# Lecture VI

Numerical methods: Euler's method, improved Euler's method

In most real situations, it is impossible to find analytical solution to the IVP

$$\frac{dy}{dx} = f(x,y), \quad y(x_0) = y_0 \tag{1}$$

Thus, we need to solve (1) numerically. Following two methods, described below, are the simplest of the numerous numerical methods that are used to solve (1). They can be derived different in many ways.

Note the (1) is equivalent to

$$y(x) = y_0 + \int_{x_0}^{x} f(t, y(t))\, dt \tag{2}$$

Suppose $x_0 = a$ and we need to find the solution at $x = b > a$. We divide the interval $[a = x_0, b]$ in to $N$ intervals of size $h = (b-a)/N$. Thus, the uniform mesh points are $a = x_0 < x_1 < x_2 < \cdots < x_N = b$.

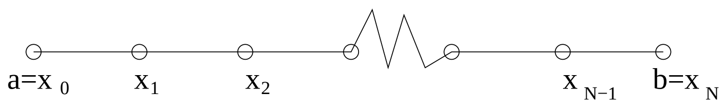


Figure 1: Mesh points

Consider arbitrary interval $[x_n, x_{n+1}]$. Using (2), we find

$$y(x_{n+1}) = y(x_n) + \int_{x_n}^{x_{n+1}} f(t, y(t))\, dt. \tag{3}$$

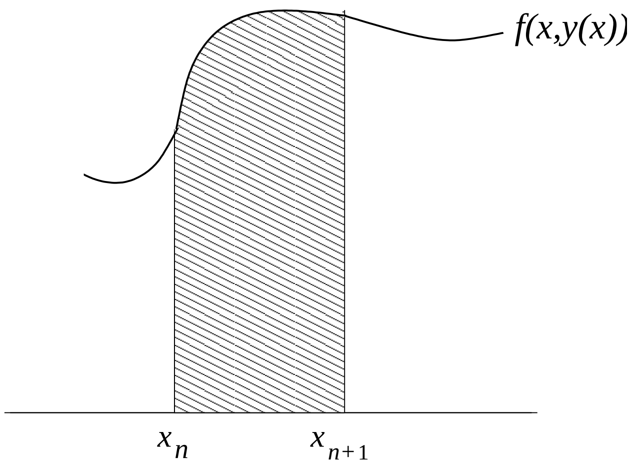


Figure 2:

The integral on the RHS is the shaded area shown in Figure 2. But since $y(t)$ is unknown, we have to approximate the integral in the RHS of (3).

Now from the mean value theorem of integral calculus, we write (3) as

$$y(x_{n+1}) = y(x_n) + (x_{n+1} - x_n) f(\zeta, y(\zeta)), \quad \zeta \in (x_n, x_{n+1})$$

or equivalently

$$y(x_{n+1}) = y(x_n) + h f(\zeta, y(\zeta)), \quad \zeta \in (x_n, x_{n+1}) \tag{4}$$

Now $f(\zeta, y(\zeta))$ is the slope of the solution curve $y = y(x)$ at the unknown point $x = \zeta$. Further $y$ is also itself unknown. We need to approximate the slope using numerical methods.

# 1 Euler's method

We assume that the slope $f$ varies so little in $[x_n, x_{n+1}]$ that we can approximate it by its value at $x = x_n$, i.e. $f(\zeta, y(\zeta)) \approx f(x_n, y(x_n))$. Hence,

$$y(x_{n+1}) \approx y(x_n) + h f(x_n, y(x_n))$$

But the exact value $y(x_n)$ is not known except $y(x_0) = y_0$. Hence, if we assume $y(x_n) \approx y_n$, then $f(x_n, y(x_n)) \approx f(x_n, y_n)$. Finally the Euler's method becomes

$$y_{n+1} = y_n + h f(x_n, y_n), \qquad n = 0, 1, 2, N - 1. \tag{5}$$

**Geometrical interpretation**

The curve between $x_0$ and $x_1$ is approximated by the straight line that passes through $(x_0, y_0)$ having slope $f(x_0, y_0)$. Similarly, the curve between $x_1$ and $x_2$ is approximated by the straight line that passes through $(x_1, y_1)$ having slope $f(x_1, y_1)$. Obviously, at each step we are making error (shown by the red segment in Figure 3). The error between the exact and computed solution at the $n$-th step is $|y(x_n) - y_n|$.

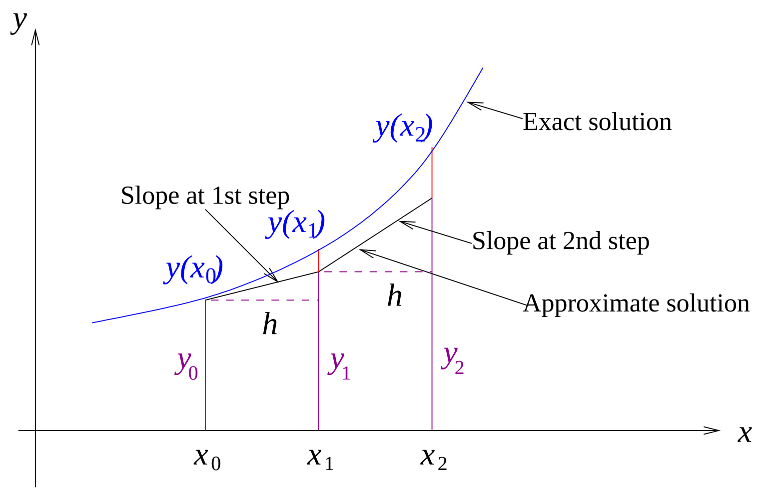


Figure 3: Euler's method

## 2 Improved Euler's method

A better approximate for $f(\zeta, y(\zeta))$ in (4) is the average of the slopes at $x_n$ and $x_{n+1}$, i.e.

$$f(\zeta, y(\zeta)) \approx \frac{f(x_n, y(x_n)) + f(x_{n+1}, y(x_{n+1}))}{2}.$$

Hence, we find

$$y(x_{n+1}) \approx y(x_n) + \frac{h}{2}\left[f(x_n, y(x_n)) + f(x_{n+1}, y(x_{n+1}))\right] \tag{6}$$

As before $y(x_n) \approx y_n$. Now the unknown $y(x_{n+1})$ also occurs in the RHS of (6). This $y(x_{n+1})$ in the RHS of (6) is approximated using the value from the Euler's method. Thus, we substitute $y(x_{n+1}) \approx y_n + hf(x_n, y_n) = y^*_{n+1}$ in the RHS of (6). Hence, finally, the improved Euler's method becomes

$$\left.\begin{aligned} y^*_{n+1} &= y_n + hf(x_n, y_n) \\ y_{n+1} &= y_n + \tfrac{h}{2}\Big[f(x_n, y_n) + f(x_{n+1}, y^*_{n+1})\Big] \end{aligned}\right\}, \ n = 0, 1, 2, N-1. \tag{7}$$

**Geometrical interpretation**

First the Euler's method is used to predict the value $y^*_1$ at $x = x_1$. Slope at $x = x_1$ is calculated from $f(x_1, y^*_1)$. Then we obtained the average slope from the slopes at $x = x_0$ and at $x = x_1$. Now the approximate solution is the straight line that passes throgh $(x_0, y_0)$ with slope equal to the average value of the slopes. This process is repeated for the next step $[x_1, x_2]$ and so on.

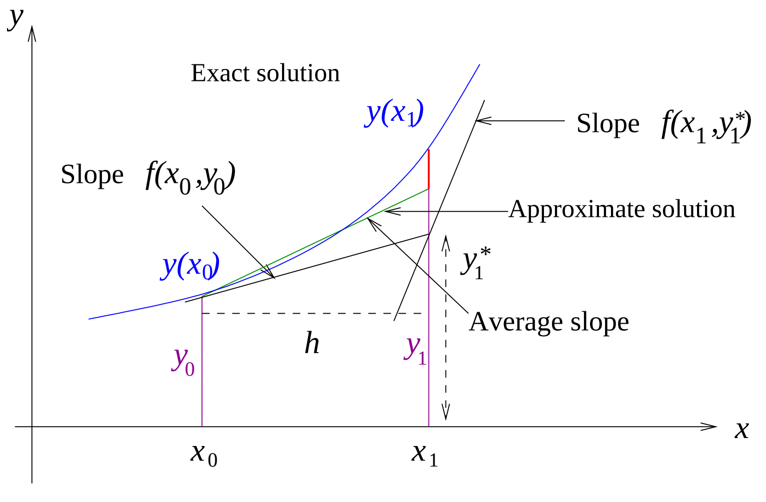


Figure 4: Improved Euler's method

**Comment 1**: Improved Euler's method has many different names such as Modified Euler's method, Heun method and Runge-Kutta method of order 2. It also belongs to the category of predictor-corrector method.

**Comment 2**: It can be proved that the accuracy of Euler's method is proportional to $h$ and that of improved Euler's method to $h^2$, where $h$ is the step size. Hence, Improved Euler's method has better accuracy than that of Euler's method.

**Example 1.** *Apply (i) Euler's method and (ii) improved Euler's method to compute $y(x)$ at $x = 0.4$ with stepsize $h = 0.2$ for the initial value problem:*

$$y' = 2x(1 - y),\ y(0) = 2.$$

*Compare the errors $e_n = |y(x_n) - y_n|$ in each step with the exact solution $y(x) = 1 + e^{-x^2}$.*

**Solution**: The values are rounded to three places of decimal in the Tables.

| $n$ | $x_n$ | $y_n$ | $y(x_n)$ | $e_n$ | $f(x_n, y_n)$ |
|---|---|---|---|---|---|
| 0 | 0 | 2 | 2 | 0 | 0 |
| 1 | 0.2 | 2 | 1.961 | 0.039 | -0.4 |
| 2 | 0.4 | 1.92 | 1.852 | 0.068 | |

Table 1: Euler's method

| $n$ | $x_n$ | $y_n$ | $y(x_n)$ | $e_n$ | $f(x_n, y_n)$ | $y^*_{n+1}$ | $f(x_{n+1}, y^*_{n+1})$ |
|---|---|---|---|---|---|---|---|
| 0 | 0 | 2 | 2 | 0 | 0 | 2 | $-0.4$ |
| 1 | 0.2 | 1.96 | 1.961 | 0.001 | $-0.384$ | 1.883 | $-0.706$ |
| 2 | 0.4 | 1.845 | 1.852 | 0.007 | | | |

Table 2: Improved Euler's method

It is clear that the improved Euler's method has better accuracy than the Euler's method which is expected. Also, decreasing the step size $h$ improves the accuracy of both of the methods.