UNIT - I

§1 BASIC TYPES:

Definition:

Fuzzy sets:

If X is a collection of objects denoted generically by x, then a fuzzy set A in X is a set of ordered pairs.

$$A = \{(x, A(x)) \mid x \in X\}$$

A(x) is the membership fn.

Example: 1

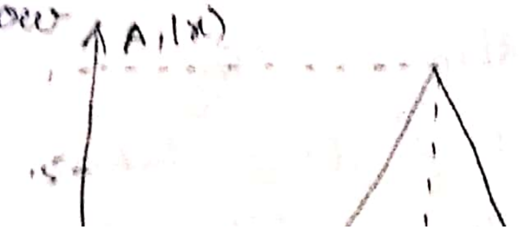
Define

$$A_1(x) = \begin{cases} x-1, & \text{when } x \in [1,2] \\ 3-x, & \text{when } x \in [2,3] \\ 0, & \text{otherwise} \end{cases}$$

It is a membership fn.

The fuzzy set determined by this fn expresses in a particular form the general conception of a class of real numbers that are close to 2.

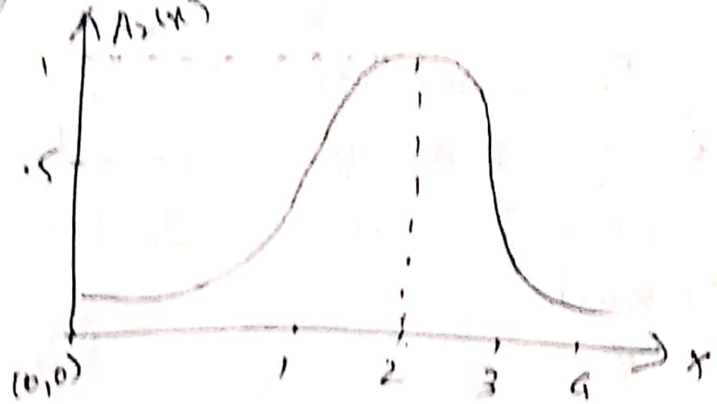
The graphical representation of the fuzzy set A_1 is given below



Ex: 2

Define $A_2(x) = \dfrac{1}{1+10(x-2)^2}$, $\forall x \in R$.

This membership fn determines a fuzzy sets A_2 to the conception of a class of real numbers that are closed to 2. The graphical representation of the fuzzy sets A_2 to is given below.



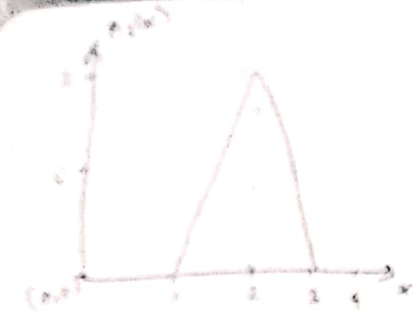
Ex: 3

Define

$$A_3(x) = e^{-|5(x-2)|} \quad \forall x \in R.$$

This membership fn yields a fuzzy sets A_3 to the connection of a class of real numbers that are close to 2.

The graphical representation of the fuzzy sets A_3 is given



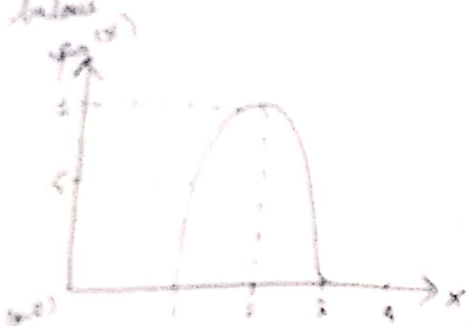
Ex:

Define

$$A_2(x) = \begin{cases} \frac{1+\cos(2\pi(x-2))}{2}, & x \in [3/2, 5/2] \\ 0, & \text{otherwise} \end{cases}$$

This membership fn is a fuzzy set A_2 to the concept of a class of real nos. that are closed to 2.

The graphical representation of the fuzzy sets A_2 is given below



Characteristic function:-

A set is defined by a fn usually called a characteristic fn that declares which element of X are members of the set & which are not. Set A is defined by its characteristic fn.

χ_A is as follows

$$\chi_A(x) = \begin{cases} 1, & \text{for } x \in A \\ 0, & \text{for } x \notin A \end{cases}$$

i.e, the characteristic fn maps elements of X. two element of the set {0,1} which is formly expressed by

$$\chi_A : X \longrightarrow \{0,1\}$$

for each $x \in X$ when $\chi_A(x) = 1$ x is declared to be a member of A. when $\chi_A(x) = 0$, x is declared as a non-member of A.

Membership function

Let A be a subset of X & the value of assigned by a fn. to the element of the universal set X fall with in a specified range and indicate the membership grade of those elements to the member of A, then the fn. is called membership fn. (or) grade fn.

UNIVERSAL SET.

Every set is a subset of itself, and every set is a subset of the universal set.

EQUAL SET.

If $A \subseteq B$ & $B \subseteq A$ then A & B contain the same member they are called equal sets this is denoted by $A = B$.

FAMILY OF SET.

A set whose elements of are themselves set is often refered to as a family of sets it can be defined in the form $\{A_i / i \in I\}$ where i & I are called the set index and the index set respectly because the index 'i' is used to reference the set A_i. the family of sets is also called an index set.

In this text, families of sets are usually denoted by script capital letter. for

Ex:

$$\mathcal{A} = \{A_1, A_2 \ldots A_n\}$$

POWER SET:

The family of all subsets of a given set A is called the power set of A. and its usually denoted by $\mathcal{P}(A)$ the family of all subsets of $\mathcal{P}(A)$ is called a IInd order power set of A its denoted by $\mathcal{P}^2(A)$, which stands for $\mathcal{P}(\mathcal{P}(A))$ & higher order power sets. $\mathcal{P}^3(A), \mathcal{P}^4(A), \mathcal{P}(A) \ldots$ can be define.

CORDINALITY:

The no. of members of a finite set A is called the cordinality of A and is denoted by $|A|$, when A is finite Then

$$|\mathcal{P}(A)| = 2^{|A|}$$

$$|\mathcal{P}^2(A)| = 2^{2^{|A|}}$$

RELATIVE COMPLEMENT:

The relative complement of a set w.r.to set B is the containing all members of B that are not also members of A. This can be written B-A thus

$$B - A = \{ x / x \in B \ \& \ x \notin A \}$$

UNION OF SETS:

The union of set A & B is the set containing all the elements that belong either to set A alone to set B alone (or) two both set A & set B

$$A \cup B = \{ x / x \in A \ (or) \ x \in B \}$$

INTERSECTION OF SETS:

The intersection of sets A & B is the set containing all the elements belonging to both set A & set B.

Thus

$$A \cap B = \{ x / x \in A \ \& \ x \in B \}$$

PARTITION:

A family of pairwise disjoint non empty subset of a set A is called a partition on A.

If the union these subset yields the regional set A be denote a partition on A' by the symbol $\pi(A)$, formly,

$$\pi(A) = \{ A_i / i \in I, A_i \subseteq A \}$$

where $A_i \neq \phi$ is a partition on A iff $A_i \cap A_j = \phi$.

For each pair $i, j \in I$, $i \neq j$ and $\bigcup_{i \in I} A_i = A$.

BLOCKS:

Members of a partition $\phi(A)$ which are subsets of A are usually referred to blocks of the partition each no/. of A belongs to one & only one block of $\phi(A)$.

CONVEX:

A set A in $\mathbb{R}^n$ is called convex iff for every pair of points.

$$r = \langle r_i / i \in \mathbb{N}_n \rangle$$

$$s = \langle s_i / i \in \mathbb{N}_n \rangle$$

In A and every real no/.. $\lambda \in [0,1]$ the pt/..

$$t = \langle \lambda r_i + (1-\lambda) s_i / i \in \mathbb{N}_n \rangle$$

is also in A.

UPPER BOUND / LOWER BOUND:

Let R denote a set of real no/.. ($R \subseteq \mathbb{R}$) if there is a real no/.. 'r' $\ni$; $x \le r$ ((or) $x \ge s$, respectively) for every $x \in R$, then r is called an upper bound of R. (or) lower bound of R. and we say that A is bounded above by r.

SUPREMUM:

For any set of real no/.. R that is bounded above the a real no/.. r is called the supremum of R iff

(a) r is an upper bound of R.

(b) No no/.., < r is then upper bound of R.

(c) If r is the supremum of R we write r = Sup R.

INFIMUM:

For any set of real no/.. R that is bounded below a real no/. δ is called the infimum of R, iff

(a) δ is a lower bound of A

(b) No no/. greater than, δ is a lower bound of R.

If δ is the infimum of R we write $\delta = \inf R$.

GENERAL TYPE OF EXAMPLES:

Each function in ... given below is a member of a parameter family of fn/.

The following are general formula describing the 4 families of membership fns/., where r denotes the real no/. for which the membership grade is require to be one, & P_i ($i = 1, 2 \ldots$) is a parameter that determines the rate at which, for each x, the fn/. decreases with increasing diffe...

$$A_1(x) = \begin{cases} P_1(x-r)+1, & x \in [r - 1/P_1, \ldots \\ P_1(r-x)+1, & x \in [r, r+1/\ldots \\ 0, & \text{otherwise.} \end{cases}$$

$$A_2(x) = \frac{1}{1 + P_2(x-r)^2} \quad \forall x \in \ldots$$

$$A_3(x) = e^{-|P_3(x-r)|}$$

$$A_4(x) = \begin{cases} 1 + \cos(P_4 \pi (x-r)/2), & x \in [\frac{r-1}{P_4}, \frac{r+}{P_4} \ldots \\ 0, & \text{otherwise.} \end{cases}$$

There are fuzzy sets definite with in the set of real no/..

EXAMPLE OF FUZZY SET DEFINE WITH IN A FINITE UNIVERSAL:

Let us consider now, 3 fuzzy sets define with in a finite universal set that consists of 7 levels of education

0 – No Education.
1 – Elementry school.
2 – high school.
3 – Higher Secondary School.
4 – Bachelor's degree
5 – Master's degree
6 – doctoral degree.

Define the 3 membership fn/. of the 3 fuzzy sets as follows to capture the concepts indicated againsed them respectively.

A_1 – Little education.

A_2 – Highely education.

A_3 – Very highely education.

For the fn/. A_1, we define the degree as

$A_1(0) = 1$
$A_1(1) = 0.8$
$A_1(2) = 0.5$
$A_1(3) = 0$.

Thus fuzzy set

$A_1 = \{(0,1), (1,.8), (2,.5)\}$

For the fn/. A_2, we define the degree $A_2(2) = 0.2$,
$A_2(3) = 0.6$,
$A_2(4) = 0.8$,
$A_2(5) = 1$

Thus fuzzy set

$A_2 = \{(2, 0.2), (3, 0.6), (4, 0.8), (5, 1)\}$

For the fn/. A_3, we define the degree as $A_3(2) = 0$,
$A_3(3) = 0.1$,
$A_3(4) = 0.5$,
$A_3(5) = 0.8$,
$A_3(6) = 1$.

The fuzzy set,

$A_3 = \{(3, 0.1), (4, 0.5), (5, 0.8), (6, 1)\}$.

⟶ §1.2

ORDINARY FUZZY SET:

Given a relevant universal set X, any arbitary fuzzy set of this type (say set A) is defined by a fn/. of the form

$$A : X \longrightarrow [0,1].$$

Fuzzy sets of this types by far the most common in the literature as well as in the successful applications of fuzzy set theory, However, several more general types of fuzzy sets have also been proposed in the literature.

Let fuzzy set of the types thus far discussed be called ordinary fuzzy sets to distinguish them from fuzzy sets of the various generalized types.

INTERVAL VALUED FUZZY SETS.

A membership fn/. based on the latter approach doesnot assign to each element of the universal set one real no/., but a closed interval of real no/. b/w the identified lower and upper bounds. Fuzzy sets defined by membership fns/. of this types are called interval valued fuzzy sets. These sets are defined formally by fn/. of the form

$$A : X \longrightarrow \mathcal{E}([0,1])$$

Where $\mathcal{E}([0,1])$ denotes the family of all cld/. interval of real no/. in $[0,1]$. clearly

$$\mathcal{E}([0,1]) \subset P([0,1])$$

Example:

For each x, $A(x)$ is represented by the segment b/w the two curves which express the indentified lower & upper bounds, Thus

$$A(a) = [\alpha_1, \alpha_2).$$

An example of an interval valued fuzzy set

$$(A(a)) = [\alpha_1, \alpha_2].$$

L – FUZZY SET:

The membership grades must be represented by no/. in the unit interval $[0,1]$ & allow them to be represented by symbols of an arbitary set 'L' that is atleast partially ordered we obtain fuzzy sets of another generalized type they are called L-fuzzy sets & their membership fns/. the form

$$A : X \longrightarrow L.$$

2.2 Basic Concepts

α-cut & strong α-cut

One of the most important concepts of fuzzy sets is the concept of an α-cut and its variant, a strong α-cut.

A fuzzy set A defined on X & any no. $\alpha \in [0,1]$.

α-cut → $^{\alpha}A$

Strong α-cut → $^{\alpha+}A$.

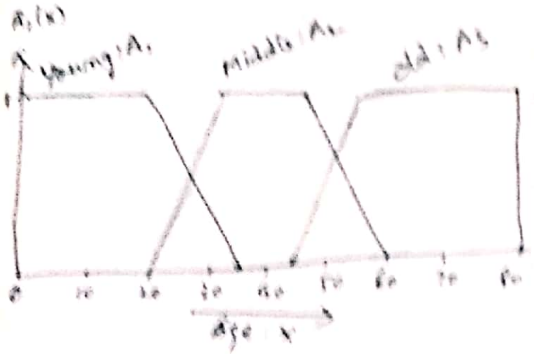
The crisp sets

$$^{\alpha}A = \{x / A(x) \geq \alpha\}$$

$$^{\alpha+}A = \{x / A(x) > \alpha\}$$

Examples:

The following is a complete characterization of all α-cuts & strong α-cuts for the fuzzy sets A_1, A_2, A_3 as given below



Consider three fuzzy sets that represent the concepts of a young age, middle age & old person.

A reasonable expression of these concepts by trapezoidal membership fn A_1, A_2, A_3 is given above.

$$A = \{0, 1, 2, \ldots, 80\} \to [0,1]$$

$$^{0}A_1 = {}^{0}A_2 = {}^{0}A_3 = [0, 80] = X$$

$$^{\alpha}A_1 = [0, 35 - 15\alpha]$$

$$^{\alpha}A_2 = [15\alpha + 20, 60 - 15\alpha]$$

$$^{\alpha}A_3 = [15\alpha + 45, 80] \quad \forall \alpha \in (0,1]$$

$$^{\alpha+}A_1 = [0, 35 - 15\alpha)$$

$$^{\alpha+}A_2 = (15\alpha + 20, 60 - 15\alpha)$$

$$^{\alpha+}A_3 = (15\alpha + 45, 80] \quad \forall \alpha \in [0,1)$$

Problem:

$$A = \frac{0.5}{x_1} + \frac{0.9}{x_2} + \frac{0.5}{x_3} + \frac{0.8}{x_4} + \frac{1}{x_5}$$

find α-cuts & strong α-cuts.

Sol:

To find α-cuts:

$$^{0.6}A = \{x_2, x_4, x_5\}$$

$$^{0.5}A = \{x_1, x_2, x_3, x_4, x_5\}$$

$$^{0.8}A = \{x_1, x_2, x_4, x_5\}$$

To find strong α-cuts:

$$^{0.1+}A = \{x_1, x_3, x_4, x_5\}$$

$$^{0.5+}A = \{x_4, x_5\}$$

$$^{0.8+}A = \{x_4, x_5\}$$

Properties of α-cuts & strong α-cuts

The total ordering of values of α in [0,1] is inversely preserved by set inclusion of the corresponding α-cuts as well as strong α-cuts.

i.e., for any fuzzy sets A and fuzzy set A and pair $\alpha_1, \alpha_2 \in [0,1]$ of distinct values s.t. $\alpha_1 < \alpha_2$, we have

$$^{\alpha_1}A \supseteq {}^{\alpha_2}A \;\&$$

$$^{\alpha_1+}A \supseteq {}^{\alpha_2+}A$$

This property can also be expressed by the eqn

$$^{\alpha_1}A \cap {}^{\alpha_2}A = {}^{\alpha_2}A$$

$$^{\alpha_1}A \cup {}^{\alpha_2}A = {}^{\alpha_1}A$$

and

$$^{\alpha_1+}A \cap {}^{\alpha_2+}A = {}^{\alpha_2+}A$$

$$^{\alpha_1+}A \cup {}^{\alpha_2+}A = {}^{\alpha_1+}A$$

HEIGHT:

The height $h(A)$ of a fuzzy set A is the largest membership grade attained by any element in that set formally,

$$h(A) = \sup_{x \in X} A(x)$$

Example

(i) Support of A

$$A = \frac{x_1}{0} + \frac{x_2}{0.1} + \frac{x_3}{1}$$

denoted as

$$^{0+}A = \{x_2, x_3\}$$

(ii) Height of A

$$A = \frac{x_1}{0} + \frac{x_2}{0.1} + \frac{x_3}{1}$$

$$A = \{x_3\}, \quad h(A) = 1$$

NORMAL & SUB NORMAL

A fuzzy set A is called normal when $h(A) = 1$ & called sub normal when $h(A) < 1$.

The height of A may also be viewed as the supremum of α for which $^{\alpha}A \neq \phi$.

Example:

(i) Normal

$A = \frac{x_1}{0} + \frac{x_2}{0.7} + \frac{x_3}{1}$

$h(A) = 1, \quad \alpha = 1$

${}^1A = \{x_3\}$

(ii) Sub-normal

$A = \frac{x_1}{0} + \frac{x_2}{0.7} + \frac{x_3}{1}$.

$h(A) < 1$

$\phi'(A) = \{x_2\}$

$\phi(A) = \{x_1\}$

Sub-normal $= \{x_1, x_2\}$.

CUT WORTHY & STRONG CUT WORTHY:

Any property generalized from classical set theory into the domain of fuzzy set theory that is preserved in all $+\alpha$ for $\alpha \in (0,1]$ in the classical sense is called a cut worthy property.

If it is preserved in all strong α-cuts for $\alpha \in [0,1]$. It is called a strong cut worthy property.

STANDARD COMPLEMENT:

DEGREE OF SUBSETHOOD:

For any pair of fuzzy subsets define on a finite universal set X, the degree of subset hood $S(A,B)$ of A in B is define by the formula

$$S(A,B) = \frac{1}{|A|}\left(|A| - \sum_{x \in X} \max[0, A(x) - B(x)]\right)$$

The Σ term in this formula describes the sum of the degree to which the subset inequality

$A(x) \leq B(x)$

is violated the difference describes the lack of the violations and the cardinal cardinality $|A|$ in the dinaminator is a normalizing factor to obtain the range.

$0 \leq S(A,B) \leq 1$,

$$S(A,B) = \frac{|A \cap B|}{|A|}$$

SCALAR CARDINALITY:

For any fuzzy sets A defined on a finite universal set X we define its scalar cardinality $|A|$ by the formula

$$|A| = \sum_{x \in X} A(x)$$

The scalar cardinality $|A|$, of A is also known as sigma count of A.

Example:

$c(x) = \frac{x}{x+1} \quad \forall x \in \{0, 1, \ldots 10\} = X$.

$\tilde{A} = \{(0,0), (1,1/2), (2,2/3), (3,3/4), (4,4/5), (5,5/6), (6,6/7), (7,7/8), (8,8/9), (9,9/10), (10,10/11)\}$

$|A| = \sum_{x \in X} A(x)$.

$|A| = 0 + \frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \frac{4}{5} + \frac{5}{6} + \frac{6}{7} + \frac{7}{8} + \frac{8}{9} + \frac{9}{10} + \frac{10}{11}$

$|A| = 8.98$.

ELEMENTS OF THE SUPPORT:

Given a fuzzy set A define on a finite universal set X, let $x_1, x_2, \ldots x_n$ denote the elements of the support $0+A$ of A and let A_i denote the grade of membership of x_i in A $\forall i \in N_n$. Then A is written as

$$A = \frac{a_1}{x_1} + \frac{a_2}{x_2} + \cdots + \frac{a_n}{x_n}$$

HAMMING DISTANCE:

Hamming distance we have

$$d(A,B) = \sum_{x \in X} |A(x) - B(x)|$$

FUZZY VARIABLE: § 1.2

General fuzzy set representing linguistic concepts such as low, medium, high and so on are often employed to define states of a variable such a variable is usually called a fuzzy variable.

Example:

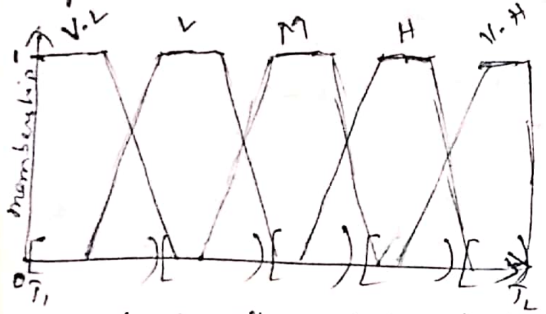
Consider the temperature with in a range $[t_1, t_2]$ this temperature interval is characterised as a fuzzy variable. States of the fuzzy variable are a fuzzy sets representing five linguistic concepts;

very low, low, medium, high, very high.

They are all defined by membership fn/. of the form

$[T_1, T_2] \to [0,1]$

Graphs of these fns/. have trapezoidal shapes is given below.



graphs of these fns/. which have trapezoidal shapes together with triangular shapes are most common in current application.

TYPE - 2 FUZZY SET:

Integrable valued fuzzy set can be generalized allowing their interval to be fuzzy.

Each interval now becomes an arbitrary fuzzy set defined within the universal set [0,1] fuzzy set with membership fn/. have the form.

$$A : X \to \mathcal{F}([0,1]).$$

where $\mathcal{F}([0,1])$ denotes the set of all ordinary fuzzy sets that can be define within the universal set [0,1] are called fuzzy set of type 2. $\mathcal{F}([0,1])$ is also called fuzzy power set [0,1].

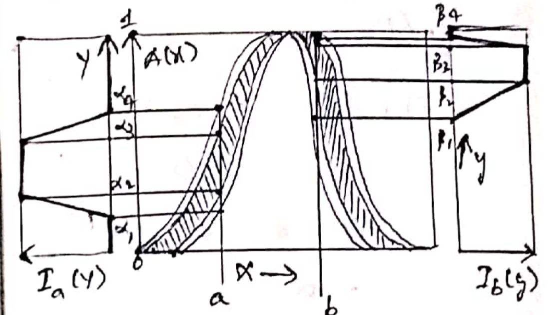
Example:

It is assumed here that membership fn/. of all fuzzy intervals that involved are of trapezoidal shapes and consequently each of them is fully defined by 4 - nos/.

For each x, there nos/. are produced by 4 - fns/. represented in the diagram given below, by the 4-curve

Thus for example if $x = a$ we obtain nos/. $\alpha_1, \alpha_2, \alpha_3$ & α_4 by which the fuzzy interval assigned to α is uniquely determined.

||ly If $x = b$, obtains nos/. $\beta_1, \beta_2, \beta_3$ & β_4 assigned in right - hand side.



LEVEL - 2 TYPE FUZZY SET:

Fuzzy set define within the universal set whose elements on ordinary fuzzy set are known as level to fuzzy sets.

Their membership fn/. have the form.

$$A : \mathcal{F}(X) \to [0,1].$$

Where $\mathcal{F}(x)$ denotes the fuzzy power set of x.

Example:

Fuzzy sets expressing proposition of the form x is close to 'r' where x is a variable whose values are real no/. and 'r' is a particular real nos/.

FUZZY SET OF TYPE - 2 & ALSO OF LEVEL 2.

Fuzzy sets of type 2 and of level to have their membership fns/. of the form

$$A : \mathcal{F}(x) \to \mathcal{F}([0,1]).$$

§ Sec: 2.

Theorem 2.1:

Statement:

A fuzzy set A on R is a convex if

$$A(\lambda x_1 + (1-\lambda)x_2) \geq \min[A(x_1), A(x_2)]$$

$\forall x_1, x_2 \in R$ & all $\lambda \in [0,1]$ where min denoted the minimum operator.

proof:

Case (i):

Assume that A is convex.

let $\alpha = A(x_1) \leq A(x_2)$

Then $x_1, x_2 \in {}^{\alpha}A$

$\lambda x_1 \in {}^{\alpha}A$

$(1-\lambda)x_2 \in {}^{\alpha}A$

$\therefore \lambda x_1 + (1-\lambda)x_2 \in {}^{\alpha}A$

$\forall \lambda \in [0,1]$.

by the convexity of A.

Consequently,

$A(x_2) \geq \alpha = A(x_1)$

$A(\lambda x_1 + (1-\lambda)x_2) \geq \alpha = A(x_1)$

$= \min[A(x_1), A(x_2)]$

$\therefore A(\lambda x_1 + (1-\lambda)x_2) \geq \min[A(x_1), A(x_2)]$ → (1)

Case (ii):

Assume that A satisfies eqn (1).

To PROVE: for any $\alpha \in [0,1]$, ${}^{\alpha}A$ is convex,

for any $x_1, x_2 \in {}^{\alpha}A$.

i.e., $A(x_1) \geq \alpha$

$A(x_2) \geq \alpha$

for any $\lambda \in [0,1]$,

by (1),

$A(\lambda x_1 + (1-\lambda)x_2)$

$\geq \min[A(x_1), A(x_2)]$

$\geq \min[\alpha, \alpha]$

$= \alpha$

$A(\lambda x_1 + (1-\lambda)x_2) \geq \alpha$

i.e, $\lambda(x_1) + (1-\lambda)x_2 \in {}^{\alpha}A$

$\therefore {}^{\alpha}A$ is convex. $\forall \alpha \in [0,1]$.

Hence, A is convex.

§2. ADDITIONAL PROPERTIES OF α - CUTS :-

Thm 2.1:

Let $A, B \in \mathcal{F}(x)$. Then the following properties hold for all $\alpha, \beta \in [0,1]$:

(i) ${}^{\alpha+}A \subseteq {}^{\alpha}A$;

(ii) $\alpha \leq \beta \Rightarrow {}^{\alpha}A \supseteq {}^{\beta}A$ & ${}^{\alpha+}A \supseteq {}^{\beta+}A$;

(iii) ${}^{\alpha}(A \cap B) = {}^{\alpha}A \cap {}^{\alpha}B$ &

${}^{\alpha}(A \cup B) = {}^{\alpha}A \cup {}^{\alpha}B$;

(iv) ${}^{\alpha+}(A \cap B) = {}^{\alpha+}A \cap {}^{\alpha+}B$ &

${}^{\alpha+}(A \cup B) = {}^{\alpha+}A \cup {}^{\alpha+}B$;

(v) ${}^{\alpha}(\bar{A}) = {}^{(1-\alpha)+}\bar{A}$.

proof:

Case (iii):

To prove: (iii). ${}^{\alpha}(A \cap B) = {}^{\alpha}A \cap {}^{\alpha}B$.

let $x \in {}^{\alpha}(A \cap B)$

$\Rightarrow (A \cap B)(x) \geq \alpha$

$\Rightarrow \min\{A(x), B(x)\} \geq \alpha$

$\Rightarrow A(x) \geq \alpha$ and $B(x) \geq \alpha$.

$\Rightarrow x \geq {}^{\alpha}A$ and $x \geq {}^{\alpha}B$

$\Rightarrow x \in {}^{\alpha}A \cap {}^{\alpha}B$

$\therefore {}^{\alpha}(A \cap B) \subseteq {}^{\alpha}A \cap {}^{\alpha}B$ → (1)

Conversely,

$x \in {}^{\alpha}A \cap {}^{\alpha}B$

$\Rightarrow x \in {}^{\alpha}A$ and $x \in {}^{\alpha}B$.

$\Rightarrow A(x) \geq \alpha$ and $B(x) \geq \alpha$

$\Rightarrow \min\{A(x), B(x)\} \geq \alpha$

$\Rightarrow (A \cap B)(x) \geq \alpha$

$\Rightarrow x \in {}^{\alpha}(A \cap B)$

$\Rightarrow {}^{\alpha}A \cap {}^{\alpha}B \subseteq {}^{\alpha}(A \cap B)$ → (2)

From (1) & (2),

${}^{\alpha}(A \cap B) \subseteq {}^{\alpha}A \cap {}^{\alpha}B$

${}^{\alpha}(A \cap B) \supseteq {}^{\alpha}A \cap {}^{\alpha}B$

$\therefore \boxed{{}^{\alpha}(A \cap B) = {}^{\alpha}A \cap {}^{\alpha}B}$

To prove: (iii) ${}^{\alpha}(A \cup B) = {}^{\alpha}A \cup {}^{\alpha}B$

let $x \in {}^{\alpha}(A \cup B)$

$\Rightarrow (A \cup B)(x) \geq \alpha$

$\Rightarrow \min\{A(x), B(x)\} \geq \alpha$

$\Rightarrow A(x) \geq \alpha$ and $B(x) \geq \alpha$.

$\Rightarrow x \in {}^{\alpha}A$ and $x \in {}^{\alpha}B$.

$\Rightarrow x \in {}^{\alpha}A \cup {}^{\alpha}B$.

$\therefore {}^{\alpha}(A \cup B) \subseteq {}^{\alpha}A \cup {}^{\alpha}B$. → (1)

Conversely,

let $x \in {}^{\alpha}A \cup {}^{\alpha}B$

$\Rightarrow x \in {}^{\alpha}A$ and $x \in {}^{\alpha}B$

$\Rightarrow A(x) \geq \alpha$ and $B(x) \geq \alpha$.

$\Rightarrow \max\{A(x), B(x)\} \geq \alpha$

$\Rightarrow (A \cup B)(x) \geq \alpha$

$\Rightarrow x \in {}^{\alpha}(A \cup B)$

$\therefore {}^{\alpha}A \cup {}^{\alpha}B \subseteq {}^{\alpha}(A \cup B) \longrightarrow (2)$

From (1) & (2),

$$\therefore \boxed{{}^{\alpha}(A \cup B) = {}^{\alpha}A \cup {}^{\alpha}B}$$

Case (iv):

To prove: (iv) ${}^{\alpha+}(A \cap B) = {}^{\alpha+}A \cap {}^{\alpha+}B$

If $x \in {}^{\alpha+}(A \cap B)$

$\Rightarrow (A \cap B)(x) > \alpha$

$\Rightarrow \min\{A(x), B(x)\} > \alpha$

$\Rightarrow A(x) > \alpha$ and $B(x) > \alpha$

$\Rightarrow x \in {}^{\alpha+}A$ and $x \in {}^{\alpha+}B$

$\Rightarrow x \in {}^{\alpha+}A \cap {}^{\alpha+}B$

$\therefore {}^{\alpha+}(A \cap B) \subseteq {}^{\alpha+}A \cap {}^{\alpha+}B. \longrightarrow (1)$

Conversely,

let $x \in {}^{\alpha+}A \cap {}^{\alpha+}B$.

$\Rightarrow x \in {}^{\alpha+}A$ and $x \in {}^{\alpha+}B$.

$\Rightarrow A(x) > \alpha$ and $B(x) > \alpha$

$\Rightarrow \min\{A(x), B(x)\} > \alpha$.

$\Rightarrow (A \cap B)(x) > \alpha$

$\Rightarrow x \in {}^{\alpha+}(A \cap B)$

$\therefore {}^{\alpha+}A \cap {}^{\alpha+}B \subseteq {}^{\alpha+}(A \cap B) \longrightarrow (2)$

From (1) & (2)

${}^{\alpha+}(A \cap B) \subseteq {}^{\alpha+}A \cap {}^{\alpha+}B$

${}^{\alpha+}(A \cap B) \supseteq {}^{\alpha+}A \cap {}^{\alpha+}B$

$$\therefore \boxed{{}^{\alpha+}(A \cap B) = {}^{\alpha+}A \cap {}^{\alpha+}B}$$

To prove (iv) ${}^{\alpha+}(A \cup B) = {}^{\alpha+}A \cup {}^{\alpha+}B$.

let $x \in {}^{\alpha+}(A \cup B)$

$\Rightarrow (A \cup B)(x) > \alpha$

$\Rightarrow \max\{A(x), B(x)\} > \alpha$

$\Rightarrow A(x) > \alpha$ & $B(x) > \alpha$

$\Rightarrow x \in {}^{\alpha+}A$ & $x \in {}^{\alpha+}B$.

$\Rightarrow x \in {}^{\alpha+}A \cup {}^{\alpha+}B$.

$\therefore {}^{\alpha+}(A \cup B) \subseteq {}^{\alpha+}A \cup {}^{\alpha+}B. \longrightarrow (1)$

Conversely,

let $x \in {}^{\alpha+}A \cup {}^{\alpha+}B$

$\Rightarrow x \in {}^{\alpha+}A$ & $x \in {}^{\alpha+}B$

$\Rightarrow A(x) > \alpha$ & $B(x) > \alpha$.

$\Rightarrow \max\{A(x), B(x)\} > \alpha$.

$\Rightarrow (A \cup B)(x) > \alpha$.

$\Rightarrow x \in {}^{\alpha+}(A \cup B)$

$\therefore {}^{\alpha+}A \cup {}^{\alpha+}B \subseteq {}^{\alpha+}(A \cup B) \longrightarrow (2)$

From (1) & (2),

${}^{\alpha+}(A \cup B) \subseteq {}^{\alpha+}A \cup {}^{\alpha+}B$

${}^{\alpha+}(A \cup B) \supseteq {}^{\alpha+}A \cup {}^{\alpha+}B$.

$$\therefore \boxed{{}^{\alpha+}(A \cup B) = {}^{\alpha+}A \cup {}^{\alpha+}B}$$

Case (v):

To prove: ${}^{\alpha}(\bar{A}) = {}^{(1-\alpha)+}\bar{A}$.

let $x \in {}^{\alpha}(\bar{A})$

$\Rightarrow \bar{A}(x) \geq \alpha$.

$\Rightarrow 1 - \bar{A}(x) \geq \alpha$.

$\Rightarrow 1 - \alpha \geq \bar{A}(x)$.

$\Rightarrow \bar{A}(x) \leq 1 - \alpha$

$\Rightarrow x \in {}^{(1-\alpha)+}\bar{A}$

$\therefore {}^{\alpha}(\bar{A}) \subseteq {}^{(1-\alpha)+}\bar{A} \longrightarrow (1)$

Conversely,

$x \in {}^{(1-\alpha)+}\bar{A}$.

$\Rightarrow A(x) \leq 1 - \alpha$.

$\Rightarrow 1 - \alpha \geq A(x)$

$\Rightarrow 1 - A(x) \geq \alpha$

$\Rightarrow \bar{A}(x) \geq \alpha$

$\Rightarrow x \in {}^{\alpha}(\bar{A})$

$\therefore {}^{(1-\alpha)+}\bar{A} \subseteq {}^{\alpha}(\bar{A}) \longrightarrow (2)$

From (1) & (2),

$$\boxed{{}^{\alpha}(\bar{A}) = {}^{(1-\alpha)+}(\bar{A}).}$$

Case (i):

To prove: ${}^{\alpha+}A \subseteq {}^{\alpha}A$

let $x \in {}^{\alpha+}A$

$\Rightarrow A(x) > \alpha$.

$\Rightarrow x \in {}^{\alpha}A$.

$$\therefore \boxed{{}^{\alpha+}A \subseteq {}^{\alpha}A.}$$

Case (ii):

To prove: $\alpha \leq \beta \Rightarrow {}^{\alpha}A \supseteq {}^{\beta}A$

${}^{\alpha+}A \supseteq {}^{\beta+}A$.

(i) ${}^{\alpha}A \supseteq {}^{\beta}A$.

let $x \in {}^{\beta}A$

$\Rightarrow A(x) \geq \beta \geq \alpha$.

$\Rightarrow A(x) \geq \alpha$

$\Rightarrow x \in {}^{\alpha}A$.

$$\therefore \boxed{{}^{\beta}A \subseteq {}^{\alpha}A}$$

(ii) let $x \in {}^{\beta+}A$.

$\Rightarrow A(x) \geq \beta \geq \alpha$

$\Rightarrow A(x) \geq \alpha$

$\Rightarrow x \in {}^{\alpha+}A$.

$$\therefore \boxed{{}^{\beta+}A \subseteq {}^{\alpha+}A}$$

Thm 3.2:

Let $A_i \in \mathcal{F}(X)\ \forall\ i \in I$, where I is an index set. Then,

(i) $\bigcup_{i\in I} {}^{\alpha}A_i \subseteq {}^{\alpha}\left(\bigcup_{i\in I} A_i\right)$ and

$\bigcap_{i\in I} {}^{\alpha}A_i = {}^{\alpha}\left(\bigcap_{i\in I} A_i\right)$;

(ii) $\bigcup_{i\in I} {}^{\alpha+}A_i = {}^{\alpha+}\left(\bigcup_{i\in I} A_i\right)$ and

$\bigcap_{i\in I} {}^{\alpha+}A_i \subseteq {}^{\alpha+}\left(\bigcap_{i\in I} A_i\right)$.

Proof:

Case (i):

Let $x \in \bigcup_{i\in I} {}^{\alpha}A_i$

iff $\exists\ i_0 \in I \ni; x \in {}^{\alpha}A_{i_0}$

i.e., $A_{i_0}(x) > \alpha$

This inequality is satisfied iff

$\sup_{i\in I} A_i(x) > \alpha$.

which is equivalent to

$\left(\bigcup_{i\in I} A_i\right)(x) > \alpha$

$\Rightarrow x \in {}^{\alpha}\left(\bigcup_{i\in I} A_i\right)$

$\therefore$ $\boxed{\bigcup_{i\in I} {}^{\alpha}A_i \subseteq {}^{\alpha}\left(\bigcup_{i\in I} A_i\right)}$

2nd proposition:

Let $x \in \bigcap_{i\in I} {}^{\alpha}A_i$

iff $\exists\ i_0 \in I \ni; x \in {}^{\alpha}A_{i_0}$

inequality satisfied iff.

$\inf_{i\in I} A_i(x) > \alpha$.

$\Rightarrow \left(\bigcap_{i\in I} A_i\right)(x) > \alpha$.

$\Rightarrow x \in {}^{\alpha}\bigcap_{i\in I} A_i$

$\therefore$ $\boxed{\bigcap_{i\in I} {}^{\alpha}A_i \subseteq {}^{\alpha}\left(\bigcap_{i\in I} A_i\right)}$ $\rightarrow$ ①

$|||^{ly}$. $\bigcap_{i\in I} {}^{\alpha}A_i \supseteq {}^{\alpha}\left(\bigcap_{i\in I} A_i\right)$ $\rightarrow$ ②

From ① & ②,

$\therefore$ $\boxed{\bigcap_{i\in I} {}^{\alpha}A_i = {}^{\alpha}\left(\bigcap_{i\in I} A_i\right)}$

Case (ii):

Let $x \in \bigcup_{i\in I} {}^{\alpha+}A_i$

iff $\exists\ i_0 \in I \ni; x \in {}^{\alpha+}A_{i_0}$

$\Rightarrow A_{i_0}(x) > \alpha$

Inequality satisfied iff.

$\sup_{i\in I} A_i(x) > \alpha$.

$\Rightarrow \bigcup_{i\in I} A_i(x) > \alpha$

$\Rightarrow x \in {}^{\alpha+}\left(\bigcup_{i\in I} A_i\right)$

$\therefore \bigcup_{i\in I} {}^{\alpha+}A_i \subseteq {}^{\alpha+}\left(\bigcup_{i\in I} A_i\right)$.

$|||^{ly}$ $\bigcup_{i\in I} {}^{\alpha+}A_i \supseteq {}^{\alpha+}\left(\bigcup_{i\in I} A_i\right)$

From above eqn/.

$\therefore$ $\boxed{\bigcup_{i\in I} {}^{\alpha+}A_i = {}^{\alpha+}\left(\bigcup_{i\in I} A_i\right)}$

2nd proposition:

Let $x \in \bigcap_{i\in I} {}^{\alpha+}A_i$

iff $\exists\ i_0 \in I \ni; x \in {}^{\alpha+}A_{i_0}$

i.e., $A_{i_0}(x) > \alpha$.

Inequality satisfied iff.

$\inf_{i\in I} A_i(x) > \alpha$

$\Rightarrow \bigcap_{i\in I} A_i(x) > \alpha$

$\Rightarrow x \in {}^{\alpha+}\left(\bigcap_{i\in I} A_i\right)$

$\therefore$ $\boxed{\bigcap_{i\in I} {}^{\alpha+}A_i \subseteq {}^{\alpha+}\left(\bigcap_{i\in I} A_i\right)}$

Thm 3.3:

Let $A, B \in \mathcal{F}(X)$. Then, for all $\alpha \in [0,1]$,

(i) $A \subseteq B$ iff ${}^{\alpha}A \subseteq {}^{\alpha}B$;

$A \subseteq B$ iff ${}^{\alpha+}A \subseteq {}^{\alpha+}B$;

(ii) $A = B$ iff ${}^{\alpha}A = {}^{\alpha}B$;

$A = B$ iff ${}^{\alpha+}A = {}^{\alpha+}B$;

Proof:

Case (i): $\Leftrightarrow$

To prove: $A \subseteq B$ iff ${}^{\alpha}A \subseteq {}^{\alpha}B$

(i) ${}^{\alpha}A \subseteq {}^{\alpha}B \Rightarrow A \subseteq B$

To prove: Contrapositive ${}^{\alpha}A \nsubseteq {}^{\alpha}B \Rightarrow A \nsubseteq B$

Let ${}^{\alpha}A \nsubseteq {}^{\alpha}B$

$\exists\ \alpha_0 \in [0,1] \ni; x_0 \in X$ ∋ $x_0 \in {}^{\alpha_0}A$ & $x_0 \notin {}^{\alpha_0}B$

$\Rightarrow A(x_0) \geq \alpha_0$ & $B(x_0) \leq \alpha$

$\Rightarrow B(x_0) < A(x_0)$

$\Rightarrow B \subseteq A$

$\Rightarrow \Leftarrow$ with $A \subseteq B$

$\therefore$ $\boxed{{}^{\alpha}A \subseteq {}^{\alpha}B \Rightarrow A \subseteq B}$ $\rightarrow$ ①

Conversely,

(ii) $A \subseteq B \Rightarrow {}^{\alpha}A \subseteq {}^{\alpha}B$

To prove: Contrapositive

$A \not\subseteq B \Rightarrow {}^{\alpha}A \not\subseteq {}^{\alpha}B$.

Let $A \not\subseteq B$

$\exists$; $x_0 \in X$, $\ni$; $B(x_0) < A(x_0)$

Let $\alpha = A(x_0)$

$\exists$; $x_0 \in {}^{\alpha}A$ & $x_0 \notin {}^{\alpha}B$.

$\Rightarrow {}^{\alpha}A \not\subseteq {}^{\alpha}B$

$\Rightarrow {}^{\alpha}B \not\supseteq {}^{\alpha}A$

$\therefore {}^{\alpha}B \not\supseteq {}^{\alpha}A$

It demonstrates that

${}^{\alpha}A \subseteq {}^{\alpha}B$

is not satisfied for all $\alpha \in [0,1]$

$\therefore$ $\boxed{A \subseteq B \Rightarrow {}^{\alpha}A \subseteq {}^{\alpha}B}$ → (2)

From (1) & (2),

$\boxed{A \subseteq B \Leftrightarrow {}^{\alpha}A \subseteq {}^{\alpha}B}$ //.

Case 2nd proposition:

To prove: $A \subseteq B \Leftrightarrow {}^{\alpha+}A \subseteq {}^{\alpha+}B$.

(i) ${}^{\alpha+}A \subseteq {}^{\alpha+}B \Rightarrow A \subseteq B$

To prove: $\Rightarrow \Leftarrow$

Let ${}^{\alpha+}A \not\subseteq {}^{\alpha+}B$

$\exists$; $\alpha_0 \in [0,1]$ $\ni$;

$x_0 \in X$,

$\Rightarrow x_0 \in {}^{\alpha_0+}A$ & $x_0 \notin {}^{\alpha_0+}B$

$\Rightarrow A(x_0) > \alpha_0$ & $B(x_0) \leq \alpha_0$

$\Rightarrow B(x_0) < A(x_0)$

$\Rightarrow B \subseteq A$.

$\Rightarrow \Leftarrow$ with $A \subseteq B$.

$\therefore$ $\boxed{{}^{\alpha+}A \subseteq {}^{\alpha+}B \Rightarrow A \subseteq B}$ → (1).

(ii) $A \subseteq B \Rightarrow {}^{\alpha+}A \subseteq {}^{\alpha+}B$.

To prove: $\Rightarrow \Leftarrow$

Let $A \not\subseteq B$

$\exists$; $x_0 \in X$, $\ni$;

$A(x_0) > B(x_0)$

i.e., $x_0 \in {}^{\alpha+}A$ & $x_0 \notin {}^{\alpha+}B$

${}^{\alpha+}A \not\subseteq {}^{\alpha+}B$.

which is clear that

${}^{\alpha+}A \subseteq {}^{\alpha+}B$

is not satisfied $\forall$ $\alpha \in [0,1]$.

$\therefore$ $\boxed{A \subseteq B \Rightarrow {}^{\alpha+}A \subseteq {}^{\alpha+}B}$. → (2)

From (1) & (2),

$\boxed{A \subseteq B \Leftrightarrow {}^{\alpha+}A \subseteq {}^{\alpha+}B}$ //.

Case (ii)

1st proposition:

To prove: $A = B \Leftrightarrow {}^{\alpha}A = {}^{\alpha}B$.

(i) ${}^{\alpha}A = {}^{\alpha}B \Rightarrow A = B$.

To prove: $\Rightarrow \Leftarrow$

Let ${}^{\alpha}A \neq {}^{\alpha}B$.

$\exists$; $x_0 \in X$, $x_0 \in {}^{\alpha}A$, $x_0 \notin {}^{\alpha}B$

$\ni$: $A(x_0) > \alpha$, $B(x_0) < \alpha$.

$B(x_0) < A(x_0)$

$B \subseteq A$

$\Rightarrow \Leftarrow$. to $A = B$.

$\therefore$ $\boxed{{}^{\alpha}A = {}^{\alpha}B \Rightarrow A = B}$ → (1)

Conversely,

(ii) $A = B \Rightarrow {}^{\alpha}A = {}^{\alpha}B$

To prove: $\Rightarrow \Leftarrow$

Let $A \neq B$

$\exists$; $x_0 \in X$ $\ni$; $B(x_0) \neq A(x_0)$

Let $\alpha = A(x_0)$, $\alpha \neq B(x_0)$

$\Rightarrow x_0 \in {}^{\alpha}A$, $x_0 \notin {}^{\alpha}B$

$\Rightarrow {}^{\alpha}A \neq {}^{\alpha}B$.

$\Rightarrow \Leftarrow$

$\therefore$ $\boxed{A = B \Rightarrow {}^{\alpha}A = {}^{\alpha}B}$ → (2)

From (1) & (2),

$\boxed{A = B \Leftrightarrow {}^{\alpha}A = {}^{\al, }B}$.

$\|^{ly}$ $\boxed{A = B \text{ iff } {}^{\alpha+}A = {}^{\alpha+}B}$

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Thm 2.4

For any $A \in \mathcal{F}(X)$, the following properties hold.

(i) $^{\alpha}A = \bigcap_{\beta<\alpha} {}^{\beta}A = \bigcap_{\beta<\alpha} {}^{\beta+}A$.

(ii) $^{\alpha+}A = \bigcup_{\alpha<\beta} {}^{\beta}A = \bigcup_{\alpha<\beta} {}^{\beta+}A$.

Proof Case (i):

For any $\beta < \alpha$,

clearly,

$$^{\alpha}A \subseteq {}^{\beta}A$$

$$\Rightarrow {}^{\alpha}A \subseteq \bigcap_{\beta<\alpha} {}^{\beta}A. \longrightarrow (A)$$

$\forall x \in \bigcap_{\beta<\alpha} {}^{\beta}A$ & $\epsilon > 0$. $\longrightarrow (1)$

$\therefore x \in {}^{\alpha-\epsilon}A$.

Since, $\alpha - \epsilon < \alpha$

$\leq A(x)$

When $\epsilon \to$ (+ve) arbitrary constant

$\epsilon \to 0$

$\alpha - 0 \leq A(x)$

$A(x) \geq \alpha$

$x \in {}^{\alpha}A$. $\longrightarrow$ (2)

from (1) & (2)

$$\bigcap_{\beta<\alpha} {}^{\beta}A \subseteq {}^{\alpha}A. \longrightarrow (B)$$

from (A) & (B)

$$^{\alpha}A = \bigcap_{\beta<\alpha} {}^{\beta}A = \bigcap_{\beta<\alpha} {}^{\beta+}A.$$

Case (ii):

For any $\alpha < \beta$

clearly, $\alpha < \beta$

$^{\alpha}A \subset {}^{\beta}A$

Hence, $^{\alpha+}A \subseteq \bigcup_{\alpha<\beta} {}^{\beta+}A$ $\longrightarrow$ (A)

Now,

$\forall x \in \bigcup_{\alpha<\beta} {}^{\beta+}A$ and $\longrightarrow (1)$

$\epsilon > 0$

$\therefore x \in {}^{\alpha-\epsilon}A$

Since, $\alpha - \epsilon < \alpha$

$< A(x)$

$\therefore \alpha - \epsilon < A(x)$

Where $\epsilon \to$ arbitrary positive constant

$\therefore \epsilon \to 0$

$\therefore \alpha - 0 < A(x)$

$A(x) > \alpha$

$x \in {}^{\alpha+}A$

$\longrightarrow (2)$

from (1) & (2),

$$\bigcup_{\alpha<\beta} {}^{\beta+}A \subseteq {}^{\alpha+}A \longrightarrow (B)$$

which concludes the proof of first eq/.

(same in position.)

First Decomposition Thm:

Statement:

For every $A \in \mathcal{F}(X)$.

$$A = \bigcup_{\alpha \in [0,1]} {}_{\alpha}A$$

where $_{\alpha}A$ is defined by $_{\alpha}A(x) = \alpha$, and $\bigcup$ denotes the standard fuzzy union.

Proof:

For each $x \in X$, let $A(x) = a$

Then

$$\left(\bigcup_{\alpha\in[0,1]} {}_{\alpha}A\right)(x) = \sup_{\alpha\in[0,1]} {}_{\alpha}A(x)$$

$$= \max\left[\sup_{\alpha\in[0,a]} {}_{\alpha}A(x), \sup_{\alpha\in(a,1]} {}_{\alpha}A(x)\right]$$

for each $\alpha \in (a,1]$, we have

$A(x) = a < \alpha$

$\therefore {}_{\alpha}A(x) = 0$

On the otherhand.

for each $\alpha \in [0,a]$,

$A(x) = a \geq \alpha$

$\therefore {}_{\alpha}A(x) = \alpha$

Hence,

$$\left(\bigcup_{\alpha\in[0,1]} {}_{\alpha}A\right)(x) = \sup_{\alpha\in[0,1]} \alpha$$

$$= \max\left[\sup_{\alpha\in[0,a]} {}_{\alpha}A(x), \sup_{\alpha\in(a,1]} {}_{\alpha}A(x)\right]$$

$$= \max\left[\sup_{\alpha\in[0,a]} \alpha, \sup_{\alpha\in(a,1]} 0\right]$$

$$= \max\left[\sup_{\alpha\in[0,a]} \alpha\right]$$

$$= \sup_{\alpha\in[0,a]} \alpha$$

$= a$

$= A(x)$

$$\boxed{\bigcup_{\alpha\in[0,1]} {}_{\alpha}A(x) = A(x)}$$

Second Decomposition Th

Statement:

For every $A \in \mathcal{F}(X)$

$$A = \bigcup_{\alpha\in[0,1]} {}_{\alpha+}A$$

where $_{\alpha+}A$ denotes a special fuzzy set defined by $_{\alpha+}A(x) = \alpha \cdot {}^{\alpha+}A(x)$

and $\cup$ denotes the std fuzzy union.

Proof

For each particular $x \in X$, let $a = A(x)$. Then

$$\left(\bigcup_{\alpha\in[0,1]} {}_{\alpha+}A\right)(x) = \sup_{\alpha\in[0,1]} {}_{\alpha+}A(x)$$

$$= \max\left[\sup_{\alpha\in[0,a)} {}_{\alpha+}A(x), \sup_{\alpha\in[a,1]} {}_{\alpha+}A(x)\right] \longrightarrow (1)$$

for each $\alpha \in (a,1)$ then

$A(x) = a < \alpha$

$\therefore A(x) = 0$.

Then,

$${}_{\alpha+}A(x) = \alpha \cdot {}_{\alpha+}A(x)$$
$$= \alpha \cdot 0$$
$$= 0.$$

On other hand,

for each $\alpha \in [0,a)$

$A(x) = a > \alpha$

$\therefore A(x) = \alpha$

$$\therefore {}_{\alpha+}A(x) = \alpha \cdot {}_{\alpha+}A(x)$$
$$= \alpha \cdot 1$$
$$= \alpha.$$

eqn (1) becomes

$$\bigcup_{\alpha\in[0,1]} {}_{\alpha+}A = \max\left[\sup_{\alpha\in[0,a)} {}_{\alpha+}A(x), \sup_{\alpha\in[a,1)} {}_{\alpha+}A(x)\right]$$

$$= \max\left[\sup_{\alpha\in[0,a)} \alpha, 0\right]$$

$$= \sup_{\alpha\in[0,a)} \alpha$$

$$= 0$$

$$= A(x)$$

$$\boxed{\bigcup_{\alpha\in[0,1]} {}_{\alpha+}A(x) = A(x)}$$

Third Decomposition Thm.

State:

For every $A \in \mathcal{F}(X)$,

$$A = \bigcup_{\alpha\in\Lambda(A)} {}_{\alpha}A,$$

where $\Lambda(A)$ is the level set of A, ${}_{\alpha}A$ is defined by $({}_{\alpha}A = \alpha \cdot {}^{\alpha}A)$, & $\cup$ denotes the std fuzzy union.

Proof:

$\forall x \in X$, let $A(x) = a$

$$\bigcup_{\alpha\in[0,1]} {}_{\alpha}A = \bigcup_{\alpha\in\Lambda(A)} {}_{\alpha}A$$

$$\left(\bigcup_{\alpha\in[0,1]} {}_{\alpha}A\right)(x) = \sup_{x\in[0,1]} \alpha A(x)$$

$$= \max\left[\sup_{\alpha\in[0,a]} \alpha A(x), \sup_{\alpha\in[a,1]} \alpha A(x)\right] \longrightarrow (1)$$

for each $\alpha \in [a,1]$ then

$A(x) = a \le \alpha$

$$\therefore \alpha A(x) = \alpha \cdot \alpha A(x)$$
$$= \alpha \cdot 0$$
$$= 0$$

On the other hand

for each $\alpha \in [0,a]$ Then.

$A(x) = a \ge \alpha$.

$$\therefore \alpha A(x) = \alpha \cdot \alpha A(x)$$
$$= \alpha \cdot 1$$
$$= \alpha.$$

$\therefore (1) \Rightarrow$

$$\bigcup_{\alpha\in[0,1]} \alpha A = \max\left[\sup_{\alpha\in[0,a]} \alpha A(x), \sup_{\alpha\in[a,1]} \alpha A(x)\right]$$

$$= \max\left[\sup_{\alpha\in[0,a]} \alpha, 0\right]$$

$$= \sup_{\alpha\in[0,a]} \alpha$$

$$= a$$

$$\bigcup_{\alpha\in[0,1]} \alpha A(x) = A(x)$$

$$\boxed{\bigcup_{\alpha\in\Lambda(A)} \alpha A = A}$$ //

§4 Extension Principle

(i) A principle of fuzzifying a crisp fn/. is called extension principle

(ii) Suppose $f : X \longrightarrow Y$ is a crisp fn/. If f is extended to act on the special fuzzy set defined on X and Y then f is said to be fuzzy field

i.e., $\mathcal{F} : \mathcal{F}(X) \longrightarrow \mathcal{F}(Y)$

The extended fn/. has the same notation

① Explain the extension principle in the special fuzzy domain.

We consider the extension of $f : X \longrightarrow Y$ to their power sets.

i.e., $\mathcal{F} : P(X) \longrightarrow P(Y)$

Suppose $A \in P(X)$ Then,

$$f(A) = \{y / y = f(x) ; x \in A\}$$

$$f^{-1} : P(Y) \longrightarrow P(X)$$

Suppose $B \in P(Y)$

$$f^{-1}(B) = \{x / f(x) \in B\}$$

We represented $f(x)$ & $f^{-1}(B)$ by means of characteristic fn/ which is viewed as a special membership fn/.

Suppose $A \in \mathcal{F}(X)$, $B \in \mathcal{F}(Y)$ Then

$$f(A)(y) = \sup_{x / y = f(x)} A(x) \;\&$$

$$f^{-1}B(x) = B(f(x))$$

Where

$$f : \mathcal{F}(X) \longrightarrow \mathcal{F}(Y) \;\&$$

$$f^{-1} : \mathcal{F}(Y) \longrightarrow \mathcal{F}(X)$$

Thm 4.1:

Let $f : X \longrightarrow Y$ be an arbitrary crisp fn/. Then, for any $A \in \mathcal{F}(X)$ & $\alpha \in [0,1]$ the following properties of f fuzzified by the extension principle hold

(i) $^{\alpha+}[f(A)] = f(^{\alpha+}A)$.

(ii) $^{\alpha}[f(A)] \supseteq f(^{\alpha}A)$.

Proof:

Case (i):

For all $y \in Y$

$$y \in {}^{\alpha+}[f(A)]$$

$$\Leftrightarrow [f(A)](y) > \alpha.$$

$$\Leftrightarrow \sup_{x / y = f(x)} A(x) > \alpha.$$

$\Leftrightarrow \exists : x_0 \in X$, $y = f(x_0)$ and $A(x_0) > \alpha$

$\Leftrightarrow \exists : x_0 \in X$, $y = f(x_0)$ and $x_0 \in {}^{\alpha+}A$.

$$\Leftrightarrow y \in f(^{\alpha+}A).$$

Hence, $\boxed{^{\alpha+}[f(A)] = f(^{\alpha+}A)}$

Case (ii):

If $y \in f(^{\alpha}A)$

$\exists : x_0 \in {}^{\alpha}A \ni : y = f(x_0)$

Hence,

$$[f(A)](y) = \sup_{x / y = f(x)} A(x) \geq A(x_0) \geq \alpha.$$

and consequently,

$$y \in {}^{\alpha}[f(A)]$$

$\therefore$ $\boxed{f(^{\alpha}A) \subseteq {}^{\alpha}[f(A)]}$ //

Thm 4.2:

Let $f : X \longrightarrow Y$ be an arbitrary crisp fn/. Then, for any $A \in \mathcal{F}(X)$, f fuzzified by the extension principle satisfies the eqn/.

$$f(A) = \bigcup_{\alpha \in [0,1]} f(_{\alpha+}A).$$

Proof:

Results: Second Decomposition

For every $A \in \mathcal{F}(X)$

$$A = \bigcup_{\alpha \in [0,1]} {}_{\alpha+}A.$$

where $_{\alpha+}A$ denotes a special fuzzy set defined by

$$_{\alpha+}A(x) = \alpha \cdot {}^{\alpha+}A(x)$$

2. Thm 4.1: $^{\alpha}[f(A)] \supseteq f($…

using above result,

$f(A)$, which is a fuzzy set on Y, we obtain.

~~$f(A) = \bigcup_{\alpha \in [0,1]} f(_{\alpha+}A)$~~

$$f(A) = \bigcup_{\alpha \in [0,1]} \alpha+[f(A)]$$

<u>by definition,</u>

$$\alpha+[f(A)] = \alpha \cdot {}^{\alpha+}[f(A)]$$

<u>using Result 2:</u>

$$f(A) = \bigcup_{\alpha \in [0,1]} \alpha \cdot f({}^{\alpha+}A)$$

$$= \bigcup_{\alpha \in [0,1]} f(\alpha+A)$$

$$= \bigcup_{\alpha \in [0,1]} f(\alpha A)$$

UNIT - III
FUZZY ARITHMETIC

Fuzzy Numbers:

Among the various types of fuzzy sets of special significance are fuzzy sets that are defined on the set $\mathbb{R}$ of real.

Membership fn. of these sets which have the form

$$A : \mathbb{R} \rightarrow [0,1]$$

clearly have a quantitative meaning and may, under certain conditions be viewed as <u>fuzzy numbers</u> (or) <u>fuzzy intervals</u>.

<u>Note:</u>

Fuzzy numbers (or) intervals such as "Numbers that are cld/. to a given real nos/."

(or)

Number that are around a given interval of real nos/.

Such concept are essential for characterizing states of fuzzy variable & Consequently, play an important roll in many applications including fuzzy control, decision making, approximate with reasoning, optimization and statistics with imprecise probabilities

<u>To qualify as a fuzzy no/.:</u>

A fuzzy set A on $\mathbb{R}$ must posses atleast the following three properties

(i) A must be a norm fuzzy set.

(ii) ${}^{\alpha}A$ must be a cld interval for every $\alpha \in [0,1]$

(iii) The support of A, ${}^{0+}A$ must be boun

Ex. **Normal fuzzy set:**

The fuzzy set must be normal since our conception of a set of "real nos. close to r" is fully satisfied by r itself. Hence the member grade of r in any fuzzy set that attempts to capture this conception (i.e., fuzzy nos.) must be 1.

The bdd. support of a fuzzy no.

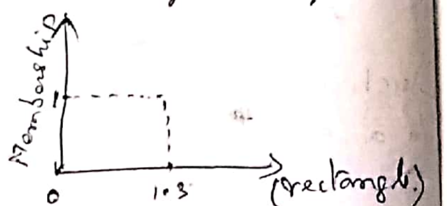
The bdd. support of a fuzzy no. be and all it's α-cuts for α ≠ 0 must be cld intervals to allow us to define meaningful arithmetic operation of fuzzy nos. in terms of std arithmetic operations on cld interval, well establish in classical interval analysis.

Convex fuzzy set:

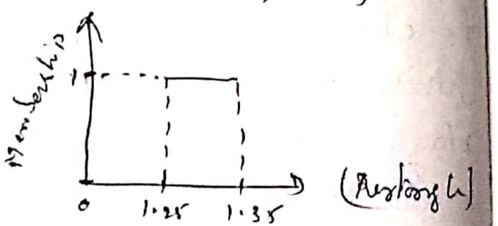
Since α-cuts of any fuzzy nos. are required to be cld intervals for all α ∈ [0,1], every fuzzy no. is a convex fuzzy set. The inverse however is not necessarily true, since α-cuts of some convex fuzzy sets may be open or half open intervals.

§ **Special cases of fuzzy nos. (Symmetric based).**

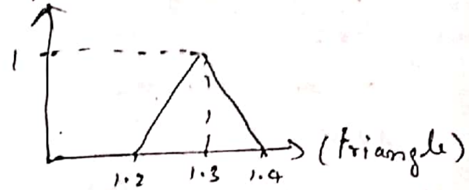
(a). An ordinary real no. 1.3.



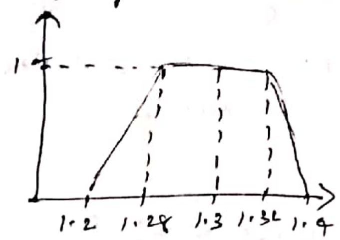
(b) An ordinary (crisp) cld. interval [1.25, 1.35).



(c) An fuzzy number expressing the proposition "cld to 1.3".



(d) A fuzzy no. with flat region (a fuzzy interval)



Def:

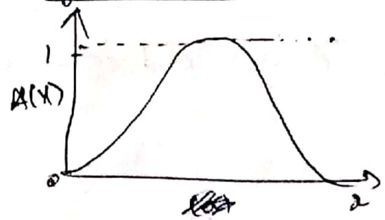
Bell shaped Membership fn.:

(or)

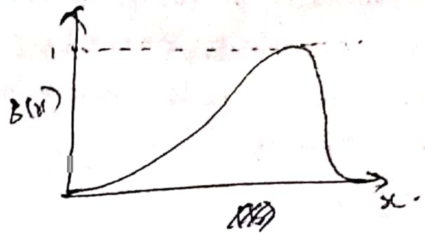
Non-Symmetric fuzzy nos.:

Membership fns. one of fuzzy no. need not be symmetric, It is called bell shaped membership fn..

(a) **Symmetric**



(b) **Asymmetric:**



Thm 1:

Let $A \in \mathcal{F}(\mathbb{R})$. Then A is a fuzzy no. iff there exists a cld. interval $[a,b] \neq \phi$ such that

$$A(x) = \begin{cases} 1 & \text{for } x \in [a,b] \\ l(x) & \text{for } x \in (-\infty, a) \\ r(x) & \text{for } x \in (b, \infty) \end{cases} \quad \longrightarrow (i)$$

When l is a fn. from $(-\infty, a)$ to $[0,1]$ that is monoton increasing, continuous from the right, and such that $l(x) = 0$ for $x \in (-\infty, \omega_1)$; r is a fn. from (b, ∞) to $[0,1]$ that is monotonic decreasing, continuous from the left, and such that $r(x) = 0$ for $x \in (\omega_2, \infty)$.

proof:

Result 1:

A fuzzy set A on $\mathbb{R}$ is convex iff.

$A(\lambda x_1 + (1-\lambda)x_2) \geq \min[A(x_1), A(x_2)]$

for all $x_1, x_2 \in \mathbb{R}$ & all $\lambda \in [0,1]$, where min denotes the minimum operator."

Main of proof:

Necessity:

$\because$ A is a fuzzy no/.

${}^{\alpha}A$ is a cld interval $\forall$ $\alpha \in [0,1]$.

For $\alpha = 1$,

${}^{1}A$ is a non empty cld interval because A is normal.

$\exists$; $a, b \in \mathbb{R}$ $\ni$: ${}^{1}A = [a,b]$ where $a \leq b$.

i.e., $A(x) = 1$, $x \in [a,b]$

$A(x) < 1$, $x \notin [a,b)$.

Now,

let $l(x) = A(x)$, $x \in (-\infty, a)$

Then $0 \leq l(x) < 1$

$\because 0 \leq A(x) < 1$, $x \in (-\infty, a)$,

let $x \leq y < a$, then

$A(y) \geq \min[A(x), A(a)]$

$= A(x)$

by Result I:

$\because$ A is convex & $A(a) = 1$.

Hence, $l(y) \geq l(x)$,

i.e. l is monotonic increasing.

Assume that $l(x)$ is not continuous from the right.

let $x_0 \in (-\infty, a)$ $\exists$; a sequence of no/. $\{x_n\}$ $\ni$; $x_n \geq x_0$ for any n.

$\lim_{n \to \infty} x_n = x_0$

but $\lim_{n \to \infty} l(x_n) = \lim_{n \to \infty} A(x_n)$

$= \alpha$

$> l(x_0)$

$= A(x_0)$.

Now, $x_n \in {}^{\alpha}A$ for any n.

$\because$ ${}^{\alpha}A$ is a cld interval

also $x_0 \in {}^{\alpha}A$.

$\therefore$ $l(x_0) = A(x_0) \geq \alpha$.

$\Rightarrow \Leftarrow$.

i.e., $l(x)$ is continuous from the right.

$\therefore$ The fn/. r in (1) is monotonic decreasing & continuous from the left is similar.

$\because$ A is a fuzzy no/.,

${}^{0+}A$ is bdd/. .

Hence,

$\exists$; $\omega_1, \omega_2 \in \mathbb{R}$ of finite nos/. $\ni$: $A(x) = 0$ $\forall$ $x \in (-\infty, \omega_1) \cup (\omega_2, \infty)$.

Sufficiency:

Every fuzzy set A defined by

$$A(x) = \begin{cases} 1, & \text{for } x \in [a,b] \\ l(x), & \text{for } x \in (-\infty, a] \\ r(x), & \text{for } x \in [b, \infty) \end{cases}$$

is clearly normal, & it's support, ${}^{0+}A$, is bdd/.

$\therefore$ ${}^{0+}A \subseteq [\omega_1, \omega_2]$

To prove: ${}^{\alpha}A$ is a cld interval for any $\alpha \in [0,1]$.

let

$x_\alpha = \inf\{x / l(x) \geq \alpha, x < a\}$

$y_\alpha = \sup\{x / r(x) \geq \alpha, x > b\}$.

for each $\alpha \in [0,1]$.

Need to prove: ${}^{\alpha}A = [x_\alpha, y_\alpha]$ $\forall$ $\alpha \in [0,1]$

let $x_0 \in {}^{\alpha}A$ if $x_0 < a$,

Then $l(x_0) = A(x_0) \geq \alpha$.

i.e., $x_0 \in \{x / l(x) \geq \alpha, x < a\}$

Consequently,

$x_0 \geq \inf\{x / l(x) \geq \alpha, x < a\}$

$= x_\alpha$

If $x_0 > b$, then $r(x_0) = A(x_0)$

$\geq \alpha$.

i.e, $x_0 \in \{x / r(x) \geq \alpha, x > b\}$

Consequently,

$x_0 \leq \sup\{x / r(x) \geq \alpha, x > b\}$

$= y_\alpha$.

obviously,

$x_\alpha \le a$ & $y_\alpha \ge b$

i.e. $[a,b] \subseteq [x_\alpha, y_\alpha]$.

$\therefore x_0 \in [x_\alpha, y_\alpha]$

Hence $^\alpha A \subseteq [x_\alpha, y_\alpha]$.

To prove: $x_\alpha, y_\alpha \in {}^\alpha A$.

By the definition of x_α, $\exists$ a sequence $\{x_n\}$ in $\{x \mid A(x) \ge \alpha, x < a\}$

$\ni$: $\lim_{n\to\infty} x_n = x_\alpha$,

where $x_n \ge x_\alpha$ $\forall$ n.

$\because$ A is continuous from the right.

$A(x_\alpha) = A(\lim_{n\to\alpha} x_n)$

$= \lim_{n\to\alpha} A(x_n)$

$A(x_\alpha) \ge \alpha$

$\therefore x \in {}^\alpha A$.

11^{ly} $y \in {}^\alpha A$.

Hence proved.

§ Linguistic Variables.

The Concept of a fuzzy no/ plays a fundamental role in formulating quantitative fuzzy variables. These are variables whose states are fuzzy nos/.

When in addition the fuzzy nos/. represent linguistic concepts, such as very small, small, medium and so on as interpreted in a particular context, the resulting constructs are usually called <u>linguistic variables</u>.

<u>Base variable:</u>

In each linguistic variables the states of which are expressed by linguistic terms interpreted as specific fuzzy nos/. is defined in terms of a <u>base variable</u> the values of which are real nos/. with in a specific range

(i) A base variables is a variable in the classical sense, exemplified by any physical variables

Ex: Temperature, Pressure, Speed, voltage, humidity etc.

(ii) Any other numerical variable

Eg: Age, interest rate, performance, salary, blood count, probability etc.

<u>Characteristic of linguistic Variable:</u>

Each linguistic variable is fully characteristic by a quintuple (v, T, X, g, m) in which v is the name of the variable T is the set of linguistic terms of v that refer to a base variable whose values range over a universal set X, g is a syntactic rule (a grammar) for generating linguistic terms and m is a semantic rule that assign to each linguistic term $t \in T$ its meaning $m(t)$. which is a fuzzy set on X that is $m: T \longrightarrow \mathcal{F}(X)$.

§ Arithmetic Operation on Interval.

Fuzzy arithmetic based on two properties of fuzzy nos/.

(i) Each fuzzy set, & thus also each fuzzy no/. can fully and uniquely be represented by its α-cuts.

(ii) α-cuts of each fuzzy no/. are cldf. intervals of real nos/. for all $\alpha \in (0,1]$

These property enable us to define arithmetic operation on fuzzy nos/. in terms of arithmetic operations on their α-cut

That is called arithmetic operation on cld/. Interval.

<u>Four arithmetic operation on cld/. Interval:</u>

Let $*$ denote any of the four arithmetic operations on cld/. Intervals these are addition(+) subtraction(−), multiplication(·) and division(/). Then

$[a,b] * [d,e]$

$= \{ f * g \,/\, a \le f \le b,\ d \le g \le e \}$

is a general property of all arithmetic operation on cld/. Intervals except that $[a,b]/[d,e]$ is not defined when $0 \in [d,e]$

i.e, the result of an arithmetic operation on cld interval is again a cld interval.

(i) $[a,b] + [d,e] = [a+d, b+e]$

eg: $[2,5] + [1,3] = [2+1, 5+3]$
$= [3,8]$

(ii) $[a,b] - [d,e] = [a-e, b-d]$

eg: $[0,1] - [-6,5] = [-5,7]$

(iii) $[a,b] \cdot [d,e]$

$= [\min(ad, ae, bd, be), \max(ad, ae, bd, be)]$

eg: $[-1,1] \cdot [-2,-0.5]$
$= [-2, 2]$

(iv) $[a,b] / [d,e]$

$= [a,b] \cdot [1/e, 1/d]$

$= [\min(a/d, a/e, b/d, b/e), \max(a/d, a/e, b/d, b/e)]$

eg: $[4, 10] / [1,2] = [2,10]$

<u>Arithmetic operations on cld/. Intervals satisfy some useful properties.</u>

Let $A = [a_1, a_2]$,
$B = [b_1, b_2]$,
$C = [c_1, c_2]$,
$0 = [0,0]$,
$1 = [1,1]$

using these symbols the properties are formulated as follows:

1. <u>(commutativity)</u>
$A + B = B + A$
$A \cdot B = B \cdot A$.

2. <u>Associativity</u>
$(A+B)+C = A+(B+C)$
$(A \cdot B) \cdot C = A \cdot (B \cdot C)$

3. <u>Identity:</u>
$A = 0 + A = A + 0$
$A = I \cdot A = A \cdot I$

4. <u>Sub distributivity:</u>
$A \cdot (B+C) \subseteq A \cdot B + A \cdot C$

5. <u>distributivity:</u>
If $b \cdot c \ge 0$ for every $b \in B$ & $c \in C$, then
$A \cdot (B+C) = A \cdot B + A \cdot C$

<u>Furthermore,</u>
if $A = [a,a]$ then
$a \cdot (B+C) = a \cdot B + a \cdot C$.

6. $0 \in A - A$ & $1 \in A/A$ ..

7. <u>Inclusion monotonicity:</u>
If $A \subseteq E$ & $B \subseteq F$, then
$A + B \subseteq E + F$
$A - B \subseteq E - F$
$A \cdot B \subseteq E \cdot F$
$A/B \subseteq E/F$

<u>Special Cases of Arithmetic operations on Interval.</u>

Assume now without any loss of generality that $b_1 \ge 0$ & $c_1 \ge 0$, then we have to consider the following three cases.

(i) If $a_1 \ge 0$, then
$A \cdot (B+C) = [a_1 \cdot (b_1 + c_1), a_2 \cdot (b_2 + c_2)]$
$= [a_1 \cdot b_1, a_2 \cdot b_2] + [a_1 c_1, a_2 c_2]$
$= A \cdot B + A \cdot C$.

(ii) If $a_1 < 0$ & $a_2 \le 0$,
then $-a_2 \ge 0$,
$(-A) = [-a_2, -a_1]$

$((-A)\cdot(B+C))$

$= (-A)\cdot B + (-A)\cdot C$

$A\cdot(B+C) = A\cdot B + A\cdot C$

(iii) If $a_1 < 0$, $a_2 > 0$, then

$A\cdot(B+C) = [a_1\cdot(b_2+c_2),\ a_2\cdot(b_2+c_2)]$

$= [a_1\cdot b_2, a_2\cdot b_2] + [a_1\cdot c_2, a_2\cdot c_2]$

$= A\cdot B + A\cdot C$.

Arithmetic Operation of fuzzy nos.:

Fuzzy numbers:-

Fuzzy nos. are represented by continuous membership fns.

Let A & B denote fuzzy no's and let $*$ denote any of the four basic arithmetic operations. Then we define fuzzy set on $\mathbb{R}$, $A*B$, by defining its

$$^{\alpha}(A*B) = {}^{\alpha}A * {}^{\alpha}B \longrightarrow (1)$$

for any $\alpha \in (0,1]$.

$A*B$ can be expressed as

$$A*B = \bigcup_{\alpha\in[0,1]} {}^{\alpha}(A*B)$$

$\because$ $^{\alpha}(A*B)$ is a cld. interval for each $\alpha \in [0,1]$ & A, B are fuzzy numbers, $A*B$ is also fuzzy no.

Triangle shape fuzzy nos.:

A & B defined as follows.

$$A(x) = \begin{cases} 0 & ; \text{for } x<-1 \text{ \& } x>3 \\ \frac{(x+1)}{2} & , \text{for } -1<x\le 1 \\ \frac{3-x}{2} & , \text{for } 1<x\le 3. \end{cases}$$

$$B(x) = \begin{cases} 0 & , \text{for } x\le 1 \\ \frac{x-1}{2} & , \text{for } 1<x\le 3 \\ \frac{5-x}{2} & , \text{for } 3<x\le 5. \end{cases}$$

Their α-cuts are,

$^{\alpha}A = [2\alpha-1, 3-2\alpha]$

$^{\alpha}B = [2\alpha+1, 5-2\alpha]$

% 10m / 5m

Theorem:

Let $* \in \{+,-,\cdot,/\}$ and let A, B denote continuous fuzzy nos. Then the fuzzy set $A*B$ defined by

$$(A*B)(z) = \sup_{z=x*y} \min[A(x), B(y)]$$

is a continuous fuzzy no.

Proof:

Prove that:

$^{\alpha}(A*B)$ is a cld interval for every $\alpha \in (0,1]$ for any $z \in {}^{\alpha}A * {}^{\alpha}B$, $\exists$ some $x_0 \in {}^{\alpha}A$ & $y_0 \in {}^{\alpha}A$ $\ni$;

$z = x_0 * y_0$.

$$(A*B)(z) = \sup_{z=x*y} \min[A(x), B(y)]$$
$$\ge \min[A(x_0), B(y_0)]$$
$$\ge \alpha.$$

Hence, $z \in {}^{\alpha}(A*B)$

Consequently,

$$^{\alpha}A * {}^{\alpha}B \subseteq {}^{\alpha}(A*B).$$

For any $z \in {}^{\alpha}(A*B)$ we have.

$$(A*B)(z) = \sup_{z=x*y} \min[A(x), B(y)] \ge \alpha.$$

Moreover, for any $n > [\frac{1}{\alpha}]+1$, where $[\frac{1}{\alpha}]$ denotes the largest integer that is less than or equal to $1/\alpha$, $\exists$: x_n & y_n $\ni$: $z = x_n * y_n$

$$\min[A(x_n), B(y_n)] > \alpha - \frac{1}{n}$$

i.e., $x_n \in {}^{\alpha-1/n}A$,

$y_n \in {}^{\alpha-1/n}B$.

We may consider two sequences, $\{x_n\}$ & $\{y_n\}$.

$\because$ $\alpha - 1/n \le \alpha - \frac{1}{n+1}$

we have,

$^{\alpha-1/(n+1)}A \subseteq {}^{\alpha-1/n}A$,

$^{\alpha-1/(n+1)}B \subseteq {}^{\alpha-1/n}B$.

Hence, $\{x_n\}$ & $\{y_n\}$ fall into some $^{\alpha-1/n}A$ & $^{\alpha-1/n}B$ respectively.

$\because$ the latter are cld intervals $\{x_n\}$ & $\{y_n\}$ are bdd. sequences.

Thus, ∃ a cgt subsequence $\{x_{n,i}\}$ ∋; $x_{n,i} \to x_0$.

To the corresponding subsequence $\{y_{n,i}\}$ there also exists a convergent subsequence $\{y_{n,i,j}\}$ ∋: $y_{n,i,j} \to y_0$.

If we take the corresponding subsequence $\{x_{n,i,j}\}$ from $\{x_{n,i}\}$ then $x_{n,i,j} \to x_0$ thus we have two sequences $\{x_{n,i,j}\}$ & $\{y_{n,i,j}\}$ ∋; $x_{n,i,j} \to x_0$, $y_{n,i,j} \to y_0$ and

$$x_{n,i,j} * y_{n,i,j} = z$$

Now, ∵ * is continuous

$$z = \lim_{j\to\infty} x_{n,i,j} * y_{n,i,j}$$

$$= \left(\lim_{j\to\infty} x_{n,i,j}\right) * \left(\lim_{j\to\infty} y_{n,i,j}\right)$$

$$= x_0 * y_0$$

Also, since,

$A(x_{n,i,j}) > \alpha - \frac{1}{n_{i,j}}$ and

$B(y_{n,i,j}) > \alpha - \frac{1}{n_{i,j}}$,

$$A(x_0) = A\left(\lim_{j\to\infty} x_{n,i,j}\right)$$

$$= \lim_{j\to\infty} A(x_{n,i,j})$$

$$\geq \lim_{j\to\infty} \left(\alpha - \frac{1}{n_{i,j}}\right)$$

$A(x_0) = \alpha$
$x_0 \in {}^\alpha A$

$$B(y_0) = B\left(\lim_{j\to\infty} y_{n,i,j}\right)$$

$$= \lim_{j\to\infty} B(y_{n,i,j})$$

$$\geq \lim_{j\to\infty} \left(\alpha - \frac{1}{n_{i,j}}\right)$$

$B(y_0) = \alpha \Rightarrow y_0 \in {}^\alpha B$

∴ ∃; $x_0 \in {}^\alpha A$, $y_0 \in {}^\alpha B$.

∋: $z = x_0 * y_0$.

i.e, $z \in {}^\alpha A * {}^\alpha B$.

Thus,

$${}^\alpha(A*B) \subseteq {}^\alpha A * {}^\alpha B.$$

and consequently,

$${}^\alpha(A*B) = {}^\alpha A * {}^\alpha B.$$

∴ $A*B$ must be continuous.

Assume $A*B$ is not continuous at z_0;

$$\lim_{z\to z_0} (A*B)(z) < (A*B)(z_0) = \sup_{z_0 = x+y} \min[A(x), B(y)]$$

∃; x_0, y_0 ∋; $z_0 = x_0 * y_0$

&

$$\lim_{z\to z_0} (A*B)(z) < \min[A(x_0), B(x_0)]. \longrightarrow (A)$$

∵ $* \in \{+, -, \cdot, /\}$ is monotonic w.r.to the 1st & the 2nd arguments, respectively.

Two sequence $\{x_n\}$ & $\{y_n\}$ ∋; $x_n \to x_0$, $y_n \to y_0$ as $n \to \infty$,

$$x_n * y_n < z_0 \quad \forall n.$$

Let $z_n = x_n * y_n$, then.

$z_n \to z_0$ as $n \to \infty$.

Thus.

$$\lim_{z\to z_0} (A*B)(z) = \lim_{n\to\infty} (A*B)(z_n)$$

$$= \lim_{n\to\infty} \sup_{z_0 = x*y} \min[A(x), B(y)].$$

$$\geq \lim_{n\to\infty} \min[A(x_n), B(y_n)].$$

$$= \min\left[A\left(\lim_{n\to\infty} x_n\right), B\left(\lim_{n\to\infty} y_n\right)\right]$$

$$= \min[A(x_0), B(y_0)].$$

⇒ ⇐ to (A).

∴ $A*B$ must be a continuous fuzzy no/..