THE WKB APPROXIMATION.

This is an approximate method of solution of ordinary differential equations of the type $\frac{d^2u}{dx^2} + Q(x)u = 0$. Though it has been known for a long time as a general mathematical method, and has in particular been studied extensively by Jeffreys, it is named after Wentzel, Kramers and Brillouin who pioneered in use in quantum mechanics. In this context, the basis of the method is an expansion of the wave function in powers of $\hbar$, and its utility is expected to be highest in situations where $\hbar$ can be considered small. It is therefore after reffered to as the "semi-classical approximation". A remarkable result is that in this approximation.

THE ONE-DIMENSIONAL SCHRODINGER'S EQUATIONS:

The solution of this equation in the WKB approximation proceeds through three main stage.

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(i) We obtain an approximate solution through a series expansion in powers of 'ħ' but recognize that the approximation breaks down near classical turning point.

(ii) The Derivation of a solution which holds near turning points.

(iii) The application of boundary conditions, including matching of the expressions valid in the different regions.

THE ASYMPTOTIC SOLUTION:

Consider Schrodinger's equation.

$$\frac{d^2u}{dx^2} + \frac{2m}{\hbar^2}(E-V)u = 0$$

The solution of the above equation.

$$A \exp[i(2m(E-V))^{1/2} x/\hbar]$$

V = constant.

$$u(x) = A(x)\, e^{iS(x)/\hbar}$$

where A and S are real function. simplyfies the equation we get the result when $(E-V)$

is negative.

$$K(x) \rightarrow i\varkappa(x), \quad \varkappa = +[2m(V-E)/\hbar^2]^{1/2}$$

The solution is therefore an asymptotic solution in the sense that it is good far from classical turning points.

SOLUTION NEAR A TURNING POINT:

Consider a particular turning point occurring for a given energy E. We can take the origin of the x-co-ordinates at this point, without loss of generality so that $(E-V)=0$ at $x=0$. a Taylor expansion of $(E-V)$

$$K^2(x) = fx^n(1+\alpha x+\beta x^2+\cdots) \quad f>0$$

if $x>0$ $\quad u(x) = c \pm \xi^{1/2} K^{-1/2} J_{\pm m}(\xi)$,

$$m = \frac{1}{n+2}, \quad \xi = \int_0^x K\,dx.$$

$J_m(x) \Rightarrow$ Bessel function of order m.

The modified equation.

$$\frac{d^2u}{dx^2} + (k^2 - \chi)\,u = 0$$

$$\chi = \frac{3k'^2}{4k^2} - \frac{k''}{2k} + (m^2 - 1/4)\frac{k^2}{\xi^2}$$

The asymptotic form of either of these solutions goes over directly into a linear combination of the asymptotic solutions. This fact simplifies the "matching" problem immensely.

MATCHING AT A LINEAR TURNING POINT.

Let us now consider the most commonly occuring situation, of a linear turning point. ie one near which the variation of $(E-V)$ or k^2 with x is linear.

For $x>0$ $(E>V)$;

$$\xi_> = \xi = \int_0^x k\,dx$$

$x<0$ $(E<V)$: $k = i\kappa$; $\xi_< = i\xi = \int_x^0 \kappa\,dx$

By forming suitable linear combinations of the independent solution U_+ and U_- .. In this combination $(U_+ + U_-)$ which decreases exponentially on going into the interior of the classically forbidden region $(x \to -\alpha)$.

Similarly by considering the combination $\sin\frac{1}{6}\pi \cos\eta(u_+ + u_-) - \cos\frac{1}{6}\pi \sin\eta(u_+ - u_-)$

The connection formula.

$$\sin\eta \, x^{-1/2} \exp(\xi_{<}) \div k^{-1/2} \cos(\xi_{>} - \frac{1}{4}\pi + \eta)$$

This is valid provided $\sin\eta$ is not so small as to reduce the left hand side to the same order of magnitude as the neglected $\exp(-\xi_{<})$ term.