Fuzzy Relations and Fuzzy Graphs

4.1 INTRODUCTION

Fuzzy relations are fuzzy subsets of $X \times Y$, i.e., a mapping from X to Y. The applications of fuzzy relations are widespread and important. We have considered only 'Binary Relations' (i.e., relation between two sets) or simply 'Relations' (unary, binary, ternary, ...).

GENERAL DEFINITIONS

Definition (1) : *Let* $X, Y \subseteq \mathbf{R}$ *be universal set, then*

$R = \{((x, y), \mu_R(x, y)) : (x, y) \in X \times Y\}$ *is called a fuzzy relation from* X *to* Y.

For Example : Let $X = Y = \mathbf{R}$, i.e., set of real numbers and $\tilde{A}$ = 'considerably larger than'.

The membership function of the fuzzy relation which is of course a fuzzy set on $X \times Y$ can be defined as follows :

$$\mu_{\tilde{A}}(x, y) = \begin{cases} 0 , & x \le y \\ (x-y)/10y , & y < x \le 11y \\ 1 , & x > 11y \end{cases}$$

or $$\mu_{\tilde{A}}(x, y) = \begin{cases} 0 , & x \le y \\ [1+(y-x)^{-1}]^{-1} , & x > y \end{cases}$$

For discrete support, fuzzy relations can also be defined by matrices.

Let $\tilde{A}$ = "x considerably larger than y". Then

	y_1	y_2	y_3	y_4
x_1	.8	1	.1	.7
x_2	0	.8	0	0
x_3	.9	1	.7	.8

and $\tilde{B}$ = "y is very close to x".

	y_1	y_2	y_3	y_4
x_1	.4	0	.9	.6
x_2	.9	.4	.5	.7
x_3	.3	0	.8	.5

Generalization of Definition (1)

Definition (2) : *Let X, Y* $\subseteq$ **R**, *and*

$$X = \{ (x, \mu_X(x)) : x \in X \}; \quad Y = \{(y, \mu_Y(y)) : y \in Y \}, \text{ be two fuzzy sets.}$$

Then, fuzzy relation $\tilde{R}(X, Y)$ *is defined as follows*

$\tilde{R} = \tilde{R}(X, Y) = \{((x, y), \mu_{\tilde{R}}(x, y)) : (x, y) \in X \times Y \}$ *is a fuzzy relation on* $\tilde{A}$ *and* $\tilde{B}$ *if*

$$\mu_{\tilde{R}}(x, y) \leq \mu_X(x), \qquad \forall (x, y) \in X \times Y$$

$$\mu_{\tilde{R}}(x, y) \leq \mu_Y(y), \qquad \forall (x, y) \in X \times Y$$

or $$\mu_{\tilde{R}}(x, y) \leq \min (\mu_X(x), \mu_Y(y))$$

Remarks

- The above definition of fuzzy relation is very useful in defining a graph.
- Fuzzy relations are obviously the fuzzy sets in the product spaces.

Definition (3) : *Let* $\tilde{R}$ *and* $\tilde{Z}$ *be two fuzzy relations in the same product spaces. Then, the union and intersection of* $\tilde{R}$ *and* $\tilde{Z}$ *is defined as follows :*

$$\tilde{R} \cup \tilde{Z} = \{[(x, y), \mu_{\tilde{R}\cup\tilde{Z}}(x, y)] : (x, y) \in X \times Z\}$$

here, $\mu_{\tilde{R}\cup\tilde{Z}}(x, y) = \max \{ \mu_{\tilde{R}}(x, y), \mu_{\tilde{Z}}(x, y) : (x, y) \in X \times Z \}$

d $$\tilde{R} \cap \tilde{Z} = \{[(x, y), \mu_{\tilde{R}\cap\tilde{Z}}(x, y)] : (x, y) \in X \times Z\}$$

ere, $\mu_{\tilde{R}\cap\tilde{Z}}(x, y) = \min \{ \mu_{\tilde{R}}(x, y), \mu_{\tilde{Z}}(x, y) : (x, y) \in X \times Z \}$

example : Let $\tilde{R}$ and $\tilde{Z}$ be the two fuzzy relations. The union of $\tilde{R}$ and $\tilde{Z}$ ich can be interpreted as "x considerably larger than y" and "x is very close to .e., "x considerably larger or very close to y", is then given by the following two rix of the relations

considerably larger than y"

$\tilde{R}$ =

	y_1	y_2	y_3	y_4
x_1	.8	1	.1	.7
x_2	0	.8	0	0
x_3	.9	1	.7	.8

ery close to x"

$\tilde{Z} =$

	y_1	y_2	y_3	y_4
x_1	.4	0	.9	.6
x_2	.9	.4	.5	.7
x_3	.3	0	.8	.5

Then,

$\tilde{R} \cup \tilde{Z} =$

	y_1	y_2	y_3	y_4
x_1	.8	.1	.9	.7
x_2	.9	.8	.5	.7
x_3	.9	.1	.8	.8

and

$\tilde{R} \cap \tilde{Z} =$

	y_1	y_2	y_3	y_4
x_1	.4	0	.1	.6
x_2	0	.4	0	0
x_3	.3	0	.7	.5

4.2 Projections And Cylindrical Fuzzy Relations

[Rohilkhand 2009 BP]

The projection and cylindrical extensions of fuzzy relations is defined as follows :

Definition : *Let* $\tilde{R} = \{((x, y), \mu_{\tilde{R}}(x, y)) : (x, y) \in X \times Y\}$ *be the fuzzy binary relation, then the first projection of* $\tilde{R}$ *is defined as*

$$\tilde{R}^{(1)} = \{(x, \max_y \mu_{\tilde{R}}(x, y)) : (x, y) \in X \times Y\}$$

and the second projection of $\tilde{R}$ *is defined as*

$$\tilde{R}^{(2)} = \{(y, \max_x \mu_{\tilde{R}}(x, y)) : (x, y) \in X \times Y\}$$

Also, the total projection is defined as

$$\tilde{R}^{(T)} = \max_x \max_y \{\mu_{\tilde{R}}(x, y) : (x, y) \in X \times Y\}$$

and the cylindrical extension is the largest fuzzy relation which is produced by first or second projection (in the sense of membership grade of elements of the extended cartesian product), i.e., compatible with the given projection.

For Example

(1) Let $\tilde{R}(X, Y)$ be a fuzzy relation defined by the following relation matrix. Then the first, second, total projection and cylindrical extensions are shown below

	y_1	y_2	y_3	y_4	y_5	First projection $\tilde{R}^{(1)}$
x_1	.8	0	.3	.4	0	.8
x_2	0	1	.5	0	.5	1
x_3	1	.4	.6	.8	.6	1
Second projection $\tilde{R}^{(2)}$	1	1	.6	.8	.6	

← max = 1 → (first projection column: ← max = 1 →)

Total projection = 1

The cylindrical extension of $\tilde{R}^{(1)}$ is

.8	.8	.8	.8	.8
1	1	1	1	1
1	1	1	1	1

The cylindrical extension of $\tilde{R}^{(2)}$ is

1	1	.6	.8	.6
1	1	.6	.8	.6
1	1	.6	.8	.6

(2) Let $\tilde{R}$ be a fuzzy relation defined by the following relation matrix. Then the first, second, total projection are shown as follows

	y_1	y_2	y_3	y_4	y_5	y_6	$\tilde{R}^{(1)}$
x_1	.3	.2	.5	1	.1	.8	1
x_2	.4	.2	.8	.1	1	.9	1
x_3	4	.8	1	.8	.4	.2	1
$\tilde{R}^{(2)}$	.4	.8	1	1	1	.9	

← max = 1 → (column $\tilde{R}^{(1)}$: ← max = 1 →)

Total projection = 1

(3) Let $\tilde{R}$ be a fuzzy relation between two sets

$X = \{\text{New York, Paris}\}$ and

$Y = \{\textit{Beijing, New York, London}\}$

which represent the relation "very far". Then

$$\tilde{R}(X, Y) \equiv \tilde{R} = \frac{1}{[N.Y., Beijing]} + \frac{0}{[N.Y., N.Y.]} + \frac{.6}{[N.Y., london]}$$

$$+\frac{.9}{\{Paris, Beijing\}}+\frac{.7}{\{Paris, N.Y.\}}+\frac{.3}{\{Paris, london\}}$$

or $\tilde{R}$ is represented as in matrix form by

	New York	Paris
Beijing	1	.9
New York	0	.7
London	.6	.3

SOME MORE DEFINITIONS

(1) Domain of a Fuzzy Relation : *If $\tilde{R}(X, Y) = \tilde{R}$ is a binary relation of X and Y, then its domain is a fuzzy set on X, denoted by dom. $\tilde{R}$ and defined by (i.e., its membership functions are)*

$$dom\ \tilde{R}(x) = \max_{y \in Y} \tilde{R}(x, y) \quad \text{for each } x \in X$$

i.e., each element of set X belongs to the domain of $\tilde{R}$ to the degree equal to the strength of its strongest relation to any member of the set Y.

(2) Range of a Fuzzy Relation : *The range of $\tilde{R}(x, y)$ is a fuzzy set on Y denoted by ran $\tilde{R}$ and defined as*

$$ran\ \tilde{R}(y) = \max_{x \in X} \tilde{R}(x, y) \quad \textit{for each } y \in Y$$

i.e., each element of set Y belongs to an element of X equal to the degree of that elements membership in the range of $\tilde{R}$.

Domain ⇒ Row wise maximum

Range ⇒ Column wise maximum

(3) Height of a Fuzzy Relation : *The height of a fuzzy relation $\tilde{R}(X, Y)$ is a number $h(\tilde{R})$, defined by*

$$h(\tilde{R}) = \max_{y \in Y} \max_{x \in X} \tilde{R}(x, y)$$

i.e., $h(\tilde{R})$ is the largest membership grade attained by any pair (x, y) in $\tilde{R}$.

Remark

- $h(\tilde{R})$ is also called the **total projection.**

(4) Inverse of a Fuzzy Relation : *The inverse of a fuzzy relation $\tilde{R}(X, Y)$ which is denoted by $\tilde{R}^{-1}(Y, X)$ is a relation on $Y \times X$ and is defined by*

$$\tilde{R}^{-1}(y, x) = \tilde{R}(x, y), \quad \forall x \in X,\ y \in Y$$

Clearly, $(\tilde{R}^{-1})^{-1} = \tilde{R}$ for the fuzzy relation.

Relational Join

Let $\tilde{P}=[p_{ik}]$ and $\tilde{Q}=[q_{kj}]$ and $\tilde{R}=[r_{ij}]$ be the membership matrices of binary relations such that $\tilde{R}=\tilde{P}\circ\tilde{Q}$, then

$$[r_{ij}] = [p_{ik}]\circ[q_{kj}], \text{ where, } \quad r_{ij} = \max_k \min[p_{ik}, q_{kj}]$$

For Example : Let $\tilde{P} = \begin{bmatrix} .3 & .5 & .8 \\ 0 & .7 & 1 \\ .4 & .6 & .5 \end{bmatrix}$, $\tilde{Q} = \begin{bmatrix} .9 & .5 & .7 & .7 \\ .3 & .2 & 0 & .9 \\ 1 & 0 & .5 & .5 \end{bmatrix}$

Then, by using $r_{11} = \max[\min[p_{11}, q_{11}], \min[p_{12}, q_{21}], \min[p_{13}, q_{31}]]$, we can find

$$\tilde{P}\circ\tilde{Q} = \begin{bmatrix} .3 & .5 & .8 \\ 0 & .7 & 1 \\ .4 & .6 & .5 \end{bmatrix} \circ \begin{bmatrix} .9 & .5 & .7 & .7 \\ .3 & .2 & 0 & .9 \\ 1 & 0 & .5 & .5 \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} & r_{13} & r_{14} \\ r_{21} & r_{22} & r_{23} & r_{24} \\ r_{31} & r_{32} & r_{33} & r_{34} \end{bmatrix}$$

where r_{ij} can be found as follows :

$$r_{11} = \max[\min(p_{11}, q_{11}), \quad \min(p_{12}, q_{21}), \quad \min(p_{13}, q_{31})]$$
$$= \max[\min(0.3, 0.9), \quad \min(0.5, 0.3), \quad \min(0.8, 1)]$$
$$= \max[0.3 \quad 0.3 \quad 0.8] = 0.8$$

$$r_{12} = \max[\min(p_{11}, q_{12}), \quad \min(p_{12}, q_{22}), \quad \min(p_{13}, q_{32})]$$
$$= \max[\min(0.3, 0.5), \quad \min(0.5, 0.2), \quad \min(0.8, 0)]$$
$$= \max[0.3 \quad 0.2 \quad 0] = 0.3$$

$$r_{13} = \max[\min(p_{11}, q_{13}), \quad \min(p_{12}, q_{23}), \quad \min(p_{13}, q_{33})]$$
$$= \max[\min(0.3, 0.5), \quad \min(0.5, 0), \quad \min(0.8, 0.5)]$$
$$= \max[0.3 \quad 0 \quad 0.5] = 0.5$$

$$r_{14} = \max[\min(p_{11}, q_{14}), \quad \min(p_{12}, q_{24}), \quad \min(p_{13}, q_{34})]$$
$$= \max[\min(0.3, 0.7), \quad \min(0.5, 0.9), \quad \min(0.8, 0.5)]$$
$$= \max[0.3 \quad 0.5 \quad 0.5] = 0.5$$

$$r_{21} = \max[\min(p_{21}, q_{11}), \quad \min(p_{22}, q_{21}), \quad \min(p_{23}, q_{31})]$$
$$= \max[\min(0, 0.9), \quad \min(0.7, 0.3), \quad \min(1, 1)]$$
$$= \max[0 \quad 0.3 \quad 1] = 1$$

$$r_{22} = \max[\min(p_{21}, q_{12}), \quad \min(p_{22}, q_{22}), \quad \min(p_{23}, q_{32})]$$
$$= \max[\min(0, 0.5), \quad \min(0.7, 0.2), \quad \min(1, 0)]$$
$$= \max[0 \quad 0.2 \quad 0] = 0.2$$

$$r_{23} = \max[\min(p_{21}, q_{13}),\ \min(p_{22}, q_{23}),\ \min(P_{23}, Q_{33})]$$
$$= \max[\min(0, 0.7),\ \min(0.7, 0),\ \min(1, 0.5)]$$
$$= \max[0\ \ 0\ \ 0.5] = 0.5$$
$$r_{24} = \max[\min(p_{21}, q_{14}),\ \min(p_{22}, q_{24}),\ \min(p_{23}, q_{34})]$$
$$= \max[\min(0, 0.7),\ \min(0.7, 0.9),\ \min(1, 0.5)]$$
$$= \max[0\ \ 0.7\ \ 0.5] = 0.7$$
$$r_{31} = \max[\min(p_{31}, q_{11}),\ \min(p_{32}, q_{21}),\ \min(p_{33}, q_{31})]$$
$$= \max[\min(0.4, 0.9),\ \min(0.6, 0.3),\ \min(0.5, 1)]$$
$$= \max[0.4\ \ 0.3\ \ 0.5] = 0.5$$
$$r_{32} = \max[\min(p_{31}, q_{12}),\ \min(p_{32}, q_{22}),\ \min(p_{33}, q_{32})]$$
$$= \max[\min(0.4, 0.5),\ \min(0.6, 0.2),\ \min(0.5, 0)]$$
$$= \max[0.4\ \ 0.2\ \ 0] = 0.4$$
$$r_{33} = \max[\min(p_{31}, q_{13}),\ \min(p_{32}, q_{23}),\ \min(p_{33}, q_{33})]$$
$$= \max[\min(0.4, 0.7),\ \min(0.6, 0),\ \min(0.5, 0.5)]$$
$$= \max[0.4\ \ 0\ \ 0.5] = 0.5$$
$$r_{34} = \max[\min(p_{31}, q_{14}),\ \min(p_{32}, q_{24}),\ \min(p_{33}, q_{34})]$$
$$= \max[\min(0.4, 0.7),\ \min(0.6, 0.9),\ \min(0.5, 0.5)]$$
$$= \max[0.4\ \ 0.6\ \ 0.5] = 0.6$$

Hence, $\tilde{P} \circ \tilde{Q} = \begin{bmatrix} 0.8 & 0.3 & 0.5 & 0.5 \\ 1 & 0.2 & 0.5 & 0.7 \\ 0.5 & 0.4 & 0.5 & 0.6 \end{bmatrix}$

The above similar operation holds into binary relations and is known as **Relational Join.**

Definition : *Let $\tilde{P}(X, Y)$ and $\tilde{Q}(Y, Z)$ be two fuzzy relations, then their relational join is denoted by $\tilde{P} * \tilde{Q}$ and is defined by a relation $\tilde{R}(X, Y, Z)$ as*

$$\tilde{R}(X, Y, Z) = [\tilde{P} * \tilde{Q}](x, y, z) = \min[\tilde{P}(x, y), \tilde{Q}(y, z)],\ \forall x \in X,\ y \in Y, z \in Z$$

Remark

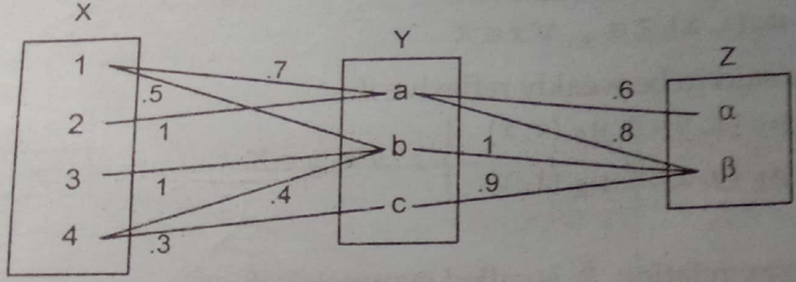
- It is a fact that the relational join produces a ternary relation from the binary relations, is a major difference from the composition because it results in another binary relation.

Max-Min Compositions

Max-Min composition is obtained by aggregating appropriate elements of the corresponding join by the Max operator such that

$$(\tilde{P} \circ \tilde{Q})(x, z) = \max_{y} [\tilde{P} * \tilde{Q}](x, y, z), \quad \forall x \in \tilde{X},\ z \in \tilde{Z}$$

For Example : Let $\tilde{X} = \{1, 2, 3, 4\}, \tilde{Y} = \{a, b, c\}, \tilde{Z} = \{\alpha, \beta\}$ be three fuzzy sets, then a relational join exists as



$\tilde{P}(X, Y) =$

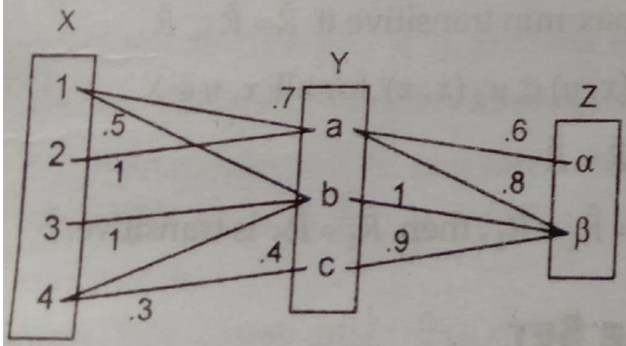
	a	b	c
1	.7	.5	0
2	1	0	0
3	0	1	0
4	0	.4	.3

$\tilde{Q}(Y, Z) =$

	α	β
a	.6	.8
b	0	1
c	0	.9

$$\tilde{P} \circ \tilde{Q} = \begin{bmatrix} .7 & .5 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & .4 & .3 \end{bmatrix} \circ \begin{bmatrix} .6 & .8 \\ 0 & 1 \\ 0 & .9 \end{bmatrix} = \begin{bmatrix} .6 & .7 \\ .6 & .8 \\ 0 & 1 \\ 0 & .4 \end{bmatrix}$$

$\tilde{S} = \tilde{P} * \tilde{Q}$ is defined as



$S = \tilde{P} * \tilde{Q} = \min[\tilde{P}(x, y), \tilde{Q}(y, z)]$

$\min(.7, .6) = .6$ $\quad = .6$

$\min(.7, .8) = .7$, $\min(.5, 1) = .5$ $\quad \max(.5, .7) = .7$

$\min(1, .6)$ $\quad = .6$

$\min(1, .8)$ $\quad = .8$

$\min(1, 1)$ $\quad = 1$

$\min(.4, 1) = .4$, $\min(.3, .9) = .3$ $\quad \max(.3, .4) = .4$

Fig. 3

SOLVED EXAMPLES

Example 1. *Show that the following matrices given $\tilde{R}_1, \tilde{R}_2 + \tilde{R}_3$ are perfectly antisymmetric, antisymmetric and non-symmetric.*

Solution. Let

$\tilde{R}_1 =$

	x_1	x_2	x_3	x_4
x_1	.4	0	.1	.8
x_2	.8	1	0	0
x_3	0	.6	.7	0
x_4	0	.2	0	0

In above relation, we see that $x_3\, \tilde{R}_1\, x_2 > 0$, i.e., $x_2\, \tilde{R}_1\, x_3 = 0$ as $x_2 \neq x_3$

$\Rightarrow$ Relation $\tilde{R}_1$ is perfectly antisymmetric.

Also, $x_3 \neq x_4$ but $x_3\, \tilde{R}\, x_4 = x_4\, \tilde{R}\, x_3 = 0$

$\Rightarrow$ Relation is antisymmetric for some $x \in X$, i.e., relation is antisymmetric.

Let

$\tilde{R}_2 =$

	x_1	x_2	x_3	x_4
x_1	.4	0	.7	0
x_2	0	1	.9	.6
x_3	.8	.4	.7	.4
x_4	0	.1	0	0

Here, $x_2\, \tilde{R}_2\, x_3 = .9, \quad x_3\, \tilde{R}_2\, x_2 = .4$

i.e., $x_2 \neq x_3 \quad \Rightarrow \quad \mu_{\tilde{R}}(x, y) \neq \mu_{\tilde{R}}(y, x)$

also, $x_3\, \tilde{R}_2\, x_4 = .4, \quad x_4\, \tilde{R}_2\, x_3 = 0$

i.e., $x_3 \neq x_4 \quad \Rightarrow \quad \mu_{\tilde{R}}(x, y) \neq \mu_{\tilde{R}}(y, x)$

and, $x_1\, \tilde{R}_2\, x_4 = .0, \quad x_4\, \tilde{R}_2\, x_1 = 0$

i.e., $x_1 \neq x_4$ but $\quad \mu_{\tilde{R}}(x, y) = \mu_{\tilde{R}}(y, x) = 0$

Hence, $\tilde{R}_2$ is antisymmetric.

Let

$\tilde{R}_3 =$

	x_1	x_2	x_3	x_4
x_1	.4	.8	.1	.8
x_2	.8	1	0	.2
x_3	1	.6	.7	.1
x_4	0	.2	0	0

Here, $x_4\, \tilde{R}_3\, x_1 = 0, \quad x_1\, \tilde{R}_3\, x_4 = .8$

i.e., $x\ \tilde{R}_3\ y \neq y\ \tilde{R}_3\ x\ ,\ \ \forall\, x, y \in X$

$\Rightarrow$ Relation $\tilde{R}_3$ is non-symmetric.

Let

$\tilde{R}_4 =$

	x_1	x_2	x_3	x_4
x_1	0	.1	0	.1
x_2	.1	1	.2	.3
x_3	0	.2	.8	.8
x_4	.1	.3	.8	1

The relation $\tilde{R}_4$ is symmetric as $x\ \tilde{R}\ y = y\ \tilde{R}\ x\ ,\ \ \forall\, x, y \in X$

Example 2. *Express the following matrix relation as similar or equivalence relation.*

$$\begin{array}{c|cccccc} & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \\ \hline x_1 & 1 & .2 & 1 & .6 & .2 & .6 \\ x_2 & .2 & 1 & .2 & .2 & .8 & .2 \\ x_3 & 1 & .2 & 1 & .6 & .2 & .6 \\ x_4 & .6 & .2 & .6 & 1 & .2 & .8 \\ x_5 & .2 & .8 & .2 & .2 & 1 & .2 \\ x_6 & .6 & .2 & .6 & .8 & .2 & .1 \end{array}$$

Solution. From the above relation matrix, we see that

$\tilde{R}\,(x, x) = 1\ ,\ \forall\, x \in X \quad \Rightarrow \quad \tilde{R}$ is reflexive.

$\tilde{R}(x, y) = \tilde{R}\,(y, x)\ ,\ \ \forall\, x, y \in X \ \Rightarrow \quad \tilde{R}$ is symmetric.

Now to check for transitivity, we find $\tilde{R}_T$. So

$$\tilde{R} \circ \tilde{R} = \begin{bmatrix} 1 & .2 & 1 & .6 & .2 & .6 \\ .2 & 1 & .2 & .2 & .8 & .2 \\ 1 & .2 & 1 & .6 & .2 & .6 \\ .6 & .2 & .6 & 1 & .2 & .8 \\ .2 & .8 & .2 & .2 & 1 & .2 \\ .6 & .2 & .6 & .8 & .2 & 1 \end{bmatrix} \circ \begin{bmatrix} 1 & .2 & 1 & .6 & .2 & .6 \\ .2 & 1 & .2 & .2 & .8 & .2 \\ 1 & .2 & 1 & .6 & .2 & .6 \\ .6 & .2 & .6 & 1 & .2 & .8 \\ .2 & .8 & .2 & .2 & 1 & .2 \\ .6 & .2 & .6 & .8 & .2 & 1 \end{bmatrix}$$

$$\text{Let } \tilde{R} \circ \tilde{R} = \begin{bmatrix} r_{11} & r_{12} & r_{13} & r_{14} & r_{15} & r_{16} \\ r_{21} & r_{22} & r_{23} & r_{24} & r_{25} & r_{26} \\ r_{31} & r_{32} & r_{33} & r_{34} & r_{35} & r_{36} \\ r_{41} & r_{42} & r_{43} & r_{44} & r_{45} & r_{46} \\ r_{51} & r_{52} & r_{53} & r_{54} & r_{55} & r_{56} \\ r_{61} & r_{62} & r_{63} & r_{64} & r_{65} & r_{66} \end{bmatrix} \qquad \text{......(A)}$$

We know that

$$[r_{11}] = [p_{1k}] \cdot [q_{k1}] = \max_k [\min(p_{1k}, q_{k1})]$$

Here, $r_{11} = \max[\min(1,1), \min(.2,.2), \min(1,1), \min(.6,.6), \min(.2,.2), \min(.6,.6)]$

$$r_{11} = \max[1, .2, 1, .6, .2, .6] = 1$$

$$r_{12} = \max[\min(1,.2), \min(.2,1), \min(1,.2), \min(.6,.2), \min(.2,.8), \min(.6,.2)]$$
$$= \max[.2,.2,.2,.2,.2,.2] = .2$$

$$r_{13} = \max[\min(1,1), \min(.2,.2), \min(1,1), \min(.6,.6), \min(.2,.2), \min(.6,.6)]$$
$$= \max[1,.2,1,.6,.2,.6] = 1$$

$$r_{14} = \max[\min(1,.6), \min(.2,.2), \min(1,.6), \min(.6,1), \min(.2,.2), \min(.6,.8)]$$
$$= \max[.6,.2,.6,.6,.2,.6] = .6$$

$$r_{15} = \max[\min(1,.2), \min(.2,.8), \min(1,.2), \min(.6,.2), \min(.2,1), \min(.6,.2)]$$
$$= \max[.2,.2,.2,.2,.2,.2] = .2$$

$$r_{16} = \max[\min(1,.6), \min(.2,.2), \min(1,.6), \min(.6,.8), \min(.2,.2), \min(.6,1)]$$
$$= \max[.6,.2,.6,.6,.2,.6] = .6$$

Similarly,

$\because$ $$[r_{21}] = \max_k [\min(p_{2k}, q_{k2})]$$

$$r_{21} = \max[\min(.2,1), \min(1,.2), \min(.2,1), \min(.2,.6), \min(.8,.2), \min(.2,.6)]$$
$$= \max[.2,.2,.2,.2,.2,.2] = .2$$

$$r_{22} = 1,\ r_{23} = .2,\ r_{24} = .2,\ r_{25} = .8,\ r_{26} = .2$$

$$r_{31} = \max[\min(1,1), \min(.2,.2), \min(1,1), \min(.6,.6), \min(.2,.2), \min(.8,.6)]$$
$$= \max[1,.2,1,.6,.2,.8] = 1$$

$$r_{32} = .2,\ r_{33} = 1,\ r_{34} = .6,\ r_{35} = .2,\ r_{36} = .6$$

$$r_{41} = .6,\ r_{42} = .2,\ r_{43} = .6,\ r_{44} = 1,\ r_{45} = .2,\ r_{46} = .8$$

$$r_{51} = .2,\ r_{52} = .8,\ r_{53} = .2,\ r_{54} = .2,\ r_{55} = 1,\ r_{56} = .2$$

$$r_{61} = .6,\ r_{62} = .2,\ r_{63} = .6,\ r_{64} = .8,\ r_{65} = .2,\ r_{66} = 1$$

Putting in (A)

$$\Rightarrow \quad \tilde{R} \circ \tilde{R} = \begin{bmatrix} 1 & .2 & 1 & .6 & .2 & .6 \\ .2 & 1 & .2 & .2 & .8 & .2 \\ 1 & .2 & 1 & .6 & .2 & .6 \\ .6 & .2 & .6 & 1 & .2 & .8 \\ .2 & .8 & .2 & .2 & 1 & .2 \\ .6 & .2 & .6 & .8 & .2 & 1 \end{bmatrix}$$

Now,

$$[\tilde{R} \circ \tilde{R}] \cup \tilde{R} = \begin{bmatrix} 1 & .2 & 1 & .6 & .2 & .6 \\ .2 & 1 & .2 & .2 & .8 & .2 \\ 1 & .2 & 1 & .6 & .2 & .6 \\ .6 & .2 & .6 & 1 & .2 & .8 \\ .2 & .8 & .2 & .2 & 1 & .2 \\ .6 & .2 & .6 & .8 & .2 & 1 \end{bmatrix} \cup \begin{bmatrix} 1 & .2 & 1 & .6 & .2 & .6 \\ .2 & 1 & .2 & .2 & .8 & .2 \\ 1 & .2 & 1 & .6 & .2 & .6 \\ .6 & .2 & .6 & 1 & .2 & .8 \\ .2 & .8 & .2 & .2 & 1 & .2 \\ .6 & .2 & .6 & .8 & .2 & 1 \end{bmatrix}$$

$$\because \quad \tilde{A} \cup \tilde{B} = \max[\mu_{\tilde{A}}(x),\ \mu_{\tilde{B}}(x)]$$

$$\tilde{R}' = \begin{bmatrix} \max(1,1) & \max(.2,.2) & \max(1,1) & \max(.6,.6) & \max(.2,.2) & \max(.6,.6) \\ \max(.2,.2) & \max(1,1) & \max(.2,.2) & \max(.2,.2) & \max(.8,.8) & \max(.2,.2) \\ \max(1,1) & \max(.2,.2) & \max(1,1) & \max(.6,.6) & \max(.2,.2) & \max(.6,.6) \\ \max(.6,.6) & \max(.2,.2) & \max(.6,.6) & \max(1,1) & \max(.2,.2) & \max(.8,.8) \\ \max(.2,.2) & \max(.8,.8) & \max(.2,.2) & \max(.2,.2) & \max(1,1) & \max(.2,.2) \\ \max(.6,.6) & \max(.2,.2) & \max(.6,.6) & \max(.8,.8) & \max(.2,.2) & \max(1,1) \end{bmatrix}$$

$$= \begin{bmatrix} 1 & .2 & 1 & .6 & .2 & .6 \\ .2 & 1 & .2 & .2 & .8 & .2 \\ 1 & .2 & 1 & .6 & .2 & .6 \\ .6 & .2 & .6 & 1 & .2 & .8 \\ .2 & .8 & .2 & .2 & 1 & .2 \\ .6 & .2 & .6 & .8 & .2 & 1 \end{bmatrix}$$

Example 3. *Show that a fuzzy relation* $\tilde{R}$ *is called max-min transitive if* $\tilde{R} \circ \tilde{R} \subseteq \tilde{R}$.

[Meerut-2005]

Solution. Let

$$R = \begin{bmatrix} .2 & 1 & .4 \\ 0 & .6 & .3 \\ 0 & 1 & .3 \end{bmatrix}$$

[Remember]

Now, firstly we shall prove that R is transitive.

$$R \circ R = \begin{bmatrix} .2 & 1 & .4 \\ 0 & .6 & .3 \\ 0 & 1 & .3 \end{bmatrix} \circ \begin{bmatrix} .2 & 1 & .4 \\ 0 & .6 & .3 \\ 0 & 1 & .3 \end{bmatrix}$$

$$= \begin{bmatrix} \max(.2,0,0) & \max(.2,.6,.4) & \max(0,.3,.3) \\ \max(0,0,0) & \max(0,.6,.3) & \max(0,.3,.3) \\ \max(0,0,0) & \max(0,.6,.3) & \max(0,.3,.3) \end{bmatrix}$$

$$R \circ R = \begin{bmatrix} .2 & .6 & .3 \\ 0 & .6 & .3 \\ 0 & .6 & .3 \end{bmatrix} \qquad \ldots\ldots(1)$$

$$(R \circ R) \cup R = \begin{bmatrix} .2 & .6 & .3 \\ 0 & .6 & .3 \\ 0 & .6 & .3 \end{bmatrix} \cup \begin{bmatrix} .2 & 1 & .4 \\ 0 & .6 & .3 \\ 0 & 1 & .3 \end{bmatrix} = \begin{bmatrix} .2 & 1 & .4 \\ 0 & .6 & .3 \\ 0 & 1 & .3 \end{bmatrix} = R'$$

Here, $(R \circ R) \cup R = R'$. Thus, $R' = R_T$, the closure membership matrix.

so R is transitive.

Now, we shall prove that $\tilde{R} \circ \tilde{R} \subseteq \tilde{R}$

$$\mu_{\tilde{R} \circ \tilde{R}}(x) \leq \mu_{\tilde{R}}(x) \qquad \text{(By (1))}$$

so $\tilde{R} \circ \tilde{R} \subseteq R$ Proved (Particular Case)

Example 4. *If $\tilde{R}_1$ is reflexive and $\tilde{R}_2$ is an arbitrary fuzzy relation from $\tilde{R}$, then show that $\tilde{R}_2 \subset \tilde{R}_1 \circ \tilde{R}_2$ and $\tilde{R}_2 \subset \tilde{R}_2 \circ \tilde{R}_1$.* **[Rohilkhand-2007]**

Solution. Let

$$R_1 = \begin{matrix} & x_1 & x_2 & x_3 \\ x_1 & 1 & .7 & .3 \\ x_2 & .4 & 1 & .8 \\ x_3 & .7 & .5 & 1 \end{matrix}$$

$$R_2 = \begin{matrix} & x_1 & x_2 & x_3 \\ x_1 & .7 & 0 & 1 \\ x_2 & .4 & .5 & 0 \\ x_3 & 0 & 1 & .5 \end{matrix}$$

$$\tilde{R}_1 \circ \tilde{R}_2 = \begin{bmatrix} 1 & .7 & .3 \\ .4 & 1 & .8 \\ .7 & .5 & 1 \end{bmatrix} \begin{bmatrix} .7 & 0 & 1 \\ .4 & .5 & 0 \\ 0 & 1 & .5 \end{bmatrix}$$

$$= \begin{bmatrix} \max(.7,.4,0) & \max(0,.5,.3) & \max(1,0,.3) \\ \max(.4,.4,0) & \max(0,.5,.8) & \max(.4,0,.5) \\ \max(.7,.4,0) & \max(0,.5,1) & \max(.7,0,.5) \end{bmatrix} \qquad \text{......(1)}$$

$$\tilde{R}_1 \circ \tilde{R}_2 = \begin{matrix} & x_1 & x_2 & x_3 \\ x_1 & .7 & .5 & 1 \\ x_2 & .4 & .8 & .5 \\ x_3 & .7 & 1 & .7 \end{matrix}$$

Here, we observe that for each value

$$\tilde{R}_2(x_i\, x_j) \leq \tilde{R}_1 \circ \tilde{R}_2(x_i\ \ x_j)$$

So, we can say that if R_1 is reflexive, R_2 is arbitrary, then

$$\tilde{R}_2 \subset \tilde{R}_1 \circ \tilde{R}_2$$

and thus in the similar way, we can show that

$$\tilde{R}_2 \subset \tilde{R}_2 \circ \tilde{R}_1$$

Decision Making in Fuzzy Environment

- Individual Decision Making
- Multiperson Decision Making
- Multicriteria Decision Making
- Fuzzy Ranking Method
- Fuzzy Linear Programming's Rule

1.4 Fuzzy Decision making

9.1 INTRODUCTION

The subject of decision making is the study of how the decision are made and how they can be made better or more successfully. There are several classes of decision making. It is classified as those involving a single decision maker and those which involve several decision makers. These problem classes are known as individual decision making and multiperson decision making respectively. We also distinguish decision problems that involve a simple optimization of a utility function, an optimization under constraints or an optimization under multiple objective criteria. Also, the decision making can be done in one stage or it can be done iteratively in several stages. In this chapter, we shall discuss the applicability of fuzzy set theory to the main classes of decision making problems.

9.2 INDIVIDUAL DECISION MAKING – 1.4.1

There are several fuzzy models decision making in which relevant goals and constraints are expressed in terms of fuzzy sets. Also, we can determine a decision by an appropriate aggregation of these fuzzy sets. Therefore, a decision situation can be characterized by the following components.

(i) *a set A of possible events/actions.*

(ii) *a set of goals* $G_i\,(i \in \mathbf{N})$, *each of which is expressed in terms of a fuzzy set defined on A.*

(iii) *A set of constraints* $C_j\ (j \in \mathbf{N})$ *each of which is expressed by a fuzzy set defined on* A.

Let $G_i^{'}$ and $C_j^{'}$ be fuzzy sets defined on sets X_i and Y_i respectively. Also, assume that these fuzzy sets represents goals and constraints expressed by the decision makers. Then, for each $i \in \mathbf{N}$, $j \in \mathbf{N}$, the meanings of actions in a set $\tilde{A}$ in terms of sets X_i and Y_j can be defined by functions

$g_i : \tilde{A} \to X_i$

$C_j = \tilde{A} \to Y_j$

and express goals G_i and constrains C_j by the compositions of g_i with G_i' and the compositions of C_j and C_j', i.e.,

$$G_i(a) = G_i'(g_i(a))$$

$$C_j(a) = C_j'(C_j(a)) \quad \text{for each } a \in A$$

Definition : *Given a decision situation characterized by fuzzy sets* $\tilde{A}$, G_i $(i \in N)$ *and* C_j $(j \in N)$, *a fuzzy decision, D is conceived as a fuzzy set on A that simultaneously satisfies the given goals* G_i *and constraints* C_j, *i.e.,*

$$D(a) = \min\left[\inf_{i \in N} G_i(a), \inf_{j \in N} C_j(a)\right]$$

for all $a \in A$, *provided that the standard operator of fuzzy intersection is employed.*

Use of Weighted Coefficients : The fuzzy model can be extended to accommodate the relative importance of the various goals and constraints by the use of weighted coefficients. Then, we define the decision D as follows

$$D(a) = \sum_{i=1}^{n} u_i G_i(a) + \sum_{j=1}^{m} v_j C_j(a) \qquad \text{for all } a \in \tilde{A} \qquad \ldots\ldots(1)$$

where, u_i and v_j are non-negative weights attached to each fuzzy goals G_i $(i \in N)$ and each fuzzy constraints C_j $(j \in N)$ respectively such that

$$\Sigma u_i + \Sigma v_j = 1$$

Remarks

- Formula (1) can also be written as

$$D(a) = \min\left[\inf_{i \in N} G_i^{u_i}(a) \inf_{j \in N} C_j^{v_j}(a)\right]$$

- In 1970, Bellman and Zadeh suggest a fuzzy model of decision making, in which relevant goals and constraints are expressed in terms of fuzzy sets.
- Once a fuzzy decision has been arrived at, it may be necessary to choose the 'best' single crisp alternative to the fuzzy set.

9.3 MULTIPERSON DECISION MAKING [Meerut-2006 BP]

Let us assume that each member of a group of n individuals decision makers is reflexive, antisymmetric and transitive preference ordering $P_k : k \in N$, which is totally or partially ordered a set X of alternatives. We must be found a 'social choice' function which gives the individual preference ordering, produces the most acceptable overall group preference ordering. To deal with the multiplicity of opinion, evidenced in the group, the social preference S may be defined as follows:

$$S : X \times X \rightarrow [0,1]$$

which assigns the membership grade $S(x_i, x_j)$, indicating the degree of the group preference of alternative x_i over x_j.

METHOD FOR FINDING $S(x_i, x_j)$ 14.2

Step Knowledge:

(1) The simple method computes the relative popularity of alternative x_i over x_j by dividing the number of persons preferring x_i to x_j, denoted by $N(x_i, x_j)$ by the total numbers of decision makers n. Therefore

$$S(x_i, x_j) = \frac{N(x_i, x_j)}{n}$$

(2) Let $\overset{k}{>}$ represents the preference ordering of one individual k who exercises complete control over the group decision. Then, a dictarial situation can be modeled by the group preference relation S for which

$$S(x_i, x_j) = \begin{cases} 1 & \text{if } x_i \overset{k}{>} x_j \text{ for some individual } k \\ 0 & \text{otherwise} \end{cases}$$

(3) When we have defined the fuzzy relationship S, then the final non-fuzzy group preference can be determined by using the following formula

$$S = \bigcup_{\alpha \in [0,1]} \alpha\, {}^{\alpha}S$$

where, ${}^{\alpha}S$ is the crisp relations comprising the α-cuts of the fuzzy relation S, scaled by α. Further, α represents the level of agreement between the individual concerning the particular crisp ordering ${}^{\alpha}S$.

Working Aid

(1) The procedure that maximizes the final agreement level consists of intersecting the classes of crisp total ordering that are compatible with the pairs in the α-cuts ${}^{\alpha}S$ for increasingly smaller values of α until a single crisp total ordering is obtained.

(2) Removed any pairs (x_i, x_j) if it leads to an intransitivity.

(3) The largest value of α for which the unique compatible ordering on $X \times X$ is found represents the maximized agreement level of the group, and the crisp ordering itself represents the group decision.

SOLVED EXAMPLE

Example 1. *Let us assume that each individual of a group of eight decision makers has a total preference ordering* P_i $(i \in N)$ *on a set of alternatives* $X = \{w, x, y, z\}$ *as follows*

$$P_1 = \{w, x, y, z\}$$
$$P_2 = P_5 = \{z, y, x, w\}$$
$$P_3 = P_7 = \{x, w, y, z\}$$
$$P_4 = P_8 = \{w, z, x, y\}$$
$$P_6 = \{z, w, x, y\}$$

Using the membership function $S(x_i, x_j) = \dfrac{N(x_i, x_j)}{n}$

for the fuzzy group performance, find the fuzzy preference relation. Also, find α-cuts of this fuzzy relation S, and group level of agreement concerning the social choice denoted by the total ordering (w, z, x, y). [Rohilkhand-2008]

Solution. We have

$$S(x_i, x_j) = \frac{N(x_i, x_j)}{n}$$

We have

$$S = \begin{array}{c} \\ w \\ x \\ y \\ z \end{array} \begin{array}{c} \begin{array}{cccc} w & x & y & z \end{array} \\ \begin{pmatrix} S(w,w) & S(w,x) & S(w,y) & S(w,z) \\ S(x,w) & S(x,x) & S(x,y) & S(x,z) \\ S(y,w) & S(y,x) & S(y,y) & S(y,z) \\ S(z,w) & S(z,x) & S(z,y) & S(z,z) \end{pmatrix} \end{array} \quad \text{......(1)}$$

$S(w,w) = 0,\ S(y,y) = 0,\ n = 8$

$S(x,x) = 0,\ S(z,z) = 0$

Now, $S(w,x) = \dfrac{N(w,x)}{n} = \dfrac{4}{8} = .5$

$S(w,y) = \dfrac{N(w,y)}{n} = \dfrac{6}{8} = .75$

$S(w,z) = \dfrac{N(w,z)}{n} = \dfrac{5}{8} = .625$

$S(x,w) = \dfrac{N(x,w)}{n} = \dfrac{4}{8} = .5$

$S(x,y) = \dfrac{N(x,y)}{n} = \dfrac{6}{8} = .75$

$S(x,z) = \dfrac{N(x,z)}{n} = \dfrac{3}{8} = .375$

$$S(y,w) = \frac{N(y,w)}{n} = \frac{2}{8} = .25$$

$$S(y,x) = \frac{N(y,x)}{n} = \frac{2}{8} = .25$$

$$S(y,z) = \frac{N(y,z)}{n} = \frac{3}{8} = .375$$

$$S(z,w) = \frac{N(z,w)}{n} = \frac{3}{8} = .375$$

$$S(z,x) = \frac{N(z,x)}{n} = \frac{5}{8} = .625$$

$$S(z,y) = \frac{N(z,y)}{n} = \frac{5}{8} = .625$$

Putting all these values in (1), the fuzzy social preference relation is given by

$$S = \begin{array}{c} \\ w \\ x \\ y \\ z \end{array} \begin{array}{c} \begin{array}{cccc} w & x & y & z \end{array} \\ \begin{bmatrix} 0 & .5 & .75 & .625 \\ .5 & 0 & .75 & .375 \\ .25 & .25 & 0 & .375 \\ .375 & .625 & .625 & 0 \end{bmatrix} \end{array}$$

The α-cuts of this fuzzy relation S are as follows

$${}^{1}S = \phi$$

$${}^{.75}S = \{(w,y),(x,y)\}$$

$${}^{.625}S = \{(w,z),(z,x),(z,y),(w,y),(x,y)\}$$

$${}^{.5}S = \{(x,w),(w,x),(w,z),(z,x),(z,y),(w,y),(x,y)\}$$

$${}^{.375}S = \{(z,w),(x,z),(y,z),(x,w),(w,x),(w,z),(z,x),(z,y),(w,y),(x,y)\}$$

$${}^{.25}S = \{(y,w),(y,x),(z,w),(x,z),(y,z),(x,w),(w,x),(w,z),(z,x),(z,y),(w,y),(x,y)\}$$

To determine the group choice, we may apply the procedure to arrive at the unique crisp ordering.

We have, all total ordering on $X \times X$ are compatible with the empty set of ${}^{1}S$. The total ordering ${}^{.75}O$ that are compatible with the pairs in the crisp relation ${}^{.75}S$ are

$${}^{.75}O = \{(z,w,x,y),(w,x,y,z),(w,z,x,y),(w,x,z,y),(z,x,w,y),(x,w,y,z),(x,z,w,y),(x,w,z,y)\}$$

Therefore

$${}^{1}O \cap {}^{.75}O = {}^{.75}O$$

Further, the ordering compatible with ${}^{.625}S$ are

$${}^{.625}O = \{(w,z,x,y)\}$$

and $\quad {}^{1}O \cap {}^{.75}O \cap {}^{.625}O = \{(w,z,x,y)\}$

Hence, the value .625 represents the group level of agreement concerning the social choice denoted by the total ordering (w, z, x, y).

Example 2. *Let each individual of four decisions makers has a total preference ordering P_i $(i \in N)$ on a set of alternatives $X = \{a,b,c,d\}$ as*

$$P_1 = (a,b,d,c);\ P_2 = (a,c,b,d);\ P_3 = (b,a,c,d);\ P_4 = (a,d,b,c)$$

Find the fuzzy preference relation. Also, find α-cuts of the fuzzy relation and group level of agreement concerning the social choice denoted by the total ordering (a, b, c, d).

[Meerut-2005]

Solution. We have

$$S(x_i, x_j) = \frac{N(x_i, x_j)}{n}, \text{ Here } n = 4$$

Proceeding same as in previous example, we get

$$S = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array}\begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} 0 & .75 & 1 & 1 \\ .25 & 0 & .75 & .75 \\ 0 & .25 & 0 & .5 \\ 0 & .25 & .5 & 0 \end{bmatrix} \end{array}$$

The α-cuts of this fuzzy relation S are as follows

$${}^{1}S = \{(a,c),(a,d)\}$$

$${}^{.75}S = \{(a,b),(b,c),(b,d),(a,c),(a,d)\}$$

$${}^{.5}S = \{(c,d),(d,c),(a,b),(b,c),(b,d),(a,c),(a,d)\}$$

$\because$ either (c,d) exists or (d,c) exists, thus, here, we consider (c,d) and neglect (d,c).

$$\Rightarrow \quad {}^{.5}S = \{(c,d),(a,b),(b,c),(b,d),(a,c),(a,d)\}$$

$${}^{.25}S = \{(b,a),(c,d),(c,b),(d,b),(a,b),(b,c),(b,d),(a,c),(a,d)\}$$

Again here also, we consider the pairs $(a,b),(b,c),(b,d)$ and neglecting $(b,a),(c,b),(d,b)$

$$\Rightarrow \quad {}^{.25}S = \{((c,d),(a,b),(b,c),(b,d),(a,c),(a,d)\}$$

$${}^{0}S = \{(a,a),(b,b),(c,c),(d,d),(c,a),(d,a),(c,d),(a,b),$$
$$(b,c),(b,d),(a,c),(a,d)\}$$

Here, considering the pairs $(a,c),(a,d)$ and neglecting $(c,a),(d,a)$

$$\Rightarrow \quad {}^{0}S = \{(a,a),(b,b),(c,c),(d,d),(c,d),(a,b),(b,c),(b,d),(a,c),(a,d)\}$$

Now, the ordering compatible with ${}^{1}S$ are

$${}^{1}O = \{(a,c,d,b),(a,d,c,b),(a,b,c,d),(b,a,c,d),(a,b,d,c),(b,a,d,c)\}$$

For ${}^{.75}S$ ordering compatible

Further, the ordering compatible with ${}^{.6}S$ are

$${}^{.6}O = \{(a,b,c,d)\}$$

and $\quad {}^{1}O \cap {}^{.8}O \cap {}^{.6}O = \{(a,b,c,d)\}$

Hence, the value 0.6 represents the grouplevel of agreement concerning the scial choice denoted by the total ordering (a,b,c,d)

Remark

➤ In the above procedure of group decision making, it is required that each group member can order the given set of alternatives.

Construction of an Ordering of Given Alternatives : There is a simple method (given by Shimura) to construct an ordering of all given alternatives on the basis of their pairwise comparisons.

Let $f(x_i, x_j)$ denotes the attractiveness grade given by the individual to x_i with respect to x_j. These evaluations, which are expressed by positive numbers in a given range, are made by individuals for all pairs of alternatives in the given set X. Then, they are converted to relative preference grades $F(x_i x_j)$ by using the following formula

$$F(x_i, x_j) = \frac{f(x_i, x_j)}{\max\left[f(x_i, x_j), f(x_j, x_i) \right]} = \min\left[1, \frac{f(x_i, x_j)}{f(x_j, x_i)}\right]$$

Remarks

➤ $F(x_i\ x_j) \in [0,1]$ for all pairs $(x_i, x_j) \in X^2$.

➤ If $F(x_i, x_j) = 1$, then x_i is considered at least as x_j

➤ For each pair of alternatives, at least one must be as atractive as the other $\max\left[F(x_i, x_j), F(x_j, x_i) \right]$.

Relative Preference Grades : *For each $x_i \in X$, the overall relative preference grades $p(x_i)$ of x_i with respect to all other alternatives in X is given by the following formula*

$$p(x_i) = \min_{x_j \in X} F(x_i, x_j)$$

Remark

➤ The preference ordering of alternatives in X induced by the numerical ordering of these grades $p(x_i)$.

9.4 MULTICRITERIA DECISION MAKING

Let $X = \{x_1, x_2, \ldots\ldots, x_n\}$ and $\tilde{C} = \{\tilde{C}_1, \tilde{C}_2, \ldots\ldots, \tilde{C}_m\}$ be a set of alternatives and a set of criteria characterizing a decision situations respectively. Then, the information in multicriteria decision making can be expressed as follows

$$R = \begin{matrix} & x_1 & x_2 & \cdots & x_n \\ \tilde{C}_1 & r_{11} & r_{12} & \cdots & r_{1n} \\ \tilde{C}_2 & r_{21} & r_{22} & \cdots & r_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \tilde{C}_m & r_{m1} & r_{m2} & \cdots & r_{mn} \end{matrix}$$

Let us assume that all entries of this matrix are needed in [0, 1] and each entry r_{ij} expresses the degree to which criteria x_i is satisfied by alternatives x_j $(i \in N, j \in N)$. Sometimes, instead of matrix R with entries in [0, 1] another matrix R' is initially given. Then, we convert R' into R by using the following formula

$$r_{ij} = \frac{r'_{ij} \min\limits_{j \in N} r'_{ij}}{\max\limits_{j \in N} r'_{ij} - \min\limits_{j \in N} r'_{ij}} \quad \text{for all } i \in N,\ j \in N$$

The multicriteria decision problems is converted into a single criteria decision problem by finding a global criterion

$$r_j = h(r_{1j}, r_{2j}, \ldots\ldots, r_{mj})$$

that for each $x_j \in X$ is an aggregate of values $r_{1j}, r_{2j}, \ldots\ldots, r_{mj}$ to which the individual criteria $C_1, C_2, \ldots\ldots, C_m$ are satisfied.

Aggregate Operator : An aggregate operator is the weighted average given by

$$r_j = \frac{\sum\limits_{i=1}^{m} w_i\, r_{ij}}{\sum\limits_{i=1}^{m} wi} \quad (j \in N)$$

where $w_1, w_2, \ldots\ldots, w_m$ are the weights that indicate the relative importance of criteria $\tilde{C}_1, \tilde{C}_2, \ldots\ldots, \tilde{C}_m$.

Class of Weighted Aggregations : A class of possible weighted aggregations is given by the following formula

$$r_j = h(r_{1j}^{w_1}, r_{2j}^{w_2}, \ldots\ldots, r_{mj}^{w_m})$$

where, h is an aggregation operator and $w_1, w_2, \ldots\ldots, w_m$ are weights.

Remark

- The weighted average $\bar{r}_j$ can also be obtained by the following formula

$$\bar{r}_j = \sum_{i=1}^{m} \bar{w}_i\, \bar{r}_{ij}, \text{ where } \bar{r}_{ij} \text{ are the fuzzy numbers on } R^+.$$

9.5 FUZZY RANKING METHOD

Sometimes, in fuzzy decision problems, the final scores of alternatives are represented in terms of fuzzy numbers. To express a crisp preference of alternatives, we need a method for constructing a crisp total ordering from fuzzy numbers. Since the lattice of fuzzy numbers {R, Min, Max} is not linearly ordered. Thus some fuzzy numbers are not directly comparable.

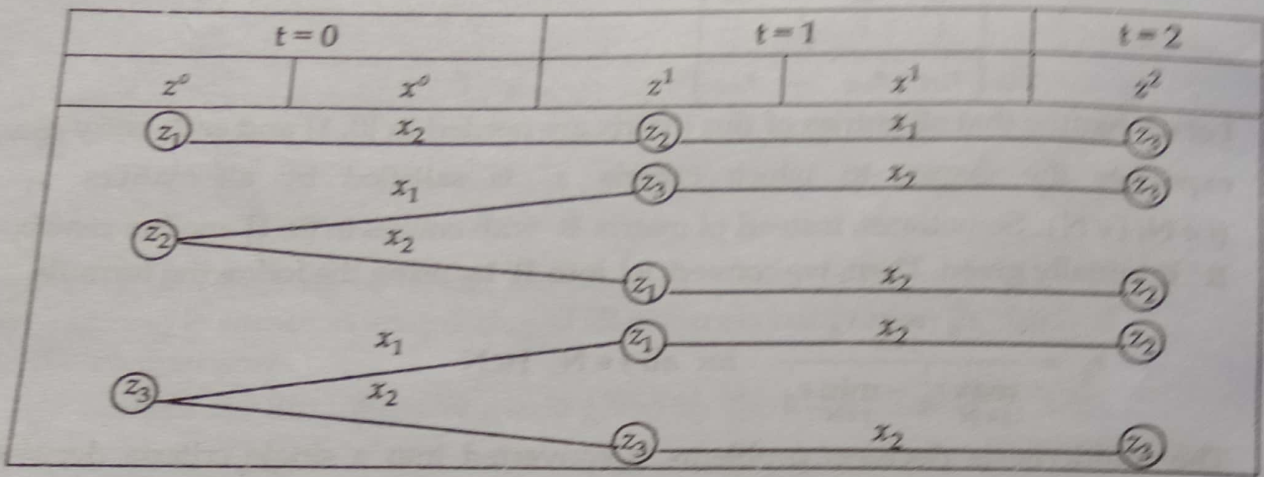


Fig. 1 : Maximizing decisions for different initial states z^0

Method of Total Ordering

(1) The first method is based upon defining the Hamming distance on the set R of all fuzzy numbers, then for fuzzy numbers A and B, the Hamming distance is given as

$$d(A, B) = \int_R |A(x) - B(x)|\, dx \qquad \text{......(1)}$$

This distance for l.u.b. (Least Upper Bound) in lattice (Max (A, B)) is defined as $d(Max(A, B), A)$ and $d(\max(A, B), B)$.

Also, $A \leq B$ if $d(Max(A, B), A) \geq d(Max(A, B), B)$

If $A \leq B$ (i.e., fuzzy numbers are directly comparable), then $\max(A, B) = B$ and hence $A \leq B$.

The ordering defined by Hamming distance is compatible with ordering of comparable fuzzy numbers in R (modifier R).

Remark

- We define similar ordering of fuzzy numbers $\tilde{A}$ and $\tilde{B}$ via the g.l.b. (Greatest Lower Bound) in lattice (Min($\tilde{A}, \tilde{B}$)) is defined as d ($\tilde{A}$, min ($\tilde{A}, \tilde{B}$)), and d ($\tilde{B}$, min ($\tilde{A}, \tilde{B}$)).

) The second method is based on α-cuts. Let two fuzzy numbers $\tilde{A}$ and $\tilde{B}$ are to be compared. We select a particular value $\alpha \in [0, 1]$ and determine the α-cuts, i.e., ${}^{\alpha}\tilde{A} = [a_1, a_2] = {}^{\alpha}\tilde{B} = [b_1, b_2]$. Then $\tilde{A} \leq \tilde{B}$ if $a_2 \leq b_2$

emark

This definition depends on the choosen value of α. Usually we take $\alpha > 0.5$.

(3) The third method is based on extension principle. This method can be used for ordering several fuzzy numbers say $\tilde{A}_1, \tilde{A}_2, \ldots, \tilde{A}_n$. The basic idea is to construct a fuzzy set P on $\{\tilde{A}_1, \tilde{A}_2, \ldots, \tilde{A}_n\}$, called priority set, such as $P(\tilde{A}_i)$ is the degree to which $\tilde{A}_i$ is ranked as the greatest fuzzy number. Using the extension principle, P is defined for each $i \in N_n$ by the formula

$$P(\tilde{A}_i) = \sup \min_{k \in N_n} \tilde{A}_k(r_k)$$

For Example : We want to compare the three fuzzy ranking methods. Let $\tilde{A}$ and $\tilde{B}$ be fuzzy numbers whose triangular type membership functions are given in figures.

The max $(\tilde{A}, \tilde{B})$ is the fuzzy number whose membership function is indicated in the figures by bolds. The Hamming distance $d(\max(\tilde{A}, \tilde{B}), \tilde{A})$, $d(\max(\tilde{A}, \tilde{B}), \tilde{B})$ are defined as

$$d(\max(\tilde{A}, \tilde{B}), \tilde{A}) = \int_{1.5}^{2} [x - 1 - \frac{x}{3}]\,dx + \int_{2}^{2.25} [-x + 3 - \frac{x}{3}]\,x\,dx$$

$$+ \int_{2.25}^{3} \left[\frac{x}{3} + x - 3\right] dx + \int_{3}^{4} [4 - x]\,dx$$

$$= \frac{1}{12} + \frac{1}{24} + \frac{3}{8} + \frac{1}{2} = 1$$

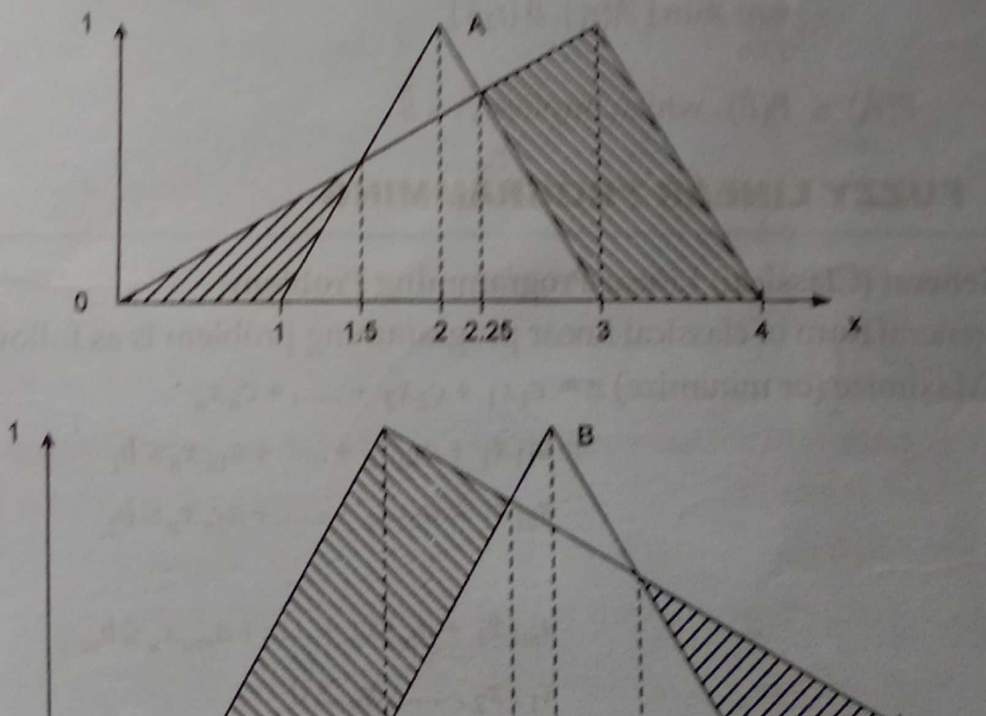


Fig. 2

$$d(\max(\tilde{A},\tilde{B}),\tilde{B}) = \int_0^{1.5} (x/3)\,dx - \int_1^{1.5} (x-1)\,dx$$

$$= \frac{3}{8} - \frac{1}{8} = 0.25$$

i.e., $d(\max(\tilde{A},\tilde{B}),\tilde{A}) > d(\max(\tilde{A},\tilde{B}),\tilde{B})$

$\Rightarrow \quad \tilde{A} < \tilde{B}$

i.e., we can find here by the first ranking method $\tilde{A} \le \tilde{B}$.

For second method, we see from figures

$\tilde{A} \le \tilde{B}$ for $\alpha \in [0,1]$

${}^{\alpha}\tilde{A} = [1.5, 2.25]$

${}^{\alpha}\tilde{B} = [2.75, 3.5]$

Here, $3.5 > 2.25$, i.e., $\tilde{A} \le \tilde{B}$

According to third method, we can construct the priority fuzzy set P on $(\tilde{A},\tilde{B})$ as follows

$$P(\tilde{A}) = \sup_{r_1 \ge r_2} \min[\tilde{A}(r_1), \tilde{B}(r_2)] = 0.75$$

$$P(B) = \sup_{r_2 \ge r_4} \min[\tilde{A}(r_1), \tilde{B}(r_2)]$$

$$= \sup_{r_2 \ge r_1} \min[\tilde{A}(r_1), \tilde{B}(r_2)]$$

i.e., $P(\tilde{A}) \le P(\tilde{B})$, which implies $\tilde{A} \le \tilde{B}$.

9.6 FUZZY LINEAR PROGRAMMING

(1) General (Classical) Linear Programming Problem

The general form of classical linear programming problem is as follows :

Maximize (or minimize) $z = c_1x_1 + c_2x_2 + \ldots\ldots + c_nx_n$

$$a_{11}x_1 + a_{12}x_2 + \ldots\ldots + a_{1n}x_n \le b_1$$
$$a_{21}x_1 + a_{22}x_2 + \ldots\ldots + a_{2n}x_n \le b_2$$
$$\ldots \quad \ldots \quad \ldots \quad \ldots$$
$$a_{m1}x_1 + a_{m2}x_2 + \ldots\ldots + a_{mn}x_n \le b_m$$
$$x_1, x_2, \ldots\ldots, x_n \ge 0$$

Here, the function z is known as objective function. The numbers C_i's are called cost coefficients. The matrix $A = [a_{ij}]$, where $i \in N$ and $j \in N$ is called a constraints matrix and the vector $b = (b_1, b_2, \ldots\ldots, b_m)$ is called a right hand side vector. Also, $x_1, x_2, \ldots, x_n$ all denote the non-negative restrictions. The set of vectors x that satisfy all the given constraints is called a feasible set.

(2) Fuzzy Linear Programming Problem

The general type of fuzzy linear programming is given as follows

$$\max z = \sum_{j=1}^{n} \tilde{C}_j X_j \text{ such that } \sum_{j=1}^{n} \tilde{A}_{ij} x_j \leq \tilde{B}_i,\ i \in N$$

$$x_j \geq 0,\ j \in N\ \lambda, x_j \geq 0$$

where $\tilde{A}_{ij}, \tilde{B}_i, \tilde{C}_j$ are fuzzy numbers and X_j are variables whose states are fuzzy numbers, the operations of addition and multiplication are operations of fuzzy arithmetic and $\leq$ denotes the ordering of fuzzy numbers.

9.7 SPECIAL CASES OF FUZZY LINEAR PROGRAMMING

(1) Symmetric Fuzzy Linear Programming

In this model, we shall assume that the decision makers can establish an aspiration level z for the value of the objective function he wants to achieve. In this case the fuzzy linear programming becomes

Find x such that

$$C^T x \geq z$$

$$A x \leq b,\ x \geq 0 \qquad \text{------(1)}$$

This problem can also be written as

$$Bx \leq d,\ x \geq 0 \qquad \text{------(2)}$$

where, $\begin{pmatrix} -C \\ A \end{pmatrix} = B$ and $\begin{pmatrix} -z \\ b \end{pmatrix} = d$

Each of the (m + 1) rows of (2) shall be represented by a fuzzy set, the membership functions of which are $\mu_i(x)$. The membership function is then defined as follows

$$\mu_{\tilde{D}}(x) = \min\{\mu_i(x)\} \qquad \text{......(3)}$$

$\mu_i(x)$ can be interpreted as the degree to which x satisfy the fuzzy inequality $B_i\, x \leq d$ (where, B_i is the i[th] row of B).

The membership function $\mu_i(x)$ should be zero if the constraints (including objective function) are strongly violated and 1 if they are well satisfied, and $\mu_i(x)$ should increase monotonically from 0 to 1, i.e.,

$$\mu_i(x) = \begin{cases} 1, & \text{if } B_i\, x \leq d \\ \in [0,1], & \text{if } d_i < B_i x \leq d_i + p_i \\ 0, & \text{if } B_i\, x > d_i + p_i \end{cases} \qquad \text{......(4)}$$

Using the simplest type of membership function, we assume them to be linearly increasing over the 'tolerance interval' p_i.

$$\mu_i(x) = \begin{cases} 1 & , \text{ if } B_i x \le d \\ 1 - \frac{B_i x - d_i}{p_i} & , \text{ if } d_i < B_i x \le d_i + p_i \\ 0 & , \text{ if } B_i x > d_i + p_i \end{cases} \quad \text{------(5)}$$

Putting this value in

$$\max_{x \ge 0} \min_i \{\mu_i(x)\} = \max_{x \ge 0} \mu_D(x)$$

and after some arrangement and with some additional assumption

$$\max_{x \ge 0} \min_i \left(1 - \frac{B_i x - d_i}{\mu_i}\right)$$

We get $\max \lambda$ such that $\lambda p_i + B_i x \le d_i + p_i$, $\quad i = 1, 2, \ldots m \neq 1$ ------(6)

$$x \ge 0$$

If the optimal solution to (6) is the vector (λ, x_o), then x_o is the maximizing solution of the given model with membership function given by (5).

General Form : $\lambda[z_u - z_l] - cx \le -z_t$

$$\lambda p_i + \sum_{j=1}^{n} a_j x_j \le d_i + p_i$$

$$\lambda, x_j \ge 0$$

SOLVED EXAMPLES

Example 1. *Solve the following fuzzy linear programming problem*

$$Max z = 0.5x_1 + 0.2x_2$$

such that

$$x_1 + x_2 \le B_1$$

$$2x_1 + x_2 \le B_2 \qquad x_1, x_2 \ge 0$$

where, $$B_1(x) = \begin{cases} 1 & ; \text{ for } x \le 300 \\ \frac{400 - x}{100} & ; \text{ for } 300 < x \le 400 \\ 0 & ; \text{ for } x > 400 \end{cases}$$

and $$B_2(x) = \begin{cases} 1 & ; \text{ for } x \le 400 \\ \frac{500 - x}{100} & ; \text{ for } 400 < x \le 500 \\ 0 & ; \text{ for } x > 500 \end{cases}$$

[Meerut-2005, 2008]

Solution. We have to find the lower and upper bound of the objective function. The classical L.P.P. corresponding to the fuzzy LPP is given by

$$\max z_l = .5x_1 + .2x_2$$

such that

$$x_1 + x_2 \le 300$$
$$2x_1 + x_2 \le 400$$

Introducing the slack variables s_1 and s_2, we can write

$$\max z_l = .5x_1 + .2x_2 + 0s_1 + 0s_2$$

such that

$$x_1 + x_2 + s_1 = 300$$
$$2x_1 + x_2 + s_2 = 400 \qquad x_1, x_2, s_1, s_2 \ge 0$$

First simplex table

B.V.	C_B	X_B	x_1	x_2	s_1	s_2	Min. Ratio
s_1	0	300	1	1	1	0	300
s_2	0	400	[2]	1	0	1	200 ←
			−.5	−.2	0	0	
s_1	0	100	0	½	1	-½	
x_1	.5	200	1	½	0	½	
	z = 100		0	.05	0	.25	

Thus, the solution is given by

$$x_1 = 200,\ x_2 = 0 \text{ and } z_l = 100$$

Also, from the given equation, we can write the second simplex problem

$$\max z_u = .5x_1 + .2x_2 + 0s_1 + 0s_2$$

such that

$$x + x_2 + s_1 = 400$$
$$2x_1 + x_2 + s_2 = 500 \qquad x_1, x_2, s_1, s_2 \ge 0$$

which can be solved by using following simplex table

B.V.	C_B	X_B	x_1	x_2	s_1	s_2	Min. Ratio
s_1	0	400	1	1	1	0	400
s_2	0	500	[2]	1	0	1	250 ←
		s_j	−.5	−.2	0	0	
s_1	0	150	0	½	1	-½	
x_1	.5	250	1	½	0	½	
			0	0.5	0	.25	