Unit - III

Geodesics on a Surface :-

Geodesic :

let A and B be two given Points on surface S and let these points be joined by curves lying on S Then any curve possessing stationary length for small variation over S is called a geodesic.

Stationary :-

If α is such that the variation in $S(\alpha)$ is atmost of order ϵ^2 for all small variations in α for different $\lambda(t)$ and $\mu(t)$, then $S(\alpha)$ is said to be stationary and α is geodesic.

Theorem:

A necessary and sufficient Condition for a curve $u = u(t)$, $v = v(t)$ on a surface.

$r = r(u,v)$ to be a geodesic is that

$$U \frac{\partial T}{\partial \dot{v}} - V \frac{\partial T}{\partial \dot{u}} = 0 \rightarrow (1).$$

where $U = \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{u}}\right) - \frac{\partial T}{\partial u} = \frac{1}{2T}\frac{dT}{dt} \cdot \frac{\partial T}{\partial \dot{u}}$

$$V = \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{v}}\right) - \frac{\partial T}{\partial v} = \frac{1}{2T}\frac{dT}{dt} \cdot \frac{\partial T}{\partial \dot{v}} \rightarrow (2)$$

Equation ③ are called geodesic equation and we use the usual method of calculus of variation to derive equations ③, with help of ②, we obtain ① and then prove that the converse of ① is also true.

To prove: ②, we need the following lemma.

If $g(t)$ is a continuous function for $0<t<1$ and if,

$$\int_0^1 v(t)\, g(t)\, dt = 0, \quad \forall \text{ admissible} \longrightarrow ⑤$$

function $v(t)$ as defined above, then $g(t)=0$.

Proof:-

Suppose, $\int_0^1 v(t)\, g(t)\, dt = 0$ for all admissible function. $v(t)$ & $g(t) \neq 0$

Then there exists a "t_0" between 0 and 1 such that $g(t_0) \neq 0$.

let us take $g(t_0) > 0$. Since $g(t)$ is continuous on $(0,1)$ and $t_0 \in (0,1)$, there exists a neighbourhood (a,b) of t_0 such that $g(t_0) > 0$ in (a,b) where $0 \leq a < t < b \leq 1$.

Now let us define a function $v(t)$ as follows.

$$v(t) = \begin{cases} (t-a)^3 (b-t)^3 & \text{for } a \le t \le b. \\ 0 & \text{for } 0 \le t < a, \\ & \text{and } b < t \le 1. \end{cases}$$

Then $v(t)$ is an admissible function in $(0,1)$, So that ③ can be written as,

$$\int_0^1 v(t)\, g(t)\, dt = \int_0^a v(t)\, g(t)\, dt + \int_a^b v(t) g(t)\, dt + \int_b^1 v(t) g(t)\, dt.$$

Using $v(t)$ in $[0,1]$ in the above step,

$$\int_0^1 v(t)\, g(t)\, dt = \int_a^b (t-a)^3 (b-t)^3\, g(t)\, dt \longrightarrow ④.$$

Since $(t-a)^3 (b-t)^3 > 0$ in (a,b) and $g(t) > 0$ for $a < t < b$ we get from ④,

$\int_0^1 v(t)\, g(t) > 0$ contradiction the hypothesis.

$\int_0^1 v(t) g(t)\, dt = 0$, for all admissible functions $v(t)$.

Hence; our assumption that there exist a t_0 such ~~that~~ that $g(t_0) \neq 0$ is false. Consequently, $g(t) = 0$

Scanned with CamScanner

for all t in $(0,1)$ and thus the lemma is proved.

Proof of theorem:-

To prove that ②, we proceed as follows.

let $f(u, v, \dot{u}, \dot{v}) = \sqrt{2T}$ where,

$$2T(u, v, \dot{u}, \dot{v}) = \dot{s}^2 = E\dot{u}^2 + 2F\dot{u}\dot{v} + G\dot{v}^2$$

in terms of f, the arc length $s(\alpha)$ is,

$$s(\alpha) = \int_0^1 \dot{s}\, dt = \int_0^1 \sqrt{2T}\, dt$$

$$= \int_0^1 f(u, v, \dot{u}, \dot{v})\, dt.$$

After a slight deformation the arc length $s'(\alpha')$ is,

$$s(\alpha') = \int_0^1 f(u + \xi\lambda, v + \xi\mu, \dot{u} + \xi\dot{\lambda}, \dot{v} + \xi\dot{\mu})\, dt$$

Hence the variation in $s(\alpha)$

$$s(\alpha') - s(\alpha) = \int_0^1 f(u + \xi\lambda, v + \xi\mu, \dot{u} + \xi\dot{\lambda} + \dot{v} + \xi\dot{\mu}) - f(u, v, \dot{u}, \dot{v})\, dt \rightarrow ⑤.$$

using taylor series,

$$f(u + \epsilon\lambda, v + \epsilon\mu, \dot{u} + \epsilon\dot{\lambda}, \dot{v} + \epsilon\dot{\mu})$$

$$f(u, v, \dot{u}, \dot{v}) + \epsilon\lambda \frac{\partial f}{\partial u} + \left(\epsilon\mu \frac{\partial f}{\partial u}\right) +$$

$\left(\dot{\lambda}\frac{\partial f}{\partial \dot{u}} + \dot{\mu}\frac{\partial f}{\partial \dot{v}}\right) + O\epsilon^2 \rightarrow (6)$.

using (6) in (5),

$$S(\alpha') - S(\alpha) = \int_0^1 \left(\epsilon\lambda\frac{\partial f}{\partial u} + \epsilon\mu\frac{\partial f}{\partial u}\right) + \left(\dot{\lambda}\frac{\partial f}{\partial \dot{u}} + \dot{\mu}\frac{\partial f}{\partial \dot{v}}\right)dt + O\epsilon)^2 \rightarrow (7)$$

Taking integration by parts,

$$\int_0^1 \dot{\lambda}\frac{\partial f}{\partial \dot{u}}\,dt = \int_0^1 d\lambda\,\frac{\partial f}{\partial \dot{u}}\,dt$$

$$= \int_0^1 \frac{\partial f}{\partial \dot{u}}\,d(\lambda t).$$

$$= \int_0^1 \frac{\partial f}{\partial \dot{u}}\,d(\lambda).$$

Now we take,

$dv = d\lambda$, $u = \frac{\partial f}{\partial \dot{u}}$

$$\int_0^1 \dot{\lambda}\frac{\partial f}{\partial \dot{u}}\,dt = \left[\lambda\frac{\partial f}{\partial \dot{u}}\right]_0^1 - \int_0^1 \lambda\frac{d}{dt}\left(\frac{\partial f}{\partial \dot{u}}\right)dt.$$

Since $\lambda = 0$ at $t = 0$, and $t = 1$,

we have, $\left(\lambda\cdot\frac{\partial f}{\partial \dot{u}}\right)_0^1 = 0$.

Hence,

$$\int_0^1 \dot{\lambda}\frac{\partial f}{\partial \dot{u}}\,dt = -\int_0^1 \lambda\frac{d}{dt}\left(\frac{\partial f}{\partial \dot{u}}\right)dt. \rightarrow (8)$$

lly, $$\int_0^1 \dot{\mu}\frac{\partial f}{\partial \dot{v}}\,dt = -\int_0^1 \mu\frac{d}{dt}\left(\frac{\partial f}{\partial \dot{v}}\right)dt \rightarrow (9)$$

Scanned with CamScanner

using ⑧ and ⑨ in ⑦ we get,

$$S(\alpha') - S(\alpha) = \epsilon \int_0^1 \Big\{ \lambda \frac{\partial f}{\partial u} + \dot{\lambda} \frac{\partial f}{\partial \dot{u}} + \mu \frac{\partial f}{\partial v} + \dot{\mu} \frac{\partial f}{\partial \dot{v}} \Big\} dt + O(\epsilon^2)$$

$$= \epsilon \int_0^1 \Big[\lambda \frac{\partial f}{\partial u} - \lambda \frac{d}{dt}\left(\frac{\partial f}{\partial \dot{u}}\right) + \mu \frac{\partial f}{\partial v} - \mu \frac{d}{dt}\left(\frac{\partial f}{\partial \dot{v}}\right) \Big] dt + O(\epsilon^2)$$

$$= \epsilon \int_0^1 \lambda \left(\frac{\partial f}{\partial u} - \frac{d}{dt}\left(\frac{\partial f}{\partial \dot{u}}\right) \right) + \mu \left[\frac{\partial f}{\partial v} - \frac{d}{dt}\frac{\partial f}{\partial \dot{v}} \right] dt + O(\epsilon^2)$$

$$= \epsilon \int_0^1 (\lambda L + \mu M)\, dt + O(\epsilon^2)$$

where $L = \frac{\partial f}{\partial u} - \frac{d}{dt}\left(\frac{\partial f}{\partial \dot{u}}\right)$, $M = \frac{\partial f}{\partial v} - \frac{d}{dt}\left(\frac{\partial f}{\partial \dot{v}}\right)$

Since $\epsilon > 0$, therefore,

$S(\alpha') - S(\alpha) < \epsilon$.

$\therefore S(\alpha') - S(\alpha) = 0$.

$$\int_0^1 (\lambda L + \mu M)\, dt = 0$$

Since $f = \sqrt{2T}$

$$L = \frac{\partial f}{\partial u} - \frac{d}{dt}\left(\frac{\partial f}{\partial \dot{u}}\right).$$

$$L = \frac{1}{\sqrt{2}\sqrt{T}}$$

$$= \frac{1}{\sqrt{2T}}.$$

$\sqrt{T}$ $d(\sqrt{T})$ $(T)^{1/2}$ $= \frac{1}{2}(T)^{-1/2}$ $= \frac{1}{2\sqrt{T}}$.

$T^{1/2}$ $(T)^{-1/2}$ $\frac{1}{2\sqrt{2}}$

$$L = \frac{1}{\sqrt{2T}} \frac{\partial T}{\partial \dot{u}} - \frac{d}{dt}\left(\frac{1}{\sqrt{2T}} \frac{\partial T}{\partial \dot{u}} \right)$$

$$L = \frac{1}{\sqrt{2T}} \left(\frac{\partial T}{\partial u} - \left\{ \frac{1}{\sqrt{2T}} \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{u}}\right) + \frac{\partial T}{\partial \dot{u}} \frac{d}{dt}\left(\frac{1}{\sqrt{2T}}\right) \right\} \right.$$

$$= \frac{1}{\sqrt{2T}} \frac{\partial T}{\partial u} - \frac{1}{\sqrt{2T}} \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{u}}\right) - \frac{\partial T}{\partial \dot{u}} \frac{d}{dt}\left(\frac{1}{\sqrt{2T}}\right)$$

$$= \frac{1}{\sqrt{2T}} \left(\frac{\partial T}{\partial u} - \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{u}}\right) \right) - \frac{\partial T}{\partial \dot{u}} \left(\frac{-1}{2\sqrt{2}\,T^{3/2}} \right) \frac{dT}{dt}$$

$$= \frac{1}{\sqrt{2T}} \left(\frac{\partial T}{\partial u} - \frac{d}{dt} \frac{\partial T}{\partial \dot{u}} \right] + \frac{1}{(2T)^{3/2}} \frac{\partial T}{\partial t} \cdot \frac{\partial T}{\partial \dot{u}}$$

Since $T \neq 0$, when $L = 0$,

$$= \frac{\partial T}{\partial u} - \frac{d}{dt} \frac{\partial T}{\partial \dot{u}} \Big] + \frac{1}{2T} \frac{\partial T}{\partial t} \cdot \frac{\partial T}{\partial \dot{u}} = 0.$$

$$\frac{d}{dt} \frac{\partial T}{\partial \dot{u}} - \frac{\partial T}{\partial u} = \frac{1}{2T} \frac{\partial T}{\partial t} \cdot \frac{\partial T}{\partial \dot{u}}$$

Example :-

prove that on the general Surface, a necessary and sufficient Condition that Curve $V = c$ be a geodesic is,

$$\boxed{E E_2 + F E_1 - 2 E F_1 = 0}$$

Soln:

we know that on a curve on a surface is geodesic iff,

$$U \frac{\partial T}{\partial \dot{v}} - V \frac{\partial T}{\partial \dot{u}} = 0 \rightarrow (1).$$

On the curve $V = C$,

U itself can be taken as a parameter so that the equation of the curve $u = t$, and $V =$ Constant

The Condition for the parametric Curve $V =$ constant to be geodesic we find U, V, $\frac{\partial T}{\partial \dot{u}}$, $\frac{\partial T}{\partial \dot{v}}$ and use theorem in (1),

$$T = \frac{1}{2} \left(E\dot{u}^2 + 2F\dot{u}\dot{v} + G\dot{v}^2 \right)$$

where E, F and G are functions of u, v.

Now,

$$\frac{\partial T}{\partial u} = \frac{1}{2}\left[E_1 \dot{u}^2 + 2F_1 \dot{u}\dot{v} + G_1 \dot{v}^2\right].$$

$$\frac{\partial T}{\partial v} = \frac{1}{2}\left[E_2 \dot{u}^2 + 2F_2 \dot{u}\dot{v} + G_2 \dot{v}^2\right].$$

$$\frac{\partial T}{\partial \dot{u}} = \frac{1}{2}\left[2E\dot{u} + 2F\dot{v}\right].$$

$$\left.\begin{aligned} \frac{\partial T}{\partial \dot{u}} &= \left[E\dot{u} + F\dot{v}\right] \\ \frac{\partial T}{\partial \dot{v}} &= \left[F\dot{u} + G\dot{v}\right] \end{aligned}\right\} \rightarrow (5).$$

According to the choice of the parameter $\dot{u} = 1$, $\dot{v} = 0$.

Hence
$$\left.\begin{aligned} \frac{\partial T}{\partial u} &= \frac{1}{2} E_1 \\ \frac{\partial T}{\partial v} &= \frac{1}{2} E_2. \\ \frac{\partial T}{\partial \dot{u}} &= E \\ \frac{\partial T}{\partial \dot{v}} &= F \end{aligned}\right\} \rightarrow (5).$$

using (3), we get,

$$\boxed{U = \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{u}}\right) - \frac{\partial T}{\partial u}.}$$

$$U = \frac{d}{dt}(E) - \frac{1}{2}E_1.$$

$$U = \frac{\partial E}{\partial u}\frac{du}{dt} + \frac{\partial E}{\partial v}\cdot\frac{dv}{dt} - \frac{1}{2}E_1.$$

$$= E_1 \dot{u} + E_2 \dot{v} - \frac{1}{2}E_1.$$

Since $\dot{u} = 1$, and $\dot{v} = 0$.

we have,

$$U = \tfrac{1}{2} E_1. \longrightarrow (4)$$

Illy,

$$V = \frac{d}{dt}\left(\frac{\partial T}{\partial v^{\circ}}\right) - \frac{\partial T}{\partial v}.$$

$$= \frac{d}{dt}(F) - \cancel{\frac{\partial T}{\partial v}} \tfrac{1}{2} E_2.$$

$$= \frac{\partial F}{\partial u} \cdot \frac{\partial u}{\partial t} + \frac{\partial F}{\partial v} \cdot \frac{dv}{dt} - \tfrac{1}{2} E_2.$$

$$V = F_1 \cdot \dot{u} + F_2 \dot{v} - \tfrac{1}{2} E_2.$$

Since $\dot{u} = 1$, $\dot{v} = 0$.

$$V = F_1 - \tfrac{1}{2} E_2. \longrightarrow (5)$$

equ (4) and (5) in (1),

$$u \frac{\partial T}{\partial v^{\circ}} - V \frac{\partial T}{\partial u^{\circ}} = 0.$$

$$\left(\tfrac{1}{2} E_1\right) F - \left(F_1 - \tfrac{1}{2} E_2\right) E = 0.$$

$$\tfrac{1}{2} E_1 F - F_1 E + \frac{E_2 E}{2} = 0.$$

$$E_1 F - 2F_1 E + E_2 E = 0$$

$$\therefore\ E E_2 + F E_1 - 2 E F_1 = 0$$

Hence the proof.

9585205128

0 43602
23 04352

Prove that every helix on a cylinder is a geodesic and conversely.

Soln: we shall

let the generators of the cylinder be parallel to a constant vector a.

let γ be the helix on the cylinder and let p be a point on γ. let t, n be the tangent and principle normal, at p to γ.

Since the helix cuts the generators at a constant angle, we have

$$t \cdot a = \text{Constant}. \longrightarrow (1)$$

Diff w.r.t (1) we have,

$$\frac{dt}{ds} \cdot a + t \cdot \frac{da}{ds} = 0. \longrightarrow (2)$$

Since a is constant vector, equ (2) gives κn is $\kappa n \cdot a = 0$. as $t \cdot n = 0$, the condition $n \cdot a = 0$ and $t \cdot n = 0$, together imply that n is perpendicular to a, and t so that n is parallel to $a \times t$.

But both a and t are tangential to the surface of the cylinder and so $a \times t$ = is parallel to the ~~normp~~ surface normal. Thus the same vector $a \times t$ gives the direction of both principle normal to γ and the surface normal so that γ is a geodesic.

To prove the converse, let us take a geodesic γ on a cylinder and show that it is a helix.

At every point on γ, ~~as~~ we have, we have $n \cdot a = N \cdot a$.
Since a is parallel to the generator of the cylinder, $N \cdot a = 0$ and hence $n \cdot a = 0$. Since,

$k \neq 0$, $n \cdot a = 0$ implies

$kn \cdot a$ so that $\frac{dt}{ds} \cdot a = 0 \rightarrow (3)$

as a is constant vector we have

$t \cdot \frac{da}{ds} = 0 \rightarrow (4)$

using (3) and (4) we get,

$$\frac{dt}{ds} \cdot a + t \cdot \frac{da}{ds} = \frac{d}{ds}(t \cdot a) = 0$$

which proves that $t \cdot a$ is constant
Hence the geodesic γ cuts the generator at constant angle, and

$\therefore$ Therefore it is a helix.

<u>Note</u>: For any curve of a surface a geodesic curvature vector is Intrinsic then

$$\lambda = \frac{1}{H^2}(UG - VF)$$

$$\mu = \frac{1}{H^2}(EV - FU)$$

$a \cdot b = 0$
$a \cdot c = 0$
$a = \lambda(b \times c)$.
$(b \cdot c \neq 0)$

where, $H^2 = EG - F^2$.

1) The condition of orthogonality of the geodesic curvature vector

Note: λ, μ with any vector u, v on a surface, is

$uu' + vv' = 0$. where $U = E\lambda + F\mu$.

$V = F\lambda + G\mu$.

definition of Gaussian curvature of the surface.

Theorem (Gauss-Bonnet). For any curve C which encloses a simply connected region R on surface, *ex* C is equal to the total curvature of R.

Proof. We shall use Liouville's formula for κ_g and find $\int \kappa_g\, ds$ with the help of Green's theorem in the plane for a simply connected region R bounded by C. So we shall quote this Green's theorem as a lemma as applicable for a region R of a surface.

Lemma. If R is a simply connected region bounded by a closed curve C, then

$$\int_C P du + Q dv = \iint_R \left(\frac{\partial Q}{\partial u} - \frac{\partial P}{\partial v}\right) du\, dv$$

where P and Q are differentiable functions of u and v in R.

Proof of the Theorem. From the Liouville's formula,

$$\kappa_g = \theta' + Pu' + Qv'$$

Integrating along the curve C, we have

$$\int_C \kappa_g\, ds = \int_C (\theta' + Pu' + Qv')\, ds = \int_C (d\theta + P\, du + Q\, dv) \qquad \ldots(1)$$

where θ is the angle between the curve C and the parametric curve v = constant and P and Q are differentiable functions of u, v.

Let us suppose the simple closed curve C contains a finite number of arcs starting from A. Then at each point of the arc there passes a curve v = constant making an angle θ with C. Hence when we describe the curve C, the tangents at various members of the family v = constant described in the positive sense returns to the starting point after increasing the angles of rotation by 2π.

This increase 2π after complete rotation in the positive sense also includes the angle between the tangents at the finite number of vertices. Hence we have

$$\int_C d\theta + \sum_{r=1}^{n} \alpha_r = 2\pi \qquad \ldots(2)$$

From the definition $ex\, C = 2\pi - \sum_{r=1}^{n} \alpha_r - \int_c \kappa_g dS$ ...(3)

Using (1) and (2) in (3), we obtain

$$ex\ C = 2\pi - \left[2\pi - \int_C d\theta\right] - \left[\int_C d\theta + P\,du + Q\,dv\right]$$

Thus $$ex\ C = -\int_C P\,du + Q\,dv \qquad \text{...(4)}$$

Since R is a simply connected region and P and Q are differentiable functions of u, v, we have by Green's theorem,

$$\int_C (P\,du + Q\,dv) = \iint_R \left(\frac{\partial Q}{\partial u} - \frac{\partial P}{\partial v}\right) du\,dv \qquad \text{...(5)}$$

Since the surface element $ds = H\,du\,dv$, we rewrite (5) as

$$\int_C (P\,du + Q\,dv) = \frac{1}{H}\iint_R \left(\frac{\partial Q}{\partial u} - \frac{\partial P}{\partial v}\right) dS \qquad \text{...(6)}$$

Using (6) in (4), we get

$$ex\ C = -\frac{1}{H}\iint_R \left(\frac{\partial Q}{\partial u} - \frac{\partial P}{\partial v}\right) dS \qquad \text{...(7)}$$

If we take $K = -\frac{1}{H}\left(\frac{\partial Q}{\partial u} - \frac{\partial P}{\partial v}\right)$, we can rewrite (7) as

$$ex\ C = \iint_R K ds \qquad \text{...(8)}$$

where K is a function of u and v and it is independent of the C and defined over the region R of the surface.

Next we shall show that the $ex\,C$ is uniquely determined by K. If K is not unique, let $\overline{K}$ be such that

$$\iint_R \overline{K}\,ds = ex\ C \qquad \text{...(9)}$$

Using (8) and (9), we have $\iint_R (\overline{K} - K)\,ds = 0$

for every region R. ...(10)

Now let $\overline{K} \neq K$ at some point A of R. Then we must have $\overline{K} > K$ or $\overline{K} < K$ at A. Let us first consider $\overline{K} > K$.

Since the given surface is of class 3, $\frac{\partial Q}{\partial u}$ and $\frac{\partial P}{\partial v}$ are continuous in R so that there exists a small region R_1 of R containinig the point A such that $\overline{K} - K > 0$ at every point of R_1. For this region R containing R_1, $\iint_{R_1} (\overline{K} - K)\,ds > 0$ which contradicts (10). We get a similar contradiction $\iint_{R_1} (\overline{K} - K)\,ds < 0$ at A when $\overline{K} < K$. These contradictions prove that $\overline{K} = K$ at every point of R. That is, K is uniquelly determined as a function of u and v.

Defining $\int_R KdS$ as the total curvature of R we have proved that the total curvature is exactly the ex C in any region R enclosed by C. This completes the proof of Gauss-Bonnet Theorem.

Note **1.** $exc = \int_R KdS$ shows that there is a certain function K of u and v which is determined by E, F and G and that the excess of any curve C which encloses a simply connected region R is the integral of K over R. Also from the uniqueness of K and the form of the integral, K is invariant in the sense that it is independent of the parametric system. Since K can be found from the metric, K is an intrinsic geometrical invariant.

Definition 4. The invariant K as defined above is called the Gaussian curvature of the surface and $\int_R KdS$ is called the total curvature or integral curvature of R where R is any region whether simply connected or not.

Note **2.** We rewrite the excess of a curve C using the integral as follows.

Since $ex\ C = \int_R Kds$, we have

$$\int_R KdS = 2\pi - \sum_{r=1}^{n} \alpha_r - \int_c \kappa_g\, ds$$

We give a few examples to illustrate Gauss-Bonnet theorem.

Example 1. Find the curvature of a geodesic triangle ABC enclosing a region R on the surface.

Since ex C gives total curvature of a region bounded by C, it is enough if we find ex C in the examples where C is known.

ABC is a geodesic triangle enclosing a region R on the surface with the interior angles A, B, C. So in this case the curve C is the geodesic triangle ABC.

space. ...is defined *in abstracto*, without ...

2. Compact surfaces whose points are umbilics

For the first few theorems of this chapter we shall use the definition of surface given in Chapter II, and assume that each point has a neighbourhood (homeomorphic to an open 2-cell) which can be described by parametric equations $\mathbf{r} = \mathbf{r}(u, v)$.

As our first theorem of differential geometry in the large we shall prove

THEOREM 2.1. *The only compact surfaces of class $\geqslant 2$ for which every point is an umbilic are spheres.*

This is an example of a global theorem since part of the hypothesis—viz. the compactness of the surface considered as a set of points in E_3—evidently involves the surface as a whole. A useful technique in proving global theorems in differential geometry is first to establish the result locally in some neighbourhood of an arbitrary point, and then to try to extend the result so that it applies globally. We employ precisely this technique in proving Theorem 2.1. By means of the local differential geometry developed in Chapters II and III we shall prove that in the neighbourhood of any point the surface is either spherical or plane. We then use the property of compactness to reject one alternative, and show that the surface must in fact be a sphere.

Let S be a compact surface of class $\geqslant 2$ for which every point is an umbilic. Let P be any point on S, and let V be a coordinate neighbourhood of S containing P, in which part of S is represented parametrically by $\mathbf{r} = \mathbf{r}(u, v)$.

Since every point of V is an umbilic, it follows that every curve lying in V must be a line of curvature. Hence, from Rodrigues' formula, at all points of V,

$$d\mathbf{N} + \kappa\, d\mathbf{r} = 0, \tag{2.1}$$

the ...

3. Hilbert's lemma

We shall make use of the following lemma due to Hilbert.

In a closed region R of a surface of constant positive Gaussian curvature without umbilics, the principal curvatures take their extreme values at the boundary.

This lemma is purely local in character and we shall use results of Chapter III to prove it. We prefer to restate the lemma in a slightly different form suggested by W. F. Newns.

If at a point P_0 of any surface, the principal curvatures κ_a, κ_b are such that either (i) $\kappa_a > \kappa_b$, κ_a *has a maximum at* P_0 *and* κ_b *has a minimum at* P_0, *or* (ii) $\kappa_a < \kappa_b$, κ_a *has minimum at* P_0 *and* κ_b *has a maximum at* P_0, *then the Gaussian curvature K cannot be positive at* P_0.

We shall prove the lemma by the method of contradiction. Suppose that the lemma is false. Then there is a point P_0 at which the principal curvatures have distinct extreme values, one maximum and the other minimum. Taking the lines of curvature as parametric curves, the principal curvatures are

$$\kappa_a = L/E, \qquad \kappa_b = N/G \tag{3.1}$$

(cf. III, (3.4)). The Codazzi equations are (cf. Chapter III, Exercise 9.1)

$$\left.\begin{aligned} L_2 &= \tfrac{1}{2}E_2\left(\frac{L}{E}+\frac{N}{G}\right) \\ N_1 &= \tfrac{1}{2}G_1\left(\frac{L}{E}+\frac{N}{G}\right) \end{aligned}\right\}, \tag{3.2}$$

from which we find

$$\left.\begin{aligned} \frac{\partial \kappa_a}{\partial v} &= \frac{1}{2}\frac{E_2}{E}(\kappa_b-\kappa_a) \\ \frac{\partial \kappa_b}{\partial u} &= \frac{1}{2}\frac{G_1}{G}(\kappa_a-\kappa_b) \end{aligned}\right\}. \tag{3.3}$$

Since the principal curvatures have extrema, the left-hand members vanish at P_0. It follows that at P_0, $E_2 = G_1 = 0$, and hence that, at P_0,

$$\left.\begin{aligned} \frac{\partial^2 \kappa_a}{\partial v^2} &= \frac{1}{2}\frac{E_{22}}{E}(\kappa_b-\kappa_a) \\ \frac{\partial^2 \kappa_b}{\partial u^2} &= \frac{1}{2}\frac{G_{11}}{G}(\kappa_a-\kappa_b) \end{aligned}\right\}. \tag{3.4}$$

There are now two possibilities: either

(i) κ_a has a maximum. In this case

$$\kappa_a-\kappa_b > 0, \qquad \partial^2\kappa_a/\partial v^2 \leqslant 0, \qquad \partial^2\kappa_b/\partial u^2 \geqslant 0; \tag{3.5}$$

or

(ii) κ_a has a minimum. Then

$$\kappa_b-\kappa_a > 0, \qquad \partial^2\kappa_a/\partial v^2 \geqslant 0, \qquad \partial^2\kappa_b/\partial u^2 \leqslant 0. \tag{3.6}$$

In either case we see that $E_{22} \geqslant 0$ and $G_{11} \geqslant 0$. (Notice that the signs of κ_a, κ_b are irrelevant.) But this contradicts the fact (see II (17.3)) that the Gaussian curvature K satisfies

$$K = -\frac{1}{2EG}(E_{22}+G_{11}),$$

since the right-hand member is negative or zero while K is assumed strictly positive. This contradiction completes the proof of the lemma.