1/09/14

UNIT - IV

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RENEWAL PROCESS & THEORY.

DEFENITION:-

Consider a sequence of frequent trials with possible outcome E_j, $j = 1, 2, \ldots$ The trail of need may not be independent & assume that the trials can be repeated infinitely.

Suppose that a certain outcome in a cetain trial (or) patern of outcomes in a no. of trial. we denoted this event by E^* whenever E^* occurs we say a renewal has occur.

If it occur a n^{th} trial we say that the renewal occur at trial no. 'n' the interval b/w occurences of two successive renewal (two successive occurance of the renewal pattern E^*). is called the renewal period of the process

Denote, $F_n = Pr\left\{ \begin{matrix} E^* \text{ occurs for the } 1^{st} \\ \text{time at the } n^{th} \text{ trial} \end{matrix} \right]$

$p_n = Pr\{E^*$ occurs at the n^{th} trial. not necessarily for the 1^{st} time$\}$.

Define $f_0 = 0$, $P_0 = 1$ and also probability generating function is.

$$F(s) = \sum_{n=0}^{\infty} f_n s^n.$$

$$\therefore \quad P(s) = \sum_{n=0}^{\infty} P_n s^n$$

Now,

$f^* = \sum f_n$ is the probability that renewal E^* occurs at some trial in a converging sequence of trials.

$$\therefore \quad f^* \leq 1$$

when $f^* = 1$ the renewal is persistent and transistent when $f^* < 1$.

Eg:-

Consider a die throwing experiment the even E^* that corresponds to the trail that six occurs in the renewal event

if the die is pair the E^* occurs in a trial with probability is $1/6$ & the probability of non-occurance at a trial is $5/6$.

$$f_n = (5/6)^{n-1} \cdot 1/6 \; ; \quad n \geq 1 \;\; \& \;\; \sum_{n=1}^{\infty} f_n = 1.$$

The renewal event is persistent and the renewal period has geometric distribution.

NOTE:-

Let T_i denotes the i^{th} renewal period then $S_n = T_1 + T_2 + \dots + T_n$

if N_m denote the number of renewal in a total number of m trial

$N_m > 1 \iff S_n < m$.

$\boxed{\text{RELATIONSHIP BETWEEN } F(s) \; \& \; P(s) :-}$

eq. 14.

(a)

If the event E^* occurs at the n^{th} trial may be compound event $\ni$: E^* occurs for the 1^{st} time at the r^{th} trial $(r < n)$ and again the trial number n (i.e) In the subsequent $(n-r)$ trial,

Thus,

$$p_n = \sum_{r=1}^{n} f_r \, p_{n-r} \;, \quad r < n \;\; \& \; n \geq 1$$

The R.H.S is convolution relation

$\{f_n^*\}\ \{p_n\}$ b/w two sequence.

multiply by s^n, $s = 1, 2, \ldots$

$\therefore p_n s^n = f_1 p_{n-1} s^n + f_2 p_{n-2} s^n + \ldots + f_{n-1} p_1 s^n + \ldots f_n p_0 s^n$

$= f_1 s p_{n-1} s^{n-1} + f_2 s^2 p_{n-2} s^{n-2} + \ldots + f_{n-1} s^{n-1} p s + f_n s^n p_0 s^0.$

$\therefore \sum_{n=0}^{\infty} p_n s^n = f_1 s \sum_{n=1}^{\infty} p_{n-1} s^{n-1} + f_2 s^2 \sum_{n=2}^{\infty} p_{n-2} s^{n-2} + \ldots + f_{n-1} s^{n-1} \sum_{n=n-1}^{\infty} p \cdot s + f_n s^n \sum p_0 s^0$

$= f_1 s P(s) + P(s) f_2 s^2 + \ldots + f_{n-1} s^n P(s) + f_n s^n P(s)$

$= P(s)\left[f_1 s + f_2 s^2 + \ldots + f_n s^n\right]$

$\Rightarrow P(s) - 1 = P(s) F(s).$

$P(s) - P(s) F(s) = 1$

$P(s)\left[1 - F(s)\right] = 1$

$P(s) = \frac{1}{1 - F(s)}$

$F(s) = \frac{P(s) - 1}{P(s)} \longrightarrow (1)$

from (1) follows that $\sum p_n = \frac{1}{1 - F(1)}$

which is convergent $\Leftrightarrow F(1) < 1.$

(ie) E^* is transistent.

In otherword E^* is trasistent $\Leftrightarrow$ $\sum p_n = P(1)$ is finite.

$$F^* = F(1) = \frac{P(1)-1}{P(1)} = \frac{\sum p_n - 1}{\sum p_n} = \frac{\sum (p_n - 1)}{p_n}$$

E^* persistent $\Leftrightarrow$ $\sum p_n$ is divergent.

Def:- Periodic ; Aperiodic..

The renewal extent E^* is said to be periodic then $\exists$ an interval $m(>1) \ni$ E^* can occurs only at the trials number $m, 2m, \ldots$ the greatest m with this property is said to be period of E^*

if it is said to be aperiodic if no such m exists.

The seqn. $\{a_n\}$ is said to be periodic with period $m(>1)$

If $a_n = 0 \ \forall \ a \neq k, \ m, k = 1, 2 \ldots$ if m is greatest integer with this property ..

DEF:- [MEAN RECCURANCE TIME].

for the persistent and a periodic Renewal even $F'(1) = \sum n f^n = E(T)$ is mean reccurance time. This $F'(1)$ may be finite (or) infinite.

RENEWAL INTERVAL:-

the renewal interval (period) T has the probability mass function.

$$Pr\{T = n\} = f_n.$$

T is a proper random variable when $\sum f_n = f(1) = 1$.

$$\sum n f_n = F'(1)$$

T is also called the waiting time for the renewal occurance of the E^*

the generating function of T is.

$$F(s) = = \sum f_n s^n.$$

The probability of $f_n(2)$ that E^* occurs for the 2^{nd} time at the n^{th} trial is

$$f_n^{(2)} = \sum_{k=1}^{n-1} f_k \, f_{n-k}^{(2)}$$

The probability $f_n^{(r)}$ that E^* occurs for the r^{th} time at the n^{th} trial is

$$f_n^{(r)} = \sum_{k=1}^{n-1} f_k \, f_{n-k}^{(r)}$$

Thus $\{ f_n^{(r)} \}$ gives the probability distribution of $T^{(r)} = T_1 + T_2 + \cdots + T_r$.

where T_i are the dependently and identically distributed random variable.

The generation fun. of $f_n^{(r)}$ is

$$F^{(r)}(s) = \sum f_n^{(r)} s^n$$

$$= [F(s)]^r$$

Put $s = 1$,

$$F^{(r)}(1) = \sum f_n^{(r)} = [F(1)]^r = (f^*)^r$$

$\sum f_n^{(r)}$ is the probability that E^* occurs atleast 'r' times if the process continued indefinitely.

$Pr\{E^*$ occurs exactly r times with process continued indefinitely $\}$.

$$= (f^*)^r - (f^*)^{r+1}$$

$$= (f^*)^r \left[1 - f^*\right]^1$$

$\boxed{\text{SECTION} - 6.4}$

GENERALIZED FORM:-

DELAYED RECCURENT EVENT:-

Assume that the renewal upto the 1^{st} occunance of the renewal event E^* has a same distribution $\{f_n\}$ ∴ the reference interval b/w sucessive diff/. of E^*

It is known as realistic. Assume that the reccunance interval be an upto the occunences E^*

The 1^{st} occunences E^* is called delayed Reccurent ~~occunence~~ event while the subsequent occunence are ordinary reccurent event

Let us denote $v_n = pr\{E^* \text{ occurs at the } n^{th} \text{ trial}\}$.

∴ suppose that the first occunences of E^* happened at the trial number with probability b_k and the renewal occunences or E^* occurs at the subsequent $P(n-k)$ trial with probability then.

$$v_n = b_n + b_{n-1} p_1 + b_{n-2} p_2 + \cdots + p_n b_0.$$

$$v_n = \{b_n\} * \{p_n\}.$$

Denoting, $V(s) = \sum v_n s^n$

$$B(s) = \sum v_n s^n$$

$$P(s) = \sum p_n s^n$$

$$r(s) = B(s)\,P(s)$$

$$r(s) = B(s)\,\frac{1}{1-F(s)} \qquad \left\{ \because P(s) = \frac{1}{1-F(s)} \right.$$

$\boxed{\text{THEOREM : 6.1}}$ $p_n \to \alpha$ If p_n Converges to α then v_n $v_n \to \alpha b$ ~~α to~~ αb converges where $b = \sum_n b_n = \beta(1)$ If $\sum p_n$ converges to β then $\sum v_n$ converges $\sum v_n \to \beta$ to ~~βb~~. βb.

Proof :-

Denote $r_k = Pr\{1^{st}$ Renewal period $> k\}$.

$$= b_{k+1} + b_{k+2} + \cdots$$

we can choose k sufficiently large $\ni$:

$r_k < \epsilon$ & $p_m \leq 1$.

$V_n = b_n + b_{n-1} P_1 + b_{n-2} P_2 + \cdots P_n b_n.$

$P_n b_0 + P_{n-1} b_1 + P_{n-2} b_2 + \cdots + P_{n-k} b_{k} \leq V_n \leq b(k+1)$

$b_0 P_n + b_1 P_{n-1} + \cdots + P_{n-k} b_k + \{ P_{n-(k+1)} \cdots + b_{n-1} P_1 \cdots \} + P_0 b_0$

$\leq b_0 P_n + P_{n-1} b_1 + \cdots + P_{n-k} b_k + \{ b_{k+1} + b_{k+2} + \cdots + b_n \}$

$= b_0 P_n + P_{n-1} b_1 + \cdots + P_{n-k} b_k + r_k.$

As $P_n \to \alpha$ then,

$P_n b_0 + P_{n-1} b_1 + \cdots + P_{n-k} b_k = \alpha \left[b_0 + b_1 + \cdots + b_k \right].$

$= \alpha (b - r_k)$

$= \alpha b - \alpha r_k.$

$= b\alpha - 2\epsilon \to (1).$

Since $P_n \to \alpha$, $\alpha \leq 1 \leq 2$, $-\alpha \geq -2\epsilon$

$\therefore r_k < \epsilon.$

Now,

$\therefore V_n \leq b_0 P_n + b_1 P_{n-1} + \cdots b_k P_{n-k} + r_k.$

$\leq (b - r_k)\alpha + r_k.$

$\leq b\alpha - \alpha r_k + r_k.$

$\leq b\alpha + r_k (1-\alpha)$

$< b\alpha + 2\epsilon \to (2).$

Choose $1-\alpha = 2.$

from (1) & (2) $b\alpha - 2\epsilon \leq V_n \leq b\alpha + 2\epsilon.$

$\because \epsilon$ is arbitrary.

$\lim_{n \to \infty} v_n = b\alpha.$

$V(s) = B(s)\, P(s)$

$V(1) = B(1)\, P(1)$

$\sum v_n = \sum b_n \sum P_n.$

Since $\sum P_n \to \beta.$

$\therefore \sum v_n \Rightarrow \beta b.$

Corollary:-

If F^* is persistent then v_n converges to b/μ that implies F^* is persistent then.

$P_n \to \mu A.$

SECTION - 6.5

RENEWAL THEORY IN DISCRETE TIME:-

Suppose that $\{f_n, n \geq 1\}$ & $\{b_n, n \geq 0\}$ are two sequence of real number $\ni$: $f_n \geq 0$

$$f = \sum f_n < \infty \to (1)$$

$$b_n \geq 0, \quad b = \sum b_n < \infty \to (2).$$

Proof:-

Define a new seqn/. $\{v_n\}$, $n = 1, 2, 3$

.... By the Convolution relation

$$v_n = b_n + v_{n-1} f_1 + v_{n-2} f_2 + \cdots + v_0 f_n.$$

$$= b_n + \sum_{r=1}^{n} v_{n-r} f_r .$$

v_n uniquely defined interms of v_n & f_n

and their probability function is.

$$V(s) = \sum v_n s^n \ ; \ B(s) = \sum b_n s^n$$

$$F(s) = \sum f_n s^n$$

$$\therefore V(s) = B(s) + V(s) F(s)$$

$$\therefore V(s) - V(s) F(s) = B(s).$$

$$V(s) [1 - F(s)] = B(s)$$

$$V(s) = \frac{B(s)}{1 - F(s)}.$$

$$\therefore \quad F(s) = \frac{V(s) - B(s)}{V(s)}.$$

Here $F(s)$ & $V(s)$ converge atleast for $0 \le s < 1$

if $F(s) \le 1$, $V(s)$ is power series in s.

Def:-

the seq $\{f_n\}$ is periodic if $\exists$ an integer $m \ni:$ $f_n = 0$ Except the pt $n = km$.

THEOREM : 6.2 [RENEWAL THM].

Suppose that the relation $b_n \ge 0$

$b_n = \sum b_n < \infty$ hold and seqn $\{f_n\}$ is not periodic.

10M

a) if $f < 1$ then $v_n \to 0$.

$\sum v_n = b/_{1-f}$.

b) If $f = 1$, then v_n converges b/μ

Proof: -

W.K.T $\quad V(s) = \dfrac{B(s)}{1-F(s)}$.

Put $s = 1$, $\quad V(1) = \dfrac{B(1)}{1-F(1)}$.

$$\sum v_n = \frac{\sum b_n}{1-\sum f_n} = \frac{b}{1-f}.$$

If $f < 1$, $\sum v_n$ converges and $v_n \to 0$.

(b) If $f = 1$, $\sum v_n$ converges. to ∞.

But by the above. corrolary $v_n \to b/\mu$.

(ie) E^* is persistent when $v_n \to b/\mu$.

RENEWAL PROCESS IN CONTINOUS TIME:-

Let seqn. $\{x_n\}$ $n = 1, 2, \ldots$ be

a. sequence of non-negative random variable

Assume that $Pr\{x_n = 0\} < 1$ and

the random vatables are identically distributed

and are continous with distributed f_n

$F(\cdot)$.

Since $\{x_n\}$ is non-negative then $E\{x_n\}$ is exists and is denoted by.

$$\therefore E(X_n) = \int_0^{\infty} x \, d\, F(x) = \mu.$$

where μb be infinite whenever $\mu = \infty$, $1/\mu$ is zero.

Let $S_0 = 0$, $S_n = S_1, S_2, \ldots S_n$.

$F_n(x) = pr(S_n \leq x)$ be the distribution of S_n, $n \geq 1$.

$$F_0(x) = 1 \text{ if } x \geq 0$$
$$= 0 \text{ if } x < 0.$$

SECTION - 6.2.1

13.03.19.

RENEWAL FUNCTION & RENEWAL DENSITY:-

The function $M(t) = E(N(t))$ is called the renewal fn of the process with distribution F. $\{N(t) \geq n\} \Leftrightarrow \{S_n \leq t\}$ (or) $\{N(t) < n\} \Leftrightarrow \{S_n > t\}$.

5m

THEOREM: 6.3.

U.Q 5M

(i)(ii) The distribution of $N(t)$ is given by $p_n(t) = pr\{N(t) = n\} = F_n(t) - F_{n+1}(t) \rightarrow (1)$.

and the expected number of renewal is.

$$m(t) = \sum_{n=1}^{\infty} F_n(t) \rightarrow (2).$$

Proof:-

$$P_n(t) = Pr\{N(t) = n\}$$

$$= Pr\{N(t) \geq n\} - Pr\{N(t) \geq n+1\}$$

$$= Pr\{S_n(x) \leq t\} - Pr\{S_{n+1}(x) \leq t\}$$

$$P_n(t) = F_n(t) - F_{n+1}(t) \longrightarrow (1).$$

Also,

$$m(t) = \sum_{n=0}^{\infty} n P_n(t) = \sum_{n=0}^{\infty} n [F_n(t) - F_{n+1}(t)]$$

$$= F_1(t) - F_2(t) + 2F_2(t) - 2F_3(t) + \cdots$$

$$m(t) = \sum_{n=1}^{\infty} F_n(t) \longrightarrow (2).$$

$$= \sum_{n=1}^{\infty} Pr\{S_n \leq t\}.$$

the relation (2) can be putting term of linear transformation as follows.

Let $F'(x) = f(x)$ be the density function of X_n & $g^*(s)$ denote the laplace transform of the fn $g(t)$ than taking linear-ation of both sides of eqn (2) have transformation

$$\mu(t) = \sum_{n=1}^{\infty} 1 \cdot F_n(t)$$

$$\mu^*(s) = 1/s \sum_{n=1}^{\infty} F_n^*(s).$$

$$= 1/s \sum_{n=1}^{\infty} [f^*(s)]^n$$

$$= 1/s [f^*(s) + f^*(s)^2 + f^*(s)^3 + \cdots].$$

$$= 1/s \left[\frac{f^*(s)}{1-f^*(s)}\right]$$

$$\mu^*(s) = \frac{f^*(s)}{s(1-f^*(s))}$$

$$s\mu^*(s) = \frac{f^*(s)}{1-f^*(s)}$$

$$f^*(s) = sm^*(s) - sm^*(s) f^*(s)$$

$$f^*(s) + sm^*(s) f^*(s) = sm^*(s)$$

$$f^*(s) = \frac{sm^*(s)}{1+sm^*(s)}$$

These L.T $m(t)$ and $F(x)$ can be determine uniquely one from the other.

RENEWAL DENSITY :-

The derivative $m(t)$ of $\mu(t)$ be $\mu'(t) = m(t)$ is called the renewal density.

$$m(t) = \lim_{\Delta t \to 0} \frac{Pr\{1 \text{ or more renewals in } t, t+\Delta t\}}{\Delta t}$$

$$= \sum_{n=1}^{\infty} \lim_{\Delta t \to 0} \frac{Pr\{n^{th} \text{ renewal occurance in } (t+\Delta t)\}}{\Delta t}$$

$$= \sum_{n=1}^{\infty} \lim_{\Delta t \to 0} \frac{f_n(t)\, \Delta t + o(\Delta t)}{\Delta t}$$

$$= \sum_{n=1}^{\infty} f_n(t) = \sum_{n=1}^{\infty} F_n'(t).$$

$m(t) = M'(t).$

Assuming that $F(x)$ is absolutely and $F_n'(t) = f(t)$.

The fn $m(t)$ specifies the mean number of renewals to be expected in an interval near t.

SECTION 6.15 (CONTINUE).

~~Def~~ [RENEWAL PROCESS].

Define the random variable $N(t) = \sup\{n : S_n \le t\}$ the process is $\{N(t), t \ge 0\}$ is called renewal process with distribution F. U.Q 2M

if for some n, $S_n = t$ then the renewal is said to be occurs at time t

$\therefore$ S_n gives the time of the n^{th} renewal and is called the n^{th} renewal time.

The renewals where x's are independently and identically distributed random variables are called palm flow events.

where x_i are independently and identi-cally distributed Exponential random variables then the renewal are called ordinary (or) poisson flow of event

Eg-1 :-

one of the simplest, Example of a renewal process is provided by life time distribution of a component such a and electric bulb either works our completely Suppose that the deduction of failure of a bulb and its replacement of new bulb instantaneously and suppose that life time of bulb are with distribution fn/.

$\therefore$ we have the renewal process with distribution.

Eg:2

Consider a stage in an industrial process relating to production of certain components in batches. Immidiately on a completion of Production of a batch that of another batches under taken. Suppose that the time taken to produce successive batches are identically & independently distributed random variables with distribution F. (i.e) renewal process with distribution F.

SECTION - 6.3.

RENEWAL EQUATION:-

An integral eqn can be obtained for the renewal function $M(t) = E\{N(t)\}$ with gives the expected number of renewals in $[0,t]$.

THEOREM : 6.4 [RENEWAL EQUATION (or) INTEGRAL eqn].

The renewal fn M satisfies the eqn $M(t) = F(t) + \int_0^t M(t-x)\, dF(x)$

PROOF:-

By londitionary on the duration of the 1^{st} renewal x.

we get $M(t) = E\{N(t)\}$.

$$= \int_0^\infty E\{N(t) \mid x_1 = x\}\, dF(x) \quad \text{(1)}.$$

Consider $x > t$, Given that $X_1 = x > t$.

no renewal occurs in by the interval $[0,t)$

So that $E\{N(t) \mid X_1 = x\} = 0$.

If $x \le t$ given that the 1st renewal occurs at x then the process starts again at each x and the expected no of renewal in the remaining intervals of length $(t-x)$ is.

$$E\{N(t-x)\} = 1 + M(t-x).$$

Thus considering the above Eqn and Sub in (1).

we get,

$$\mu(t) = \int_0^t \{1 + \mu(t-x)\} \, dF(x)$$

$$= \int_0^t dF(x) + \int_0^t \mu(t-x) \, dF(x).$$

$$M(t) = F(t) + \int_0^t \mu(t-x) \, dF(x) \rightarrow (2).$$

which is called the inequal Eqn of renewal theory and the argument used to derive it is known as renewal argument. The renewal Eqn is also expressed as

$$M = F + m * f.$$

The renewal Equation can also be established as given bellow

$$M(t) = \sum_{n=1}^{\infty} F_n(t)$$

$$M(t) = F_1(t) + \sum_{n=1}^{\infty} \left\{ \int_0^t F_n(t-x) \, dF(x) \right\}$$

$\because$ by (1).

F_{n+1} being the convolution s of F_n and $F_1 = F$. Thus assuming the validity the change of order of Integration and summation ∴

$$M(t) = F(t) + \int_0^t \left[\sum_{n=1}^{\infty} F_n(t-x) \right] dF(x).$$

(or)

$$M(t) = F(t) + \int_0^t M(t-x)\, dF(x).$$

∴ The renewal equation,

$$M(t) = F(t) + \int_0^t M(t-x)\, dF(x)$$ can be

generalized as follows.

$$v(t) = g(t) + \int_0^t v(t-x)\, dF(x) \quad t \geq 0 \quad \text{③}$$

where g & F are known and v is unknown

The Equation ③ is called a renewal type of eqn.

A unique soln of $v(t)$ exist in terms of g & F as can be seen from the following.

THEOREM: - 6-5.

If $v(t) = g(t) + \int_0^t v(t-x)\, dF(x)$,

then $v(t) = g(t) + \int_0^t g(t-x)\, dm(x)$ where

$m(t) = \sum_{n=1}^{\infty} F_n(t)$.

Proof:-

Taking laplace transformation of ①

we get,

$$v^*(s) = g^*(s) + v^*(s) F^*(s).$$

$$v^*(s) - v^*(s) F^*(s) = g^*(s)$$

$$v^*(s)(1 - F^*(s)) = g^*(s).$$

$$\therefore \quad v^*(s) = \frac{g^*(s)}{1 - F^*(s)}.$$

$$= g^*(s)\left\{1 + \frac{F^*(s)}{1 - F^*(s)}\right\}$$

$$= g^*(s)(1 + sm^*(s)).$$

$$v^*(s) = g^*(s) + g^*(s)\, sm^*(s).$$

Inverse the laplace transformations,

we get.

$$L^*(v^*(s)) = L^*(g^*(s)) + L^*(g^*(s)\, sm^*(s))$$

$$v(t) = g(t) + \int_0^t g(t-x)\, dm(x).$$

and the soln $v(t)$ is unique. since fun is uniquely determined by its laplace transformation.

2M.

6.4. Stopping time : Wald's EQUATION:-

6.4.1 STOPPING TIME:-

Before going to R.T we discuss a special type of non-negative random variable associated with a seqn of random variable X (or a S.T {X(b)}) such a variable st considered by Wald (1947) while formulating sequential analytic is known as random variable independence of future (or) a markov time (or) a S.T for ∴ an elabrated discussion on markov time and its application sec A.N. Siryeev (1973).

Def:- An Integral valued random variable

consider a coin tossing experiment. Let the outcome of ith toss be denoted by $x_n = 1$ (or) 0 depending on the result being Head (or) tail respectively and

Let $pr\{X_1 = 1\} = p = 1 - pr\{X_1 = 0\}$

and then $E(X_1) = p$.

B. Pea[illegible]...

The sum $S_n = X_1 + \cdots + X_n$ denotes the commmunicate no! of heads in the 1st n tosses. Suppose that m is a given integers then

$\therefore E(S_n) - m$ & $E(N) = m/p$.

Consider the number $N(t)$ of renewals by time t w.r.to a sequence of inter arrival time $\{v_i\}$

Now $N(t) = m$ depends not only on $X_1, \ldots, X_n$ but also on X_{n+1}

Consider the variable $N(t)+1$

Now, $N(t)+1 = n$ implies then $S_{n-1} \leq t$ and $S_n > t$. so that the event $\{N(t)+1 = n\}$ is independent of $X_{n+1}, X_{n+2}, \ldots$

Thus $N(t)+1$ is a stopping time for the $\{X_p\}$ while $N(t)$ is not.

If where X_i is independent identic random variables and N is a random variable (Independent of X_i's) having finite expectation then.

$$E\left\{\sum_{i=1}^{n} x_i\right\} = E(x_i)\,E(N)$$

The same result holds also when

N is stopping time for the seq $\{x_i\}$

WALD's EQUATION:- Let $\{x_i\}$ be a seqn of, independent random variable having the same expectation and let N be a stopping time $\{x_i\}$ and $E(N) < \infty$ then

$$E\left\{\sum_{i=1}^{N} x_i\right\} = E(x_i)\,E(N) \rightarrow (1).$$

Proof:-

$$\text{Let } z_i = \begin{cases} 1 & \text{if } N \geq i \text{ dependence.} \\ 0 & \text{if } N < i \text{ independent} \end{cases}$$

So that

$$\sum_{i=1}^{N} x_i = \sum_{i=1}^{\infty} x_i z_i \quad \text{and that}$$

$$E\left\{\sum_{i=1}^{N} x_i\right\} = E\left\{\sum_{i=1}^{\infty} x_i z_i\right\}.$$

$$= \sum_{i=1}^{\infty} E\{x_i z_i\}$$

Assuming the validity of the change of order of expectation & the summation

Now,

z_i is determined by $\{N < i\}$.

(i.e) By $x_1, x_2, \ldots x_{i-1}$ and is independent of x.

Thus,

$$E\left\{\sum_{i=1}^{N} x_i\right\} = \sum_{i=1}^{\infty} E\{x_i\} E\{z_i\}.$$

$$= E\{x_i\} \sum_{i=1}^{\infty} E\{z_i\}.$$

$$= E\{x_i\} \sum_{i=1}^{\infty} Pr\{N \geq i\}$$

$$\therefore E\left\{\sum_{i=1}^{N} x_i\right\} = E\{x_i\} E\{N\}.$$

Corollary :-

Since $N(t)+1$, is a stopping time for the seqn $\{x_i\}$ we have,

$$E\{S_{N(t)+1}\} = E\left\{\sum_{i=1}^{N(t)+1} x_i\right\}.$$

$$= E\{x_i\} E\{N(t)+1\}.$$

Remark :-

1. For wald's eqn to hold the ran variable x_i's need not be identically distribution but x_i's must be independent and have the same mean

(i.e) $E(x_i) = E(x) \ \forall i$

2) if N is independent of $\{x_i\}$

then

$$E\left\{ \sum_{i=1}^{N} x_i(N) \right\} = N \cdot E\{x_i\} \text{ and.}$$

$$E\{x_i\} = E\left[E\left\{ \sum_{i=1}^{N} x_i / N \right\} \right].$$

$$= E(N)\, E(x_i).$$

6.5 RENEWAL THEOREM..

Poisson process is the renewal process having Exponential inter arrival time x_n, we have,

$$M(t) = at \quad \& \quad \frac{M(t)}{t} = a = \frac{1}{E(x_n)}$$

In general case.

The results $\frac{M(t)}{t}$ converges to $1/\mu$.

where $\mu = E(x_n) \leq \infty$. holds as $t \to \infty$.

This result is known as elementary renewal thm..

THEOREM: 6.6.

with probability, $\frac{N(t)}{t} \to 1/N$ as $t \to \infty \to (1)$ where $\mu = E(x_n) \leq \infty$.

Proof:- consider an Interval $[0, t]$.

$S_{N(t)} \le t < S_{N(t)+1} \to$ (2).

Now the strong law of large number holds for $\{S_n\}$ so that as $n \to \infty$.

$$\frac{S_n}{n} = \frac{x_1 + x_2 + \dots + x_n}{n} \to E(x_n) = \mu.$$

with probability.

Again as $t \to \infty$, $N(t) \to \infty$ with proba-bility $\frac{S_{N(t)}}{N(t)}$ converges to μ as $t \to \infty$ $\to$ (3).

Thus,

$$E\{\cancel{\sum_{i=1}^{N}} x_i\} = \cancel{\sum_{i=1}^{\infty}} E(x_i) \cancel{E}\{\cancel{z_i}\}.$$

$$\ne \cancel{E}\{\cancel{x_i}$$

Similarly with prob

$$\frac{S_{N(t)+1}}{N(t)} + \frac{S_{N(t)+1}}{N(t)+1} \cdot \frac{N(t)+1}{N(t)} \to \mu \text{ as } t \to \infty \to (4).$$

Thus from the three relations, we get that with probability $\frac{N(t)}{t} \to 1/\mu$. as $t \to \infty$

SECTION - 6.5.1

U.Q. 10M

ELEMENTARY RENEWAL THEOREM:-

Elementary renewal thm we have $\frac{M(t)}{t} \to 1/\mu$ as $t \to \infty$ where

$\mu = E(x_n) < \infty$ the limit being interupted as 0 when $\mu = \infty$.

proof:-

Let $\left.\begin{array}{l} N = N(t) + 1 \\ S_N = t + Y(t) \end{array}\right\} \longrightarrow (a)$.

where $Y(t)$ is the residual life time of the unit in use at time t.

Let $\{X_i^{(j)}, j = 1, 2, \dots\}$ be the sequence of independent realisations of the renewal process $\{x_i\}$ $\{S_n^{(j)}\}$ be the corresponding partial sums and $N^{(j)}(t)$ the corresponding number of renewals in $[0, t]$.

Let $T = S_N$, $M_K = N^{(1)}t + \dots + N^{(k)}t$.

$T_K = T^{(1)} + \dots + T^{(k)}$

Now,

T_K is the sum of $(K + M_K)$

(ie) random variable X_i.

$\frac{S_n}{n} = [x_1, x_2 \dots$

$\therefore$ Thus by the strong law of large numbers if ~~$\mu = E(X_i) < \infty$~~ then $\mu = E(X_i) < \infty$

at $k \to \infty$.

$$\frac{T_k}{k+M_k} = \frac{\sum_{i=1}^{k+M_k} x_i}{k+M_k} = E(x_n) \to \mu \quad \text{as } k \to \infty \to (1)$$

with probability.

By the same law, we get as

$k \to \infty$

$$\frac{M_k}{k} = \frac{\sum_{j=1}^{k} N^{(j)}(t)}{k} \to E\{N(t)\} = M(t) \to (2)$$

and.

$$\frac{T_k}{k} = \sum_{j=1}^{k} \frac{T^{(j)}}{k} \to E(T) = E(S_N)$$

$$= E\{t + y(t)\}.$$

$$= t + E\{y(t)\} \to (3).$$

with probabil

$\therefore$ Eqn (a) and $T = S_n$ we get.

combining Eqn (1) (2) + (3)

as $k \to \infty$.

$$\frac{T_k}{k+M_k} = T_k\left(\frac{1}{k\left(1+\frac{M_k}{k}\right)}\right)$$

$$\therefore \quad = \frac{T_k}{k}\left(\frac{1}{1+\frac{M_k}{k}}\right)$$

$\because \frac{T_k}{k} = t + E\{y(t)\}$

$\therefore \frac{M_k}{k} = M(t)$

$$\mu = t + E\{y(t)\} \cdot \frac{1}{1+M(t)}$$

$$\therefore \quad \mu = \frac{t+E(y(t))}{1+M(t)} \to (b) \quad \Rightarrow \mu(1+\mu(t)) = t + \cancel{E(y(t))}$$

$$\Rightarrow \mu(1+M(t)) = t + E(y(t)).$$

$$\frac{1+E(y(t))}{t} = \mu\left(\frac{1}{t} + \frac{\mu(t)}{t}\right)$$

and now $y(t)$ is +ve, $E(y(t))$ finite and hence as $t \to \infty$

$$\lim_{t\to\infty} \inf \frac{\mu(t)}{t} = \frac{1}{\mu} \longrightarrow (*)$$

To prove the theorem we have to show that.

$$\lim_{t\to\infty} \sup \frac{M(t)}{t} \le \frac{1}{\mu}.$$

for doing this we define a new renewal process as follows. Let A be a constant >0 and for $n = 1, 2, \ldots$ &

Let
$$X_n^* = \begin{cases} X_n & \text{if } X_n \le A. \\ A & \text{if } X_n > A. \end{cases}$$

Let
$$S_n^* = \sum_{i=1}^{n} X_i^*$$

$$N^*(t) = \sup\{n : S_n^* \le t\}.$$

$$M^*(t) = E\{N^*(t)\}.$$

Now, $S_n^* \le S_n$ & hence $N^*(t) \ge N(t)$ &

$M^*(t) \ge M(t)$

Again

$$E(X_n^*) = \mu_A \le \mu \;\&\; \mu_A \to \mu \text{ as } A \to \infty.$$

$S^*_{N(t)+1} \leq t \leq S^*_{N(t)}$

$\Rightarrow S^*_{N(t)+1} \leq t \leq t+A.$

$\left(S^*_{N(t)+1} \right) \leq t+A$ (from ⓐ where $X(t)=A$

$E\left(S^*_{N(t)+1} \right) = E(t+A)$

$= M_A \left(1 + M^*(t) \right).$

($\because$ from with $y(t)=A$).

& hence.

$$\lim_{t \to \infty} \frac{\sup m^*(t)}{t} \leq \frac{1}{\mu}$$

and as $A \to \infty$, $\mu_A \to \mu$, $M^*(t) \to M(t)$

$$\therefore \lim_{n \to \infty} \sup \frac{M(t)}{t} \leq \frac{1}{\mu} \to (*)$$

from (*) & (**) the result follows.

2M Q.

5.3 DEFINITION:-

A non negative random variable is said to be a lattice variable if it takes an value nd ($n = 0, \pm 1, \pm 2, \ldots$) (i.e) Integral multiples of a non-negative number d.

The largest 'd' is said to the period of the distribution when

∴ a lattice variable become an integer value variable. The distribution is non-lattice.

DEFENITION:-

Let $F(x)$ be the function defined on $[0, \infty]$ for a fixed $h>0$ $n=1,2,\ldots$

Let $\overline{m}_n = \max\{f(x) : (n-1)h \le x \le nh\}$.

$\underline{m_n} = \min\{f(x) : (n-1)h \le x \le nh\}$.

and if $\overline{\sigma} - \underline{\sigma} \to 0$ as $n \to \infty$.

the $f(x)$ is said to be directly Riemann integrable.

6.8 THEOREM.

BLACK WELL'S THEOREM:-

for X_i be non-lattice and for fixed $h \ge 0$, $M(t) - M(t-h) \to h/\mu$. as $t \to \infty$ and for a lattice X_i with period d.

2M Q

$\lim_{n\to\infty} P_r\{\text{Renewal at } nd\}$ converges to d/μ

6.9 THEOREM:-

SMITH'S THEOREM OR KEY-RENEWAL THEOREM:-

Let $H(t)$ be a directly riemann integrable & $H(t)=0$ for $t<0$ if X_i is non lattice.

$$\int_0^{\infty} H(t-x)\, dm(x) \to \frac{1}{\mu} \int_0^{\infty} H(t)\, dt.$$

the limit being interrupted as 0 when μ ...

if X_i is the lattice with period d

$$H(c+nd) \to \frac{d}{\mu} \cdot \sum_{k=0}^{\infty} h(c+kd)$$

THEOREM: 6.9(a). Let $M(t)$ be a non-negative, non-...-asing function of $t \geq 0$ Such that $\int_0^{\infty} H(t)dt$ ... and Let X_i be the non-lattice then as t...

$$\int_0^t H(t-x)\, d\, M(x) \to \frac{1}{\mu} \int_0^{\infty} H(t)\, dt \to (1).$$

the limit being interrupted as 0 when $\mu = \infty$

PROOF:-

$$\int_0^t H(t-x)\, d\, M(x) = \int_0^{t/2} H(t-x)\, dM(x) + \int_{t/2}^{t} H(t-x)\, d...$$

$$= I_1 + I_2 \to (2).$$

$\because$ $H(\cdot)$ is non-negative & non-increasing function then $H(t-x) \leq H(t/2)$

for $0 \leq x \leq t/2$ & So $I_1 = \int_0^{t/2} H(t-x)\, dM(x)$

$$\leq H(t/2) \int_0^{t/2} dM(x).$$

$$\leq H(t/2)\, [M(x)]_0^{t/2}.$$

$$I_1 = H(t/2)\, M(t/2)$$

$$= t/2\, H(t/2)\, \frac{M(t/2)}{t/2}$$

Now as $t \to \infty$.

$\therefore$

$$\frac{M(t/2)}{t/2} \to \frac{1}{\mu} \quad (\because \text{Elementary renewal thm})$$

$\therefore$

$$(t/2) \cdot (H(t/2)) \to 0 \text{ as } t \to \infty.$$

$\therefore$ Thus $I_1 \to 0$ as $t \to \infty \to (3)$.

Let $$J = \int_0^\infty H(t)\,dt.$$

$$= \sum_{n=0}^{\infty} \int_{nh}^{nh+h} H(t)\,dt \to (4).$$

Now,

$$H(nh+h) \leq H(t) \leq H(nh) \text{ for } nh \leq t \leq (n+1)h$$

$$\sum_{n=0}^{\infty} \int_{nh}^{nh+h} H(nh+h)\,dt \leq \sum_{n=0}^{\infty} \int_{nh}^{nh+h} H(t)\,dt.$$

$$\leq \sum_{n=0}^{\infty} \int_{nh}^{nh+h} H(nh)\,dt.$$

$$\sum_{n=0}^{\infty} H(nh+h) \int_{nh}^{nh+h} dt \leq J \leq \sum_{n=0}^{\infty} H(nh) \int_{nh}^{nh+h} dt.$$

$$\sum_{n=0}^{\infty} H(nh+h)(t)_{nh}^{nh+h} \leq J \leq \sum_{n=0}^{\infty} H(nh)(t)_{nh}^{nh+h}$$

$$\sum_{n=0}^{\infty} H(nh+h)[nh+h-nh] \leq J \leq \sum_{n=0}^{\infty} H(nh)[nh+h-nh]$$

$$h \sum_{n=0}^{\infty} H(nh+h) \leq J \leq \sum_{n=0}^{\infty} H(nh) \cdot h.$$

$$hH(h) + h\sum_{n=1}^{\infty} H(nh+h) \leq J \leq hH(0) + h\sum_{n=1}^{\infty} H(nh)$$

Subtract both sides by $h\sum_{n=1}^{\infty} H(nh)$.

$$hH(h) + h\sum_{n=1}^{\infty}\left[H(nh+h) - H(nh)\right] \leq J - h\sum_{n=1}^{\infty} H(nh)$$

$$\therefore \qquad \leq hH(0).$$

$$-\epsilon \leq J - h\sum_{n=1}^{\infty} H(nh) < \epsilon$$

$$h\sum_{n=1}^{\infty} H(nh) < J+\epsilon.$$

$$J - h\sum_{n=1}^{\infty} H(nh) < \epsilon.$$

$$J - \epsilon < h\sum_{n=1}^{\infty} H(nh) \quad (\text{or})$$

$$h\sum_{n=1}^{\infty} H(nh) > J-\epsilon.$$

Again

$$I_2 = \int_{t/2}^{t} H(t-x)\, dM(x).$$

$$= \int_{t/2}^{0} H(y)\, dM(t-y)$$

$$= -\int_{0}^{t/2} H(y)\left[dM(t-y)\right].$$

$$= \sum_{n=0}^{N-1} \int_{nh}^{nh+h} H(y)\left[-dM(t-y)\right].$$

where 'N' is the greatest integer Contain in $t/2h$

(i.e) $t/2h = N$.

we have

$$H(nh+h) \leq H(y) \leq H(nh) \text{ for } nh \leq y \leq (n+1)$$

$$\sum_{n=0}^{N-1} \int_{nh}^{\cdots} H(nh+h)\,[-dM(t-y)] \le \sum_{n=0}^{N-1} \int_{nh}^{\cdots} H(y)\,[-dM(t-y)]$$

$$\le \sum_{n=0}^{N-1} \int_{nh}^{nh+h} H(nh)\,[-dM(t-y)]$$

$$\sum_{n=0}^{N-1} H(nh+h) \int_{nh}^{nh+h} -dM(t-y) \le I_2 \le \sum_{n=0}^{N-1} H(nh) \int_{nh}^{nh+h} -dM(t-y)$$

$$\sum_{n=0}^{N-1} H(nh+h)\,\big[-M(t-y)\big]_{nh}^{nh+h} \le I_2 \le \sum_{n=0}^{N-1} H(nh)\big[-M(t-y)\big]_{nh}^{nh+h}$$

$$\sum_{n=0}^{N-1} H(nh+h)\big[-M(t-(nh+h))+M(t-nh)\big] \le I_2 \le$$

$$\sum_{n=0}^{N-1} H(nh)\big[-M(t-nh-h)+M(t-nh)\big].$$

Multiply & divide by h.

$$\Rightarrow h\sum_{n=0}^{N-1} H(nh+h)\left[\frac{M(t-nh)-M(t-nh-h)}{h}\right] \le$$

$$I_2 \le h\sum_{n=0}^{N-1} H(nh)\left[\frac{M(t-nh)-M(t-nh-h)}{h}\right]$$

for largest 't' we can make

$$\left|\frac{M(t-nh)-M(t-nh-h)}{h} - \frac{1}{\mu}\right| < \epsilon.$$

$$\Rightarrow \frac{1}{\mu} - \epsilon < \frac{M(t-nh)-M(t-nh-h)}{h.} < \frac{1}{\mu} + \epsilon$$

Also,

$$h\sum_{n=0}^{N-1} H(nh) = h\sum_{n=0}^{\infty} H(nh) - h\sum_{n=N}^{\infty} H(nh)$$

$$= hH(0) + h\sum_{n=1}^{\infty} H(nh) - h\sum_{n=N}^{\infty} H(nh)$$

$$= hH(0) + h\sum_{n=1}^{\infty} H(nh) - hH(Nh) - h\sum_{n=N+1}^{\infty} H(nh)$$

$$h\sum_{n=0}^{N-1} H(nh+h) = hH(h) + hH(2h) + \cdots + hH(Nh)$$
$$+ hH(0) - hH(0)$$
$$= h\sum_{n=0}^{N-1} H(nh) - [hH(0) - hH(Nh)]$$
$$> J - \epsilon - \epsilon.$$
$$> J - 2\epsilon.$$
$$h\sum_{n=0}^{N-1} H(nh+h) > J - 2\epsilon.$$

Sub in (A)

$$\therefore (J - 2\epsilon)\left(\frac{1}{\mu} - \epsilon\right) \le I_2 \le \left(\frac{1}{\mu} + \epsilon\right)(J + \epsilon)$$

($\because$ ϵ is arbitrary)

$$\frac{1}{\mu} J \le I_2 \le \frac{1}{\mu} J.$$
$$I_2 = \frac{1}{\mu} J.$$
$$I = I_1 + I_2.$$
$$= 0 + \frac{1}{\mu} J.$$
$$\therefore I = \frac{1}{\mu}\int_0^\infty H(t)\,dt$$

EXAMPLE :- 2(a) :-

U.Q SM LOM.

Let X_n have a gamma distribution having density $f(x) = \dfrac{a^k x^{k-1} e^{-ax}}{(k-1)!}$, $x \ge 0$

$= 0$, $x < 0$

then $f^*(s) = \left(\dfrac{a}{s+a}\right)^k$.

Solution: -

The density $F_n'(x)$ of

$$S_n = X_1 + X_2 + \cdots + X_n$$

have the laplace transform is $\left(\frac{a}{s+a}\right)^{nk}$.

then $f_n'(x) = \frac{a^{nk} \cdot x^{nk-1} e^{-ax}}{(nk-1)!}$

using density function of gamma distribution, we have.

$$g(x) = \frac{\lambda^n x^{n-1} e^{-\lambda x}}{\Gamma n}, \quad x > 0.$$

$$F_{N_n}(t) = 1 - F_N(t)^{n-1}$$

using, we get

$$F_n(x) = Pr\{n \leq x\}.$$

$$= 1 - Pr\{N(x) < n\}.$$

$$= 1 - Pr\{N(x) \leq n-1\}.$$

$$F_n(x) = 1 - F_N(x)^{n-1}$$

Hence the distribution $F_N(x)^{n-1} = 1 - F_n(x)$

$$= 1 - \int_0^x \frac{a^k x^{k-1} e^{-ax}}{(k-1)!} dx.$$

$$= 1 - \frac{1}{\Gamma k} \int_0^x a^{k-1} \cdot a \cdot x^{k-1} e^{-ax} dx.$$

$$= \frac{1}{\Gamma k} \int_0^\infty (ax)^{k-1} a e^{-ax} dx - \frac{1}{\Gamma k} \int_0^{ax} (ax)^{k-1} a e^{-ax} dx$$

$$= \frac{1}{\Gamma k} \int_0^\infty y^{k-1} a e^{-y} \frac{dy}{a} - \int_0^{ax} y^{k-1} a e^{-y} \frac{dy}{a}.$$

$$= \frac{1}{\Gamma k}\int_0^\infty y^{k-1}e^{-y}\,dy + \int_{ax}^{0} y^{k-1}e^{-y}\,dy.$$

$$= \frac{1}{\Gamma k}\int_{ax}^{\infty} y^{k-1}e^{-y}\,dy.$$

$$= \frac{1}{\Gamma k}\left[\left(-y^{k-1}e^{-y}\right)_{ax}^{\infty} + \int_{ax}^{\infty} e^{-y}(k-1)y^{k-2}\,dy\right]$$

$$= \frac{1}{\Gamma k}\left[\left[(ax)^{k-1}e^{-ax}\right] + (k-1)\int_{ax}^{\infty} e^{-y}y^{k-2}\,dy\right]$$

$$= \frac{1}{\Gamma k}\left[\left[(ax)^{k-1}e^{-ax}\right] + (k-1)\int_{ax}^{\infty} y^{k-2}\,d(-e^{-y})\right]$$

$$= \frac{1}{\Gamma k}\left[\left[(ax)^{k-1}e^{-ax}\right] + (k-1)\left[\left[-y^{k-2}e^{-y}\right]_{ax}^{\infty} + \int_{ax}^{\infty} e^{-y}(k-2)y^{k-3}\,dy\right.\right.$$

$$= \frac{1}{\Gamma k}\left[\left[(ax)^{k-1}e^{-ax}\right] + (k-1)(ax)^{k-2}e^{-ax} + (k-1)(k-2)\int_{ax}^{\infty} e^{-y}y^{k-3}\,dy\right]$$

$$= \frac{1}{\Gamma k}\left[(ax)^{k-1}e^{-ax} + (k-1)(ax)^{k-2}e^{-ax} + (k-1)(k-2)(ax)^{k-3}e^{-ax} + \right.$$

$$= \frac{e^{-ax}}{(k-1)!}\left[(ax)^{k-1} + (k-1)(ax)^{k-2} + \cdots\right]$$

$$F_N(x)^{n-1} = e^{-ax}\left[\frac{(ax)^{k-1}}{(k-1)!} + \frac{(ax)^{k-2}}{(k-2)!} + \cdots\right]$$

$$= e^{-ax}\sum_{r=0}^{nk-1}\frac{(ax)^r}{r!}$$

$$\therefore F_n(x) = 1 - F_N(x)^{n-1}$$

$$F_n(x) = 1 - e^{-ax}\sum_{r=0}^{nk-1}\frac{(ax)^r}{r!}, \quad n \geq 1 \longrightarrow (1).$$

$$P_n(t) = F_n(t) - F_{n+1}(t).$$

dx

put $x = t$ in Eqn ①.

$$P_n(t) = 1 - e^{-at}\sum_{r=0}^{nk-1}\frac{(at)^r}{r!} - 1 + e^{-at}\sum_{r=0}^{k-1}\frac{(at)^r}{r!}$$

$$P_n(t) = e^{-at}\sum_{r=0}^{k-1}\frac{(at)^r}{r!} \longrightarrow ①.$$

W.k.T.

$$M^*(s) = \frac{f^*(s)}{s(1-f^*(s))} \longrightarrow ②.$$

Since $f^*(s) = \frac{a^k}{(s+a)^k}$ in Eqn ②.

$$M^*(s) = \left[\frac{a^k}{(s+a)^k} \Big/ s\left(1 - \frac{a^k}{(s+a)^k}\right)\right]$$

$$= \frac{\frac{a^k}{(s+a)^k}}{\frac{s((s+a)^k - a^k)}{(s+a)^k}}$$

$$M^*(s) = \frac{a^k}{s((s+a)^k - a^k)} \longrightarrow ③.$$

PARTICULAR CASE:-

(i) MARKOVIAN CASE:-

when $k=1$, X_n has negative exponential distribution and the renewal process then reduces to poisson process.

put $k=1$, in Eqn ①

$$P_n(t) = e^{-at}\cdot\frac{(at)^n}{n!}$$

when $k=1$ in eqn 3

$$M^*(s) = \frac{a}{s(s+a-a)} = \frac{a}{s^2}$$

when $k=2$ in eqn 3.

$$M^*(s) = \frac{a^2}{s[(s+a)^2-a^2]}$$

$$= \frac{a^2}{s[s^2+\cancel{a^2}+2as-\cancel{a^2}]}$$

$$= \frac{a^2}{s^3+2as^2}$$

$$\frac{a^2}{s^3+2as^2} = \frac{a^2}{s^2(s+2a)}$$

$$\frac{a^2}{s^2(s+2a)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s+2a} = a^2.$$

$$As(s+2a) + B(s+2a) + Cs^2 = a^2 \longrightarrow (4).$$

put $s=0$ in eqn 4

$$B(2a) = a^2$$

$$B = a^2/2a$$

$$\boxed{B = a/2}$$

Put $s=-2a$.

$$C(4a^2) = a^2$$

$$\boxed{C = 1/4.}$$

put $s = 2a$

$$A(2a)(4a) + B(4a) + C(4a^2) = a^2$$

$\therefore$

$$A(8a^2) + 2a^2 + a^2 = a^2$$

$$8Aa^2 = a^2 - 3a^2$$

$$8a^2 A = -2a^2$$

$\therefore$

$$A = \frac{-2a^2}{8a^2}$$

$$\boxed{A = -1/4}$$

The inversion formula for laplace transform we have

$$M^*(s) = -\frac{1}{4}\left[L^{-1}(1/s)\right] + \frac{a}{2}L^{-1}(1/s^2) + \frac{1}{4}L^{-1}\left(\frac{1}{s+2a}\right)$$

$$= -\frac{1}{4}(1) + \frac{a}{2}t + \frac{1}{4}e^{-2at}$$

$$M^*(s) = \frac{a}{2}t - 1/4 + \frac{1}{4}e^{-2at}$$

EXAMPLE 2(b):-

HYPER - EXPONENTIAL DISTRIBUTION.

Let X_n have density

$$f(t) = pe^{-at}\cdot a + (1-p)be^{-bt} \rightarrow (1) \quad 0 \le p \le 1,\ a>b>0$$

Such a model may be used to describe a system which has two kinds of components, a proportion 'p' components having say a high failure rate 'a' and the remaining proportion (1-p) of components have a lower failure rate 'b'.

$$f^*(s) = \frac{pa}{s+a} + \frac{(1-p)b}{s+b} \qquad \left[\because L(e^{-at}) = \frac{1}{s+a}\right]$$

By thm 6.3,

$$M^*(s) = \frac{f^*(s)}{s(1-f^*(s))}$$

$$= \frac{\frac{pa}{s+a} + \frac{(1-p)b}{s+b}}{s\left[1 - \frac{pa}{s+a} - \frac{(1-p)b}{s+b}\right]}$$

$$= \frac{(s+b)pa + (1-p)b(s+a)}{s\left[(s+a)(s+b) - pa(s+b) - (1-p)b(s+a)\right]}$$

$$= \frac{pas + pab + (s+a-ps-pa)b}{s\left[s^2 + \cancel{bs} + as + \cancel{ab} - pas - \cancel{pab} - \cancel{bs} - \cancel{ab} + psb + \cancel{pab}\right]}$$

$$= \frac{pas + \cancel{pab} + bs + ab - pbs - \cancel{pab}}{s\left[s^2 + s(a - pa + pb)\right].}$$

$$\therefore \quad M^*(s) = \frac{ab + s\left[pa + (1-p)b\right]}{s^2\left[s + (1-p)a + pb\right]}$$

where,

$$A = pa + (1-p)b \quad \& \quad B = (1-p)a + pb.$$

$$M^*(s) = \frac{ab + As}{s^2[s+B]}$$

$$= \frac{ab}{s^2(s+B)} + \frac{A}{s(s+B)}$$

$$M^*(s) = \frac{A}{B}\left\{\frac{1}{s} - \frac{1}{s+B}\right\} + \frac{ab}{B}\left\{\frac{1}{s^2} - \left\{\frac{1}{s} - \frac{1}{s+B}\right\}\frac{1}{B}\right\}$$

Inverting the laplace transform,

$$M(t) = \frac{A}{B}\left\{1 - e^{-Bt}\right\} + \frac{ab}{B}\left\{t - \left[\frac{1-e^{-Bt}}{B}\right]\right\}.$$

$$= \frac{A}{B}(1-e^{-Bt}) + \frac{ab}{B}t - \frac{ab}{B^2}(1-e^{-Bt}).$$

$$= \left[\frac{A}{B} - \frac{ab}{B^2}\right](1-e^{-Bt}) + \frac{abt}{B} \longrightarrow (3).$$

$$M(t) = c(1-e^{-Bt}) + \frac{abt}{B}.$$

where $c = \frac{A}{B} - \frac{ab}{B^2}$

$$= \frac{pa+(1-p)b}{(1-p)a+pb} - \frac{ab}{((1-p)a+pb)^2}$$

$$= \frac{(pa+(1-p)b)((1-p)a+pb)-ab}{((1-p)a+pb)^2}$$

$$= \frac{(pa+b-pb)(a-pa+pb)-ab}{((1-p)a+pb)^2}$$

$$= \frac{pa^2-p^2a^2+p^2ab+ab-pab+pb^2-pab+p^2ab-p^2b^2-ab}{((1-p)a+pb)^2}.$$

$$= \frac{p(a^2+b^2-2ab)-p^2(a^2+b^2-2ab)}{B^2}$$

$$= \frac{p((a-b)^2)-p^2(a-b)^2)}{B^2}$$

$$= \frac{(p-p^2)(a-b)^2}{B^2}$$

$$c = \frac{p(1-p)(a-b)^2}{B^2} \geq 0.$$

MARKOVIAN CASE:-

when $p=1$ (or) $p=0$, the distribution of X_n reduces to negative exponential distribution and $c=0$, the first term of ③ vanishes

$$\therefore M(t) = at \text{ (or) } bt.$$

EXAMPLE : 5(a):—

AGE AND BLOCK REPLACEMENT POLICIES:-

The usual replacement policy impl[ies] replacement of a component, as and when it fails by a similar new one. Therfore there are other policies besides this the two most impor-ant replacement policies that are in use in general, are the age and block replacement polic[ies] under age replacement policy a component replace[d] upon failiure or when it attains a specifi[ed] age T. which ever occurs earlier. under a bl[ock] replacement policy component is replaced upon failiure and also regularity at times T, 2T, ... we shall call T the replacement interv[al] under these two policies.

we assume as usual that compon[ents] or units fail permenently independently and that th[e] direction of a failiure and replacement of the failed item are instantaneous. Suppose that the successive life time of the units are random variables with a comman distribution function F(t) having mean μ. Considering failiure as renewal the number of failiure in $[0, t]$ under the three replacement policies, the usual one, the age replacement and the block replacement can be denoted by three renewal processes as indicated bellow.

$N(t)$ = The number of failures in $[0,t]$ for an ordinary renewal process.

$N_A(t,T)$ = The number of failure in $[0,t]$ under the age replacement policy with replacement interval T.

$N_B(t,T)$ = The number of failure in $[0,T]$ under the block replacement policy with replacement interval T.

SOLN:-

Let the corresspording renewal function be.

$M(t) = E\{N(t)\}$.

$M_A(t,T) = E\{N_A(t,T)\}$.

$M_B(t,T) = E\{N_B(t,T)\}$.

Now

$$\lim_{t\to\infty} \frac{E\{N_A(t,T)\}}{E(T)} = \lim_{t\to\infty} \frac{M_A(t,T)}{t}$$

$$= \frac{1}{\mu_A} \qquad \left[\because \lim_{t\to\infty}\frac{M(t)}{t} = \frac{1}{\mu}\right]$$

Similarly

T.P $$\lim_{t\to\infty} \frac{E\{N_B(t,T)\}}{E(T)} = \lim_{t\to\infty} \frac{M_B(t,T)}{t}$$

$$= \frac{M(T)}{T}.$$

Now, consider $\{N_A(t,T)\, t \geq 0\}$ is a renewal process with distribution

$$\therefore \quad F_A(t) \leq pr\{y_i \leq t\}.$$

Such that in $nT \leq t \leq (n+1)T, \quad n = 0,1,2 \ldots$

$$1 - F_A(t) = 1 - pr\{y_i \leq t\}.$$

$$= pr\{y_i > t\}.$$

$$= [1 - F(T)]^n [1 - F(t - nT)].$$

$$\therefore \mu_A = E(y_i) = \int_0^\infty pr(y_i \geq t)\, dt.$$

$$= \sum_{n=0}^{\infty} \int_{nT}^{(n+1)T} \{(1 - F(T))^n [1 - F(t - nT)]\} dt.$$

$$= \sum_{n=0}^{\infty} [1 - F(T)]^n \int_{nT}^{(n+1)T} [1 - F(t - nT)] dt.$$

Let $x = t - nT.$

$dx = dt.$

$t = nT \Rightarrow x = 0.$

$t = (n+1)T \Rightarrow x = T.$

$$= \sum_{n=0}^{\infty} [1 - F(T)]^n \int_0^T [1 - F(x)] dx.$$

$$= \int_0^T [1 - F(x)] dx \sum_{n=0}^{\infty} [1 - F(t)]^n.$$

$$= \int_0^T [1 - F(x)] [1 + [1 - F(T)]^n + [1 - F(T)]^2 + \cdots]$$

$$= \int_0^T [1 - F(x)] dx \, [1 - (1 - F(T))]^{-1}].$$

$$= \int_0^T [1 - F(x)] \, dx \, [F(T)]^{-1}$$

$$\mu_n = \int_0 \frac{1-F[x]}{F(T)}\,dx.$$

$$\therefore \lim_{t\to\infty} \frac{E\{N_A(t,T)\}}{E(T)} = \frac{1}{\mu_A}$$

$$= \frac{F(T)}{\int_0^T [1-F(x)]\,dx} \longrightarrow (A)$$

Now consider, the block replacement at $T, 2T, \ldots$ with process states a new and if $N_B(T)$ ~~[illegible]~~ is the number of failure in the interval $[(r-1)T, rT]$, $r = 1, 2, \ldots$ following block replacements,

Then

$$N_{Br}(T) = N(T)$$

Suppose that for some specified integral value of n, $nT \le t' \le (n+1)T$ (i.e) for some n and $0 \le \tau \le T$.

Let $t = nT + \tau$

Then

$$N_B(t,T) = \sum_{r=1}^{n} N_{Br}(T) + N(\tau)$$

$\therefore$ Divide by t and take $\lim_{t\to\infty}$

$$\lim_{t\to\infty} \frac{N_B(t,T)}{t} = \lim_{t\to\infty} \sum_{r=1}^{n} \frac{N(T)}{t} + \frac{N(\tau)}{t}$$

$$= \lim_{t\to\infty} \sum_{r=1}^{n} \frac{N(T)}{nT+\tau} + \frac{N(\tau)}{nT+\tau}$$

As $\tau \to \infty$, $\frac{N(\tau)}{nT+\tau} \longrightarrow 0$.

$$= \lim_{t\to\infty} \sum_{\gamma=1}^{n} \frac{N(\tau)}{n\tau}$$

$$= \frac{E\{N(\tau)\}}{\tau}$$

$$\therefore \lim_{t\to\infty} \frac{N_B(t,\tau)}{t} = \frac{M(\tau)}{\tau} \text{ is proved.}$$

PARTICULAR CASES:-

(i) when $F(x) = 1 - e^{-x/a}$ with mean $\mu = a$ then

$$\mu_A = \int_0^\tau \frac{[1 - F(x)]}{F(\tau)}\,dx.$$

$$= \int_0^\tau \frac{1 - (1 - e^{-x/a})}{F(\tau)}\,dx.$$

$$= \int_0^\tau \frac{e^{-x/a}}{F(\tau)}\,dx.$$

$$= \frac{1}{F(\tau)}\left[\frac{e^{-x/a}}{-1/a}\right]_0^\tau$$

$$= \frac{-a}{F(\tau)}\left[e^{-\tau/a} - 1\right].$$

$$= \frac{a}{F(\tau)}\left[1 - e^{-\tau/a}\right].$$

$$= \frac{a}{F(\tau)} F(\tau) = a.$$

$$\mu_A = a = \mu.$$

$$\therefore \frac{M(\tau)}{\tau} = \frac{a\tau}{\tau} = a = \mu.$$

(ii) LIMITING CASE:-

Suppose that μ is finite, then f

large T we have asymptotically

$$\frac{1}{\mu_0} \to \frac{1}{\mu}$$

By elementary renewal theory.

$$\frac{M(T)}{T} \to \frac{1}{\mu}$$

Hence as $T \to \infty$, the renewal processes under the age and block replacement policies are both equivalent to the ordinary process $N(t)$

THEOREM : 6.10:-

Let $\{x_n, n = 1, 2, \dots\}$ be a renewal process with distribution F which the mean $\mu = E(x_i)$ and variance $\sigma^2 = E\{x_i - \mu\}^2$ exists and are finite. Let $\{N(t), t \geq 0\}$ be the renewal process generated by F.

then $$\lim_{t \to \infty} \left\{ \frac{N(t) - t/\mu}{\sqrt{t\sigma^2/\mu^3}} < x \right\} = \phi(x).$$

where $\phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left[-t^2/2\right] dt$ is the derivative function of the standard normal distribution.

Proof:-

from the central limit theorem applied to $S_n = x_1 + x_2 + \dots + x_n$

$\therefore$ we find that $n \to \infty$, S_n is ansymptotically with mean μ and variance $n\sigma^2$.

that is.

$$\lim_{n\to\infty} Pr\left\{\frac{S_n - n\mu}{\sqrt{n}} \leq x\right\} = \phi(x) \longrightarrow (1).$$

Let x be fixed and Let $n\to\infty$ and $t\to\infty$ in such a way that,

$$\lim_{\substack{n\to\infty \\ t\to\infty}} \frac{t-n\mu}{\sigma\sqrt{n}} \longrightarrow -x.$$

$t \to n\mu - x\sigma\sqrt{n}$ as $n\to\infty$, $t\to\infty$ we have

$$\lim_{\substack{n\to\infty \\ t\to\infty}} Pr\{S_n > t\} = \lim_{\substack{n\to\infty \\ t\to\infty}} Pr\left\{\frac{S_n - n\mu}{\sigma\sqrt{n}} > \frac{t-n\mu}{\sigma\sqrt{n}}\right\}.$$

$$= \lim_{\substack{n\to\infty \\ t\to\infty}} Pr\left\{\frac{S_n - n\mu}{\sigma\sqrt{n}} > -x\right\} \longrightarrow (2).$$

$$= 1 - \lim_{n\to\infty} Pr\left\{\frac{S_n - n\mu}{\sigma\sqrt{n}} < -x\right\}.$$

$$= 1-(1-\phi(x))$$

$$= 1 - 1 + \phi(x).$$

$\therefore$ $= \phi(x).$

Again since renewal function & renewal density $\because$

$\{N(t) < n\} \Longleftrightarrow \{S_n > t\}.$

$\therefore$ taking limits on both side we get,

$$\lim_{\substack{n\to\infty \\ t\to\infty}} Pr\{S_n > t\} = \lim_{\substack{n\to\infty \\ t\to\infty}} Pr\{N(t) < n\}$$

Oliver linn[kal]

Subtract t/μ & divide $\sqrt{t\sigma^2/\mu^3}$, we get.

$$\phi(x) = \lim_{\substack{n\to\infty \\ t\to\infty}} Pr\left\{ \frac{N(t) - t/\mu}{\sqrt{t\sigma^2/\mu^3}} < \frac{n - t/\mu}{\sqrt{t\sigma^2/\mu^3}} \right\}$$

$$= \lim_{\substack{n\to\infty \\ t\to\infty}} Pr\left\{ \frac{N(t) - t/\mu}{\sqrt{t\sigma^2/\mu^3}} < x \right\}.$$

Then $N(t)$ is asymptotically normal with mean t/μ and variance $t\sigma^2/\mu^3$

Hence the proof.

2M

- Stopping time
- Lattice variable
- State equilibrium
- Pasta
- little formula
- Obsorbing state
- homogenity time of Poisson
- Markov chain
- order of Markovchain
- mean reccurance time

5M

1) (a) $(at)^n/n-1$ is not stationary.
(b) $X_n = k$ be are in succesive in a tossing
(2) (a) state and prove ergodic theorem for reducible.
(b) k persistent $P_{jk}^n \to 0$ state it $\frac{F_{jk}}{\mu_{kk}} \to 0$
(3) (a) additive property of poisson process
(b) sum - 163
(4) (a) wald's theorem
(b) $M(t) = F(t) + \int_0^\infty M(t-x)\, dx$.
(5) (a) waiting time for M/M/1 Queue.
(b) PK formula.

10M

(1) j persitent iff $\sum p_{jj}^{(n)} = 0$
(2) steady state solution M/M/1 case
(3) Renewal theorem
(4) Find the tran M/G/∞ model
(5) Explaine Corrleated random walk Correlation X_n & X_{n-1}