

$$\frac{1}{2}\log x = -\log u_1 + \log A$$

$$\log x = -2\log u_1 + \log A$$

$$\log x = -\log u_1^2 + \log A$$

$$\log x + \log u_1^2 = \log A$$

$$\log(xu_1^2) = \log A$$

$$xu_1^2 = A$$

$$\Rightarrow u_1 = \sqrt{A/x}$$

Taking 2nd & 5th term

$$\frac{dy}{2u_2y} = -\frac{du_2}{u_2^2}$$

$$\frac{dy}{2y} = -\frac{du_2}{u_2}$$

Integrating

$$\frac{1}{2}\int \frac{dy}{y} = -\int \frac{du_2}{u_2}$$

$$\frac{1}{2}\log y = -\log u_2 + \log B$$

$$\log y = -2\log u_2 + \log B$$

$\log y + \log u_2^2 = \log B$

$\log(y u_2^2) = \log B$

$y u_2^2 = B$

$u_2^2 = B/y$

$u_2 = \sqrt{B/y} \longrightarrow (5)$

Taking 3rd & 6th term

$u_1^2 x + u_2^2 y - z u_3^2 = 0$

$\left(\sqrt{A/x}\right)^2 x + \left(\sqrt{B/y}\right)^2 y - z u_3^2 = 0$

$\frac{A}{x} \cdot x + \frac{B}{y} \cdot y - z u_3^2 = 0$

$- z u_3^2 = -(A+B)$

$u_3^2 = \frac{A+B}{z}$

$u_3 = \sqrt{\frac{A+B}{z}}$

w.k.t,

$du = u_1 dx + u_2 dy + u_3 dz$

$= \sqrt{\frac{A}{x}} dx + \sqrt{\frac{B}{y}} dy + \sqrt{\frac{A+B}{z}} dz$

$\int du = \int \sqrt{\frac{A}{x}} dx + \int \sqrt{\frac{B}{y}} dy + \int \sqrt{\frac{A+B}{z}} dz$

$u = \sqrt{A} \int \frac{1}{\sqrt{x}} dx + \sqrt{B} \int \frac{1}{\sqrt{y}} dy + \sqrt{A+B} \int \frac{1}{\sqrt{z}} dz$

$= \sqrt{A} \cdot 2\sqrt{x} + \sqrt{B} \cdot 2\sqrt{y} + \sqrt{A+B} \cdot 2\sqrt{z}$

$= 2\sqrt{Ax} + 2\sqrt{By} + 2\sqrt{(A+B)z}$

2. Find the
Method pq
Sol) Giv
w.k.t
Sub (3
1
dx
fi
d
u

2. Find the complete integral by using Jacobi method $pqxy = z^2$

Sol) Given: $pqxy = z^2 \rightarrow (1)$

w.k.t, by using Jacobi method

$$p = -\frac{u_1}{u_3}, \quad q = -\frac{u_2}{u_3} \rightarrow (2)$$

Sub (2) in (1)

$$\left(-\frac{u_1}{u_3}\right)\left(-\frac{u_2}{u_3}\right) xy = z^2$$

$$\frac{u_1 u_2}{u_3^2} xy = z^2$$

$$u_1 u_2 xy = z^2 u_3^2$$

$$u_1 u_2 xy - z^2 u_3^2 = 0$$

Let $f(x, y, z, u_1, u_2, u_3) = u_1 u_2 xy - z^2 u_3^2 \rightarrow (3)$

$f_x = u_1 u_2 y$ $\qquad f_{u_1} = u_2 xy$

$f_y = u_1 u_2 x$ $\qquad f_{u_2} = u_1 xy$

$f_z = -2z u_3^2$ $\qquad f_{u_3} = -2u_3 z^2$

The auxiliary equ.,

$$\frac{dx}{f_{u_1}} = \frac{dy}{f_{u_2}} = \frac{dz}{f_{u_3}} = \frac{du_1}{-f_x} = \frac{du_2}{-f_y} = \frac{du_3}{-f_z}$$

$$\frac{dx}{u_2 xy} = \frac{dy}{u_1 xy} = \frac{dz}{-2u_3 z^2} = \frac{du_1}{-u_1 u_2 y} = \frac{du_2}{-u_1 u_2 x} = \frac{du_3}{2z u_3^2}$$

Taking 3rd & 6th term

$$\frac{dz}{-2u_3 z^2} = \frac{du_3}{2z u_3^2}$$

$$\frac{dz}{z} = \frac{-du_3}{u_3}$$

$\int$ ing

$$\int \frac{dz}{z} = -\int \frac{du_3}{u_3}$$

$$\log z = -\log u_3 + \log A$$

$$\log z + \log u_3 = \log A$$

$$z u_3 = A$$

$$\boxed{u_3 = \frac{A}{z}}$$

Taking 1st & 4th term

$$\frac{dx}{u_2 xy} = \frac{du_1}{-u_1 u_2 y}$$

$$\frac{dx}{x} = \frac{du_1}{-u_1}$$

$$\int \frac{dx}{x} = -\int \frac{du_1}{u_1}$$

$$\log x = -\log u_1 + \log B$$

$$\log x + \log u_1 = \log B$$

$$\log(x u_1) = \log B$$

$$\Rightarrow x u_1 = B$$

$$\boxed{u_1 = B/x}$$

Taking 2nd & 5th term

$$\frac{dy}{u_1 xy} = \frac{du_2}{-u_1 u_2 x}$$

$$\frac{dy}{y} = \frac{du_2}{-u_2}$$

$$\int \frac{dy}{y} = -\int \frac{du_2}{u_2}$$

$$\log y = -\log u_2 + \log c$$

$$\log y + \log u_2 = \log c$$

$$\log (y u_2) = \log c$$

$$y u_2 = c$$

$$\Rightarrow \boxed{u_2 = c/y}$$

W.k.t,

$$du = u_1\, dx + u_2\, dy + u_3\, dz$$

$$du = \frac{B}{x}\, dx + \frac{c}{y}\, dy + \frac{A}{z}\, dz$$

$$\int du = \int \frac{B}{x}\, dx + \int \frac{c}{y}\, dy + \int \frac{A}{z}\, dz$$

$$u = B \log x + c \log y + A \log z$$

$$= \log x^B + \log y^c + \log z^A$$

$$=$$

Theorem:

Show that a complete integral of the equation $f\left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z}\right) = 0$ is $u = ax + by + \theta(a,b) z + c$ and when a, b, c are arbitrary constant and

$f(a, b, \theta) = 0$. Find C.I of the equation

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = \frac{\partial u}{\partial x} \cdot \frac{\partial u}{\partial y} \cdot \frac{\partial u}{\partial z}$$

Sol Given: $f(u_1, u_2, u_3) = 0 \rightarrow$ ①

W.k.t,

The A.E is

$$\frac{dx}{f_{u_1}} = \frac{dy}{f_{u_2}} = \frac{dz}{f_{u_3}} = \frac{du_1}{-f_x} = \frac{du_2}{-f_y} = \frac{du_3}{-f_z}$$

$$\frac{dx}{f_{u_1}} = \frac{dy}{f_{u_2}} = \frac{dz}{f_{u_3}} = \frac{du_1}{0} = \frac{du_2}{0} = \frac{du_3}{0}$$

$$\frac{dx}{f_{u_1}} = \frac{du_1}{0}$$

$$du_1 = 0$$

$$\int du_1 = 0$$

$$\boxed{u_1 = a}$$

$$\frac{dy}{u_2} = \frac{du_2}{0} \Rightarrow \int du_2 = 0$$

$$\boxed{u_2 = b}$$

Apply the values of u_1 and u_2 in equ①

$f(a, b, u_3) = 0$

In hypothesis $f(a, b, \theta) = 0$

$$u_3 = \theta(a, b)$$

W.k.t,

$$du = u_1\, dx + u_2\, dy + u_3\, dz$$

$$\int du = a\int dx + b\int dy + \theta(a,b)\int dz$$

$$u = ax + by + \phi(a,b)z + c$$

Given, $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = \frac{\partial u}{\partial x} \cdot \frac{\partial u}{\partial y} \cdot \frac{\partial u}{\partial z}$

$$u_1 + u_2 + u_3 - u_1 u_2 u_3 = 0$$

Let $f = u_1 + u_2 + u_3 - u_1 u_2 u_3 \rightarrow ②$

$f_{u_1} = 1 - u_2 u_3 \qquad f_x = 0$

$f_{u_2} = 1 - u_1 u_3 \qquad f_y = 0$

$f_{u_3} = 1 - u_1 u_2 \qquad f_z = 0$

$$\frac{dx}{1-u_2u_3} = \frac{dy}{1-u_1u_3} = \frac{dz}{1-u_1u_2} = \frac{du_1}{0} = \frac{du_2}{0} = \frac{du_3}{0}$$

$\frac{dx}{1-u_2u_3} = \frac{du_1}{0}$

$\int du_1 = 0$

$\boxed{u_1 = a}$

$\frac{dy}{1-u_1u_3} = \frac{du_2}{0}$

$\int du_2 = 0$

$\boxed{u_2 = b}$

$② \Rightarrow u_1 + u_2 + u_3 - u_1 u_2 u_3 = 0$

$(a+b) + u_3 - (ab)u_3 = 0$

$(a+b) + u_3(1-ab) = 0$

$u_3(1-ab) = -(a+b)$

$u_3 = -\frac{(a+b)}{(1-ab)}$

$u_3 = \frac{(a+b)}{(ab-1)}$

$$u = ax + by + \frac{(a+b)}{(ab-1)} z + c.$$