4 FIELD EQUATIONS AND CONSERVATION LAWS

INTRODUCTION :

[In chapter 1 and 3 we have summarised the mathematical description of static electric and magnetic fields. We now wish to consider the more general situation in which the field quantities may depend upon time. Under such conditions there is an interdependence of the field quantities and it is no longer possible to discuss separately the electric and magnetic fields and we are forced to consider the general concept of an electromagnetic field. The time dependent electromagnetic field equation are called Maxwell's equations. These equations are mathematical abstractions of experimental results.

In this chapter we seek to establish the formation of the field equations, to show that their solutions are unique, to discuss the scalar and vector potentials of the field and to consider the law of conservation of charge, energy and momentum].

§ 4.1. Equation of Continuity.

Under steady-state conditions the charge density in any given region will remain constant. We now relax the requirement of steady-state conditions and allow the charge density to become a function of time. It is experimentally verified that the net amount of electric charge in a closed system remains constant. Therefore if the net charge within a certain region decreases with time, this implies that a like amount of charge must appear in some other region. This transport of charge constitutes a current

i.e. $$I = -(dq/dt). \quad \text{...(1)}$$

–ive sign here indicates that charge contained in a specified volume decreases with time.

However by definition of current density

$$I = \oint \mathbf{J} \cdot d\mathbf{s}. \quad \text{...(2)}$$

So from equation (1) and (2) we have—

$$\oint_S \mathbf{J} \cdot d\mathbf{s} = -\frac{dq}{dt}$$

i.e. $$\oint_S \mathbf{J} \cdot ds = -\frac{d}{dt}\int_\tau \rho \, d\tau. \quad \left[as\ q = \int_\tau \rho \, d\tau\right] \quad \text{...(2A)}$$

If we hold the surface S fixed in space, the time variation of the volume integral must be solely due to the time variation of ρ. Thus

$$\oint_S \mathbf{J} \cdot ds = \int \frac{\partial \rho}{\partial t} \, d\tau. \quad \text{...(3)}$$

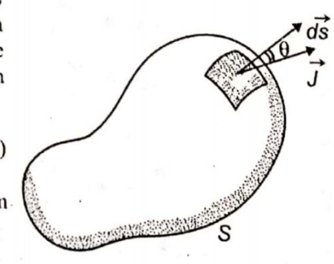

Fig. 4.1

But from Gauss's theorem

$$\oint_S \mathbf{J} \cdot ds = -\int_\tau (\text{div}\, \mathbf{J})\, d\tau. \quad \text{...(4)}$$

So comparing expression (3) and (4) we get

$$\oint_\tau (\text{div}\, \mathbf{J})\, d\tau = -\int_\tau \frac{\partial \rho}{\partial t}\, d\tau.$$

i.e. $$\oint_\tau \left(\text{div}\, \mathbf{J} + \frac{\partial \rho}{\partial t}\right) d\tau = 0. \quad \text{...(5)}$$

Since equations (5) is true for any arbitrary finite volume, the integral must vanish *i.e.*

$$\text{div}\ \mathbf{J} + \frac{\partial \rho}{\partial t} = 0. \quad \text{...(A)}$$

Equation (A) is called the *equation of continuity* and is an expression of the experimental fact that electric charge is conserved.

Example 1. *Starting with the equation of continuity and assuming Ohm's law, show that the charge density in a conductor obeys the equation*

$$\frac{\sigma}{\varepsilon} \rho + \frac{\partial q}{\partial t} = 0.$$

Solving the above equation discuss the final result.

Solution. As according to continuity equation

$$\text{div}\ \mathbf{J} + \frac{\partial \rho}{\partial t} = 0. \quad \text{...(1)}$$

And according to Ohm's law

$$\mathbf{J} = \sigma \mathbf{E}. \quad \text{...(2)}$$

So
$$\text{div}\,(\sigma \mathbf{E}) + \frac{\partial \rho}{\partial t} = 0$$
or
$$\sigma\,\text{div}\,\mathbf{E} + \frac{\partial \rho}{\partial t} = 0$$
or
$$\frac{\sigma}{\varepsilon}\,(\text{div}\,\varepsilon\,\mathbf{E}) + \frac{\partial \rho}{\partial t} = 0$$
or
$$\frac{\sigma}{\varepsilon}\,(\text{div}\,\mathbf{D}) + \frac{\partial \rho}{\partial t} = 0 \qquad (\text{as } \mathbf{D} = \varepsilon \mathbf{E})$$
or
$$\frac{\sigma}{\varepsilon}\,\rho + \frac{\partial \rho}{\partial t} = 0. \ (\text{as div } \mathbf{D} = \rho). \qquad ...(3)$$

This is the required result. To solve the above equation, let
$$\frac{\varepsilon}{\sigma} = \text{constt} = \tau^*$$

So the equation (3) becomes
$$\frac{\partial \rho}{\partial t} = -\frac{1}{\tau}\rho \quad i.e. \quad \frac{\partial \rho}{\rho} = -\frac{\partial t}{\tau}.$$

On integrating it, we get
$$\log \rho = -\frac{t}{\tau} + \log C$$
i.e.
$$\rho = C e^{-(t/\tau)}. \qquad ...(4)$$

Now if at $t = 0$ $\rho = \rho_0$ then.
$$C = \rho_0. \qquad ...(5)$$

So in the light of equation (5), (4) becomes
$$\rho = \rho_0\, e^{-(t/\tau)} \qquad ...(6)$$

This is the desired result and from this result it is clear that—

(*i*) The charge density in the volume decreases exponentially with time at a rate such that after a time t

it reduces to $t = \tau$
$$\rho = \rho_0 e^{-1} = \frac{1}{e}(\rho_0)$$

* As the unit of constant τ is
$$\tau = \frac{\varepsilon}{\sigma} = \frac{\text{Farad}}{\text{meter}} \times \frac{(\text{ohm} \times \text{meter})}{1}$$
i.e.
$$\tau = \text{Farad} \times \text{ohm} = \frac{\text{coul} \times \text{volt}}{\text{volt} \times \text{amp.}}$$
i.e.
$$\tau = \frac{\text{coul}}{\text{amp.}} = \frac{\text{coul}}{\text{coul/sec.}} = \text{sec.}$$

So its dimensions are that of time. The constant τ is called *relaxation time* and is a characteristic of the given medium.

i.e. 36.8% of its original value. So the relaxation time is that time in which charge density reduces to 36.8% of its initial value.

(ii) As for good conductors
$$\tau = \frac{\varepsilon}{\sigma} = \frac{\varepsilon_0 \varepsilon_r}{\sigma} \to 0 \qquad [\text{as } \sigma \to \infty \text{ and } \varepsilon_r = 1]^*.$$
So
$$\rho = \rho_0 e^{-(t/\tau)} \to \rho_0 e^{-\infty} \to 0$$

i.e. in case of good conductors, the free volume charge density is practically zero, and any net charge must be situated at the surface *i.e.* charge disappears almost instantaneously from the interior of a good conductor (such as copper) and resides on its surface.

(iii) As for good insulator
$$\tau = \frac{\varepsilon}{\sigma} = \frac{\varepsilon_r \varepsilon_0}{\sigma} \to \infty \qquad (\text{as } \sigma \to 0)$$
So
$$\rho = \rho_0 e^{-(t/\tau)} \to \rho_0 e^{-0} \to \rho_0$$

i.e. in case of good insulators, the free volume charge density practically remains unchanged as time passes, *i.e.* charge remains for an extremely long period within good insulators such as quartz.

§ 4.2. Displacement Current.

We know that Ampere's circuital law in its most general form is given by
$$\oint_C \mathbf{H}.d\mathbf{l} = \int_S \mathbf{J} \bullet d\mathbf{s} \quad [\text{see equation } c \text{ of § 3.10 (a)}]$$
i.e.
$$\int_S \text{curl } \mathbf{H} \bullet d\mathbf{s} = \int_S \mathbf{J} \bullet d\mathbf{s}$$
or
$$\text{curl } \mathbf{H} = \mathbf{J} \qquad ...(1)$$

Let us now examine the validity of this equation in the event that the fields are allowed to vary with time. If we take the divergence of both sides of equation (1) then
$$\text{div (curl } \mathbf{H}) = \text{div } \mathbf{J}. \qquad ...(2)$$

Now as *div of curl* of any vector is zero, we get from equation (2)
$$\text{div } \mathbf{J} = 0 \qquad ...(3)$$

* In note (ii) of application (d) in § 2.3 we have shown that for electrostatic effects a conductor acts like a material of infinite dielectric constant. However in case of steady current as polarisation effects are completely overshadowed by dispersion of metals this result does not hold good. For purpose of estimation, in case of conduction through metal we usually take $\varepsilon_r \to 1$ as discussed in § 7.9.

Now the continuity equation in general states

$$\text{div}\,\mathbf{J} = -\frac{\partial\rho}{\partial t} \quad ...(4)$$

and will therefore vanish only in the special case that the charge density is static. Consequently we must conclude that Ampere's law as stated in equation (1) is valid only for steady state conditions and is insufficient for the case of time-dependent fields. Because of this Maxwell assumed that equation (1) is not complete but should have something else to it. Let this 'something' be denoted be $\mathbf{J}_d$, then equation (1) can be rewritten as

$$\text{cur}\,\mathbf{H} = \mathbf{J} + \mathbf{J}_d. \quad ...(5)$$

In order to identify $\mathbf{J}_d$, we calculate the divergence of equation (2) again and get

$$\text{div curl}\,\mathbf{H} = \text{div}\,(\mathbf{J}+\mathbf{J}_d)$$

i.e. $$\text{div}\,(\mathbf{J}+\mathbf{J}_d) = 0 \quad (\text{as div curl}\,\mathbf{H} = 0)$$

or $$\text{div}\,\mathbf{J} + \text{div}\,\mathbf{J}_d = 0$$

or $$\text{div}\,\mathbf{J}_d = -\,\text{div}\,\mathbf{J}$$

i.e. $$\text{div}\,\mathbf{J}_d = \frac{\partial\rho}{\partial t} \quad [\text{from equation (4)}]$$

i.e. $$\text{div}\,\mathbf{J}_d = \frac{\partial}{\partial t}(\text{div}\,\mathbf{D}) \quad [\text{as div}\,\mathbf{D} = \rho]$$

i.e. $$\text{div}\left(\mathbf{J}_d - \frac{\partial\mathbf{D}}{\partial t}\right) = 0 \quad ...(6)$$

As equation (6) is true for any arbitrary volume

$$\mathbf{J}_d = \frac{\partial\mathbf{D}}{\partial t}. \quad ...(A)$$

And so the modified form of Ampere's circuital law becomes

$$\text{curl}\,\mathbf{H} = \mathbf{J} + \frac{\partial\mathbf{D}}{\partial t}. \quad ...(B)$$

The term which Maxwell added to Ampere's law *viz.* $(\partial\mathbf{D}/\partial t)$ is called the *displacement* current to distinguish it from **J**, the conduction current. By adding this term to Ampere's law, Maxwell assumed that the time rate of change of displacement produces a magnetic field just as a conduction current does.

Regarding displacement current it is worthy to note that :

(i) Displacement current is a current only in the sense that it produces a magnetic field. It has none of the other properties of current.

For example displacement current can have a finite value in perfect vacuum where there are no charges of any type.

(ii) The magnitude of the displacement current is equal to the time rate of change of electric displacement vector **D**.

(iii) Displacement current serves to make the total current continuous across discontinuities in conduction current. (See example 2 and problem 4).

(iv) The displacement current in a good conductor is negligible as compared to the condition current at any frequency lower than the optical frequencies (~ 10^{15} Hertz). (See example 3).

(v) The addition of displacement current *i.e.* $(\partial\mathbf{D}/\partial t)$ to Ampere's law *i.e.* curl **H** = **J** results in

$$\text{curl}\,\mathbf{H} = \mathbf{J} + \frac{\partial\mathbf{D}}{\partial t}$$

i.e. displacement current relates the electric field vector **E** (as $\mathbf{D} = \varepsilon\mathbf{E}$) to the magnetic field vector **H**. This in turn implies that in case of time dependent fields it is not possible to deal with electric and magnetic fields separately, but the two fields are interlinked and give rise to what are know as electromagnetic fields *i.e. The addition of displacement current to Ampere's law result, in the unification of electric and magnetic phenomena.*

It must be emphasized here that the ultimate justification for Maxwell's assumption of displacement current is in the experimental verification. Indeed the effects of the displacement current are difficult to observe directly except at very high frequencies. However indirect verification is afforded by predictions of many effects particularly in electromagnetic theory of light which are confirmed by experiments. We may therefore consider that Maxwell's form of Ampere's law has been subjected to experimental tests and has been found to be generally valid.

Note : *Different types of current densities—*

(a) Conduction current density J : It arises due to the physical motion of true charge and is given by

$$\mathbf{J} = \sigma\mathbf{E} = nqv \quad [\S\ 3.1]$$

(b) Polarisation current density $\mathbf{J}_p$: It arises due to the polarisation of the material and is given by

$$\mathbf{J}_p = (\partial\mathbf{P}/dt) \quad [\text{note in problem—7}]$$

$$J_d = \varepsilon_r \varepsilon_0 \frac{\partial}{\partial t}(E_0 \cos \omega t) \qquad [\text{as } \varepsilon = \varepsilon_r \varepsilon_0]$$

i.e.

$$J_d = -\omega \varepsilon_r \varepsilon_0 E_0 \sin \omega t$$

i.e.

$$J_d = \omega \varepsilon_r \varepsilon_0 E_0 \cos\left(\omega t + \frac{\pi}{2}\right). \qquad ...(4)$$

i.e.

So from equation (3) and (4) it is clear that

$$\frac{J_d}{J} = \frac{\omega \varepsilon_r \varepsilon_0 E_0 \cos\left(\omega t + \frac{\pi}{2}\right)}{\sigma E_0 \cos \omega t}$$

$$\left|\frac{J_d}{J}\right| = \frac{\omega \varepsilon_r \varepsilon_0}{\sigma}. \qquad ...(5)$$

But as for a good conductor

$$\varepsilon_r \approx 1^* \text{ and } \sigma \sim 10^7 \text{ mohs/meter}$$

$$\left|\frac{J_d}{J}\right| \approx \frac{2\pi \times f \times 9 \times 10^{-12}}{10^7} \sim f \times 10^{-17}$$

i.e. the displacement current in a good conductor is completely neligible compared to the conduction current at any frequency lower than optical frequencies ($f_{op} \sim 10^{15}$ Hertz).

Note : It is interesting to note that although conduction current (equation 3) is in phase with the electric field intensity (equation 1), the displacement current (equation 4) leads the electric field by $\pi/2$ radians *i.e.* displacement current leads the conduction current by $\pi/2$ radians.

§ 4.3. Maxwell's equations.

(A) The equation :

These are four fundamental equations of electromagnetism and corresponds to a generalisation of certain experimental observations-regarding electricity and magnetism. The following four laws of electricity and magnetism constitutes the so called '*differential form*' of Maxwell's equations :

* In note (ii) of application (d) in § 2.3 we have shown that for electrostatic effects a conductor acts like a material of infinite dielectric constant. However in case of steady current as polarisation effects are completely overshadowed by dispersion of metals this result does not hold good. For purpose of estimation, in case of conduction through metal we usually take $\varepsilon_r = n^2 \to 1$, from eqn. (A) in § 7.7.

(i) *Gauss' law* for the electric field of charge yields

$$\text{div } \mathbf{D} = \nabla . \mathbf{D} = \rho$$

where **D** is electric displacement in coulombs/m^2 and ρ is the free charge density in coul/m^3.

(ii) *Gauss' law* for magnetic field yields

$$\text{div } \mathbf{B} = \nabla . \mathbf{B} = 0$$

where **B** is the magnetic induction in web/m^2.

(iii) *Ampere's law* in circuital form for the magnetic field accompanying a current when modified by Maxwell yields

$$\text{curl } \mathbf{H} = \nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$$

where **H** is the magnetic field intensity in amperes/m and **J** is the current density in amp/m^2.

(iv) *Faraday's law* in circuital form for the induced electromotive force produced by the rate of change of magnetic flux linked with the path yields

$$\text{curl } \mathbf{E} = \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

where **E** is the electric field intensity in volts/m.

(B) Derivations :

(i) Let us consider a surface S bounding a volume τ within a dielectric. Originally the volume τ contains no net charge but we allow the dielectric to be polarised say by placing it in an electric field. We also deliberately place some charge on the di-electric body. Thus we have two type of charges :

(a) real charge of density ρ (b) bound charge density ρ'.

Gauss's law then can be written as

$$\oint_s \mathbf{E}\, ds = \frac{1}{\varepsilon_0} \int (\rho + \rho')\, d\tau$$

i.e.

$$\varepsilon_0 \oint_s \mathbf{E}. ds = \int_\tau \rho\, d + \tau \int \rho'\, d\tau. \qquad ...(1)$$

But as the bound charge density ρ' is defined as $\rho' = -\text{div } \mathbf{P}$ and

$$\oint_s \mathbf{E}. ds = \int_\tau \text{div } \mathbf{E}\, d\tau$$

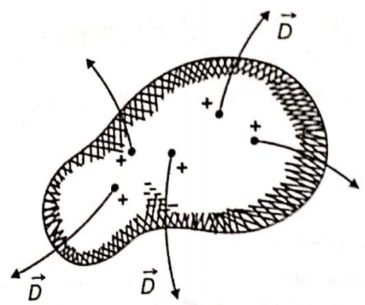

Fig. 4.3

So equation (1) becomes

$$\varepsilon_0 \int_\tau \text{div } \mathbf{E}\, d\tau = \int_\tau \rho\, d\tau - \int_\tau \text{div } \mathbf{P}\, d\tau$$

i.e. $$\int_\tau \text{div } (\varepsilon_0 \mathbf{E} + \mathbf{P})\, d\tau = \int_\tau \rho\, d\tau$$

or $$\int_\tau \text{div } \mathbf{D}\, d\tau = \int_\tau \rho\, d\tau \qquad (\text{as } \mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P})$$

or $$\int_\tau (\text{div } \mathbf{D} - \rho)\, d\tau = 0.$$

Since this equation is true for all volumes, the integrand must vanish. Thus we have

$$\text{div } \mathbf{D} = \nabla \bullet \mathbf{D} = \rho. \qquad \text{...(A)}$$

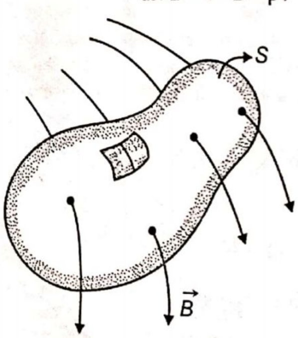

Fig. 4.4

(ii) Experiments to-date have shown that magnetic monopoles do not exist. This in turn implies that the magnetic lines of force are either closed group or go off to infinity. Hence the number of magnetic lines of force entering any arbitrary closed surface is exactly the same leaving it. Therefore the flux of magnetic induction **B** across any closed surface is always zero *i.e.*

$$\oint_S \mathbf{B} . d\mathbf{s} = 0.$$

Transforming this surface integral into volume integral by Gauss's theorem, we get

$$\int_\tau \text{div } \mathbf{B}\, d\tau = 0.$$

But as the surface bounding the volume is quite arbitrary the above equation will be true only when the integrand vanishes *i.e.*

$$\text{div } \mathbf{B} = \nabla \bullet \mathbf{B} = 0. \qquad \text{...(B)}$$

Note : For alternative methods of proving div **B** = 0 see example 1 in chapter 3.

(iii) From Ampere's circuital law the work done in carrying unit magnetic pole once round a closed arbitrary path linked with the current I is expressed by

$$\oint_C \mathbf{H} \bullet d\mathbf{l} = I$$

i.e. $$\oint_C \mathbf{H} \bullet d\mathbf{l} = \int_S \mathbf{J} \bullet d\mathbf{s} \qquad \left(\text{as } I = \int \mathbf{J} \bullet d\mathbf{s}\right)$$

where S is the surface bounded by the closed path C.

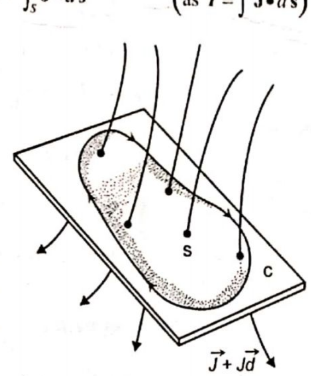

Fig. 4.5

Now changing the line integral into surface integral by Stoke's theorem, we get

$$\int_S \text{curl } \mathbf{H} \bullet d\mathbf{s} = \int_S \mathbf{J} \bullet d\mathbf{s}$$

i.e. $$\text{curl } \mathbf{H} = \mathbf{J}. \qquad \text{...(2)}$$

But Maxwell found it to be incomplete for changing electric fields and assumed that a quantity

$$\mathbf{J}_d = \frac{\partial \mathbf{D}}{\partial t}$$

called displacement current must also be included in it so that it may satisfy the continuity equation *i.e.* **J** must be replaced in equation (2) by $\mathbf{J} + \mathbf{J}_d$ so that the law becomes

$$\text{curl } \mathbf{H} = \mathbf{J} + \mathbf{J}_d$$

i.e. $$\text{curl } \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \qquad \text{...(C)}$$

(iv) According to Faraday's law of electromagnetic induction we know that the induced e.m.f. is proportional to the rate of change of flux *i.e.*

$$\epsilon = -\frac{d\phi_B}{dt}. \qquad ...(3)$$

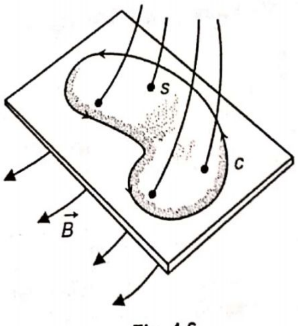

Fig. 4.6

Now if $\mathbf{E}$ be the electric intensity at a point the work done in moving a unit charge through a small distance $d\mathbf{l}$ is $\mathbf{E} \bullet d\mathbf{l}$. So the work done in moving the unit charge once round the circuit is $\oint_c \mathbf{E} \bullet d\mathbf{l}$.

Now as e.m.f. is defined as the amount of work done in moving a unit charge once round the electric circuit.

$$\epsilon = \oint_c \mathbf{E} \bullet d\mathbf{l} \qquad ...(4)$$

So comparing equation (3) and (4), we get

$$\oint_c \mathbf{E} \bullet d\mathbf{l} = -\frac{d\phi_B{}^*}{dt}. \qquad ...(5)$$

But as

$$\phi_B = \int_S \mathbf{B} \bullet ds$$

So

$$\oint_c \mathbf{E} \bullet d\mathbf{l} = -\frac{d}{dt}\int \mathbf{B} \bullet ds.$$

Transformating the line integral by Stoke's theorem into surface integral we get

$$\int_S \text{curl } \mathbf{E} \bullet ds = -\frac{d}{dt}\int \mathbf{B} \bullet ds.$$

Assuming that surface S is fixed in space and only $\boldsymbol{B}$ changes with time, above equation yields

$$\int_S \left(\text{curl } \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t}\right) \bullet ds = 0$$

* This equation shows that a changing magnetic field produces an electric field. However it has also been found that a changing electric field also produces a magnetic field. The magnetic field produced by changing electric field is given by

$$\oint_c \mathbf{B} \bullet d\mathbf{l} = \mu_0 \varepsilon_0 = \frac{d\phi_E}{dt} = \frac{1}{c^2}\frac{d\phi_E}{dt} \qquad \left[as\ \mu_0\varepsilon_0 = 1/c^2\right]$$

As the above integral is true for any arbitrary surface the integrand must vanish

i.e.

$$\text{curl } \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}. \qquad ...(D)$$

(C) Particular Cases :

(i) In a *conducting medium* of relative permitivity ε_r and permeability μ_r as

$$\mathbf{D} = \varepsilon\mathbf{E} = \varepsilon_r \varepsilon_0 \mathbf{E}$$

And

$$\mathbf{B} = \mu\mathbf{H} = \mu_r \mu_0 \mathbf{H}.$$

Maxwell's equation reduce to

(*i*) $\nabla \bullet \mathbf{E} = \rho/\varepsilon_r\varepsilon_0$ (*ii*) $\nabla \bullet \mathbf{H} = 0$

(*iii*) $\nabla \times \mathbf{H} = \mathbf{J} + \varepsilon_r\varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}$ (*iv*) $\nabla \times \mathbf{E} = -\mu_r\mu_0 \frac{\partial \mathbf{H}}{\partial t}$.

(ii) In a *non-conducting media* of relative permitivity ε_r and permeability μ_r as

$$\rho = \sigma = 0$$

so

$$\mathbf{J} = \sigma\mathbf{E} = 0$$

and hence Maxwell's equations become

(*i*) $\nabla \bullet \mathbf{E} = 0$ (*ii*) $\nabla \bullet \mathbf{H} = 0$

(*iii*) $\nabla \times \mathbf{H} = \varepsilon_r\varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}$ (*iv*) $\nabla \times \mathbf{E} = -\mu_r\mu_0 \frac{\partial \mathbf{H}}{\partial t}$.

(iii) *In free space as*

$$\varepsilon_r = \mu_r = 1$$

$$\rho = \sigma = 0.$$

Maxwell's equations become

(*i*) $\nabla \bullet \mathbf{E} = 0$ (*ii*) $\nabla \bullet \mathbf{H} = 0$

(*iii*) $\nabla \times \mathbf{H} = \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}$ (*iv*) $\nabla \times \mathbf{E} = -\mu_0 \frac{\partial \mathbf{H}}{\partial t}$.

(D) Discussion :

1. These equations are based on experimental observations. The equations : (A) and (C) correspond to electricity while (B) and (D) to magnetism.

2. These equations are general and apply to all electromagnetic phenomena in media which are at rest w.r.t. the coordinate system.

3. These equations are not independent of each other as from equation (D) we can derive (B) and from (C), (A) (see example-4). This is why equations (B) and (D) are called the first pair of Maxwell's equations while (A) and (C) are called the second pair.

4. The equation (A) represents Coulomb's law while (C) the law of conservation of charge *i.e.* continuity equation (see example 5).

5. If we compare equation (A) with (B) and (C) with (D) we find that left hand sides are identical while right hand sides are not. This is turn implies that electric and magnetic phenomena are assymmetric and this asymmetry arises due to the non-existance of monopoles.

Note : This asymmetry of electro-magnetism suggests that monopoles (a particle having either north or south magnetic charge) should exist as the concept of magnetic monopoles would bring to electricity and magnetism a symmetry to which nature loves and is lacking in our present picture. Dirac has also proved on theoretical grounds that monopole should exist and predicted their properties. But so far the magnetic monopoles has frustated all its investigators. The experiments have failed to find any sign of these. The theorists on the other hand have failed to find any good reason why monopoles should not exist.

Recently, American Institute of Physics and the University of California at Berkley jointly announced that monopole has been observed by a group of physists. If confirmed, the detection of monopoles will have a major impact on Physics and Technology.

(6) The correspondence of **B** and **H** with **E** and **D** through Maxwell equations (D) and (C) respectively implies that in case of time dependent fields the electric and magnetic fields are inseparably linked with each other giving rise to what is known as electromagnetic field and it is not possible to deal separately with electric and magnetic fields in this situation.

(E) Physical Significance (or Integral Form)

By means of Gauss's and Stoke's Theorems we can write the Maxwell's field equations in integral form and hence obtain their physical significance.

(i) Integrating Maxwell's first equation div $\mathbf{D} = \rho$ over an arbitrary volume τ we get

$$\int_\tau \nabla \bullet \mathbf{D}\, d\tau = \int_\tau \rho\, d\tau$$

changing the vol. integral of L. H. S. into surface integral by Gauss's divergence theorem and keeping in mind that $\int \rho\, d\tau = q$ we get

$$\oint \mathbf{D} \bullet d\mathbf{s} = q \qquad \text{...}(A_1)$$

So Maxwell's first equation signifies that *the total flux of electric displacement linked with a closed surface is equal to the total charge enclosed by the closed surface.*

(ii) Integrating Maxwell's second equation div $\mathbf{B} = 0$ over an arbitrary vol. τ we get

$$\int_\tau \nabla \bullet \mathbf{B}\, d\tau = 0.$$

Converting the vol. integral into surface integral with the help of Gauss's theorem we get

$$\oint \mathbf{B} \bullet d\mathbf{s} = 0 \qquad \text{...}(B_1)$$

So Maxwell's **II** equation signifies that *the total flux of magnetic induction linked with a closed surface is zero.*

(iii) Integrating Maxwell's III equation curl $\mathbf{H} = \mathbf{J} + (\partial \mathbf{D}/\partial t)$ over a surface S bounded by the loop C we get

$$\int \text{curl } \mathbf{H} \bullet d\mathbf{s} = \int \left(\mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \right) \bullet d\mathbf{s}$$

Converting the surface integral of **L.H.S.** into line integral with the help of stoke's theorem we get

$$\oint \mathbf{H} \bullet d\mathbf{l} = \int \left(\mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \right) \bullet d\mathbf{s} \qquad \text{...}(C_1)$$

which signifies that *magnetomotive force around a closed path* $\left[\oint \mathbf{H} \bullet d\mathbf{l}\right]$ *is equal to the conduction current plus displacement current linked with that path.*

(iv) Integrating Maxwell's IV equation curl $\mathbf{E} = -(\partial \mathbf{B}/\partial t)$ over a surface S bounded by the loop C we get

$$\int \text{curl } \mathbf{E} \bullet d\mathbf{s} = -\int \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{s}$$

Converting the surface integral of **L.H.S.** into line integral with the help of Stoke's Theorem we get

$$\oint \mathbf{E} \bullet d\mathbf{l} = -\frac{\partial}{\partial t} \int \mathbf{B} \bullet d\mathbf{s} \qquad \text{...}(D_1)$$

which signifies that *the electromotive force i.e. line integral of electric intensity around a closed path is equal to the negative rate of change of magnetic flux linked with the path.*

Example 4. *Starting with Maxwell's equations*

$$curl\ \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \quad and \quad curl\ \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$$

respectively show that

$$div\ \mathbf{B} = 0 \quad and \quad div\ \mathbf{E} = \rho$$

Solution. Taking the divergence of given Maxwell's equations we get

$$\text{div curl } \mathbf{E} = -\text{div} \frac{\partial}{\partial t} (\mathbf{B}) \text{ and div curl } \mathbf{H} = \text{div}\left(\mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}\right)$$

Now as the div curl of any vector is zero and space and time operations are interchangable, above equations reduce to

$$\frac{\partial}{\partial t}(\text{div } \mathbf{B}) = 0 \text{ and } \frac{\partial}{\partial t}(\text{div } \mathbf{D}) = -\text{div } \mathbf{J}$$

or $$\frac{\partial}{\partial t}(\text{div } \mathbf{B}) = 0 \text{ and } \frac{\partial}{\partial t}(\text{div } \mathbf{D}) = \frac{\partial \rho}{\partial t} \qquad \left(\text{as div } \mathbf{J} + \frac{\partial \rho}{\partial t} = 0\right)$$

Now if for each point of space div **B** and div **D** become zero at any time either in the past or in future, above equation on integration yields

$$\text{div } \mathbf{B} = 0 \text{ and div } \mathbf{D} = \rho$$

Example 5. *Starting from Maxwell's equations prove (a) Coulomb's law, (b) Continuity equation.*

Solution. (*a*) From Maxwell's first equation we have

$$\text{div } \mathbf{D} = \rho \qquad ...(1)$$

Integrating this equation over a sphere of radius r we get

$$\int_\tau \text{div } \mathbf{D}\, d\tau = \int_\tau \rho\, d\tau$$

i.e. $$\oint_s \mathbf{D} \bullet d\mathbf{s} = q$$

i.e. $$\varepsilon_0 \oint_s \mathbf{E} \bullet d\mathbf{s} = q \qquad (\text{as } \mathbf{D} = \varepsilon_r \varepsilon_0 \mathbf{E} = \varepsilon_0 \mathbf{E})$$

i.e. $$\varepsilon_0 \mathbf{E}(4\pi r^2) = q$$

i.e. $$\mathbf{E} = \frac{1}{4\pi\varepsilon_0} \frac{q}{r^2}$$

i.e. $$\mathbf{E} = \frac{1}{4\pi\varepsilon_0} \frac{q}{r^3} \mathbf{r} \qquad (\text{as } \mathbf{E} \text{ is radial}) \qquad ...(2)$$

So the force experienced by a test charge q_0

$$\mathbf{F} = q_0 \mathbf{E} = \frac{1}{4\pi\varepsilon_0} \frac{q q_0}{r^3} \mathbf{r}$$

This is the required result.

(b) From Maxwell's third equation we have

$$\text{curl } \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}. \qquad ...(3)$$

Taking divergence of both sides of this equation

$$\text{div curl } \mathbf{H} = \text{div}\left(\mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}\right).$$

Now as the div curl of any vector is zero, above equation reduces to

$$\text{div}\left(\mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}\right) = 0$$

i.e. $$\text{div } \mathbf{J} + \text{div}\left(\frac{\partial \mathbf{D}}{\partial t}\right) = 0$$

i.e. $$\text{div } \mathbf{J} + \frac{\partial}{\partial t}(\text{div } \mathbf{D}) = 0 \qquad ...(4)$$

or $$\text{div } \mathbf{J} + \frac{\partial \rho}{\partial t} = 0 \quad (\text{as from eqn. (1) } \nabla \bullet \mathbf{D} = \rho) \qquad ...(5)$$

This is the required result.

§ 4.4. Energy in Electromagnetic fields. (Poynting's theorem)

From Maxwell's equations it is possible to derive an important expression which we shall recognise at the energy principle in an electromagnetic field.

For this consider Maxwell's equations (C) and (D) *i.e.* Ampere's and Faraday's laws in differential forms

$$\text{curl } \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}. \qquad ...(1)$$

and $$\text{curl } \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}. \qquad ...(2)$$

If we take the scalar product of equation (1) with **E** and of equation (2) with (−**H**) we get

$$\mathbf{E} \bullet \text{curl } \mathbf{H} = \mathbf{E} \bullet \mathbf{J} + \mathbf{E} \bullet \frac{\partial \mathbf{D}}{\partial t}. \qquad ...(3)$$

and
$$-\mathbf{H}\bullet\text{curl}\,\mathbf{E} = +\mathbf{H}\bullet\frac{\partial \mathbf{D}}{\partial t}. \qquad ...(4)$$

adding equation (3) and (4) we get

$$-\mathbf{H}\bullet\text{curl}\,\mathbf{E} + \mathbf{E}\bullet\text{curl}\,\mathbf{H} = \mathbf{J}\bullet\mathbf{E} + \left[\mathbf{E}\bullet\frac{\partial \mathbf{D}}{\partial t} + \mathbf{H}\bullet\frac{\partial \mathbf{B}}{\partial t}\right]$$

But by the vector identity

$$\mathbf{H}\bullet\text{curl}\,\mathbf{E} - \mathbf{E}\bullet\text{curl}\,\mathbf{H} = \text{div}\,(\mathbf{E}\times\mathbf{H})$$

The above equation reduces to

$$-\text{div}\,(\mathbf{E}\times\mathbf{H}) = \mathbf{J}\bullet\mathbf{E} + \left[\mathbf{E}\bullet\frac{\partial \mathbf{D}}{\partial t} + \mathbf{H}\bullet\frac{\partial \mathbf{B}}{\partial t}\right]. \qquad ...(5)$$

Now as $\mathbf{E}\bullet\frac{\partial \mathbf{D}}{\partial t} = \varepsilon_r\varepsilon_0\,\mathbf{E}\frac{\partial \mathbf{E}}{\partial t} = \frac{1}{2}\varepsilon_r\varepsilon_0\frac{\partial}{\partial t}(\mathbf{E}\bullet\mathbf{E}) = \frac{1}{2}\frac{\partial}{\partial t}(\mathbf{E}\bullet\mathbf{D})$

and $\mathbf{H}\bullet\frac{\partial \mathbf{B}}{\partial t} = \mu_r\mu_0\,\mathbf{H}\bullet\frac{\partial \mathbf{H}}{\partial t} = \frac{1}{2}\mu_r\mu_0\frac{\partial}{\partial t}(\mathbf{H}\bullet\mathbf{H}) = \frac{1}{2}\frac{\partial}{\partial t}(\mathbf{H}\bullet\mathbf{B})$

So equation (5) reduces to

$$\mathbf{J}\bullet\mathbf{E} + \frac{1}{2}\frac{\partial}{\partial t}(\mathbf{E}\bullet\mathbf{D} + \mathbf{H}\bullet\mathbf{B}) + \text{div}\,(\mathbf{E}\times\mathbf{H}) = 0. \qquad ...(6)$$

Each term in the above equation can be given some physical meaning if it is multiplied by an element of volume $d\tau$ and integrated over a volume τ whose enclosing surface is S. Thus the result is

$$\int_\tau (\mathbf{J}\bullet\mathbf{E})d\tau + \int_\tau \frac{1}{2}\frac{\partial}{\partial t}(\mathbf{E}\bullet\mathbf{D} + \mathbf{H}\bullet\mathbf{B})\,d\tau + \int_\tau \text{div}\,(\mathbf{E}\times\mathbf{H})\,d\tau = 0$$

But as $\int_\tau \text{div}\,(\mathbf{E}\times\mathbf{H})\,d\tau = \oint_s (\mathbf{E}\times\mathbf{H})\cdot d\mathbf{s}$

so
$$\int_\tau (\mathbf{J}\bullet\mathbf{E})\,d\tau + \int_\tau \frac{1}{2}\frac{\partial}{\partial t}(\mathbf{E}\bullet\mathbf{D} + \mathbf{H}\bullet\mathbf{B})\,d\tau + \oint_s (\mathbf{E}\times\mathbf{H})\cdot d\mathbf{s} = 0 \qquad ...(A)$$

To understand what equation (A) means, let us now interpret various term in it—

(A) Interpretation of $\int_\tau (\mathbf{J}\bullet\mathbf{E})\,d\tau$:

The current distribution represented by the vector **J** can be considered as made up of various changes q_i moving with velocity $\mathbf{v}_i$ so that

$$\int \mathbf{J}\bullet\mathbf{E}\,d\tau = \int I\,d\mathbf{l}\bullet\mathbf{E} \qquad [\text{as } \mathbf{J}\,d\tau = I\,d\mathbf{l}]$$

$$= \int dq\,\mathbf{v}\cdot\mathbf{E} \qquad [\text{as } I\,d\mathbf{l} = (dq/dt)d\mathbf{l} = dq\,\mathbf{v}]$$

$$= \sum q_i\,(\mathbf{v}_i\cdot\mathbf{E}_i) \qquad ...(7)$$

where $\mathbf{E}_i$ denotes the electric field at the position of charge q_i.

Now electromagnetic force on the *i*th charged particle is given by the *Lorentz expression*

$$\mathbf{F}_i = q_i\,(\mathbf{E}_i + \mathbf{v}_i \times \mathbf{B}_i).$$

So the work done per unit time on the charge q_i by the field will be

$$\frac{\partial W_i}{dt} = \mathbf{F}_i\bullet\mathbf{v}_i \qquad \left[\frac{\partial W}{dt} = \frac{\mathbf{F}.d\mathbf{l}}{dt} = \mathbf{F}\bullet\mathbf{v}\right]$$

$$= q_i\,(\mathbf{E}_i + \mathbf{v}_i \times \mathbf{B}_i)\bullet\mathbf{v}_i \qquad (\text{as } \mathbf{F}_i = q_i\,(\mathbf{E}_i + \mathbf{v}_i \times \mathbf{B}_i))$$

i.e.
$$\frac{dW_i}{dt} = q_i\mathbf{v}_i\bullet\mathbf{E}_i \qquad [\text{as } \mathbf{v}_i\bullet(\mathbf{v}_i \times \mathbf{B}_i) = (\mathbf{v}_i \times \mathbf{v}_i)\bullet\mathbf{B}_i = 0]$$

So the rate at which the work is done by the field on the charges is

$$\frac{\partial W}{\partial t} = \sum\frac{\partial W_i}{\partial t} = \sum q_i\mathbf{v}_i\bullet\mathbf{E}_i. \qquad ...(8)$$

Comparing equation (7) and (8) we find that

$$\int \mathbf{J}\bullet\mathbf{E}\,d\tau = \frac{dW}{dt} \qquad ...(9)$$

i.e. the first term $\int(\mathbf{J}\bullet\mathbf{E})\,d\tau$ *represents the rate at which work is done by the field on the charges*

Note : It is worthly to note here that—

(i) In case of charged particles moving in free space with no external force acting, the work done by the field on the charges appears as kinetic energy of the particles as

$$\frac{\partial W}{\partial t} = \sum\frac{\partial W_i}{\partial t} = \sum\mathbf{F}_i\bullet\mathbf{v}_i$$

i.e.
$$\frac{\partial W}{\partial t} = \sum m_i\frac{\partial \mathbf{v}_i}{\partial t}\bullet\mathbf{v}_i\left(\text{as } \mathbf{F}_i = m_i\frac{\partial \mathbf{v}_i}{\partial t}\right)$$

i.e.
$$\frac{\partial W}{\partial t} = \frac{\partial}{\partial t}\left(\tfrac{1}{2}m_i\mathbf{v}_i\bullet\mathbf{v}_i\right) = \sum\frac{\partial}{\partial t}\left(\tfrac{1}{2}m_i\mathbf{v}_i^2\right)$$

i.e.
$$\frac{\partial W}{\partial t} = \frac{\partial}{\partial t}\sum\tfrac{1}{2}m_i\mathbf{v}_i^2 = \frac{\partial \mathrm{T}}{\partial t}.$$

(ii) Inside matter, the work done by the field on the charge *i.e.* the kinetic energy is transferred to random motion, where it is described as heat energy or ohmic loss and is given by

$$\frac{\partial W}{\partial t} = \int_\tau \mathbf{J} \bullet \mathbf{E}\, d\tau = \int_\tau \frac{J^2}{\sigma}\, d\tau \qquad (\text{as } \mathbf{E} = \mathbf{J}/)$$

i.e.
$$\frac{\partial W}{\partial t} = \frac{J^2}{\sigma} \times S \times l\left(\text{as} \int_\tau d\tau = Sl\right)$$

i.e.
$$\frac{\partial W}{\partial t} = \rho \frac{I^2}{S^2} Sl\left(\text{as } \sigma = 1/\rho \text{ and } J = I/S\right).$$

i.e.
$$\frac{\partial W}{\partial t} = I^2 \rho \frac{l}{S} = I^2 R\ (\text{as } R = \rho\ l/S).$$

(B) Interpretation of $\int_\tau \frac{1}{2}\frac{\partial}{\partial t}(\mathbf{E}\bullet\mathbf{D} + \mathbf{H}\bullet\mathbf{B})\ d\tau$

If we allow the volume τ to be arbitrary large the surface integral in eqn. (A) can be made to vanish by placing the surface S sufficiently far away so that the field cannot propagate to this distance in any finite time *i.e.* $\oint_s (\mathbf{E}\times\mathbf{H})\bullet d\mathbf{s} = 0.$ So under these circumstances equation (A) reduces to

$$\frac{\partial}{\partial t}\int_{\text{all space}} \tfrac{1}{2}(\mathbf{E}\bullet\mathbf{D} + \mathbf{H}\bullet\mathbf{B})\ d\tau + \frac{\partial W}{\partial t} = 0$$

i.e.
$$\frac{\partial}{\partial t}\left[\int_{\text{all space}} \tfrac{1}{2}(\mathbf{E}\bullet\mathbf{D} + \mathbf{H}\bullet\mathbf{B})\ d\tau + W\right] = 0$$

Thus the quantity in the square bracket is conserved. Now consider a closed system in which the total energy is assumed to be constant. The system consists of the electromagnetic field and of all the charged particles present in the field. The term W represents the total kinetic energy of the particles. We are therefore led to associate the remaining energy term

$$\int_{\text{all space}} \tfrac{1}{2}(\mathbf{E}\bullet\mathbf{D} + \mathbf{H}\bullet\mathbf{B})d\tau$$

with the energy of the electro-magnetic field, *i.e.*

$$U = \int_{\text{all space}} \tfrac{1}{2}(\mathbf{E}\bullet\mathbf{D} + \mathbf{H}\bullet\mathbf{B})\ d\tau. \qquad ...(10)$$

The quantity U may be considered to be a kind of potential energy. One need not ascribe this potential energy to the charged particles and must consider this term as a field energy. A concept such as energy stored in the field itself rather than residing with the particles is a basic concept of the theory of electromagnetism.

Note : If we write equation (10) as

$$U = \int_{\text{all space}} u\ d\tau$$

where $u = \frac{1}{2}(\mathbf{E}\bullet\mathbf{D} + \mathbf{H}\bullet\mathbf{B})$ may be thought of as the energy density of the electro-magnetic field.

Further as

$$u = \tfrac{1}{2}\mathbf{E}\bullet\mathbf{D} + \tfrac{1}{2}\mathbf{H}\bullet\mathbf{B}$$

first term on R.H.S. contains only electrical quantities while the second, one magnetic, we can have

$$u = u_e + u_m$$

with $u_e = \frac{1}{2}\mathbf{E}\bullet\mathbf{D} = \frac{1}{2}\varepsilon_r\varepsilon_0 E^2 =$ energy density of electric field

and $u_m = \frac{1}{2}\mathbf{H}\bullet\mathbf{B} = \frac{1}{2}\mu_r\mu_0 H^2 =$ energy density of magnetic field

(C) Interpretation of $\oint_s (\mathbf{E}\times\mathbf{H})\bullet d\mathbf{s}$.

Instead of taking the volume integral in equations (A) over all space, let us now consider a finite volume. In this case the surface integral of $(\mathbf{E}\times\mathbf{H})$ will not in general vanish and so this term must be retained. Let us construct the surface S in such a way that in the interval of time under consideration, none of the charged particles will cross this surface. Then for the conservation of energy

$$\frac{\partial U}{\partial t} + \frac{\partial W}{\partial t} = -\oint_s (\mathbf{E}\times\mathbf{H})\bullet d\mathbf{s} \qquad ...(11)$$

The left hand side is the time rate of change of the energy of the field and of the particles contained within the volume τ. Thus the surface intergral $\oint_s (\mathbf{E}\times\mathbf{H})\bullet d\mathbf{s}$ must be considered as the energy flowing out of the volume bounded by the surface S per sec. But by hypothesis no particles are crossing the surface, so the vector $(\mathbf{E}\times\mathbf{H})$ is to be interpreted as the amount of the field energy passing through unit area of the surface in unit time which is normal to the direction of energy flow. The vector $(\mathbf{E}\times\mathbf{H})$ is called Poynting vector* and is represented by $\boldsymbol{S}$ *i.e.*

$$\mathbf{S}=(\mathbf{E}\times\mathbf{H}). \qquad ...(12)$$

Interpretation of the Energy Equation.

In the light of above, equation (6) in differential form can be written as

$$\mathbf{J}\cdot\mathbf{E}+\frac{\partial u}{\partial t}+\nabla\cdot\mathbf{S}=0. \qquad ...(13)$$

In the event that the medium has zero conductivity *i.e.* $\mathbf{J}=\sigma\mathbf{E}=0$, the above equation becomes exactly of the same form as the continuity equation which expresses the law of conservation of charge. We are led by this analogy that the physical meaning of equation 13, 11 or (A) s to represent the law of conservation of energy for electromagnetic phenomena. According to equation (11) *the time rate of change of electromagnetic energy within a certain volume plus the rate at which the work is done by the field on the charges is equal to the energy flowing into the system through its bounding surface per unit time.*

Example 6. *Show that for a cylindrical resistor of length l, radius r and resistivity ρ, the rate of flow of energy P, at which energy flows into the resistor through its cylindrical surface (calculated by integrating the Poynting vector over this surface) is equal to the rate at which Joule heat is produced i.e.*

$$P=I^2R.$$

Solution. As in case of a conductor

$$\mathbf{J}=\sigma\mathbf{E} \quad i.e. \quad \mathbf{E}=\rho\mathbf{J}. \quad (\text{as } \rho=1/\sigma) \qquad ...(1)$$

The electric field is parallel to the direction $\mathbf{J}$. In addition to the above electric field there is also a magnetic field whose direction is given by 'right hand thumb' rule and magnitude by Ampere's law *i.e.*

$$\oint_C \mathbf{H}\cdot d\mathbf{l}=\int_S \mathbf{J}\cdot d\mathbf{s}$$

i.e.
$$H\,2\pi r=J\,\pi r^2$$

i.e.
$$H=\tfrac{1}{2}Jr \qquad ...(2)$$

* For details see Article 4.5.

The direction of $\mathbf{E}$ and $\mathbf{H}$ are perpendicular to each other as shown in fig. 4.7, so the Poynting vector $\mathbf{S}=\mathbf{E}\times\mathbf{H}$ will be perpendicular to the plane of E and H and into the paper. Its magnitude will be

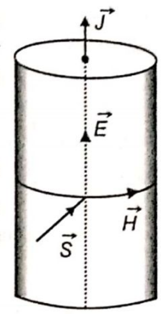

Fig. 4.7

$$S=EH=\rho J\times\tfrac{1}{2}Jr$$

i.e.
$$S=\tfrac{1}{2}\rho J^2 r \qquad ...(3)$$

As a consequence of this rate of flow of field energy into the resistor through its cylindrical surface will be

$$P=-\oint_S \mathbf{S}\cdot d\mathbf{s}=\oint_S S\,ds$$

[as S and ds are anti-parallel]

i.e.
$$\int \tfrac{1}{2}\rho J^2 r\,ds=\tfrac{1}{2}\rho J^2 r\times 2\pi r\,l \qquad \left[\text{as} \int ds=2\pi rl\right]$$

i.e.
$$P=\rho J^2(\pi r^2 l)=I^2\rho\frac{l}{\pi r^2} \qquad \left[\text{as } J=\frac{I}{\pi r^2}\right]$$

i.e.
$$P=I^2R\left[\text{as } R=\rho\frac{l}{\pi r^2}\right].$$

Note : (i) This example clearly shows that according to field energy point of view, the energy that appears in a resistor as Joule heat does not enter it through the connecting wires but through the space around the wires and the resistor.

(ii) As in this example

$$U=\left(\tfrac{1}{2}\varepsilon_0E^2+\tfrac{1}{2}\mu_0H^2\right)\pi r^2 l$$

i.e.
$$U=\left[\tfrac{1}{2}\varepsilon_0(\rho J)^2+\mu_0\left(\tfrac{1}{2}Jr\right)^2\right]\pi r^2 l$$

i.e.
$$U=\left[\tfrac{1}{2}\varepsilon_0\rho^2+\tfrac{1}{2}\mu_0 r^2\right]J^2\pi r^2 l=\text{constant}$$

or
$$(\partial U/\partial t)=0$$

So this example is an illustration of

$$\frac{\partial W}{\partial t}=-\oint_S \mathbf{S}\cdot d\mathbf{s}$$

of the general equation for conservation of field energy

i.e.
$$\frac{\partial U}{\partial t}+\frac{\partial W}{\partial t}=-\oint_S \mathbf{S}\cdot d\mathbf{s}.$$

Example 7. *A parallel plate capacitor as shown in fig. 4.8 is being charged. Show that the rate at which energy flow into the capacitor from the surrounding space. (Calculated by integrating the Poynting* *the surface of the earth, the radius of the sun is* 7×10^8 *m and the average distance between earth and sun is* $1{\cdot}5\times10^{11}$ *m.*

(b) If an interplanetary 'sail plane' had a sail of mass 10^{-4} *gm/cm*^{2} *and other weights neglible, what would be its maximum acceleration in centimeters per sec*^{2} *due to solar radiation. (Solar constant is 2 cal/cm*^{2}*-min).*

Solution :

(a) As $S_E = \dfrac{2\text{cal}}{\text{cm}^2\times\text{min}} = \dfrac{2\times4{\cdot}2}{10^{-4}\times60} = 1{\cdot}5\times10^3 \dfrac{\text{Joules}}{\text{m}^2-\text{sec}}$

$$\therefore\quad (P_{\text{rad}})_E = \frac{S_E}{c} = \frac{1{\cdot}5\times10^3}{3\times10^8} = 5\times10^{-6} N/m^2 \qquad \textbf{Ans.}$$

Further as (See example 8)

$$S_S r_s^2 = S_E r_{ES}^2$$

i.e.
$$S_S = S_E \frac{r_{ES}^2}{r_s^2} = 1{\cdot}5\times10^3\times\left(\frac{1{\cdot}5\times10^{11}}{7\times10^8}\right)$$

$$\approx 6\times10^7 \text{ Watt/m}^2$$

$$\therefore\quad (P_{\text{rad}})_S = \frac{S_S}{c} = \frac{6\times10^7}{3\times10^8} = 0{\cdot}2\ N/m^2 \qquad \textbf{Ans.}$$

(b) As in part (a)

$$(P_{\text{rad}})_E = 5\times10^{-6}\ N/m^2 = 5\times10^{-6}\times\frac{10^5\text{ dynes}}{10^4\text{ cm}^2} = 5\times10^{-5}\frac{\text{dynes}}{\text{cm}^2}$$

Now as pressure $= \dfrac{\text{Force}}{\text{area}} = \dfrac{\text{mass}\times\text{acc}}{\text{area}}$

i.e.
$$\text{acc} = \frac{\text{pressure}}{\text{mass per unit area}} = \frac{5\times10^{-5}}{10^{-4}} = 0{\cdot}5\text{ cm/sec}^2.$$

§ 4.7. Electromagnetic Potentials A and ϕ.

The analysis of an electromagnetic field is often facilitated by the use of auxiliary functions known as electromagnetic potentials. At every point of space the field vectors satisfy the equations

$$\text{div}\,\mathbf{D} = \rho. \qquad \text{...(A)}$$

$$\text{div}\,\mathbf{B} = 0. \qquad \text{...(B)}$$

$$\text{curl}\,\mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}. \qquad \text{...(C)}$$

$$\text{curl}\,\mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}. \qquad \text{...(D)}$$

According to equation (B) the field of vector **B** is always *solenoidal*, consequently **B** can be represented as the curl of another vector say **A** *i.e.*

$$\mathbf{B} = \text{curl}\,\mathbf{A} \qquad \text{...(1)}$$

where **A** is a vector which is function of space (x, y, z) and time (t) both.

Now substituting the value of **B** in equation (D) from (1), we get

$$\text{curl}\,\mathbf{E} = -\frac{\partial}{\partial t}(\text{curl}\,\mathbf{A})$$

or
$$\text{curl}\left(\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t}\right) = 0$$

i.e. the field of the vector $\mathbf{E} + \dfrac{\partial \mathbf{A}}{\partial t}$ is irrotational and must be equal to the gradient of some scalar function *i.e.*

$$\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t} = -\text{grad}\,\phi$$

or
$$\mathbf{E} = -\text{grad}\,\phi - \frac{\partial \mathbf{A}}{\partial t}. \qquad \text{...(2)}$$

Thus we have introduced a vector **A** and a scalar ϕ both being functions of position and time. These are called electromagnetic potentials. The scalar ϕ is called the scalar potential and the vector **A**, vector potential. Regarding electromagnetic potentials it is worth noting that

(i) These are mathematical functions which are not physically measurable.

(ii) These are not independent of each other.

(iii) These define the field vector **E** and **B** uniquely but themselves are non-unique (See § 4.7).

(iv) These play an important role in relativistic electrodynamics (See chapter 10).

Example 11. *Show that the potentials at the position defined by the vector* **r** *in uniform electric and magnetic fields may be written as*

$$\phi = -\mathbf{E}\cdot\mathbf{r} \text{ and } \mathbf{A} = \tfrac{1}{2}\,(\mathbf{B}\times\mathbf{r})$$

Solution. (a) We know that

$$\mathbf{E} = \mathbf{i}\,E_x + \mathbf{j}E_y + \mathbf{k}\,E_z = (\mathbf{E}\cdot\nabla)\,\mathbf{r}$$

§ 4.8. Maxwell equations in terms of Electromagnetic Potentials.

Now consider the Maxwell's equation (C) *i.e.*

$$\mu \operatorname{curl} \mathbf{H} = \mu \mathbf{J} + \mu \frac{\partial \mathbf{D}}{\partial t} \qquad ...(1)$$

or

$$\operatorname{curl} \mathbf{B} = \mu \mathbf{J} + \varepsilon\mu \frac{\partial \mathbf{E}}{\partial t} \qquad ...(2)$$

substituting **B** and **E** from equations (1) and (2) of § 4.7 in above we get

$$\operatorname{curl}(\operatorname{curl} \mathbf{A}) = \mu \mathbf{J} + \mu\varepsilon \frac{\partial}{\partial t}\left(-\operatorname{grad}\phi - \frac{\partial \mathbf{A}}{\partial t}\right)$$

i.e.

$$\operatorname{grad} \operatorname{div} \mathbf{A} - \nabla^2 \mathbf{A} = \mu \mathbf{J} - \mu\varepsilon \frac{\partial}{\partial t}(\operatorname{grad}\phi) - \mu\varepsilon \frac{\partial^2 \mathbf{A}}{\partial t^2}$$

i.e.

$$\nabla^2 \mathbf{A} - \mu\varepsilon \frac{\partial^2 \mathbf{A}}{\partial t^2} - \operatorname{grad}\left(\operatorname{div} \mathbf{A} + \mu\varepsilon \frac{\partial \phi}{\partial t}\right) = -\mu \mathbf{J}. \qquad ...(3)$$

Similarly if we consider equation (A) *i.e.*

$$\operatorname{div} \mathbf{D} = \rho$$

i.e.

$$\varepsilon \operatorname{div} \mathbf{E} = \rho$$

i.e.

$$\operatorname{div}\left(-\operatorname{grad}\phi - \frac{\partial \mathbf{A}}{\partial t}\right) = \frac{\rho}{\varepsilon}$$

i.e.

$$\nabla^2 \phi + \frac{\partial}{\partial t}(\operatorname{div} \mathbf{A}) = -\frac{\rho}{\varepsilon}.$$

Adding and subtracting $\mu\varepsilon \frac{\partial^2 \phi}{\partial t^2}$ it becomes

$$\nabla^2 \phi - \mu\varepsilon \frac{\partial^2 \phi}{\partial t^2} + \frac{\partial}{\partial t}\left(\operatorname{div} \mathbf{A} + \mu\varepsilon \frac{\partial \phi}{\partial t}\right) = -\frac{\rho}{\varepsilon}. \qquad ...(4)$$

Equation (3) and (4) are field equations in terms of electromagnetic potentials, as equations (B) and (D) are satisfied in defining the scalar and vector potentials. So Maxwell equations are reduced from four to two by electromagnetic potentials, however these are coupled.

Note : The reader may be happy to note that the concept of electromagnetic potential, reduces the number of Maxwell's equations from four to two, and even more or less of the same form. Actually this all is not a real simplicity but a trick which hides the complexity in the definition of the symbols **A** and ϕ.

To make the above statement more clear authors want to put something more interesting and fascinating that has been discovered recently. All the laws of physics can be contained in a single equation

$$U = 0$$

where U is a physical quantity called "*unwordlines*" of the situation and we have formula for it.

For example suppose we take the law of mechanics $\mathbf{F} = m\mathbf{a}$ and rewrite it as $\mathbf{F} - m\mathbf{a} = 0$ then $(\mathbf{F} - m\mathbf{a})$ is called the "*mismatch*" of mechanics and the square of 'mismatch' *i.e.* $(\mathbf{F} - m\mathbf{a})^2$ is called the unwordliness of mechanics, in other words

$$U_1 = (\mathbf{F} - m\mathbf{a})^2$$

In the same way we can calculate the unwordlines for every law of physics, then the total unwordlines will be

$$U = \sum U_i$$

and the greatest law of nature will be

$$U = 0.$$

Obviously this law though appears to be beautifully simple, in notation is most complex.

§ 4.9. Non-uniqueness of Electromagnetic Potentials and concept of Gauge :

In terms of electromagnetic potentials field vectors are given by

$$\mathbf{B} = \text{curl}\,\mathbf{A} \qquad ...(1)$$

and

$$\mathbf{E} = -\text{grad}\,\phi - \frac{\partial \mathbf{A}}{\partial t}. \qquad ...(2)$$

From equations (1) and (2) it is clear that for a given $\mathbf{A}$ and ϕ, each of the field vectors $\mathbf{B}$ and $\mathbf{E}$ has only one value *i.e.* $\mathbf{A}$ and ϕ determine $\mathbf{B}$ and $\mathbf{E}$ uniquely. However the converse is not true *i.e.* field vectors do not determine the potentials $\mathbf{A}$ and ϕ completely. This is turn implies that for a given $\mathbf{A}$ and ϕ there will be only one $\mathbf{E}$ and $\mathbf{B}$ while for a given $\mathbf{E}$ and $\mathbf{B}$ there can be an infinite number of $\mathbf{A}$'s and ϕ's. This is because the curl of the gradient of any scalar vanishes identically and hence we may add to $\mathbf{A}$ the gradient of a scalar Λ without affecting $\mathbf{B}$. That is $\mathbf{A}$ may be replaced by

$$\mathbf{A}' = \mathbf{A} + \text{grad}\,\Lambda. \qquad ...(3)$$

But if this is done, equation (2) becomes

$$\mathbf{E} = -\text{grad}\,\phi - \frac{\partial}{\partial t}(\mathbf{A}' - \text{grad}\,\Lambda)$$

i.e.

$$\mathbf{E} = -\text{grad}\left(\phi - \frac{\partial \Lambda'}{\partial t}\right) - \frac{\partial \mathbf{A}'}{\partial t}.$$

So if we make the transformation given by eqn. (3) we must also replace ϕ by

$$\phi' = \phi - \frac{\partial \Lambda}{\partial t}. \qquad ...(4)$$

The expressions for field vectors $\mathbf{E}$ and $\mathbf{B}$ remain unchanged under transformations (3) and (4) *i.e.*

$$\mathbf{B} = \text{curl}\,\mathbf{A} = \text{curl}\,(\mathbf{A}' - \text{grad}\,\Lambda) = \text{curl}\,\mathbf{A}'$$

and

$$\mathbf{E} = -\text{grad}\,\phi - \frac{\partial \mathbf{A}}{\partial t} = -\text{grad}\left(\phi' + \frac{\partial \Lambda}{\partial t}\right) - \frac{\partial \Lambda}{\partial t}(\mathbf{A}' - \text{grad}\,\Lambda)$$

$$= \text{grad}\,\phi' - \frac{\partial \mathbf{A}'}{\partial t}$$

i.e. we get the same field vectors whether we use the set $(\mathbf{A}, \phi)$ or $(\mathbf{A}', \phi')$. *So electromagnetic potentials define the field vectors uniquely though they themselves are non-unique.*

The transformations given by equations (3) and (4) are called gauge transformations and the arbitrary scalar Λ gauge function. From the above it is also clear that even though we add the gradient of a scalar function, the field vectors remain unchanged. Now it is the field quantities and not the potentials that possess physical meaningfulness. We therefore say that the *field vectors are invarient to gauge transformations i.e.* they are gauge invarient.

Because of the arbitrariness in the choice of gauge *i.e.* non-uniqueness of potentials, we are free to impose an additional condition on $\mathbf{A}$. We may state this in other terms : *a vector is not completely specified by giving only its curl but if both the curl and the divergence of a vector are specified the vector is uniquely determined.* Clearly it is to our advantage to make a choice for div $\mathbf{A}$ in any convenient manner that will provide a simplification for the particular problem under consideration. Generally div $\mathbf{A}$ is chosen in two ways (described in following articles) according as the field contains charge or not.

§ 4.10. Lorentz Gauge :

The Maxwell's field equations in terms of electromagnetic potentials are

$$\nabla^2\mathbf{A} - \mu\varepsilon\frac{\partial^2 \mathbf{A}}{\partial t^2} - \text{grad}\left(\text{div}\,\mathbf{A} + \mu\varepsilon\frac{\partial \phi}{\partial t}\right) = -\mu\mathbf{J} \qquad ...(1)$$

$$\nabla^2\phi - \mu\varepsilon\frac{\partial^2 \phi}{\partial t^2} + \frac{\partial}{\partial t}\left(\text{div}\,\mathbf{A} + \mu\varepsilon\frac{\partial \phi}{\partial t}\right) = -\frac{\rho}{\varepsilon} \qquad ...(2)$$

A casual glance at equations (1) and (2) reveals that these equations will be much more simplified (*i.e.* will become identical and uncoupled) if we choose

$$div\,\mathbf{A} + \mu\varepsilon \frac{\partial \phi}{\partial t} = 0. \quad ...(3)$$

This requirement is called the Lorentz condition and when the vector and scalar potential satisfy it, the gauge is known as Lorentz gauge.

So with Lorentz condition field equation reduce to

$$\nabla^2 \mathbf{A} - \mu\varepsilon \frac{\partial^2 \phi}{\partial t^2} = -\mu \mathbf{J} \quad ...(4)$$

and

$$\nabla^2 \phi - \mu\varepsilon \frac{\partial^2 \phi}{\partial t^2} = -\rho/\varepsilon. \quad ...(5)$$

But as $\mu\varepsilon = 1/\upsilon^2$ *.

So equations (4) and (5) can be written as

$$\Box^2 \mathbf{A} = -\mu \mathbf{J} \quad ...(6)$$

$$\Box^2 \phi = -\rho/\varepsilon \quad ...(7)$$

with

$$\Box^2 = \nabla^2 - \frac{1}{v^2} \frac{\partial^2}{\partial t^2}.$$

Equations (6) and (7) are inhomogeneous wave equations and are known as **D' Alembertian** equations and can be solved in general by a method similar to that we use to solve Poisson's equation. The potentials obtained by solving these equations are called retarted potentials and are discussed in § 8.1.

In order to determine the requirement that Lorentz condition places on Λ, we substitute **A'** and ϕ' from equations (3) and (4) of § 4.8 in eqn. (3) *i.e.*

$$\text{div}(\mathbf{A}' - \text{grad}\,\Lambda) + \mu\varepsilon \frac{\partial}{\partial t}\left(\phi' + \frac{\partial \Lambda}{\partial t}\right) = 0$$

i.e.

$$\text{div}\,\mathbf{A}' + \mu\varepsilon \frac{\partial \phi'}{\partial t} = \nabla^2 \Lambda - \varepsilon\mu \frac{\partial^2 \Lambda}{\partial t^2}$$

So A' and ϕ' will also satisfy equation (3) *i.e.* Lorentz condition provided that

$$\nabla^2 \Lambda - \varepsilon\mu \frac{\partial^2 \Lambda}{\partial t^2} = 0$$

i.e.

$$\Box^2 \Lambda = 0. \quad ...(8)$$

i.e., Lorentz condition is invariant under those gauge transformations for which the gauge functions are solutions of the homogeneous wave equations.

* See § 5.1 and 5.2 for details.

The advantages of this particular gauge are :

(i) It makes the equations for **A** and ϕ independent of each other.

(ii) It leads to the wave equations which treat ϕ and **A** on equivalent footings.

(iii) It is a concept which is independent of the co-ordinate system chosen and so fits naturally into the considerations of special theory of relativity.

Example 14. *Show that a potential called Hertz potential* $\vec{\pi}$ *defined by*

$$\phi = -\text{div}\,\vec{\pi}$$

And

$$\mathbf{A} = \frac{1}{c^2} \frac{\partial \vec{\pi}}{\partial t}$$

automatically satisfies the Lorentz condition. Obtain also **E** *and* **B** *in terms of* $\vec{\pi}$.

Solution. We know that Lorentz condition is

$$\text{div}\,\mathbf{A} + \mu\varepsilon \frac{\partial \phi}{\partial t} = 0.$$

As for free space

$$\mu\varepsilon = \mu_0\varepsilon_0 = 1/c^2$$

the Lorentz condition in free space is

$$\text{div}\,\mathbf{A} + \frac{1}{c^2}\frac{\partial \phi}{\partial t} = 0$$

Substituting the given values of **A** and ϕ in **L.H.S.**

$$\text{div}\,\mathbf{A} + \frac{1}{c^2}\frac{\partial \phi}{\partial t} = \text{div}\left(\frac{1}{c^2}\frac{\partial \vec{\pi}}{\partial t}\right) + \frac{1}{c^2}\frac{\partial}{\partial t}(-\text{div}\,\vec{\pi})$$

i.e.

$$= \text{div}\left(\frac{1}{c^2}\frac{\partial \vec{\pi}}{\partial t}\right) - \text{div}\left(\frac{1}{c^2}\frac{\partial \vec{\pi}}{\partial t}\right) = 0$$

i.e. the electromagnetic potentials given by

$$\phi = -\text{div}\,\vec{\pi} \quad \text{and} \quad \mathbf{A} = \frac{1}{c^2}\frac{\partial \vec{\pi}}{\partial t}$$

satisfy the Lorentz condition.

Further

$$\mathbf{B} = \text{curl}\,\mathbf{A} = \text{curl}\left(\frac{1}{c^2}\frac{\partial \vec{\pi}}{\partial t}\right) = \frac{1}{c^2}\text{curl}\left(\frac{\partial \vec{\pi}}{\partial t}\right) \quad ...(1)$$

$$\text{and} \qquad \mathbf{E} = -\text{grad}\,\phi - \frac{\partial \mathbf{A}}{\partial t} = \text{grad div}\,\vec{\pi} - \frac{1}{c^2}\frac{\partial^2 \vec{\pi}}{\partial t^2}$$

$$\text{i.e.} \qquad \mathbf{E} = \text{grad div}\,\vec{\pi} - \nabla^2 \vec{\pi} \qquad \left(\text{As } \nabla^2 \vec{\pi} - \frac{1}{c^2}\frac{\partial^2 \vec{\pi}}{\partial^2 t} = 0\right)$$

$$\text{i.e.} \qquad \mathbf{E} = \text{curl curl}\,\vec{\pi}. \qquad \text{...(2)}$$

So equation (1) and (2) can be written as

$$\mathbf{B} = (1/c^2)(\partial \mathbf{G}/\partial t) \text{ and } \mathbf{E} = \text{curl}\,\mathbf{G} \text{ with } \mathbf{G} = \text{curl}\,\vec{\pi}.$$

§ 4.11. Coulomb Gauge :

An inspection of field equations in terms of electromagnetic potentials *i.e.*

$$\nabla^2 \mathbf{A} - \mu\varepsilon \frac{\partial^2 \mathbf{A}}{\partial t^2} - \text{grad}\left(\text{div}\,\mathbf{A} + \mu\varepsilon\frac{\partial \phi}{\partial t}\right) = -\mu \mathbf{J}. \qquad \text{...(1)}$$

$$\text{And} \qquad \nabla^2 \phi - \mu\varepsilon \frac{\partial^2 \phi}{\partial t^2} + \frac{\partial}{\partial t}\left(\text{div}\,\mathbf{A} + \mu\varepsilon\frac{\partial \phi}{\partial t}\right) = -\frac{\rho}{\varepsilon}$$

$$\text{i.e.} \qquad \nabla^2 \phi + \frac{\partial}{\partial t}(\text{div}\,\mathbf{A}) = -\frac{\rho}{\varepsilon} \qquad \text{...(2)}$$

shows that if we assume

$$\text{div}\,\mathbf{A} = 0. \qquad \text{...(3)}$$

equation (2) reduces to Poisson's equation

$$\nabla^2 \phi_{(r,t)} = -\frac{\rho(r',t)}{\varepsilon} \qquad \text{...(4)}$$

whose solution is

$$\phi_{(r,t)} = \frac{1}{4\pi\varepsilon}\int \frac{\rho(r',t)^*}{R}\,d\tau' \qquad \text{...(5)}$$

i.e. the scalar potential is just the instantaneous Coulombian potential due to charge $\rho(x', y', z', t)$. This is the origin of the name Coulomb gauge.

Equation (1) in the light of (3) reduced to

$$\nabla^2 \mathbf{A} - \frac{1}{v^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} = -\mu \mathbf{J} + \mu\varepsilon \nabla\left(\frac{\partial \phi}{\partial t}\right) \qquad \text{...(6)}$$

Now to express equation (6) in more convenient way we use Poisson's equation (4) which with the help of (5) can be written as

$$\nabla^2 \left\{\frac{1}{4\pi\varepsilon}\int \frac{\rho(r',t)}{R}\,d\tau'\right\} = -\frac{\rho(r',t)}{\varepsilon} \qquad \text{...(7)}$$

* For the solution of Poisson's equation see Appendix–II.

Now as Poisson's equation holds good for scalar and vectors both, replacing $\rho(r', t)$ by $\mathbf{J}'$ we get

$$\nabla^2 \left\{\frac{1}{4\pi\varepsilon}\int \frac{\mathbf{J}'}{R}\,d\tau'\right\} = -\frac{\mathbf{J}'}{\varepsilon} \qquad \text{...(8)}$$

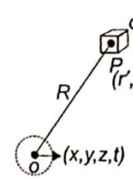

Fig. 4.14

Now from the vector identity

$$\nabla \times \nabla \times \mathbf{G} = \nabla(\nabla \cdot \mathbf{G}) - \nabla^2 \mathbf{G}$$

$$\nabla^2 \mathbf{G} = \nabla(\nabla \cdot \mathbf{G}) - \nabla \times \nabla \times \mathbf{G}$$

Taking $\mathbf{G} = \int (\mathbf{J}'/R)\,d\tau'$, we get

$$\nabla^2 \int \frac{\mathbf{J}'}{R}\,d\tau' = \nabla\left(\nabla \cdot \int \frac{\mathbf{J}'}{R}\,d\tau'\right) - \nabla \times \nabla \times \int \frac{\mathbf{J}'}{R}\,d\tau'$$

which in the light of equation (8) reduces to

$$-4\pi\,\mathbf{J}' = \nabla\left(\nabla \cdot \int \frac{\mathbf{J}'}{R}\,d\tau'\right) - \nabla \times \nabla \times \int \frac{\mathbf{J}'}{R}\,d\tau'$$

$$\text{i.e.} \qquad \mathbf{J}' = -\frac{1}{4\pi}\nabla\left[\nabla \cdot \int \frac{\mathbf{J}'}{R}\,d\tau'\right] + \frac{1}{4\pi}\nabla \times \nabla \times \int \frac{\mathbf{J}'}{R}\,d\tau' \qquad \text{...(9)}$$

Now as $\nabla \cdot \int (\mathbf{J}'/R)\,d\tau'$

$$= \int \left[\frac{1}{R}\nabla \cdot \mathbf{J}' + \mathbf{J}' \cdot \nabla\left(\frac{1}{R}\right)\right] \quad [\text{as } \nabla(S\mathbf{V}) = S\nabla \cdot \mathbf{V} + \mathbf{V} \cdot \Delta S]$$

$$= \int \mathbf{J}' \cdot \nabla(1/R)\,d\tau' \quad [\text{as } \mathbf{J}' \text{ is not a function of } x, y \text{ and } z]$$

$$= -\int \mathbf{J}' \cdot \nabla'(1/R)\,d\tau' \quad [\text{as } \nabla(1/R) = -\nabla'(1/R)]$$

$$= \int \left[\frac{\nabla' \cdot \mathbf{J}'}{R} - \nabla' \cdot \left(\frac{\mathbf{J}'}{R}\right)\right] d\tau'$$

$$\left[\text{as } \nabla' \cdot \left(\frac{\mathbf{J}'}{R}\right) = \left(\frac{1}{R}\right)\nabla' \cdot \mathbf{J}' + \mathbf{J}' \cdot \nabla'\left(\frac{1}{R}\right)\right]$$

$$= \int \frac{\nabla' \cdot \mathbf{J}'}{R}\,d\tau' - \oint_S \left(\frac{\mathbf{J}'}{R}\right) \cdot d\mathbf{s}$$

$$\left[as \int \nabla'\left(\frac{\mathbf{J}'}{R}\right) d\tau' = \oint_S \frac{\mathbf{J}'}{R} \cdot d\mathbf{s}\right]$$

As $\mathbf{J}'$ is confined to the vol τ', the surface contribution will vanish so

$$\nabla' \cdot \int \left(\frac{\mathbf{J}'}{R}\right) d\tau' = \int \frac{\nabla' \cdot \mathbf{J}'}{R}\,d\tau' \qquad \text{...(10)}$$

And $\nabla \times \int (\mathbf{J}'/R)\,d\tau'$

$$=\int\left[\frac{\nabla\times\mathbf{J}'}{R}-\mathbf{J}'\times\nabla\left(\frac{1}{R}\right)\right]d\tau'$$

[as curl $S\mathbf{V} = S$ curl $\mathbf{V} - \mathbf{V} \times$ grad S]

$$=-\int\left[\mathbf{J}'\times\nabla'(1/R)\,d\tau'\right]$$

[as $\mathbf{J}'$ is not a function of x, y and z]
[as $\nabla(1/R) = \nabla'(1/R)$]

$$=\int \mathbf{J}'\times\nabla'(1/R)\,d\tau'$$

$$=\int\left[\frac{\nabla'\times\mathbf{J}'}{R}-\nabla'\times\left(\frac{\mathbf{J}'}{R}\right)\right]d\tau'$$

$$\left[\text{as } \nabla'\times\left(\frac{\mathbf{J}'}{R}\right)=\frac{1}{R}\nabla'\times\mathbf{J}'-\mathbf{J}'\times\nabla'\left(\frac{1}{R}\right)\right]$$

$$=\int\frac{\nabla'\times\mathbf{J}'}{R}d\tau'+\oint\frac{\mathbf{J}'}{R}\times d\mathbf{s}$$

$$\left[\text{As } \int\nabla\times\mathbf{V}\,d\tau'=-\oint\mathbf{V}\times d\mathbf{s}^{*}\right]$$

As $\mathbf{J}'$ is confined to vol τ', surface contribution will vanish so

$$\nabla\times\int\left(\frac{\mathbf{J}'}{R}\right)d\tau'=\int\frac{\nabla'\times\mathbf{J}'}{R}d\tau' \quad ...(11)$$

So eqn. (9) becomes

$$\mathbf{J}'=-\frac{1}{4\pi}\nabla\int\frac{\nabla'\cdot\mathbf{J}'}{R}d\tau'+\frac{1}{4\pi}\nabla\times\int\frac{\nabla'\times\mathbf{J}'}{R}d\tau'$$

i.e. $$\mathbf{J}'=\mathbf{J}'_l+\mathbf{J}'_T \quad ...(12)$$

with $$\mathbf{J}'_l=-\frac{1}{4\pi}\nabla\int\frac{\nabla'\cdot\mathbf{J}'}{R}d\tau' \text{ and } \mathbf{J}'_T=\frac{1}{4\pi}\nabla\times\int\frac{\nabla'\times\mathbf{J}'}{R}d\tau' \quad ...(13)$$

Now as

$$\nabla\times\mathbf{J}'_l=\nabla\times\left[-\frac{1}{4\pi}\nabla\int\frac{\nabla'\cdot\mathbf{J}'}{R}d\tau'\right]$$

i.e. $$\nabla'\times\mathbf{J}'_l=0 \quad \text{(as curl grad}\,\phi=0) \quad ...(14)$$

and $$\nabla\cdot\mathbf{J}'_T=\nabla\cdot\left[\nabla\times\int\frac{\nabla'\times\mathbf{J}'}{R}d\tau'\right]$$

i.e. $$\nabla\cdot\mathbf{J}'_T=0 \quad \text{(as div curl }\mathbf{V}=0) \quad ...(15)$$

the first term on **R.H.S.** of equation (12) is irrotational and second is solenoidal. The first term is called longitudinal current and the other transverse current.

Note. From equations (12) and (13) it is clear that any well behaved vector function which vanishes at infinity can be expressed as the sum of an irrotational part and a solenoidal part. Further more we have also proved by (12) that a vector is completely specified if its divergence and curl are given at all points of space.

* See note in § 3.7.

So in the light of equation (12), (6) can be written as

$$\nabla^2\mathbf{A}-\frac{1}{v^2}\frac{\partial^2\mathbf{A}}{\partial t^2}=-\mu\left(\mathbf{J}_T+\mathbf{J}_l\right)+\mu\varepsilon\nabla\left(\frac{\partial\phi}{\partial t}\right)$$

i.e. $$\nabla^2\mathbf{A}-\frac{1}{v^2}\frac{\partial^2\mathbf{A}}{\partial t^2}=-\mu\mathbf{J}_T-\mu\mathbf{J}_l+\mu\varepsilon\nabla\frac{\partial}{\partial t}\left[\frac{1}{4\pi\varepsilon}\int\frac{\rho_{(r',t)}}{R}d\tau'\right]$$

[substituting ϕ from equation (5)]

i.e. $$\nabla^2\mathbf{A}-\frac{1}{v^2}\frac{\partial^2\mathbf{A}}{\partial t^2}=-\mu\mathbf{J}_T-\mu\mathbf{J}_l+\mu\frac{1}{4\pi}\nabla\int\frac{-\nabla'\cdot\mathbf{J}}{R}d\tau'$$

$$\left[\text{as from continuity eqn. } \frac{\partial\rho_{(r',t)}}{\partial t}=-\nabla'\cdot\mathbf{J}\right]$$

or $$\nabla^2\mathbf{A}-\frac{1}{v^2}\frac{\partial^2\mathbf{A}}{\partial t^2}=-\mu\mathbf{J}_T-\mu\mathbf{J}_l+\mu\mathbf{J}_l \quad \text{[From equation (13)]}$$

i.e. $$\nabla^2\mathbf{A}-\frac{1}{v^2}\frac{\partial^2\mathbf{A}}{\partial t^2}=-\mu\mathbf{J}_T$$

i.e. $$\Box^2\mathbf{A}=-\mu\mathbf{J}_T. \quad ...(16)$$

i.e. the equation for **A** can be expressed entirely in terms of the transverse current. So this gauge sometimes is also called transverse gauge.

The Coulomb gauge has a certain advantage. In it the scalar potential is exactly the electrostatic potential (equation 5) and electric field

$$\mathbf{E}=-\text{grad}\,\phi-\frac{\partial\mathbf{A}}{\partial t}$$

is separable into an electrostatic field $V=\phi$ and a wave field given by $-(\partial\mathbf{A}/\partial t)$.

This gauge is often used when no source are present. Then according to equation (5) $\phi=0$ and **A** satisfies the homogeneous wave equation (16). The fields are given by

$$\mathbf{E}=-\frac{\partial\mathbf{A}}{\partial t} \text{ and } \mathbf{B}=\nabla\times\mathbf{A}.$$

Example 15. *In a source free region if*

$$\mathbf{A}=\mathbf{i}x^4+z^2t^2\mathbf{k}$$

compute field vectors **E** *and* **B** *and transverse current* $\mathbf{J}_T$.

Solution. As in a source free region $\rho=0$.

So $$\phi=\frac{1}{4\pi\varepsilon_0}\int\frac{\rho_{(r',t)}}{R}d\tau=0$$